Stronger uncertainty relations

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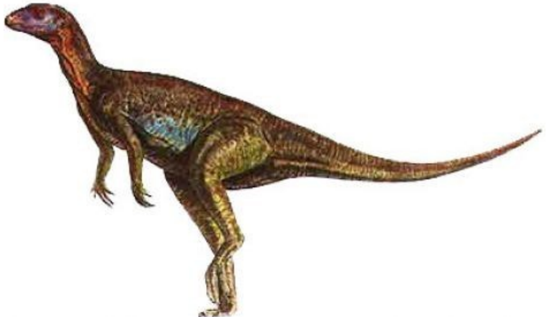
What I'm going to talk about

Heisenberg uncertainty relations do NOT convey fully the notion of complementarity



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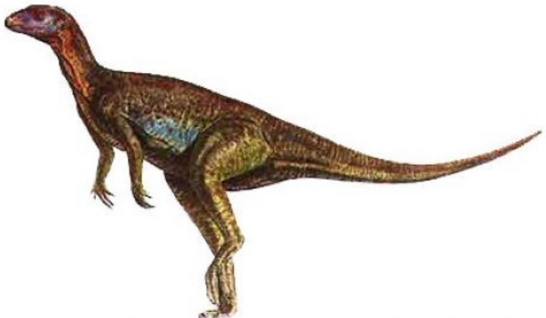
both terms can be zero, even if A and B are incompatible



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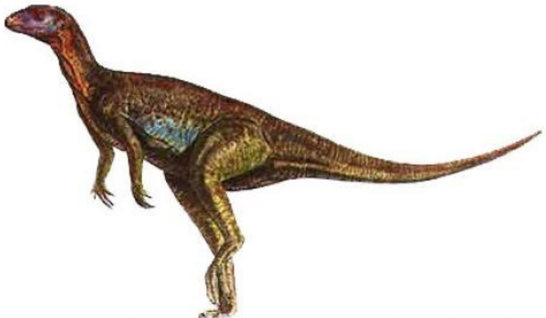
e.g. if the state of the system is an **eigenstate of A**

$$0 \geqslant 0$$



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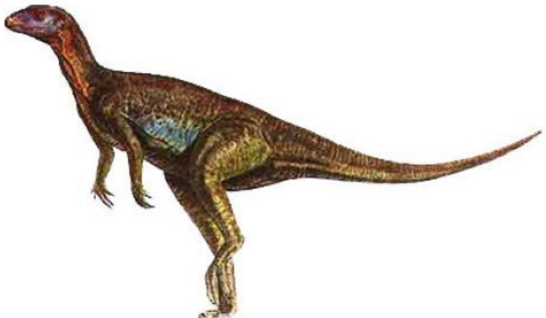

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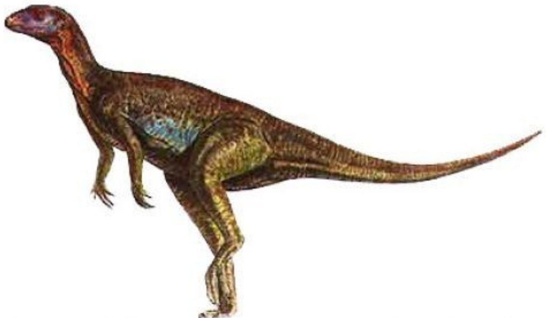

Can we do better? ➘ YES!

Reminder

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Heisenberg-Robertson UR

$$\Delta A^2 \Delta B^2 \geqslant \left| \frac{1}{2} \langle [A, B] \rangle \right|^2$$

variances and expectations calculated on system state $|\psi\rangle$

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both trivial if $|\psi\rangle$=eigenstate of *A* **<u>OR</u>** *B*

we need to give a bound to the ***sum***

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(the product $\Delta A^2 \Delta B^2$ is trivially zero if one of the two is)

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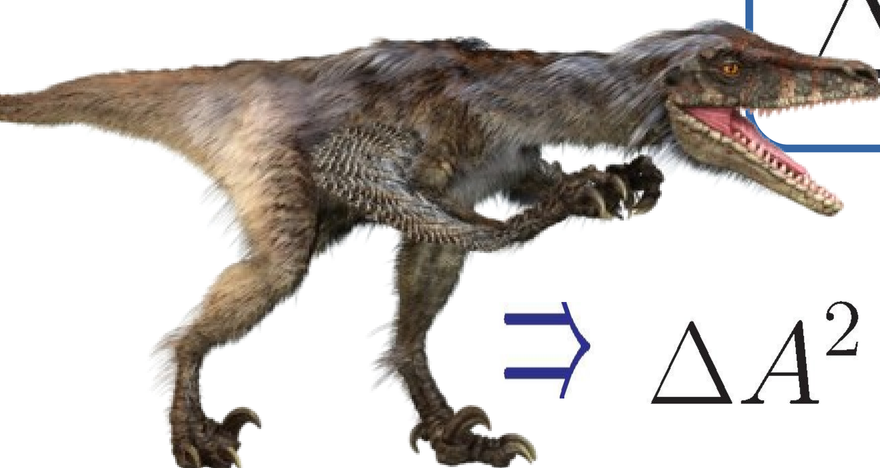

$$\boxed{\Delta A^2 + \Delta B^2 \geqslant \mathcal{M}}$$

$$\Rightarrow \quad \Delta A^2 + \Delta B^2 \geqslant \max(\mathcal{L}, \mathcal{M})$$

