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Conversion Witness for OxO Invariant States

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Basic aim

- To discuss about the convertibility of two bipartite quantum states.
- Usually it is done by some entanglement monotone.
- However in most of the cases entanglement monotones are not computable and also not fully informative.
- Concept of conversion witness is developed (Gour, et. al., NJP, 2015) that includes entanglement monotones.
- We provide the form of conversion witness for $O \times O$ invariant states that includes other available results of some class of symmetric states.

Entanglement: A novel resource

- This is possibly the most wonderful invention of quantum mechanics. Initially everyone thinks the correlation which is responsible for non-local behavior of quantum systems is nothing but the entanglement.
- However, findings in different quantum systems show there are other candidates also. e.g., the local-indistinguishibility of a complete set of orthonormal product states in 3×3 system.
- Entanglement is used as a resource in many information processing and computational tasks, (e.g, teleportation, dense coding, quantum cryptography, etc.). Therefore, the characterization and quantification problems are of the some fundamental issues generated in the last two decades. However, there are lot of difficulties.

Entanglement

Definition

A bipartite quantum state, represented by density matrix ρ, is called *separable* if and only if it can be represented as a convex combination of the product of projectors on local states, i.e., if

$$\rho = \sum_{i=1}^{k} p_i \rho_i^1 \otimes \rho_i^2, \sum_i p_i = 1, p_i \geq 0$$

The states, which can not be written in the above forms, are called entangled states.

LOCC operations

- Physical operations are represented by completely positive maps and they have the Krause representation of the form,

$$\mathcal{E}(\rho) = \sum A_k \rho A_k^\dagger,$$

where all A_k are linear operators, satisfy the relation $\sum A_k^\dagger A_k \leq I$.

- On composite quantum sysytems if the operators A_k have the decompostion $L_k^A \otimes L_k^B \otimes L_k^C \ldots$, we call them as separabale superoperators.
- A particular type of separable superoperators are the class of local operations alongwith classical communications (LOCC), where individual subsystems are allowed to do quantum operations locally and they can classically communicate their results to others.

PPT operations

- A larger class of operations that include separable superoperators are the class of PPT (positive under partial transposition) operations.
- Suppose C_Γ is the class of PPT operations. It is the set of completely positive maps $\mathcal{E}$ such that the partially transposed map $\mathcal{E}^\Gamma$ defined by

$$\mathcal{E}^\Gamma(\rho) = [\mathcal{E}(\rho^\Gamma)]^\Gamma,$$

  is also completely positive.

Entanglement Monotones

Entanglement is basically characterized by entanglement monotones. They quantify the resourcefulness of states.

Definition

An entanglement monotone is a real valued function f on quantum states that does not increase under LOCC operation. Mathematically,

$$\rho \longrightarrow \sigma \text{ under LOCC}$$

$$\Rightarrow f(\rho) \geq f(\sigma)$$

LOCC operation induces a partial order structure on state space. Finding useful entanglement monotones enables the description of possible state transformation. However most monotones are hard to calculate and they do not provide enough information for state transformation.

Entanglement Monotones

A family of monotones f_i is said to be complete if $\rho \longrightarrow \sigma$ under LOCC iff $f_i(\rho) \geq f_i(\sigma)$ for all $i \in \mathcal{I}$. $\mathcal{I}$ is the index set.

Note:

- For bipartite pure states a computable and complete family of monotone(constructed from Schmidt coefficients) exists.
- Negativity is the only known computable monotone for arbitrary states.
- Concurrence is a computable monotone for two-qubit pure states but fails to detect complete convertibility in two-qubit mixed states.
- Computable monotone for arbitrate states in general does not exist.

Since, LOCC$\subset$ PPT, PPT monotones are also LOCC monotone and PPT monotones has good mathematical structure.

Conversion Witness

Definition

Let W be a real valued function on pair of quantum states. If $W(\rho,\sigma) \geq 0 \Rightarrow \rho \longrightarrow \sigma$ under LOCC then W is said to be a go witness. If $W(\rho,\sigma) < 0 \Rightarrow \rho \nrightarrow \sigma$, then W is said to be a no-go witness. W is said to be a complete witness if it is both a go and no-go witness.

