Maximally entangled states and combinatorial designes

Karol Życzkowski

in collaboration with

Dardo Goyeneche (Concepcion/ Cracow/ Warsaw)

Jagiellonian University (Cracow)
& Polish Academy of Sciences (Warsaw)

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Combinatorial designes

Take 4 **aces**, 4 **kings**, 4 **queens** and 4 **jacks**
and arrange them into an 4×4 array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

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a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

$$\begin{array}{cccc}
A\spadesuit & K\diamondsuit & Q\heartsuit & J\clubsuit \\
K\heartsuit & A\clubsuit & J\spadesuit & Q\diamondsuit \\
Q\clubsuit & J\heartsuit & A\diamondsuit & K\spadesuit \\
J\diamondsuit & Q\spadesuit & K\clubsuit & A\heartsuit
\end{array}$$

Two **mutually orthogonal** **Latin squares** of size $N = 4$

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Entanglement measures

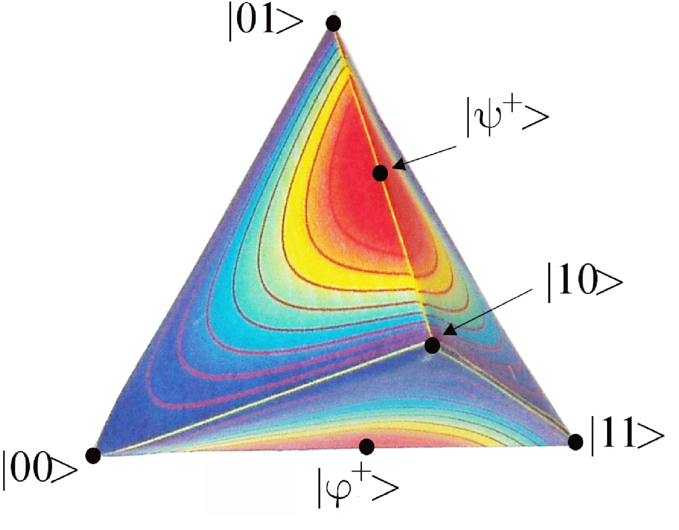
For any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ define its partial trace $\sigma = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.
Definition: Entanglement entropy of $|\psi\rangle$ is equal to von Neuman entropy of the partial trace

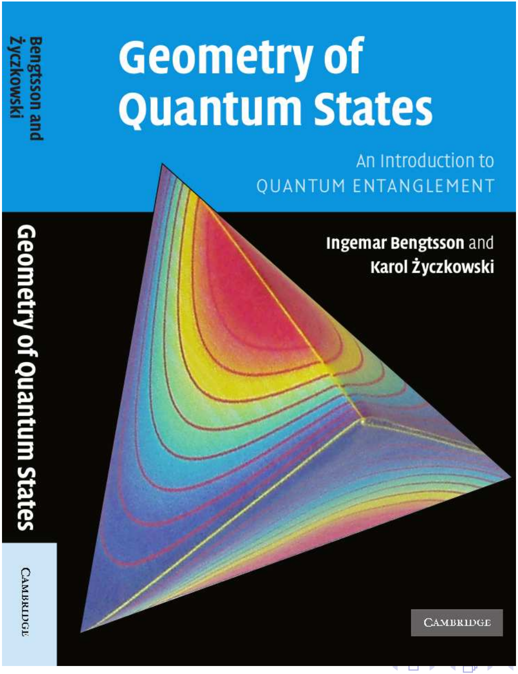
$$E(|\psi\rangle) := \ -\mathrm{Tr}\ \sigma \ln \sigma$$

The more mixed partial trace, the more entangled initial pure state...

Entanglement of two real qubits

Entanglement entropy at the thetrahedron of 2×2 real pure states

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Maximally entangled bi–partite quantum states

Bipartite systems $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B = \mathcal{H}_d \otimes \mathcal{H}_d$

generalized Bell state (for two qu$d$its),

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \tag{1}$$

distinguished by the fact that reduced states are **maximally mixed**,
e.g. $\rho_A = \mathrm{Tr}_B |\psi_d^+\rangle\langle\psi_d^+| = \mathbb{1}_d/d$.
This property holds for all locally equivalent states, $(U_A \otimes U_B)|\psi_d^+\rangle$.

