Tomography of Gaussian states

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n-mode boson Fock space

$$\Gamma(\mathbb{C}^n) = \mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{C}^{n\circledS^2} \oplus \cdots \oplus \mathbb{C}^{n\circledS^r} \oplus \cdots .$$

$\mathbb{C}^n$ = n-dimensional complex Hilbert space of all column vectors

$$\boldsymbol{z} = (z_1, z_2, \cdots, z_n)^T, \quad z_j \in \mathbb{C}.$$

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$$\boldsymbol{e}(\boldsymbol{u}) = 1 \oplus \boldsymbol{u} \oplus \frac{\boldsymbol{u}^{\otimes 2}}{\sqrt{2!}} \oplus \cdots \oplus \frac{\boldsymbol{u}^{\otimes r}}{\sqrt{r!}} \oplus \cdots$$

$$\langle \boldsymbol{e}(\boldsymbol{u}) | \boldsymbol{e}(\boldsymbol{v}) \rangle = \exp \langle \boldsymbol{u} | \boldsymbol{v} \rangle .$$

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Coherent states

$$|\psi(\boldsymbol{u})\rangle = e^{-\frac{\|\boldsymbol{u}\|^2}{2}} |\boldsymbol{e}(\boldsymbol{u})\rangle .$$

Weyl Displacement operators

$$W(\boldsymbol{u})\,|\psi(\boldsymbol{v})\rangle = e^{-\imath\,\mathrm{Im}\,\langle \boldsymbol{u}|\boldsymbol{v}\rangle}\,|\psi(\boldsymbol{u}+\boldsymbol{v})\rangle$$

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(ii) $W(\boldsymbol{u}) = e^{a(\boldsymbol{u})-a^{\dagger}(\boldsymbol{u})}$. $a(\boldsymbol{u})$, $a^{\dagger}(\boldsymbol{u})$ are annihilation, creation (operator) fields with

$$\begin{aligned}
&[a(\boldsymbol{u}), a(\boldsymbol{v})] = 0,\\
&\left[a(\boldsymbol{u}), a^{\dagger}(\boldsymbol{v})\right] = \langle \boldsymbol{u}|\boldsymbol{v}\rangle\\
&\boldsymbol{u} \to a(\boldsymbol{u}) \text{ is antilinear}\\
&\boldsymbol{u} \to a^{\dagger}(\boldsymbol{u}) \text{ is linear}
\end{aligned}$$

(Canonical commutation relations)

The symplectic group $Sp(2n,\mathbb{R})$

$$J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

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$\Gamma(L)$ unitary operator in $\Gamma(\mathbb{C}^n)$, unique upto a scalar multiplication of modulus unity satisfying

$$\Gamma(L)W(\boldsymbol{u})\Gamma(L)^{-1} = W(L\boldsymbol{u}), \quad \forall \boldsymbol{u}.$$
$$\Gamma(L_1)\Gamma(L_2) = \sigma(L_1, L_2)\Gamma(L_1L_2), \quad \sigma(L_1, L_2) = \pm 1$$

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$$(\boldsymbol{u}, L) \to W(\boldsymbol{u})\Gamma(L)$$

Projective unitary representation of $G = \mathbb{C}^n \ltimes Sp(2n, \mathbb{R})$ (semidirect product) describing all symmetries of Gaussian states!

Characteristic functions

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(i) $\hat{\rho}(\mathbf{0}) = 1$, $\hat{\rho}$ is a bounded continuous function.

(ii) For any $z_1, z_2, \cdots, z_m,\ m = 1, 2, \cdots$ the matrix inequality

$$\left[\left[\hat{\rho}(z_r - z_s) e^{\imath \operatorname{Im} \langle z_s | z_r \rangle}\right]\right] \geq 0 \tag{1}$$

holds.

Theorem (Quantum Bochner's theorem: M. D. Srinivas)

if $\phi(\mathbf{z})$ is continuous, $\phi(\mathbf{0}) = 1$ and inequality (1) is fulfilled with $\hat{\rho}$ replaced by ϕ, then there exists a unique state ρ such that $\phi = \rho$.