The main goal of no-go witness is to determine if a pair (ρ,σ) is non-convertible. Let us consider the pair of states which are LOCC non-convertible

$$N = \{(\rho,\sigma) : \rho \nrightarrow \sigma\}$$

Also consider the set where the convertibility is detected by a no-go witness W i.e.

$$D_W = \{(\rho,\sigma) : W(\rho,\sigma) < 0\}$$

clearly,

$$D_W \subseteq N$$

Conversion Witness

Now consider two no-go witnesses W_1 and W_2. Their sets of detected pairs must be subsets of N, i.e. $D_{W_1} \subseteq N$ and $D_{W_2} \subseteq N$. Suppose that $D_{W_1} \subseteq D_{W_2}$. Then all of the pairs detected by W_2 are already detected by W_1, but W_1 might detect more pairs than W_2 does. In this case, we might as well use only W_1, since W_1 tells us all of the information given by W_2, and also more! So we write $W_1 \succeq W_2$, and this is indeed a partial order!!

Partial Order

Explicitly, given two no-go witness W_1, W_2, we say

$$W_1 \succeq W_2 \text{ if } W_2(\rho, \sigma) < 0 \Rightarrow W_1(\rho, \sigma) < 0$$

. The relation is a partial order relation.

Conversion Witness

Few Notes:

1. Any complete witness is stronger than any go witness or no-go witness
2. Given two witness, they may be incomparable
3. Given a monotone f, a no-go witness can be defined as $W_f(\rho, \sigma) = f(\rho) - f(\sigma)$. Hence entanglement monotones are special types of conversion witness.
4. Similar partial order can be defined for go witness as well.

Computable no-go Conversion Witness

Necessary ideas for the construction of conversion witness

1. Negativity($:=Tr[\rho^{\Gamma}-]$) is a PPT monotone, i.e., $\rho \longrightarrow \sigma$ under PPT $\Rightarrow N(\rho) \geq N(\sigma)$. Generally, for a PPT operation $\mathcal{E} \in \mathcal{C}_\Gamma$, it can be shown $\mathcal{E}(\rho)^{\Gamma}- \leq \mathcal{E}^{\Gamma}(\rho^{\Gamma}-)$. We have used the symbol $\rho^{\Gamma}\pm = (\rho^{\Gamma})_{\pm}$ for positive and negative part of ρ.
2. Support function of a subset C is defined as

$$h_C(\rho) := \sup_{\gamma \in C} Tr[\gamma\rho]$$

3. Consider the sets

$$\mathcal{N}_c = \{\gamma : N(\gamma) \leq c\} \, \text{with} c \geq 0$$
$$\mathcal{C}(\rho) = \{\mathcal{E}(\rho) : \mathcal{E} \in \mathcal{C}\}$$

where $\mathcal{C}$ is a class of CPTP operations and $\mathcal{C}(\rho)$ is the orbit of states that are reachable from ρ via the operations from $\mathcal{C}$, we then have for PPT operation

$$C_\Gamma(\rho) \subset \mathcal{N}_{N(\rho)}$$

forall normalized states ρ.

Computable no-go Conversion Witness

The above result enables us to get for any quantum state τ,

$$Tr[\tau\sigma^{\Gamma}{}_{-}] \leq Tr[\tau\mathcal{E}^{\Gamma}(\rho^{\Gamma}{}_{-})] \leq h_{C_{\Gamma}(\rho^{\Gamma}{}_{-})}(\tau)$$

No-go conversion witness for PPT operation is defined as

$$\widehat{W}_{\tau}(\rho,\sigma) := h_{C_{\Gamma}(\rho^{\Gamma}{}_{-})}(\tau) - Tr[\tau\sigma^{\Gamma}{}_{-}]$$

using normalization $\tilde{\rho} = \frac{\rho^{\Gamma}{}_{-}}{N(\rho)}$, the witness reads

$$\widehat{W}_{\tau}(\rho,\sigma) := N(\rho) h_{C_{\Gamma}(\tilde{\rho})}(\tau) - Tr[\tau\sigma^{\Gamma}{}_{-}]$$

whenever $\tau = \frac{1}{n}I$, $\widehat{W}_{\frac{I}{n}}(\rho,\sigma) = \frac{1}{n}(N(\rho) - N(\sigma)) = \frac{1}{n}W_N(\rho,\sigma)$.
If we define,

$$\widehat{W}(\rho,\sigma) := n \min_{\tau} \widehat{W}_{\tau}(\rho,\sigma)$$

then we can at once see

$$\widehat{W} \succeq \widehat{W}_{\tau}$$

Computable no-go Conversion Witness

Replacing $C_\Gamma(\tilde{\rho})$ by $\mathcal{N}_{N(\tilde{\rho})}$ we still can have

$$Tr[\tau\sigma^{\Gamma}-] \leq N(\rho) h_{\mathcal{N}_{N(\tilde{\rho})}}(\tau)$$