Define bi-partite pure state by a matrix of coefficients, $|\psi\rangle = \sum_{i,j} G_{ij} |i,j\rangle$.
Then reduced state $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi| = GG^\dagger$.
It represents a **maximally entangled** state if $\rho_A = GG^\dagger = \mathbb{1}_d/d$, which is the case if the matrix $U = G/\sqrt{d}$ of size d is **unitary**.

Multipartite pure quantum states

Three qubits, $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C = \mathcal{H}_2^{\otimes 3}$

GHZ state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$ has a similar property: all three one-partite reductions are **maximally mixed**,
$\rho_A = Tr_{BC}|GHZ\rangle\langle GHZ| = \mathbb{1}_2 = \rho_B = Tr_{AC}|GHZ\rangle\langle GHZ|$.

(what is **not** the case e.g. for $|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$

Genuinely multipartite entangled states

k-uniform states of N qu$d$its

Definition. State $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ is called ***k*-uniform**
if for all possible splittings of the system into k and $N-k$ parts the reduced states are maximally mixed (**Scott 2001**),
(also called **MM**-states (maximally multipartite entangled)
Facchi et al. (2008,2010), **Arnaud & Cerf** (2012)

Applications: quantum error correction codes, ...

Example: 1–uniform states of N qu$d$its

Observation. A generalized, N–qu$d$it **GHZ** state,

$$|GHZ_N^d\rangle := \frac{1}{\sqrt{d}}\big[|1,1,...,1\rangle + |2,2,....,2\rangle + \cdots + |d,d,...,d\rangle\big]$$

is 1–**uniform** (but not 2–uniform!)

Examples of k–uniform states

Observation: k–uniform states may exist if $N \geq 2k$ (**Scott 2001**) (traced out ancilla of size $(N-k)$ cannot be smaller than the principal k–partite system).

Hence there are no 2-uniform states of 3 **qubits**.

However, there exist no 2-uniform state of 4 qubits either!

Higuchi & Sudbery (2000) - **frustration** like in spin systems –
Facchi, Florio, Marzolino, Parisi, Pascazio (2010) –
it is not possible to satisfy simultaneously so many constraints...

2-uniform state of 5 and 6 qubits

$$
\begin{aligned}
|\Phi_5\rangle \;=\; & |11111\rangle + |01010\rangle + |01100\rangle + |11001\rangle + \\
& + |10000\rangle + |00101\rangle - |00011\rangle - |10110\rangle,
\end{aligned}
$$

related to 5–qubit error correction code by **Laflamme et al.** (1996)

$$
\begin{aligned}
|\Phi_6\rangle \;=\; & |111111\rangle + |101010\rangle + |001100\rangle + |011001\rangle + \\
& + |110000\rangle + |100101\rangle + |000011\rangle + |010110\rangle.
\end{aligned}
$$

Hadamard matrices (real)

definition

matrix of order N with mutually orthogonal row and columns,

$$HH^* = N\mathbb{1} \ , \qquad H_{ij} = \pm 1. \tag{2}$$

given by

Hadamard matrices (real)

definition

matrix of order N with mutually orthogonal row and columns,

$$HH^* = N\mathbb{1} \ , \qquad H_{ij} = \pm 1. \tag{2}$$

given by **Sylvester** (1867)

The simplest example: one qubit, $N = 2$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{3}$$

m qubit case, $N = 2^m$

$$H_{2^m} = H_2^{\otimes m}, \ . \tag{4}$$

works e.g. for $N = 2, 4, 8, 16, 32, \ldots.$
Furthermore, there exist such matrices for $N = 12, 20, 24, 28, 36, \ldots$

Hadamard matrices II

Hadamard conjecture

Hadamard matrices do exist for $N = 2$ and $N = 4n$ for any $n = 1, 2, \ldots$

After a discovery of $N = 428$ Hadamard matrix
(**Kharaghani** and **Tayfeh-Razaie**, 2005)
this conjecture is known to hold up to $N = 664$

see: Catalogue of Hadamard matrices of **Sloane**
http://neilsloane.com/hadamard

Great challenge in combinatorics

Prove the **Hadamard conjecture**:

Construct Hadamard matrices for every $N = 4n$! !