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Theorem (Inversion theorem)

$$\begin{aligned}\rho &= \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} \hat{\rho}(z) W^{\dagger}(z) \prod_1^n \mathrm{d}x_j\ \mathrm{d}y_j \\ &= \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} \overline{\hat{\rho}(z)} W(z) \prod_1^n \mathrm{d}x_j\ \mathrm{d}y_j\end{aligned}$$

for any state ρ.

Theorem

Let $\mathcal{B}_2(\Gamma(\mathbb{C}^n))$ be the Hilbert space of Hilbert Schmidt operations on $\Gamma(\mathbb{C}^n)$. There is a Hilbert space isomorphism

$$\mathbb{F} : \mathcal{B}_2(\Gamma(\mathbb{C}^n)) \longrightarrow L^2\left(\frac{1}{\pi^n}\prod_1^n \mathrm{d}x_j\ \mathrm{d}y_j\right)$$

such that for any state ρ, $\mathbb{F}(\rho) = \hat{\rho}$. In particular, for any $\rho_1,\ \rho_2 \in \mathcal{B}_2(\Gamma(\mathbb{C}^n))$

$$\mathrm{Tr}\,\rho_1^\dagger \rho_2 = \langle \mathbb{F}(\rho_1) | \mathbb{F}(\rho_2) \rangle$$

A state ρ satisfying

$$\begin{aligned}\hat{\rho}(z) &= \operatorname{Tr}\rho W(z) \\ &= \exp\left[-\imath\sqrt{2}(\mathbf{l}^T\mathbf{x} - \mathbf{m}^T\mathbf{y}) - \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}^T S \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}\right]. \qquad (2)\end{aligned}$$

for all $\boldsymbol{z} = \boldsymbol{x} + \imath\boldsymbol{y} \in \mathbb{C}^n,\ \boldsymbol{x} = \operatorname{Re}\boldsymbol{z},\ \boldsymbol{y} = \operatorname{Im}\boldsymbol{z}$, where $\boldsymbol{l}, \boldsymbol{m} \in \mathbb{R}^n$, and S is a symmetric $2n \times 2n$ real matrix, is called *n-mode Gaussian state*.

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Theorem 2.1

There exists a state ρ such that $\hat{\rho}$ satisfies (2) if and only if the matrix S satisfies the matrix inequality $S + \frac{\imath}{2}J_{2n} \geq 0$. In such a case $\boldsymbol{l} = \operatorname{Tr}\rho\boldsymbol{p},\ \boldsymbol{m} = \operatorname{Tr}\rho\boldsymbol{q}$ and $S =$ covariance matrix of $(p_1, p_2, \cdots, p_n, -q_1, -q_2, \cdots, -q_n)$.

Here $\frac{\imath}{\sqrt{2}}(a^\dagger(\boldsymbol{z}) - a(\boldsymbol{z})) = \sum(x_j p_j - y_j q_j),\ p_j,\ q_j$ being momentum and position observables.

Corollary 2.2

In a Gaussian state any real linear combinations of the momentum and position observables has a normal distribution.

Remark

In (2), $\boldsymbol{l}$ is the mean momentum, $\boldsymbol{m}$ is the mean position and S is the covariance matrix of $(p_1, \cdots, p_n, -q_1, \cdots, -q_n)$ and the corresponding Gaussian state is completely determined by $n + n + \frac{2n(2n+1)}{2} = 2n^2 + 3n$ real parameters. We denote the corresponding Gaussian state satisfying (2) by

$$\rho_g(\boldsymbol{l}, \boldsymbol{m}; S).$$

Theorem 2.2

Marginals and products of Gaussian states are Gaussian. If ρ is Gaussian, so is

$$W(\boldsymbol{u})\Gamma(L)\rho\Gamma(L)^{\dagger}W(\boldsymbol{u})^{\dagger}.$$

Conversely, if U is a unitary operator and $U\rho U^{\dagger}$ is Gaussian whenever ρ is, then there exists $\boldsymbol{u} \in \mathbb{C}^n$, $L \in Sp(2n, \mathbb{R})$ and scalar λ of modulus unity such that $U = \lambda W(\boldsymbol{u})L$.