Hence we have the computable PPT conversion witness

$$W_\tau(\rho,\sigma) := N(\rho) h_{\mathcal{N}_{N(\tilde{\rho})}}(\tau) - Tr[\tau\sigma^{\Gamma}-]$$

Clearly $\widehat{W}_\tau(\rho,\sigma) \leq W_\tau(\rho,\sigma)$ and defining

$$W(\rho,\sigma) := n \min_\tau W_\tau(\rho,\sigma)$$

The hierarchy $\widehat{W} \succeq W \succeq W_N$ holds. However the support function is hard to evaluate.

Computable no-go Conversion Witness

- The support function is evaluated for Werner and Isotropic class of states

$$W_{wer}(\rho,\sigma) = \min\{W_N(\rho,\sigma), \frac{2}{d(d-1)} W'_{wer}(\rho,\sigma)\}$$

$$W_{iso}(\rho,\sigma) = \min\{W_N(\rho,\sigma), W'_{iso}(\rho,\sigma)\}$$

where

$$W'_{wer}(\rho,\sigma) = \frac{dN(\tilde{\rho})+1}{2} N(\rho) - Tr[F_- \sigma^{\Gamma}-]$$

$$W'_{iso}(\rho,\sigma) = \frac{2N(\tilde{\rho})+1}{2} N(\rho) - \langle\Phi|\sigma^{\Gamma}-|\Phi\rangle$$

the sub-witness W'_{wer}, W'_{iso} and W_N are incomparable, however $W_{iso}, W_{wer} \succeq W_N$. The new witness $W_\Gamma = \min\{W_N, W'_{wer}, W'_{iso}\}$ is an computable witness better than negativity.

- Any pure entangled state with negativity less than $1/3$ cannot be converted to an entangled Werner state in any dimension and with same negativity by PPT operation.

Orthogonally Invariant Class of states

Any $\mathcal{O} \otimes \mathcal{O}$ invariant state from a $n \otimes n$ system can be taken as

$$\rho_{oo} = a\,\mathbb{I}_n + b\,\mathbb{F} + c\,\hat{\mathbb{F}}$$

with $n(na + b + c) = 1$ (trace condition) and proper positivity constraints. $\mathbb{I}$ is the identity operator, $\mathbb{F} = d|\Phi\rangle\langle\Phi|^{\Gamma}$ is the flip operator and $\hat{\mathbb{F}}$ is the projection on maximally entangled state.

The operators satisfy the algebra,

$$\mathbb{F}^2 = \mathbb{I}$$
$$\mathbb{F}\hat{\mathbb{F}} = \hat{\mathbb{F}}\mathbb{F} = \hat{\mathbb{F}}$$
$$\hat{\mathbb{F}}^2 = n\,\hat{\mathbb{F}}$$

n is the dimension of each subsystem. The state can also be written in terms of orthogonal projectors

$$\rho_{oo} = \frac{\hat{f}}{d}U + \frac{1-\hat{f}}{d(d-1)}V + \frac{1-\hat{f}}{d(d+2)(d-1)}W$$

with $U = \frac{\hat{\mathbb{F}}}{d}$, $V = \frac{I-\mathbb{F}}{2}$, $W = \frac{I+\mathbb{F}}{2} - \frac{\hat{\mathbb{F}}}{d}$

Conversion Witness for $O \otimes O$

In terms of the parameters $f = \mathrm{tr}[\mathbb{F}\rho_{oo}]$ and $\hat{f} = \mathrm{tr}[\hat{\mathbb{F}}\rho_{oo}]$, negativity of the above state can be written as

$$N(\rho_{oo}) = \frac{1}{2}\left(\left[\frac{|f|}{d} + \frac{|1-\hat{f}|}{2} + \frac{d + d\hat{f} - 2f}{2d}\right] - 1\right)$$