Orthogonal Arrays

Combinatorial arrangements introduced by **Rao** in 1946 used in statistics and design of experiments, OA(r, N, d, k)

0	0
1	1

0	0	0
0	1	1
1	0	1
1	1	0

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

Orthogonal arrays OA(2,2,2,1), OA(4,3,2,2) and OA(8,4,2,3).

Definition of an **Orthogonal Array**

An array A of size $r \times N$ with entries taken from a d–element set S is called **Orthogonal array** OA(r, N, d, k) with r runs, N factors, d levels, strength k and index λ **if** every $r \times k$ subarray of A contains each $k-$tuple of symbols from S exactly λ times as a row.

Each OA is determined by 4 independent parameters r, N, d, k satisfying **Rao bounds**

$$r \geq \sum_{i=0}^{k/2} \binom{N}{i} (d-1)^i \quad \text{if } k \text{ is even}, \tag{5}$$

$$r \geq \sum_{i=0}^{\frac{k-1}{2}} \binom{N}{i} (d-1)^i + \binom{N-1}{\frac{k-1}{2}} (d-1)^{\frac{k-1}{2}} \quad \text{if } k \text{ is odd}. \tag{6}$$

The **index** λ satisfies relation $r = \lambda d^k$ see **Hedayat, Sloane, Stufken** *Orthogonal Arrays: Theory and Applications* (1999)

Orthogonal Arrays & k-uniform states

A link between them

	orthogonal arrays	multipartite quantum state $\|\Phi\rangle$
r	Runs	Number of terms in the state
N	Factors	Number of qudits
d	Levels	dimension d of the subsystem
k	Strength	class of entanglement (k–uniform)

holds

provided an **orthogonal array** $\mathrm{OA}(r, N, d, k)$

satisfies additional constraints !

(this relation is NOT one–to–one)

k–uniform states and Orthogonal Arrays I

Consider a **pure state** $|\Phi\rangle$ of N qudits,

$$|\Phi\rangle = \sum_{s_1,\ldots,s_N} a_{s_1,\ldots,s_N} |s_1,\ldots,s_N\rangle,$$

where $a_{s_1,\ldots,s_N} \in \mathbb{C}$, $s_1,\ldots,s_N \in S$ and $S = \{0,\ldots,d-1\}$. Vectors $\{|s_1,\ldots,s_N\rangle\}$ form an orthonormal basis.

Density matrix ρ reads

$$\rho_{AB} = |\Phi\rangle\langle\Phi| = \sum_{\substack{s_1,\ldots,s_N \\ s'_1,\ldots,s'_N}} a_{s_1,\ldots,s_N} a^*_{s'_1,\ldots,s'_N} \, |s_1,\ldots,s_N\rangle\langle s'_1,\ldots,s'_N|.$$

We split the system into **two** parts $\mathcal{S}_A$ and $\mathcal{S}_B$ containing N_A and N_B qudits, respectively, $N_A + N_B = N$. and obtain the **reduced state**

$\rho_A = \mathrm{Tr}_B(\rho_{AB})$

$= \sum_{\substack{s_1\cdots s_N \\ s'_1\cdots s'_N}} a_{s_1\ldots s_N} a^*_{s'_1\ldots s'_N} \langle s'_{N_A+1},\ldots,s'_N|s_{N_A+1}\ldots s_N\rangle \, |s_1\ldots s_{N_A}\rangle\langle s'_1\ldots s'_{N_A}|.$

k–uniform states and Orthogonal Arrays II

A simple, **special case**: coefficients $a_{s_1,\ldots,s_N}$ are *zero* or *one*. Then

$$|\Phi\rangle = |s_1^1, s_2^1, \ldots, s_N^1\rangle + |s_1^2, s_2^2, \ldots, s_N^2\rangle + \cdots + |s_1^r, s_2^r, \ldots, s_N^r\rangle,$$

upper index i on s denotes the $i-th$ term in $|\Phi\rangle$. These coefficients can be arranged in an **array**

