Effective time-independent analysis for quantum time-periodic systems

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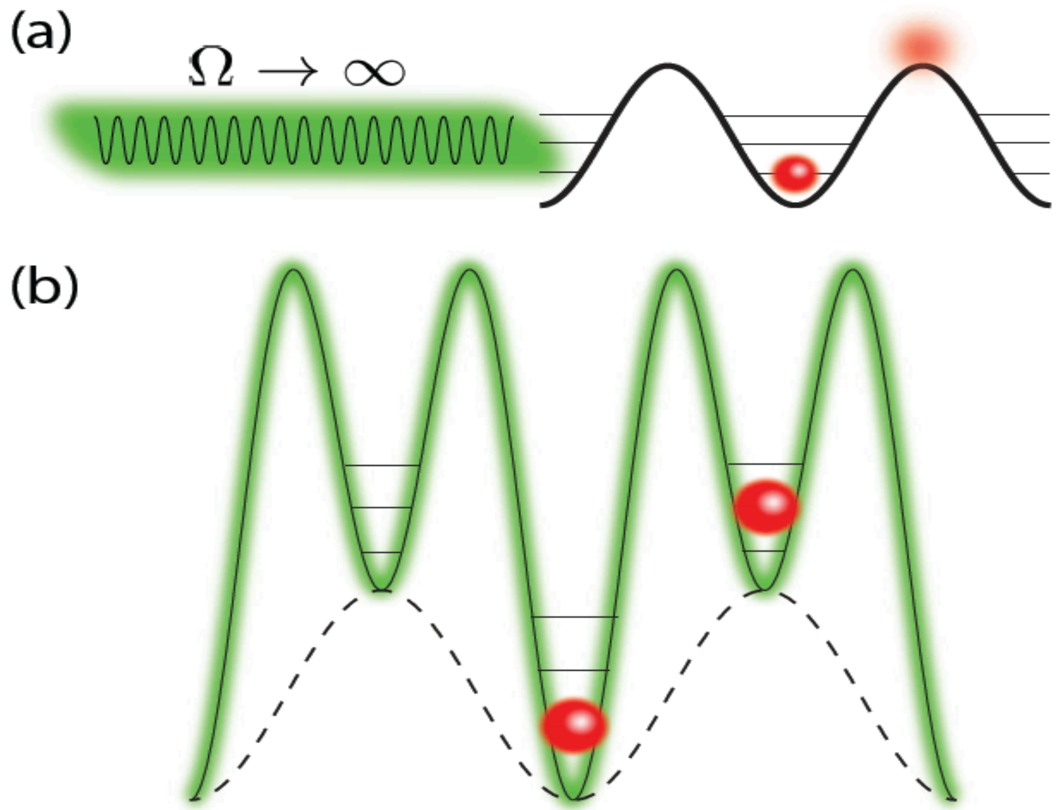
Plan

- General discussion on periodically driven systems
- Our recent work
- Future directions
- Conclusion

- Periodically driven systems have a long history. Eg. Kicked rotor (Chirikov Map)
- These systems may display very rich dynamics: integrability-to-chaos transition
- Recently, it has been shown that periodic perturbations can be used as a flexible experimental tool to realize new phases of matter, which may not easily accessible in equilibrium systems
- This new line of research may be called “Floquet engineering”

- In the simplest possible case, one considers a single monochromatic driving scheme, characterized by a coupling scheme (driving amplitude) and a single frequency $\Omega = 2\pi / T$.



- Usually, these systems are analyzed for the two extreme regimes: slow driving and fast driving
    - In the former regime, the system almost adiabatically follows the instantaneous Hamiltonian
    - In the later regime, where the driving frequency is faster than the natural frequencies of the non-driven model, the system typically feels an effective static potential dependent on the driving amplitude.
- Away from the adiabatic limit, the analysis of periodically driven system relies on the Floquet theorem

- The Floquet theorem is very similar to Bloch theorem, but in time domain

- Consider a time-periodic Hamiltonian of the form:

$$H(t) = H_0 + V(t), \text{ where } V(t) = V(t+T)$$

- The Floquet theorem states that the time evolution operator can be expressed as:

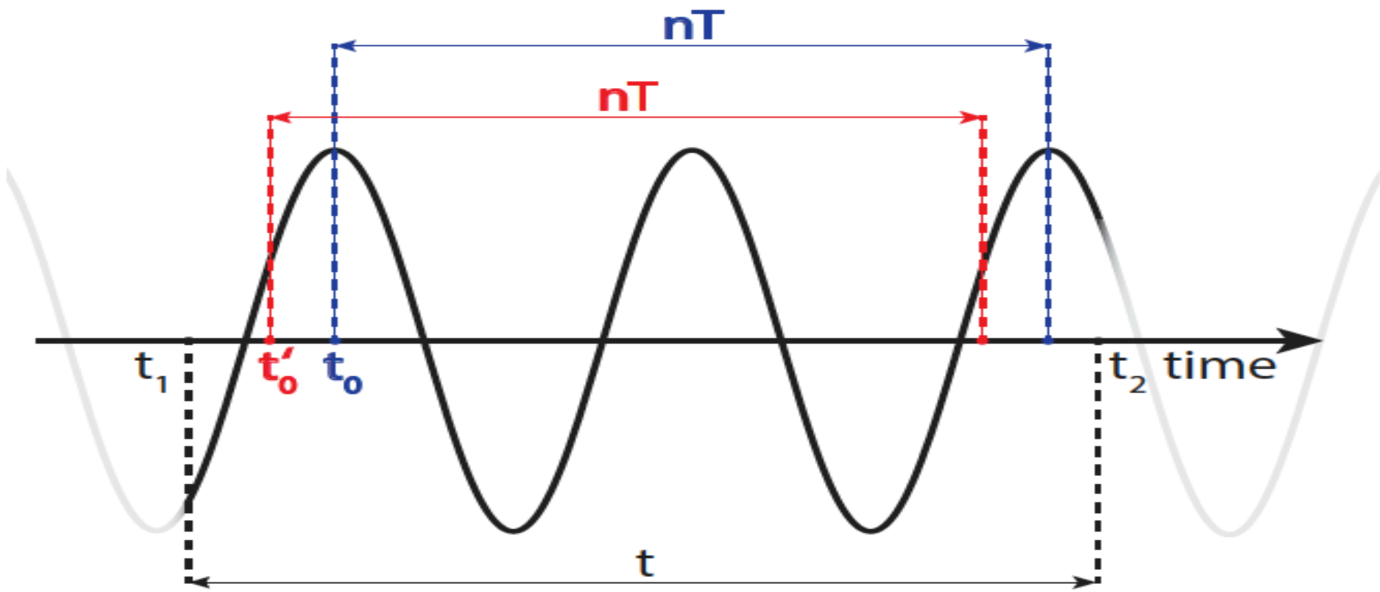
$$U(t_2, t_1) = e^{-iF(t_2)} e^{-iH_F(t_2 - t_1)} e^{iF(t_1)}$$