Theorem 2.3

$$\begin{aligned} & \mathrm{Tr}\;\; \rho_g(\mathbf{l_1}, \mathbf{m_1}; S)\rho_g(\mathbf{l_2}, \mathbf{m_2}; T) \\ = \;\; & \frac{1}{\sqrt{\det(S+T)}} \exp\left[-\frac{1}{2}\begin{bmatrix} \mathbf{l_1} - \mathbf{l_2} \\ -(\mathbf{m_1} - \mathbf{m_2}) \end{bmatrix}^T (S+T)^{-1} \begin{bmatrix} \mathbf{l_1} - \mathbf{l_2} \\ -(\mathbf{m_1} - \mathbf{m_2}) \end{bmatrix}\right]. \end{aligned}$$

Proof: By evaluation of Gaussian integrals. $\square$

Corollary 2.4

Choose $\boldsymbol{l}_1 = \boldsymbol{l},\ \boldsymbol{m_1} = \boldsymbol{m},\ S_1 = S,\ S_2 = T,$

$$\begin{aligned} T &= \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}, \\ D &= \operatorname{diag}\left\{\frac{1}{2}\left(\frac{1+e^{-t_j}}{1-e^{-t_j}}\right), j = 1, 2, \cdots, n\right\} \end{aligned}$$

so that

$$\rho_g(\mathbf{0}, \mathbf{0}; T) = \prod_{j=1}^{n} (1 - e^{-t_j}) e^{-\sum_{j=1}^{n} t_j a_j^\dagger a_j}$$

where $a_j = a(e_j)$, (product of thermal states)

$$\begin{aligned} & \operatorname{Tr} \rho_g(\mathbf{l}, \mathbf{m}; S) e^{-\sum_{j=1}^n t_j a_j^\dagger a_j} \\ = & \frac{\exp\left[-\frac{1}{2}\begin{bmatrix}\mathbf{l} \\ -(\mathbf{m})\end{bmatrix}^T \left(S + \begin{bmatrix} D & 0 \\ 0 & D\end{bmatrix}\right)^{-1} \begin{bmatrix}\mathbf{l} \\ -(\mathbf{m})\end{bmatrix}\right]}{\prod_1^n (1 - e^{-t_j}) \sqrt{\det\left(S + \begin{bmatrix} D & 0 \\ 0 & D\end{bmatrix}\right)}}. \end{aligned}$$

Putting $e^{-t_j} = x_j$, $1 \leq j \leq n$, RHS is the joint probability generating function of the distribution of number of particles in different modes in Gaussian state $\rho_g(\boldsymbol{l}, \boldsymbol{m}; S)$.

Corollary 2.5

Let $N = \sum_{j=1}^{n} a_j^\dagger a_j$ (total number of particles). In the state $\rho_g(\boldsymbol{l}, \boldsymbol{m}; S)$,

$$
\begin{aligned}
\langle N \rangle &= \frac{1}{2}\left\{ \mathrm{Tr}\left(S - \frac{1}{2} \right) + \boldsymbol{l}^T \boldsymbol{l} + \boldsymbol{m}^T \boldsymbol{m} \right\} \\
\langle N^2 \rangle - \langle N \rangle^2 &= \frac{1}{2}\mathrm{Tr}\left(S - \frac{1}{2} \right)\left(S + \frac{1}{2} \right) + \begin{pmatrix} \boldsymbol{l} \\ -\boldsymbol{m} \end{pmatrix}^T S \begin{pmatrix} \boldsymbol{l} \\ -\boldsymbol{m} \end{pmatrix},
\end{aligned}
$$

Proof: Compute from probability generating function of N. □

Theorem 2.6

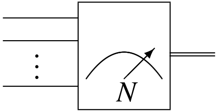
In the state $\rho_g(\boldsymbol{l}, \boldsymbol{m}; S)$