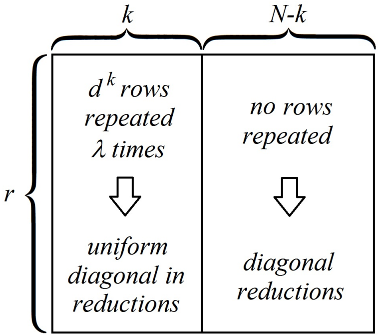
$$A = \begin{array}{cccc} s_1^1 & s_2^1 & \ldots & s_N^1 \\ s_1^2 & s_2^2 & \ldots & s_N^2 \\ \vdots & \vdots & \ldots & \vdots \\ s_1^r & s_2^r & \ldots & s_N^r \end{array}.$$

i). If A forms an **orthogonal array** for any partition the diagonal elements of the reduced state ρ_A are equal.

ii). If the sequence of N_B symbols appearing in every row of a subset of N_B columns **is not repeated** along the r rows (**irredundant OA**), the reduced density matrix ρ_A becomes diagonal.

How to construct a k–uniform state of N qudits ?

a) Take an **orthogonal array** OA(r, N, d, k) of **strength** k.



b) check if condition **ii)** is satisfied, so the array is **irredundant**.

c) If **yes**, write the corresponding k–uniform state!

Very simple examples

a) Two qubit, 1–uniform state

Orthogonal array

$$OA(2,2,2,1) = \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}$$

leads to the **Bell state** $|\Psi_2^+\rangle = |01\rangle + |10\rangle$, which is 1–uniform

b) Three–qubit, 1–uniform state

Orthogonal array

$$OA(4,3,2,{\color{red}2}) = \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

leads to the **balanced, 1–uniform state**,

$$|\Phi_3\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle.$$

Hadamard matrices & Orthogonal Arrays

A Hadamard matrix $H_8 = H_2^{\otimes 3}$ of order $N = 8$ implies OA(8,7,2,2)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \longrightarrow \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$$

This OA allows us to construct a **2–uniform state** of 7 qubits:

$$\begin{aligned} |\Phi_7\rangle \quad = \quad & |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + \\ & |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle . \end{aligned}$$

a **simplex** state $|\Phi_7\rangle$

Examples of 2–uniform states obtained form H_{12}

8 qubits

$$
\begin{aligned}
|\Phi_8\rangle \quad = \quad & |00000000\rangle + |00011101\rangle + |10001110\rangle + |01000111\rangle + \\
& |10100011\rangle + |11010001\rangle + |01101000\rangle + |10110100\rangle + \\
& |11011010\rangle + |11101101\rangle + |01110110\rangle + |00111011\rangle.
\end{aligned}
$$

9 qubits

$$
\begin{aligned}
|\Phi_9\rangle \quad = \quad & |000000000\rangle + |100011101\rangle + |010001110\rangle + |101000111\rangle + \\
& |110100011\rangle + |011010001\rangle + |101101000\rangle + |110110100\rangle + \\
& |111011010\rangle + |011101101\rangle + |001110110\rangle + |000111011\rangle.
\end{aligned}
$$

10 qubits

$$
\begin{aligned}
|\Phi_{10}\rangle \quad = \quad & |0000000000\rangle + |0100011101\rangle + |1010001110\rangle + |1101000111\rangle + \\
& |0110100011\rangle + |1011010001\rangle + |1101101000\rangle + |1110110100\rangle + \\
& |0111011010\rangle + |0011101101\rangle + |0001110110\rangle + |1000111011\rangle,
\end{aligned}
$$

Higher dimensions: uniform states of qutrits and ququarts

From OA(9,4,3,2) we get a 2–**uniform** state of **4 qutrits**:

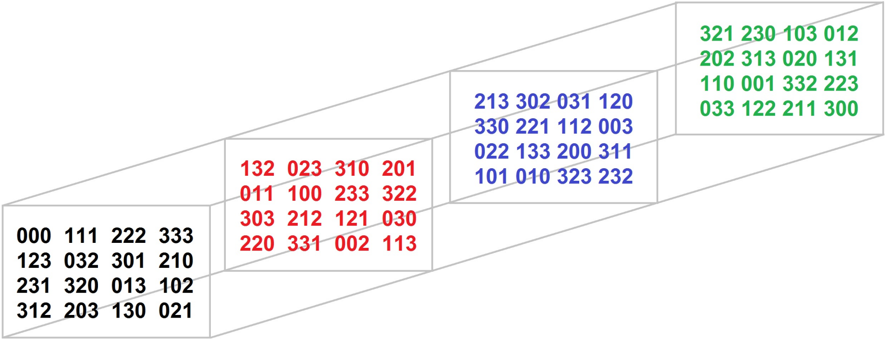
$$
\begin{aligned}
|\Psi_3^4\rangle \quad = \quad & |0000\rangle + |0112\rangle + |0221\rangle + \\
& |1011\rangle + |1120\rangle + |1202\rangle + \\
& |2022\rangle + |2101\rangle + |2210\rangle .
\end{aligned}
$$

This state is also encoded in a pair of orthogonal Latin squares of size 3,

$$
\begin{array}{|c|c|c|}
\hline
0\alpha & 1\beta & 2\gamma \\
\hline
1\gamma & 2\alpha & 0\beta \\
\hline
2\beta & 0\gamma & 1\alpha \\
\hline
\end{array}
\quad = \quad
\begin{array}{|c|c|c|}
\hline
A\spadesuit & K\clubsuit & Q\diamondsuit \\
\hline
K\diamondsuit & Q\spadesuit & A\clubsuit \\
\hline
Q\clubsuit & A\diamondsuit & K\spadesuit \\
\hline
\end{array}
\ .
$$

Corresponding **Quantum Code**:

$$
\begin{aligned}
|0\rangle &\rightarrow |\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle \\
|1\rangle &\rightarrow |\tilde{1}\rangle := |011\rangle + |120\rangle + |202\rangle \\
|2\rangle &\rightarrow |\tilde{2}\rangle := |022\rangle + |101\rangle + |210\rangle
\end{aligned}
$$



State $|\Psi_4^6\rangle$ of **six ququarts** can be generated by three mutually orthogonal **Latin cubes of order four!**

(three quarts + three address quarts = 6 quarts in $4^3 = 64$ terms)

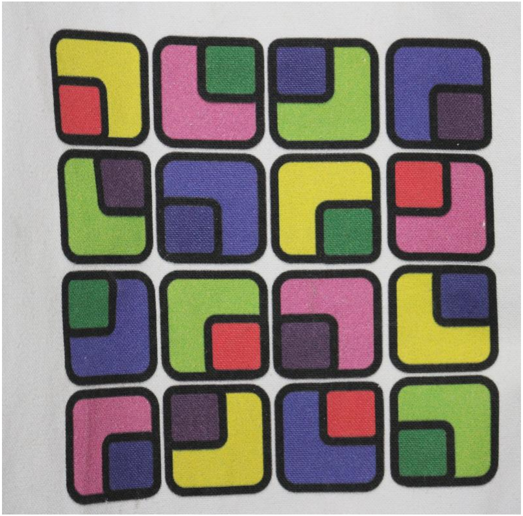
Six ququarts

The same 3–uniform state of **6 ququarts**: read from three **Mutually orthogonal Latin cubes**