Consider the regions

$$R_0 = \{(f, \hat{f}) : f \geq 0, \hat{f} <= 1\}$$
$$R_1 = \{(f, \hat{f}) : f \geq 0, \hat{f} >= 1\}$$
$$R_2 = \{(f, \hat{f}) : f \leq 0, \hat{f} <= 1\}$$
$$R_3 = \{(f, \hat{f}) : f \leq 0, \hat{f} <= 1\}$$

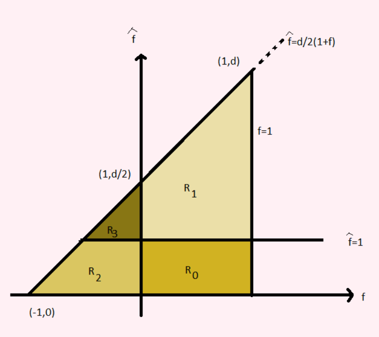

Figure: The set of all physical orthogonal invariant states lie in the large triangle ABC (determined by $\hat{f} \geq 0, f <= 1, \hat{f} \leq d/2(1+f)$) which is itself distributed in four parts. Each part R_i is determined by the value of negativity

$N(\rho)|_{R_0} = 0$, $\max N(\rho)|_{R_1} = \frac{d-1}{2}$, $\max N(\rho)|_{R_2} = \frac{1}{d}$, $\max N(\rho)|_{R_3} = \frac{d-2}{4}$.

Conversion Witness for $O \otimes O$

Support function for any orthogonally invariant state $\tau = \rho_{s,\hat{s}}$,

$$\begin{aligned} h_{\mathcal{N}_{c'}}(\tau) &= \max_{\gamma \in \mathcal{N}_c} tr[\gamma\tau] \\ &= tr[\gamma^* \tau] \\ &= tr[\tau_{o\otimes o}(\gamma^*)\tau] \\ &= \max_{N(\rho_{f,\hat{f}}) \leq c'} tr[\rho_{s,\hat{s}}\rho_{f,\hat{f}}] \end{aligned}$$

the above simplification is based on the assumption that twirling operation $\tau_{o\otimes o}$ is a PPT operation and two results $\tau_{o\otimes o}(\tau) = \tau$ and $Tr[\tau_{o\otimes o}(A)B] = Tr[A\tau_{o\otimes o}(B)]$ for hermitian operator A and B.
We have calculated the support function for orthogonally invariant class of states with maximization over all orthogonally invariant class of states.

Conversion Witness for $O \otimes O$

Clearly, the optimizing function is a linear one: $h_{\mathcal{N}_{c'}}(\tau) = Af + B\hat{f} + E$ where A, B, E are function of $s, \hat{s}$. With $d \geq 4$, this linear optimization over f, $\hat{f}$ can be performed depending upon the sign of A and B and the range of $c_1 = \min\{c', \frac{tr(\rho_{oo}^{\Gamma_-\Gamma_-})}{tr[\rho_{oo}^{\Gamma_-}]}\}$. We have shown below the value of witness at the extreme points of the $(s, \hat{s})$ plane for $c_1 >= \frac{d-1}{2}$.

Extreme points	Witness
$A1(1, 0)$	$\frac{2}{(d-1)(d+2)}\left[N(\rho) - Tr[W\sigma^{\Gamma}-]\right]$
$A2(1, \frac{2}{1+d})$	$\frac{2}{d(d+1)}\left[N(\rho) - Tr[(U+W)\sigma^{\Gamma}-]\right]$
$A3(1, d)$	$\left[N(\rho) - Tr[U\sigma^{\Gamma}-]\right]$
$A4(-1, 0)$	$\frac{2}{d(d-1)}\left[N(\rho) - Tr[V\sigma^{\Gamma}-]\right]$
$A5(\frac{1}{d}, \frac{1}{d})$	$\frac{1}{d^2}\left[N(\rho) - Tr[(W+V+U)\sigma^{\Gamma}-]\right]$
$A6(\frac{1}{1+d}, 0)$	$\frac{1}{(d^2-1)}\left[N(\rho) - Tr[(W+V)\sigma^{\Gamma}-]\right]$

The witness at $A5$ is nothing but the negativity witness. Hence, our witness is also a betterment of Negativity. Conversion witness for $d = 3$ can be evaluated separately.