1. $\left\langle W(\boldsymbol{x} + \imath\boldsymbol{y})^\dagger N\, W(\boldsymbol{x} + \imath\boldsymbol{y})\right\rangle - \langle N\rangle = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + \sqrt{2}(\boldsymbol{y}^T\boldsymbol{l} + \boldsymbol{x}^T\boldsymbol{m})$
2. $$\begin{aligned} & \left\langle \Gamma(L)^\dagger N \Gamma(L)\right\rangle - \langle N\rangle \\ = \;& \frac{1}{2}\left[\mathrm{Tr}\; S\left(L^{-1}L^{-1^T} - I_{2n}\right) + \begin{pmatrix} \boldsymbol{l} \\ -\boldsymbol{m}\end{pmatrix}^T \left(L^{-1}L^{-1^T} - I_{2n}\right)\begin{pmatrix} \boldsymbol{l} \\ -\boldsymbol{m}\end{pmatrix}\right]. \end{aligned}$$

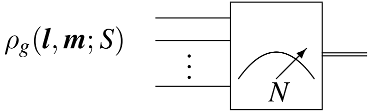
Proof: It follows from Corollary 2.5 and the transformation property of a Gaussian state under conjugation by $W(\boldsymbol{u})$ and $\Gamma(L)$, $\boldsymbol{u} \in \mathbb{C}^n$, $L \in Sp(2n, \mathbb{R})$.

Remark: Theorem 2.6 tells us that the parameters $\boldsymbol{l}, \boldsymbol{m}, S$ can be estimated from measurements of the *total number observable* after applying the *gates* $W(\boldsymbol{u})$ and $\Gamma(L)$ for suitably chosen elements $\boldsymbol{u} \in \mathbb{C}^n$ and $L \in Sp(2n, \mathbb{R})$. This can be expressed in the language of *circuits* in a quantum computer.

$\rho_g(\boldsymbol{l}, \boldsymbol{m}; S)$



Estimate $\langle N \rangle = \frac{1}{2}\left\{ \mathrm{Tr}\,\left(S - \frac{1}{2}\right) + \boldsymbol{l}^T\boldsymbol{l} + \boldsymbol{m}^T\boldsymbol{m} \right\}$



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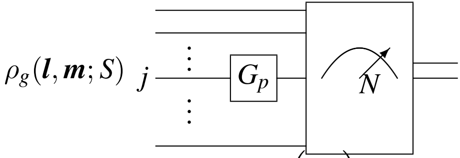
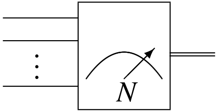


Estimate $\langle N \rangle + \frac{1}{2} + l_j$

$$G_p = W\left(\frac{\imath}{\sqrt{2}}\right)$$

$\rho_g(\boldsymbol{l}, \boldsymbol{m}; S)$



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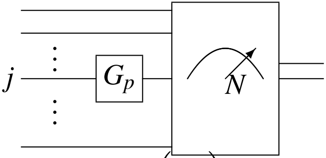
$\rho_g(\boldsymbol{l}, \boldsymbol{m}; S)$



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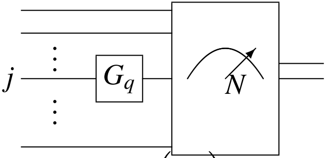
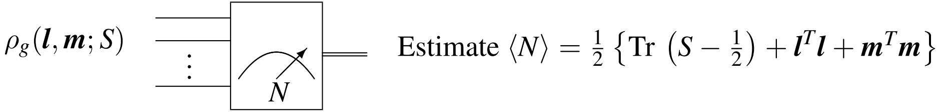
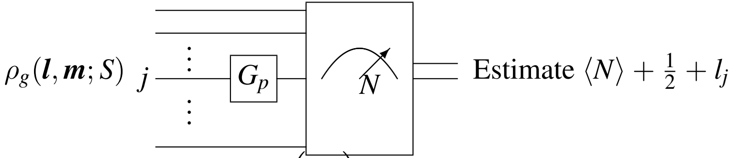