$$|\Psi_4^6\rangle =$$

$$|000000\rangle + |001111\rangle + |002222\rangle + |003333\rangle + |010123\rangle + |011032\rangle +$$
$$|012301\rangle + |013210\rangle + |020231\rangle + |021320\rangle + |022013\rangle + |023102\rangle +$$
$$|030312\rangle + |031203\rangle + |032130\rangle + |033021\rangle + |100132\rangle + |101023\rangle +$$
$$|102310\rangle + |103201\rangle + |110011\rangle + |111100\rangle + |112233\rangle + |113322\rangle +$$
$$|120303\rangle + |121212\rangle + |122121\rangle + |123030\rangle + |130220\rangle + |131331\rangle +$$
$$|132002\rangle + |133113\rangle + |200213\rangle + |201302\rangle + |202031\rangle + |203120\rangle +$$
$$|210330\rangle + |211221\rangle + |212112\rangle + |213003\rangle + |220022\rangle + |221133\rangle +$$
$$|222200\rangle + |223311\rangle + |230101\rangle + |231010\rangle + |232323\rangle + |233232\rangle +$$
$$|300321\rangle + |301230\rangle + |302103\rangle + |303012\rangle + |310202\rangle + |311313\rangle +$$
$$|312020\rangle + |313131\rangle + |320110\rangle + |321001\rangle + |322332\rangle + |323223\rangle +$$
$$|330033\rangle + |331122\rangle + |332211\rangle + |333300\rangle.$$

A quick quiz



What **quantum state** can be associated with this design ?

Hints

$$\begin{array}{cccc}
A\spadesuit & K\diamondsuit & Q\heartsuit & J\clubsuit \\
K\heartsuit & A\clubsuit & J\spadesuit & Q\diamondsuit \\
Q\clubsuit & J\heartsuit & A\diamondsuit & K\spadesuit \\
J\diamondsuit & Q\spadesuit & K\clubsuit & A\heartsuit
\end{array}$$

Two **mutually orthogonal** **Latin squares** of size $N = 4$

Hints

$A\spadesuit$	$K\diamondsuit$	$Q\heartsuit$	$J\clubsuit$
$K\heartsuit$	$A\clubsuit$	$J\spadesuit$	$Q\diamondsuit$
$Q\clubsuit$	$J\heartsuit$	$A\diamondsuit$	$K\spadesuit$
$J\diamondsuit$	$Q\spadesuit$	$K\clubsuit$	$A\heartsuit$

Two **mutually orthogonal Latin squares** of size $N = 4$

$A\spadesuit$	$K\diamondsuit$	$Q\heartsuit$	$J\clubsuit$
$K\heartsuit$	$A\clubsuit$	$J\spadesuit$	$Q\diamondsuit$
$Q\clubsuit$	$J\heartsuit$	$A\diamondsuit$	$K\spadesuit$
$J\diamondsuit$	$Q\spadesuit$	$K\clubsuit$	$A\heartsuit$

Three mutually orthogonal Latin squares of size $N = 4$

The answer

Bag shows **three mutually orthogonal** **Latin squares** of size $N = 4$ with three attributes A, B, C of each of $4^2 = 16$ squares.
Appending two indices, $i, j = 0, 1, 2, 3$ we obtain a 16×5 table,
$A_{00}, B_{00}, C_{00}, 0, 0$
$A_{01}, B_{01}, C_{01}, 0, 1$
....
$A_{33}, B_{33}, C_{33}, 3, 3.$
It forms an **orthogonal array OA(16,5,4,2)**
leading to the 2–**uniform** state of **5 ququarts**,

$$
\begin{aligned}
|\Psi_4^5\rangle = \quad & |00000\rangle + |12301\rangle + |23102\rangle + |31203\rangle \\
& |13210\rangle + |01111\rangle + |30312\rangle + |22013\rangle + \\
& |21320\rangle + |33021\rangle + |02222\rangle + |10123\rangle + \\
& |32130\rangle + |20231\rangle + |11032\rangle + |03333\rangle
\end{aligned}
$$

related to the **Reed–Solomon code** of length 5.

Absolutely maximally entangled state (AME)

Definition. A k–uniform state of $N = 2k$ qudits is called **absolutely maximally entangled**

Examples: 2-uniform state $|\Psi_3^4\rangle$ of 4 qutrits,
3-uniform state $|\Psi_4^6\rangle$ of 6 ququarts,

AME state of four parties A, B, C, D, $\quad |\psi\rangle = \sum_{i,j,l,m} G_{ijlm}|i,j,l,m\rangle$
is **maximally entangled** with respect to all **three** partitions:

$AB|CD$ and $AC|BD$ and $AD|BC$.