where $K(t) = K(t+T)$ is a periodic Hermitian operator

and H_F is the time-independent Floquet Hamiltonian.

- The choice of K and H_F operators are not unique, there is some freedom in defining them

• Different choices correspond to different “Floquet” gauge



• The system evolves from time t_1 to t_2. The evolution operator will be:

$$U(t_2, t_1) = \mathcal{T}_t \exp\left[-i \int_{t_1}^{t_2} H(\tilde{t}) \mathrm{d}\tilde{t}\right] = \prod_{j:\, t_1 \leq \tilde{t}_j \leq t_2} \mathrm{e}^{-iH(\tilde{t}_j)(\tilde{t}_{j+1} - \tilde{t}_j)}$$

•This is invariant under: $(t_1, t_2) \rightarrow (t_1 + nT, t_2 + nT)$

- The stroboscopic evolution starts at time t_0 , which can be chosen to be anywhere within the first period $[t_1, t_1 + T)$
- It is convenient to formally define the evolution within one period as $U(t_0 + T, t_0) = \exp\left[-iH_F[t_0]T\right]$
- The choice of the Floquet gauge, i.e., the choice of t_0 , in general affects the form of the Floquet Hamiltonian $H_F[t_0]$

$$U(t_2, t_1) = e^{-iF(t_2)} e^{-iH_F(t_2 - t_1)} e^{iF(t_1)}$$

- However, different Floquet Hamiltonians are gauge equivalent to a H_F which is t_0 independent

- Generally, it is very difficult to evaluate H_F exactly, one has to rely on approximation schemes
- Different approximations can give very much different Floquet Hamiltonians for different “gauge”.
- The most popular expansion scheme is based on Magnus expansion

$$U(T+t_0,t_0)=\mathcal{T}_t\exp\left(-\frac{i}{\hbar}\int_{t_0}^{T+t_0}\mathrm{d}tH(t)\right)=\exp\left(-\frac{i}{\hbar}H_F[t_0]T\right)$$

$$H_F[t_0]=\frac{i}{T}\log\left[\mathcal{T}_t\exp\left(-i\int_{t_0}^{t_0+T}\mathrm{d}tH(t)\right)\right]$$

- The RHS is then expanded in power of Ω^{-1} using Cambell-Baker-Hausdorff formula

- We followed another method:

$$\widehat{H}_{\text{eff}} = \sum_{n=0}^{\infty} \frac{1}{\omega^n} \widehat{H}_{\text{eff}}^{(n)}, \quad \widehat{F}(t) = \sum_{n=1}^{\infty} \frac{1}{\omega^n} \widehat{F}^{(n)}$$

$$\widehat{V}(t) = \widehat{V}_0 + \sum_{n=1}^{\infty} (\widehat{V}_n e^{in\omega t} + \widehat{V}_{-n} e^{-in\omega t})$$

- Comparing this with $U(t_2, t_1) = e^{-iF(t_2)} e^{-iH_F(t_2 - t_1)} e^{iF(t_1)}$, the perturbation series an be obtained upto any desired accuracy.
- At each order, the averaged time-independent coefficient is retained in H_{eff} and all time dependence push into the $F(t)$ operator

$$\widehat{H}_{\text{eff}} = \widehat{H}_0 + \widehat{V}_0 + \frac{1}{\omega}\sum_{n=1}^{\infty}\frac{1}{n}[\widehat{V}_n, \widehat{V}_{-n}] + \frac{1}{2\omega^2}\sum_{n=1}^{\infty}\frac{1}{n^2}([[\widehat{V}_n, \widehat{H}_0], \widehat{V}_{-n}] + \text{H.c.})$$

$$+ \frac{1}{3\omega^2}\sum_{n,m=1}^{\infty}\frac{1}{nm}([\widehat{V}_n, [\widehat{V}_m, \widehat{V}_{-n-m}]] - 2[\widehat{V}_n, [\widehat{V}_{-m}, \widehat{V}_{m-n}]] + \text{H.c.}),$$

$$\widehat{F}(t) = \frac{1}{i\omega}\sum_{n=1}^{\infty}\frac{1}{n}(\widehat{V}_n e^{in\omega t} - \widehat{V}_{-n} e^{-in\omega t}) + \frac{1}{i\omega^2}\sum_{n=1}^{\infty}\frac{1}{n^2}([\widehat{V}_n, \widehat{H}_0 + \widehat{V}_0] e^{in\omega t} - \text{H.c.})$$

$$+ \frac{1}{2i\omega^2}\sum_{n,m=1}^{\infty}\frac{1}{n(n+m)}([\widehat{V}_n, \widehat{V}_m] e^{i(n+m)\omega t} - \text{H.c.}) + \frac{1}{2i\omega^2}\sum_{n\neq m=1}^{\infty}\frac{1}{n(n-m)}([\widehat{V}_n, \widehat{V}_{-m}] e^{i(n-m)\omega t} - \text{H.c.})$$