Estimate $\langle N \rangle + \frac{1}{2} + m_j$

$$G_q = W\left(\frac{\imath}{\sqrt{2}}\right)$$





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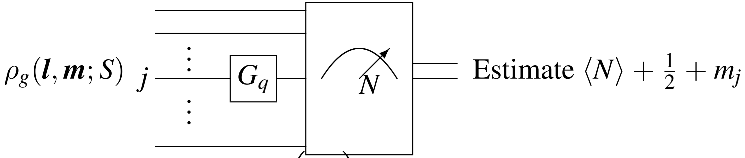

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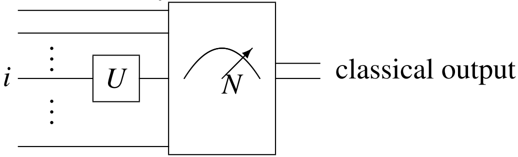
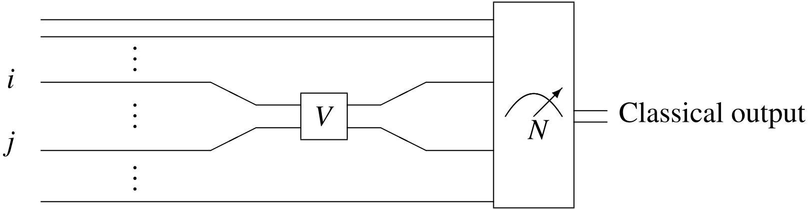
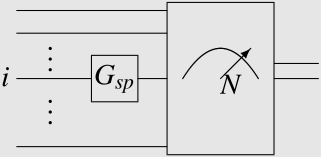
$2n + 1$measurements in all. 2 displacement gates $G_p,\ G_q$ used.
$2n + 1$ parameters $l_j,\ m_j,\ \mathrm{Tr}\, S,\ j = 1, 2, \cdots, n$ estimated.

Writing the covariance matrix S in the form $S = [[S_{ij}]]$ with 2×2 block matrices where S_{ii} is the 2×2 covariance matrix of the i-th mode and applying 1-mode gates of the form $\Gamma(\tau(L)),\ \tau(L) = (L^{-1})^T$ in the i-th mode and using the estimated values of $\boldsymbol{l}$, $\boldsymbol{m}$one obtains the estimation of blocks S_{ii} for each i. Then using the suitable $L \in Sp(4, \mathbb{R})$ with different L it is possible to estimate the off diagonal blocks S_{ij}. Pictorially





Using 3 one mode symplectic gates G_{sp} and circuit in $n-1$ modes and only two of them in n-th mode



Estimate (2) in Theorem 2.6

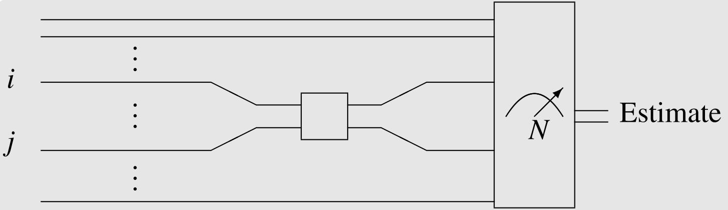
$G_{sp} \in Sp(2, \mathbb{R})$

Get an estimate of all diagonal blocks S_{ii}, $1 \leq i \leq n$

$$\# \text{ measurements } 2(n-1) + 2 = 3n - 1$$

Up to this stage # measurements $= 3n - 1 + 2n + 1 = 5n$.

Then we use 4 two mode symplectic gates which are suitably chosen and the earlier estimations of $\boldsymbol{l}$, $\boldsymbol{m}$, S_{ii}, $i = 1, 2, \cdots, n$ and obtain estimates of the off diagonal blocks S_{ij}



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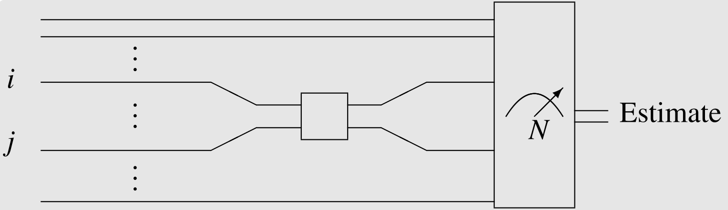

$\#$ measurements $4n(1 + 2 + \cdots + n - 1) = 2n^2 - 2n$

Resources used

1. Two one mode displacement gates.
2. Three one mode symplectic gates.
3. Four two mode symplectic gates.

Total number of measurements $= 2n^2 + 3n$.
Total number of parameters $= 2n^2 + 3n$.