Let $\rho_{ABCD} = |\psi\rangle\langle\psi|$. Hence its three reductions:
$\rho_{AB} = \mathrm{Tr}_{CD}\rho_{ABCD}$ and $\rho_{AC} = \mathrm{Tr}_{BD}\rho_{ABCD}$ and $\rho_{AD} = \mathrm{Tr}_{BC}\rho_{ABCD}$ are maximally mixed.
Thus matrices $U_{\mu,\nu}$ obtained by reshaping the tensor G_{ijkl}/d are **unitary** for three reorderings: a) $\mu, \nu = ij, lm$, b) $\mu, \nu = im, jl$, c) $\mu, \nu = il, jm$.

Such a tensor G is called **perfect**.

Corresponding unitary matrix U of order d^2 is called **multi-unitary** if reordered matrices U^{R_1} and U^{R_2} remain unitary.

Exemplary multiunitary matrices

multi–unitary permutation matrix of size 9
associated to the AME state $|\Psi_3^4\rangle$ of 4 qutrits

$$U = U_{\substack{ij \\ ml}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in U(9)$$

Furthermore, also two reordered matrices
(by partial transposition and reshuffling) remain **unitary**:

$$U^{T_1} = U_{\substack{il \\ mj}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in U(9)$$

$$U^{R} = U_{\substack{im \\ jl}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in U(9)$$

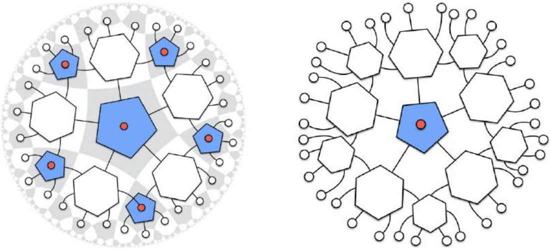
Constructive results

1. Basing on multi–qubit Hadamard matrices, $H_{2^m} = H_2^{\otimes m}$, we constructed 2–**uniform states** of N qubits for any $N \geq 6$.
2. Every orthogonal array of **index unity**, $\mathrm{OA}(d^k, N, d, k)$ allows us to generate a k–**uniform** state of N **qudits** of d levels if and only if $k \leq N/2$.
3. Making use of known results on **orthogonal matrices** we demonstrate existence of show following k–**uniform states**:
   - *(i)* k–uniform states of $d+1$ qudits with d levels, where $d \geq 2$ and $k \leq \frac{d+1}{2}$.
   - *(ii)* 3–uniform states of $2^m + 2$ qudits with 2^m levels, where $m \geq 2$.
   - *(iii)* $(2^m - 1)$–uniform states of $2^m + 2$ qudits with 2^m levels, where $m = 2, 4$.
4. From every k–**uniform** state generated from an OA we construct an entire **orbit** of **maximally entangled** states.
   Three–qubit example: a 3–parameter orbit of 1–uniform states
   $|\Phi_3\rangle(\alpha_1, \alpha_2, \alpha_3) = |000\rangle + e^{i\alpha_1}|011\rangle + e^{i\alpha_2}|101\rangle + e^{i\alpha_3}|110\rangle$,

Cracow

and Tatra mountains in the background

Post Scriptum



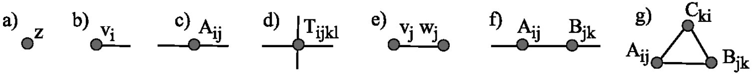
(a) Pentagon/Hexagon code. (b) One qubit code.

possible applications:

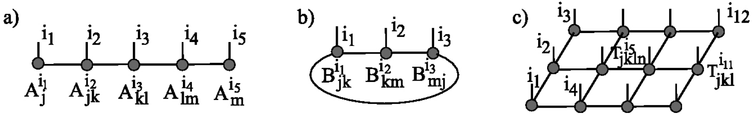
- tensor networks,
- holographic codes,
- toy models for the boundary–bulk correspondence