Conversion Witness for $O \otimes O$

$\frac{d-2}{4} < c_1 <= \frac{d-1}{2}$

Extreme points	Witness
$A1(1,0)$	$\frac{2}{(d-1)(d+2)}\left[N(\rho) - Tr[W\sigma^{\Gamma}-]\right]$
$A2(1,\frac{2}{1+d})$	$\frac{2}{d(d+1)}\left[N(\rho) - Tr[(U+W)\sigma^{\Gamma}-]\right]$
$A3(1,d)$	$\left[\frac{2c_1+1}{d}N(\rho) - Tr[U\sigma^{\Gamma}-]\right]$
$A4(-1,0)$	$\frac{2}{d(d-1)}\left[N(\rho) - Tr[V\sigma^{\Gamma}-]\right]$
$A5(\frac{1}{d},\frac{1}{d})$	$\frac{1}{d^2}\left[N(\rho) - Tr[(W+V+U)\sigma^{\Gamma}-]\right]$
$A6(\frac{1}{1+d},0)$	$\frac{1}{(d^2-1)}\left[N(\rho) - Tr[(W+V)\sigma^{\Gamma}-]\right]$

Conversion Witness for $O \otimes O$

$\frac{1}{d} < c_1 <= \frac{d-2}{4}$

Extreme points	Witness
$A1(1,0)$	$\frac{2}{(d-1)(d+2)}\left[N(\rho) - Tr[W\sigma^{\Gamma}-]\right]$
$A2(1,\frac{2}{1+d})$	$\frac{2}{d(d+1)}\left[N(\rho) - Tr[(U+W)\sigma^{\Gamma}-]\right]$
$A3(1,d)$	$\left[\frac{d(2c_1+1)-2}{d^2-4}N(\rho) - Tr[U\sigma^{\Gamma}-]\right]$
$A4(-1,0)$	$\frac{2}{d(d-1)}\left[N(\rho) - Tr[V\sigma^{\Gamma}-]\right]$
$A5(\frac{1}{d},\frac{1}{d})$	$\frac{1}{d^2}\left[N(\rho) - Tr[(W+V+U)\sigma^{\Gamma}-]\right]$
$A6(\frac{1}{1+d},0)$	$\frac{1}{(d^2-1)}\left[N(\rho) - Tr[(W+V)\sigma^{\Gamma}-]\right]$

Conversion Witness for $O \otimes O$

$c_1 <= \frac{1}{d}$

Extreme points	Witness
$A1(1,0)$	$\frac{2}{(d-1)(d+2)}\left[N(\rho) - Tr[W\sigma^{\Gamma}-]\right]$
$A2(1,\frac{2}{1+d})$	$\frac{2}{d(d+1)}\left[N(\rho) - Tr[(U+W)\sigma^{\Gamma}-]\right]$
$A3(1,d)$	$\left[\frac{1}{d}\max\{\frac{2c_1+1, d(1-c_1d)}{2}, \frac{d^2(2c_1+1)-2d}{d^2-4}\}N(\rho) - Tr[U\sigma^{\Gamma}-]\right]$
$A4(-1,0)$	$\frac{1}{d(d-1)}\left[\max\{c_1d+1, \frac{2d^2-2d(2c_1+1)-4}{d^2-4}\}N(\rho) - Tr[V\sigma^{\Gamma}-]\right]$
$A5(\frac{1}{d},\frac{1}{d})$	$\frac{1}{d^2}\left[N(\rho) - Tr[(W+V+U)\sigma^{\Gamma}-]\right]$
$A6(\frac{1}{1+d},0)$	$\frac{1}{(d^2-1)}\left[N(\rho) - Tr[(W+V)\sigma^{\Gamma}-]\right]$

Conclusion

- Conversion witness is a generalization of monotones and can determine convertibility when monotones fail.
- we have obtained the explicit form of conversion witness for some symmetric class of states and they provide betterment over negativity. Our witness includes all the results provided by Gour et.al.
- Conversion witness can be studied in other resource theories (such as coherence+LICC restriction), possibly beyond quantum theory too.

Reference

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Thank You