- Our special interest is in periodically kicked systems. For example, kicked top Hamiltonian:

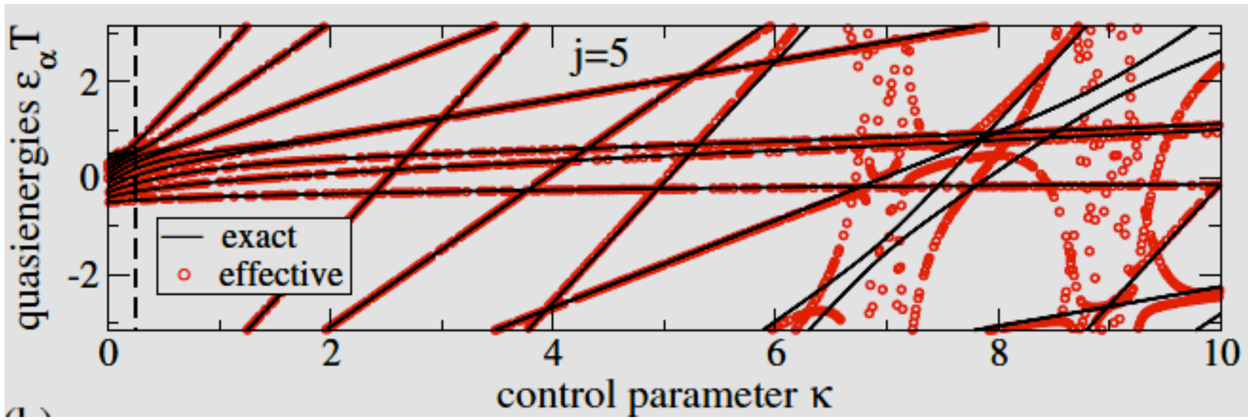
$$\hat{H}(t) = pJ_x \sum_{n=-\infty}^{\infty} \delta(t - nT) + \frac{\kappa}{2jT} J_z^2$$

- Corresponding Floquet operator:

$$\hat{\mathcal{F}} = e^{-ipJ_x} e^{-i(\kappa/2j)J_z^2}$$

- CBH based (Magnus) expansion gives:

$$\hat{H}_E = \frac{\kappa}{2j} J_z^2 + \frac{p}{2} \left[\frac{-i \frac{\kappa}{2j} J_+ (2J_z + 1)}{\exp[-i \frac{\kappa}{2j}(2J_z + 1)] - 1} + \text{H.c.} \right]$$



Bastidas et al,
PRL, **112** 140408 (2014)

- Following the alternate method, we got the effective Hamiltonian for the kicked systems upto $O\left(\omega^{-2}\right)$:

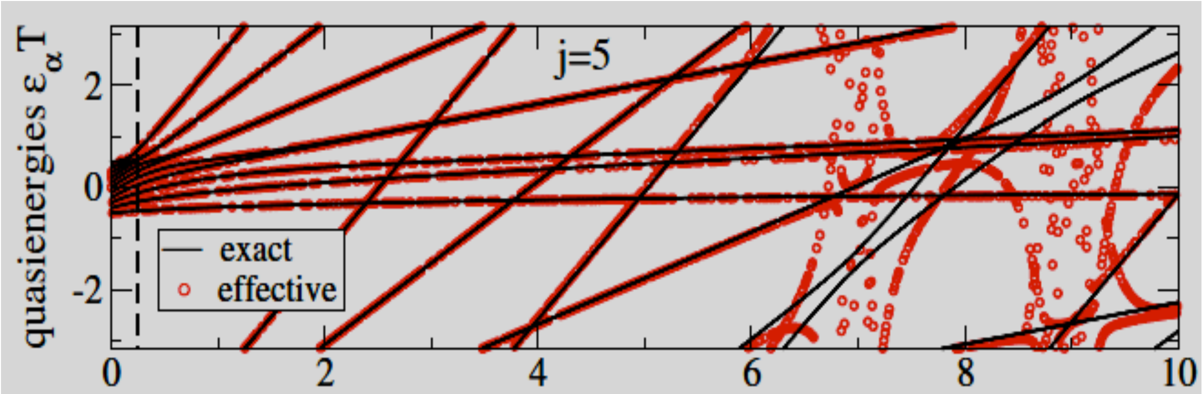
$$\widehat{H}_{\text{eff}} = \widehat{H}_0 + \frac{\widehat{V}}{T} + \frac{1}{\omega^2 T^2}[[\widehat{V}, H_0], \widehat{V}]\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) = \widehat{H}_0 + \frac{\widehat{V}}{T} + \frac{1}{24}[[\widehat{V}, H_0], \widehat{V}],$$

$$\widehat{F}(t) = \frac{2\widehat{V}}{\omega T} \sum_{n=1}^{\infty} \frac{\sin(n\omega t)}{n} + \frac{2}{i\omega^2 T}[\widehat{V}, \widehat{H}_0] \sum_{n=1}^{\infty} \frac{\cos(n\omega t)}{n^2} = \frac{\widehat{V}}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\omega t)}{n} - i \frac{T}{2\pi^2} [\widehat{V}, \widehat{H}_0] \sum_{n=1}^{\infty} \frac{\cos(n\omega t)}{n^2}$$

- For the kicked top:

$$\widehat{H}_{\text{eff}} = \frac{\alpha}{2j} \widehat{J_z^2} + \beta \widehat{J_x} - \frac{\alpha \beta^2}{24 j} (\widehat{J_z^2} - \widehat{J_y^2}),$$

$$\widehat{F}(t_i) = \widehat{F}(t_f) = -\frac{\alpha \beta}{24 j} (\widehat{J_y} \widehat{J_z} + \widehat{J_z} \widehat{J_y}).$$