Tomography of Gaussian channels

A Gaussian channel $\mathcal{K}(A, B)$ is described by a pair of real $2n \times 2n$ matrices $A,\ B$ where B is positive and the matrix inequality

$$B + \imath(A^T J_{2n} A - J_{2n}) \geq 0$$

holds.

Input $\rho_g(\boldsymbol{l}, \boldsymbol{m}; S)$ ——— $\boxed{\mathcal{K}(A, B)}$ — Output $\rho_g(\boldsymbol{l}', \boldsymbol{m}'; S')$

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Problem

Using Gaussian state tomography on output states for a few Gaussian input states, estimate A and B.

$$\# \text{ parameters involved} = (2n)^2 + \frac{2n(2n+1)}{2} = 6n^2 + n.$$

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A coherent state is $|\psi(\boldsymbol{u})\rangle\langle\psi(\boldsymbol{u})|$, $\boldsymbol{u} = \boldsymbol{x} + \imath\boldsymbol{y}$ is also

$$\rho_g\left(\sqrt{2}\boldsymbol{y}, \sqrt{2}\boldsymbol{x}; \frac{1}{2}I_{2n}\right).$$

The output is

$$\rho_g\left(\boldsymbol{y}', \boldsymbol{x}'; \frac{1}{2}(A^T A + B)\right),$$

where

$$\begin{bmatrix} \boldsymbol{y}' \\ -\boldsymbol{x}' \end{bmatrix} = A^T \begin{bmatrix} \sqrt{2}\boldsymbol{y} \\ \sqrt{2}\boldsymbol{x} \end{bmatrix}.$$

Use input states

$$\left|\psi\left(\tfrac{\imath}{\sqrt{2}}e_j\right)\right\rangle\left\langle\psi\left(\tfrac{\imath}{\sqrt{2}}e_j\right)\right| \quad j=1,2,\cdots,n$$
$$\left|\psi\left(\tfrac{1}{\sqrt{2}}e_j\right)\right\rangle\left\langle\psi\left(\tfrac{1}{\sqrt{2}}e_j\right)\right| \quad j=1,2,\cdots,n.$$

Then the corresponding output states are

$$\rho_g\left(\boldsymbol{l}_j,\boldsymbol{m}_j;\tfrac{1}{2}(A^TA+B)\right) \quad j=1,2,\cdots,n$$
$$\rho_g\left(\boldsymbol{l}'_j,\boldsymbol{m}'_j;\tfrac{1}{2}(A^TA+B)\right) \quad j=1,2,\cdots,n.$$

where

$$\begin{pmatrix}\boldsymbol{l_j}\\ -\boldsymbol{m_j}\end{pmatrix}=A^T\begin{pmatrix}\boldsymbol{e_j}\\ \boldsymbol{0}\end{pmatrix}, \quad \begin{pmatrix}\boldsymbol{l'_j}\\ -\boldsymbol{m'_j}\end{pmatrix}=A^T\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{e_j}\end{pmatrix}$$

Perform full Gaussian tomography with $2n^2+3n$ measurements on the first output with $j=1$ and determine the first row of A and the matrix A^TA+B.

On the next $2n-1$ outputs perform the tomography only for momentum position means with only $2n+1$ measurements each so that $(2n-1)(2n+1)$ measurements are used. Then we get all the remaining $2n-1$ rows of A. This determines A. Subtract A^TA from A^TA+B to get B. Thus

$$2n^2+3n+(2n-1)(2n+1)=6n^2+3n-1$$

measurements yield all the $6n^2+n$ parameters of the channel.