Tensor networks

i) Tensor network diagrams:

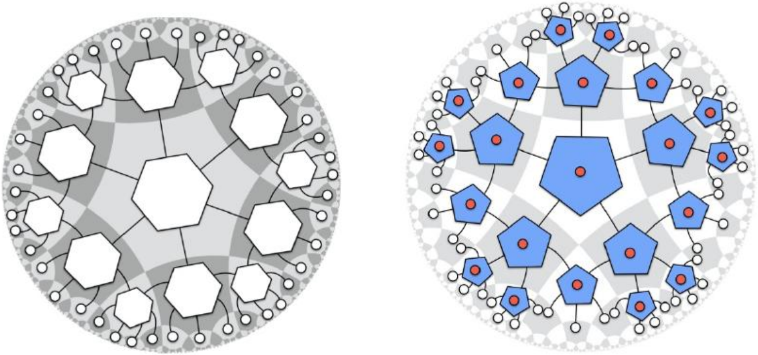


a) scalar z, b) vector v_i, c) matrix A_{ij}, d) 4-index tensor T_{ijkl},
e) scalar product $v_j w_j$, f) product of two matrices, $C_{ik} = A_{ij} B_{jk}$,
g) trace of a three matrix product, $A_{ij} B_{jk} C_{ki} = \mathrm{Tr} ABC = \mathrm{Tr} BCA$.



ii) tensor networks describing matrix product states and projected entangled pair state

Holographic quantum error correction codes:
Pastawski, Yoshida, Harlow, Preskill, JHEP 2015

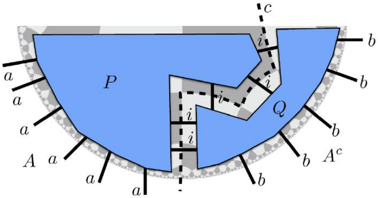


(a) Holographic hexagon state. **(b)** Holographic pentagon code.

Holographic state Holographic code

If in each sites the tensors T_{ijklmn} are **perfect** the code provides a **partial isometry** between the boundary and the bulk !

Area Law



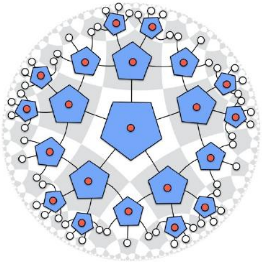
For any subset P defining a partition the entropy S of the corresponding reduced state

$$S\big(\mathrm{Tr}_Q|\psi\rangle\langle\psi|\big) \;=\; M \log d$$

is proportional to the **area** of the set P measured in the number M of edges cut!

Key idea:

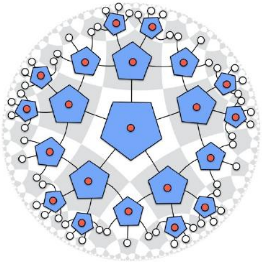
Making use of **absolutely maximally entangled states** (**multiunitary matrices** or **perfect tensors**) one can construct holographic codes, which map the Hilbert space corresponding to the **boundary** into the Hilbert space corresponding to the **bulk**.



(b) Holographic pentagon code.

Key idea:

Making use of **absolutely maximally entangled states** (**multiunitary matrices** or **perfect tensors**) one can construct holographic codes, which map the Hilbert space corresponding to the **boundary** into the Hilbert space corresponding to the **bulk**.



(b) Holographic pentagon code.

Key issue:

Assuming any dynamics (Hamiltonian/theory) at the **boundary**, which dynamics (Hamiltonian/theory) will it imply for the **bulk** ?

CENTRO DE CIENCIAS
DE BENASQUE
PEDRO PASCUAL

AV. FRANCIA 17, 22440 BENASQUE, SPAIN

Multipartite Entanglement

May 22 -- May 27, 2016

Organizers:

G. Gour (Calgary, Canada)
B. Kraus (Innsbruck, Austria)
J. I. Latorre (Barcelona, Spain)
K. Zyczkowski (Kraków, Poland)

Application deadline is March 20, 2016