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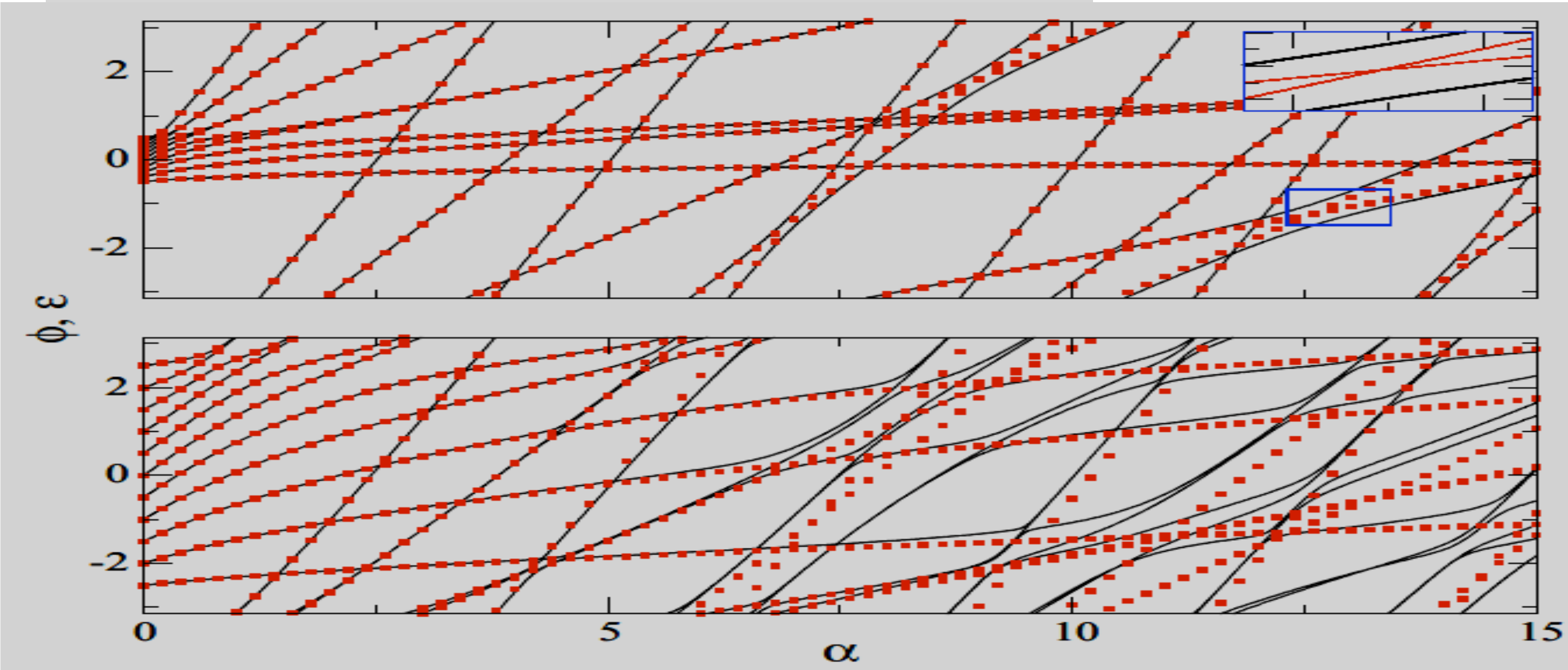
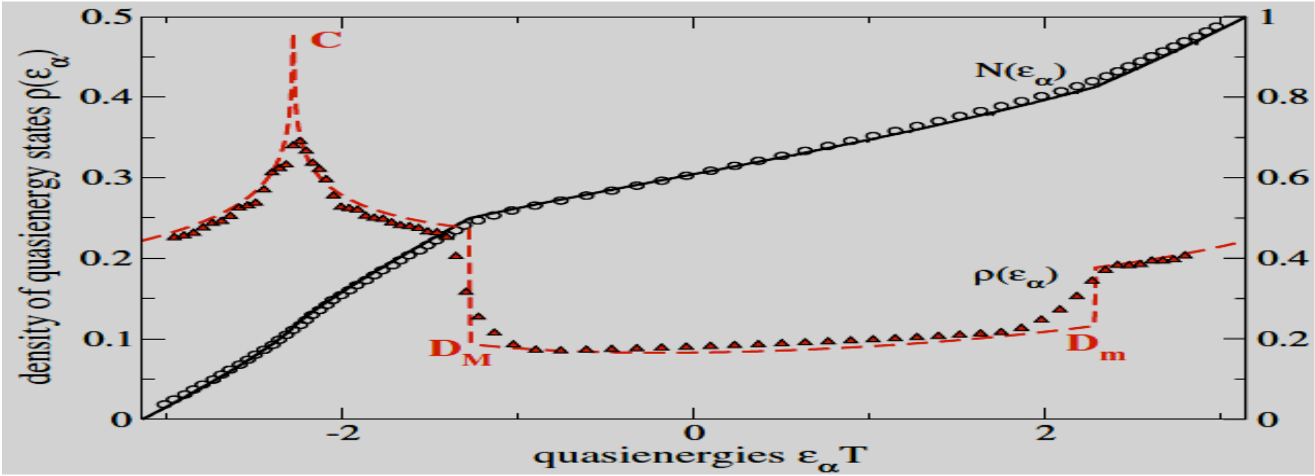
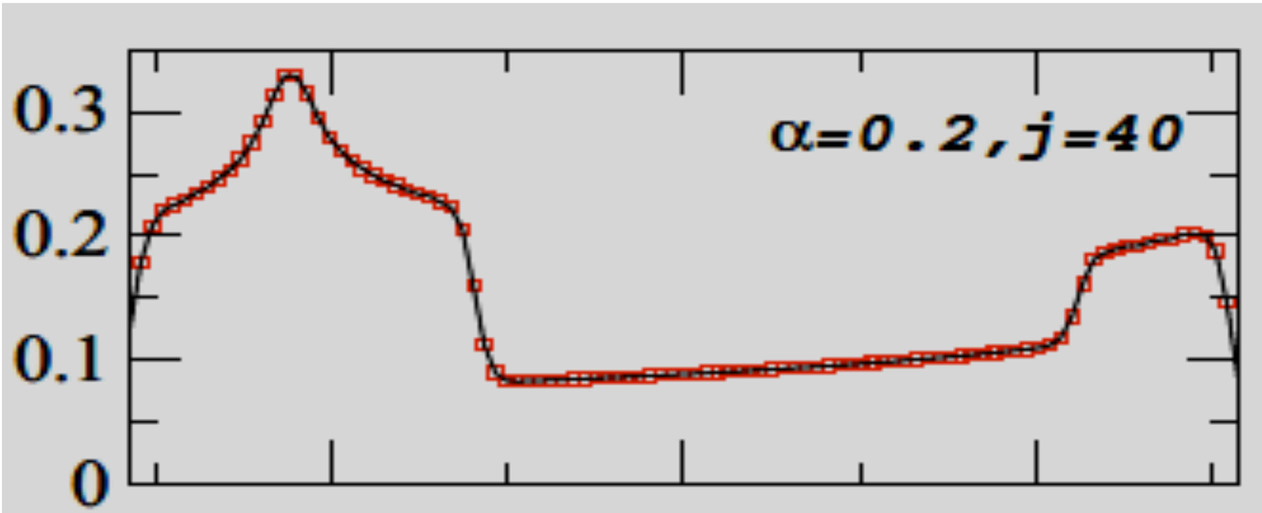


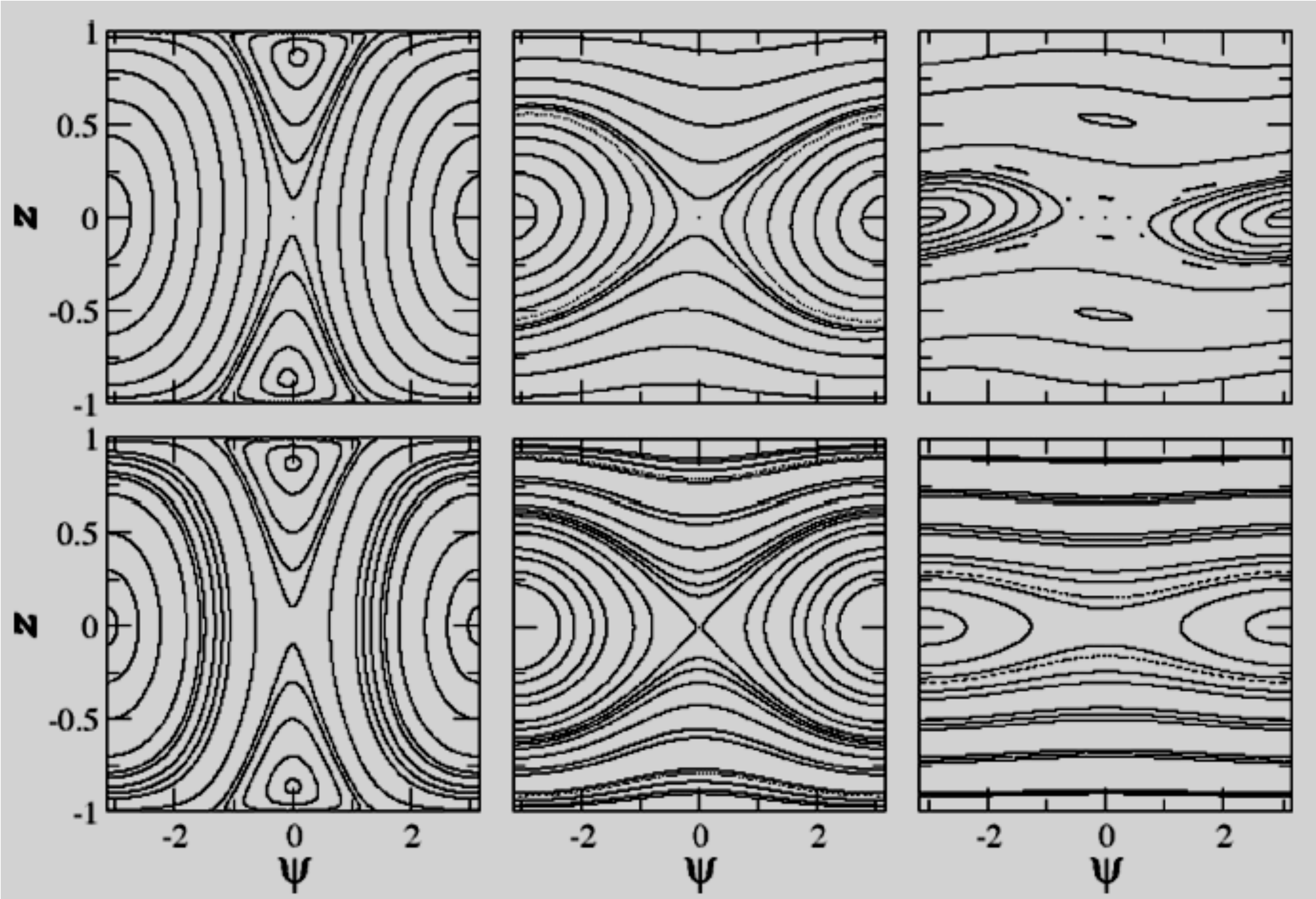
Our work. PRE **91** 032923 (2015)



density of quasienergy states ρ(ε_α)
quasienergies ε_αT
N(ε_α)
ρ(ε_α)
C
D_M
D_m
0.5
0.4
0.3
0.2
0.1
0
-2
2
1
0.8
0.6



α=0.2,j=40
0.3
0.2
0.1
0



1
0.5
z
0
-0.5
-1
1
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z
0
-0.5
-1
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2
ψ
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0
2
ψ
-2
0
2
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- Double kicked top Hamiltonian:

$$\widehat{H} = \frac{2\alpha}{T}\widehat{J}_x + \frac{\eta}{2j}\widehat{J}_z^2 \sum_{n=-\infty}^{+\infty}\left[\delta\left(t - nT - \frac{T}{2}\right) - \delta(t - nT)\right]$$

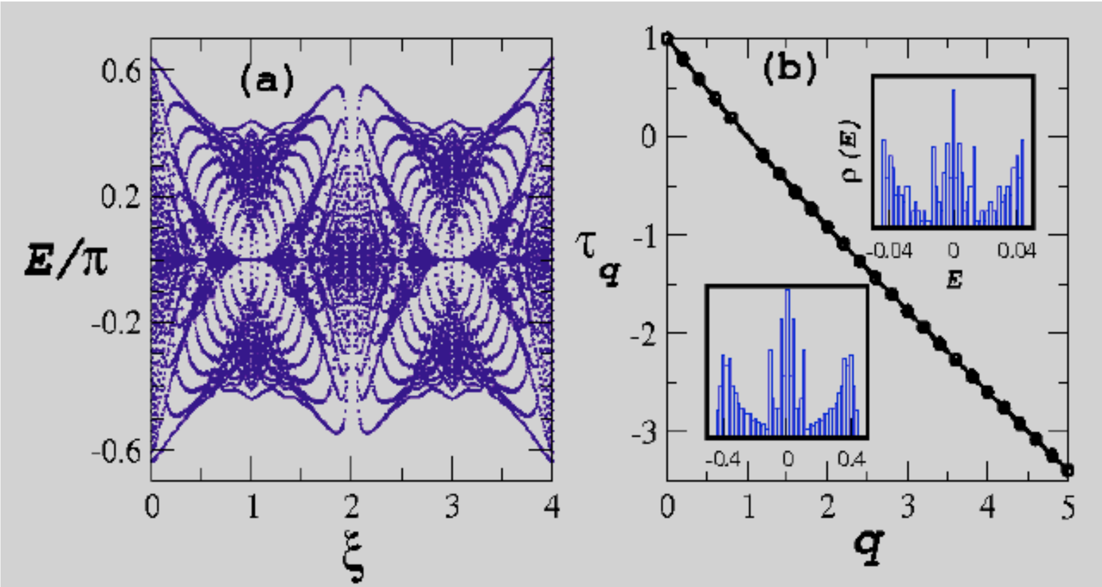
- Floquet operator:

$$\widehat{\mathcal{F}} = \exp\left\{-i\alpha\widehat{J}_+ e^{i[\eta(2\widehat{J}_z+\mathbb{1})/2j]} + \text{h.c.}\right\}\exp\left(-i\alpha\widehat{J}_x\right)$$

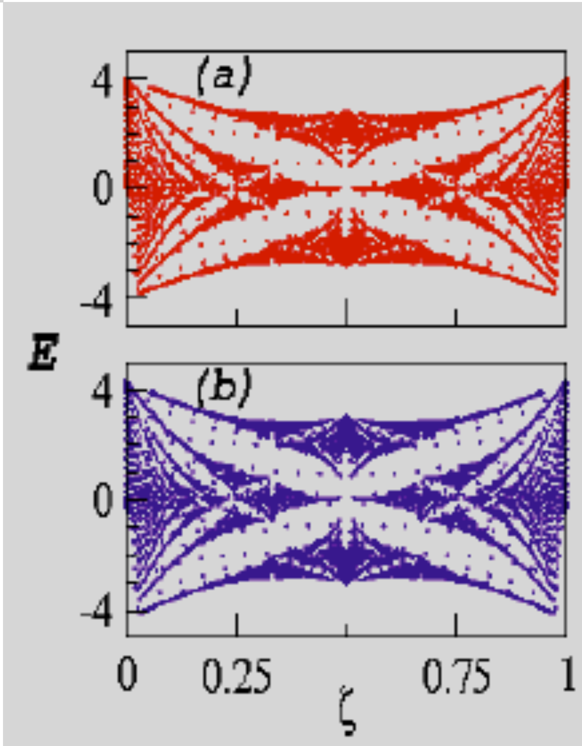
- The effective Hamiltonian:

$$\begin{aligned}\widehat{\mathcal{H}}_{\text{eff}} &= \widehat{\mathcal{H}}_0 + \frac{\widehat{\mathcal{V}}}{T} + \frac{1}{\omega^2 T^2}[[\widehat{\mathcal{V}}, \widehat{\mathcal{H}}_0], \widehat{\mathcal{V}}]\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right) \\ &= \widehat{\mathcal{H}}_0 + \frac{\widehat{\mathcal{V}}}{T} + \frac{1}{24}[[\widehat{\mathcal{V}}, \widehat{\mathcal{H}}_0], \widehat{\mathcal{V}}]\end{aligned}$$

where $\widehat{\mathcal{H}}_0 = \alpha\frac{\widehat{J}_+}{2T}\exp\left[i\frac{\eta}{2j}\left(2\widehat{J}_z + \mathbb{1}\right)\right] + \text{h.c.}$ and $\widehat{\mathcal{V}} = \alpha\widehat{J}_x$



Here, $\xi = \eta/\pi j$ for $\alpha = 1/j$



Kicked Harper and its effective time-independent Hamiltonian

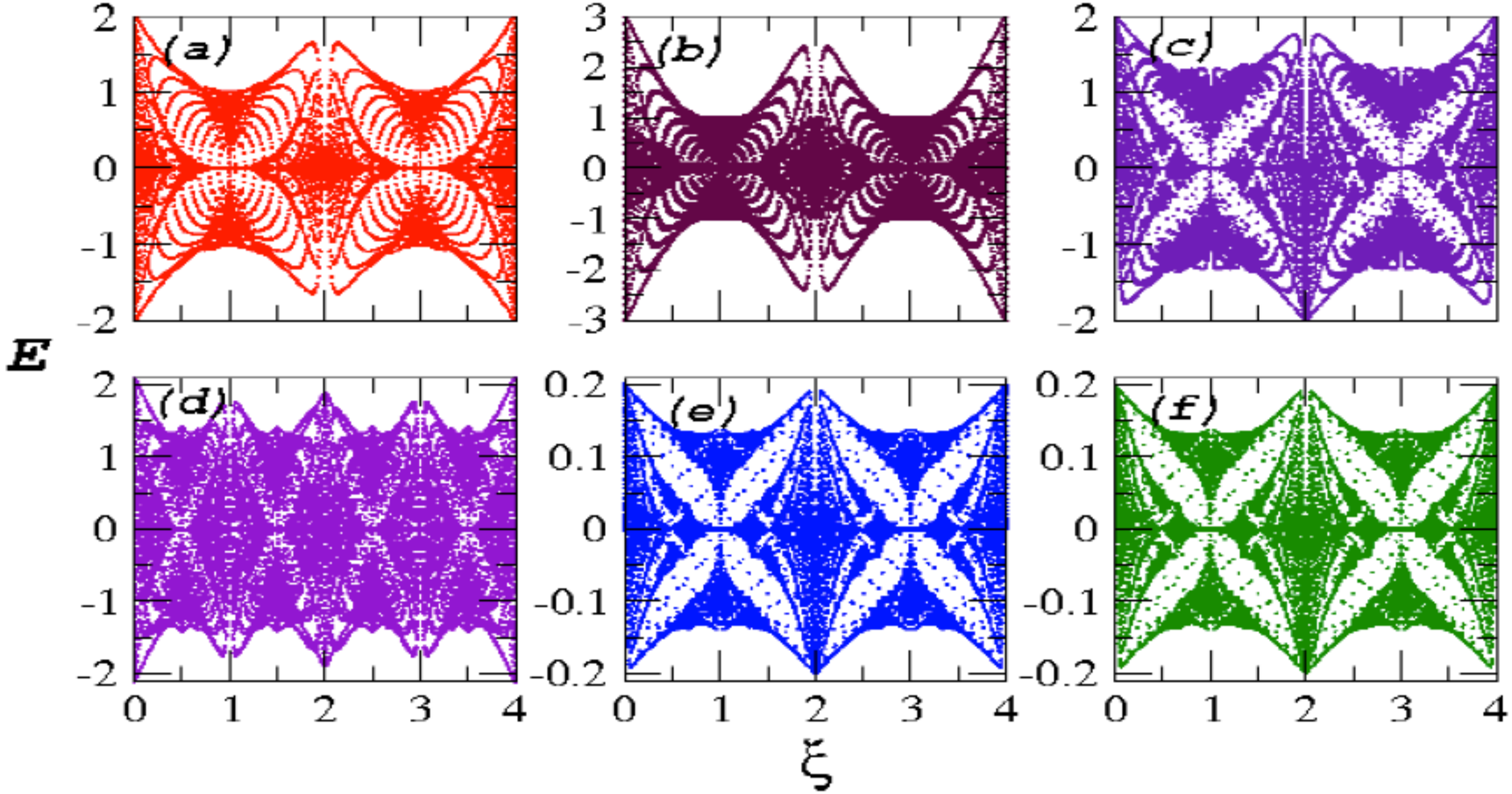
- A class of Hamiltonians

$$\widehat{H} = a\widehat{J}_x + b\widehat{\mathbb{A}} + \left[\widehat{C}\cos(\widehat{X}) + \text{h.c.}\right]$$

where $\widehat{\mathbb{A}} = \sum_{m=-j}^{+j} (|m\rangle\langle m+1| + |m+1\rangle\langle m|)$

and $\widehat{X} = \eta(2\widehat{J}_z + \mathbb{1})/2j$

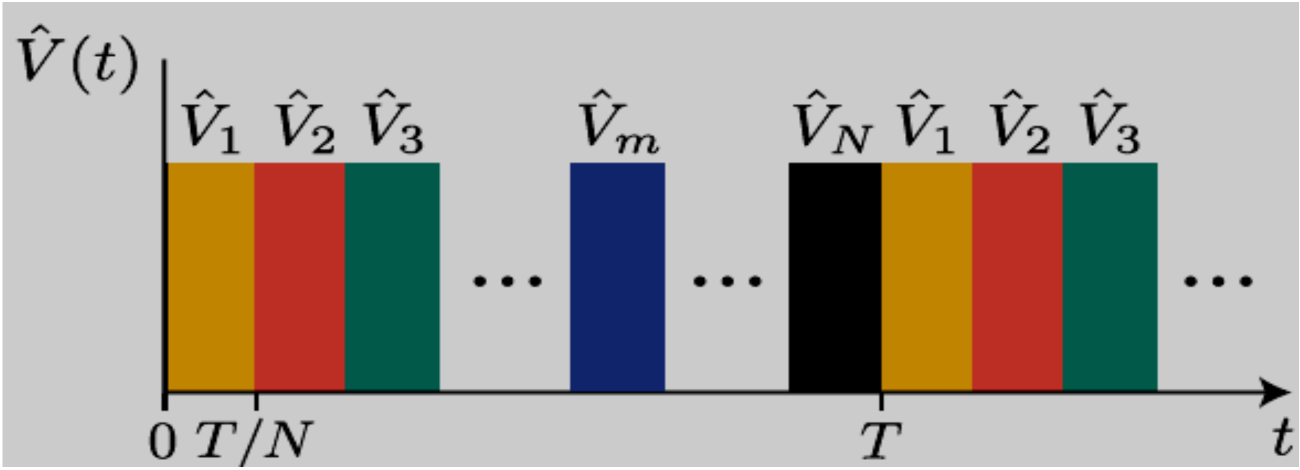
Fig.	a	b	$\widehat{C}$
3(a)	α	0	$\frac{\alpha}{2}(\widehat{J}_x + i\widehat{J}_y)$
3(b)	α	0	$\alpha\widehat{J}_x$
3(c)	α	0	$\frac{1}{2}\mathbb{1}$
3(d)	0	α	$\alpha\widehat{J}_x$
3(e)	α	$\epsilon\alpha$	$\alpha\mathbb{1}$
3(f)	0	α	$\alpha\mathbb{1}$


(a)
(b)
(c)
(d)
(e)
(f)
E
ξ

- Massless Dirac equation in curved (2+1) D space = Rippled graphene
- Metric: $ds^2 = dt^2 - e^{-2\Lambda(x,y)}\left(dx^2 + dy^2\right)$
- 2D spatial part of this metric is the most general
- Dirac equation in this space

$$i\frac{\partial\psi}{\partial t} = e^{\Lambda(x,y)}\left[-i\sigma^j\partial_j - \frac{i}{2}\left(\frac{\partial\Lambda(x,y)}{\partial y}\sigma^y + \frac{\partial\Lambda(x,y)}{\partial x}\sigma^x\right)\right]\psi$$

- Pulse scheme:



- 4-Pulse scheme: $\mathcal{P}_4 : \{H_0 + A, H_0 + B, H_0 - A, H_0 - B\}$
- The effective Hamiltonian:

$$H_{eff} = H_0 + \frac{i}{2\omega}[A, B]\frac{1}{4\omega^2}([[A, H_0], A] + [[B, H_0], B]) + \mathcal{O}(1/\omega^3)$$

$H_0 = -i\sigma^j\partial_j$: Dirac equation in flat space

$A = \sigma^j\alpha_j$ $\quad$ $B = \sigma^k\beta_k$ $\quad$ $\alpha_j = [i\partial_y, -i\partial_x, 0]$ and $\beta_k = [0, 0, -f(x,y)]$

$$H_{eff} = \frac{1}{2}\left[-i\left(1+\frac{f(x,y)}{\omega}\right)\sigma^j\partial_j\right] - \frac{1}{2}\left[i\sigma^j\partial_j\left(1+\frac{f(x,y)}{\omega}\right)\right]$$

- Substitute: $$e^{\Lambda(x,y)} = \left(1+\frac{f(x,y)}{\omega}\right)$$

$$H_{eff} = e^{\Lambda(x,y)}\left[-i\sigma^j\partial_j - \frac{i}{2}\left(\frac{\partial\Lambda(x,y)}{\partial y}\sigma^y + \frac{\partial\Lambda(x,y)}{\partial x}\sigma^x\right)\right]$$

- A quantity of geometrical interest describing 2D curved surface is Gauss curvature:

$$K(x,y) = e^{2\Lambda}\nabla^2(\Lambda)$$

and $\Lambda = \ln(1+\frac{f(x,y)}{\omega})$,

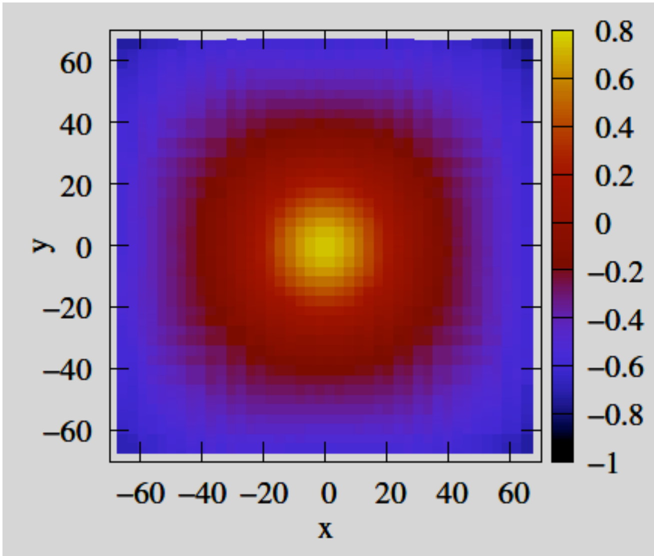
depends directly on the driving scheme f(x,y) and the driving frequency ω

- We analyze the effect on the LDOS for graphene like optical lattice under a periodic driving
- LDOS: $$\rho(\epsilon, \mathbf{r}) = -\frac{1}{\pi}\mathrm{Im}\sum_n \frac{|\psi(\mathbf{r})|^2}{(\epsilon + i\delta - E_n)}$$
- This approach has similar motivations to earlier studies on LDOS in "Rippled Graphene"
- We consider: $f(x, y) = x^2 + y^2$
- LDOS correction:

$$\frac{\rho}{\rho_o} - 1$$

$\rho \equiv$ the LDOS for pulsed graphene

$\rho_0 \equiv$ the LDOS for ordinarygraphene



Future directions

- Realization of quantum system in non-commutative space
- Realization of non-associative quantum mechanical system

Conclusion

- Floquet engineering is a powerful experimental knob to realize many different static Hamiltonians, which otherwise may not be realized easily

References

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Tapomoy

Tridev

Rashmi

Thanks!