SUM of variances

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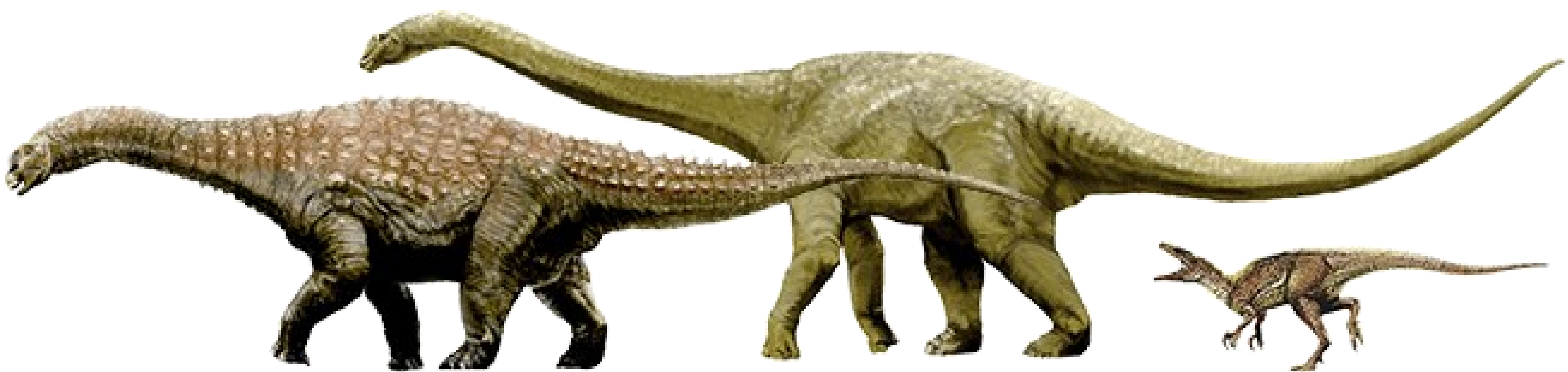
dimensional regularization needed
if *A* and *B* have different units



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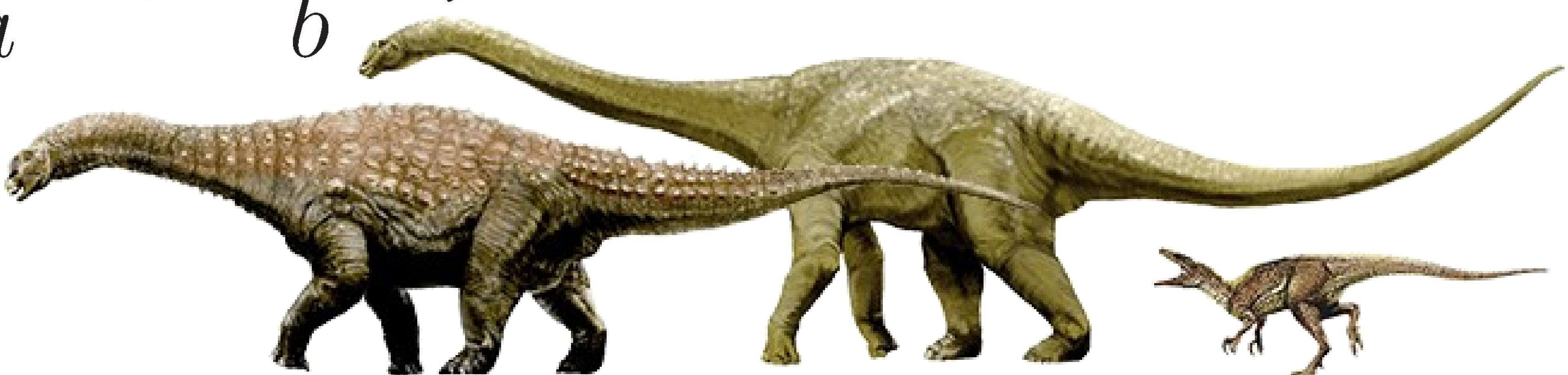
$$\frac{\Delta A^2}{a} + \frac{\Delta B^2}{b} \geqslant \ldots$$



First bound:

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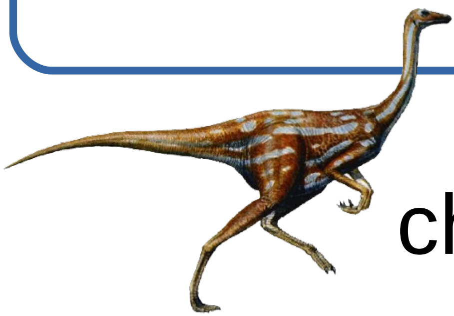

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constructive procedure to choose $|\psi^\perp\rangle$, or just choose a random one

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Green: Heisenb-Robertson UR

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Red: new part, always $\neq 0$ except if $|\psi\rangle$ is joint eigenstate of *A* **and** *B*

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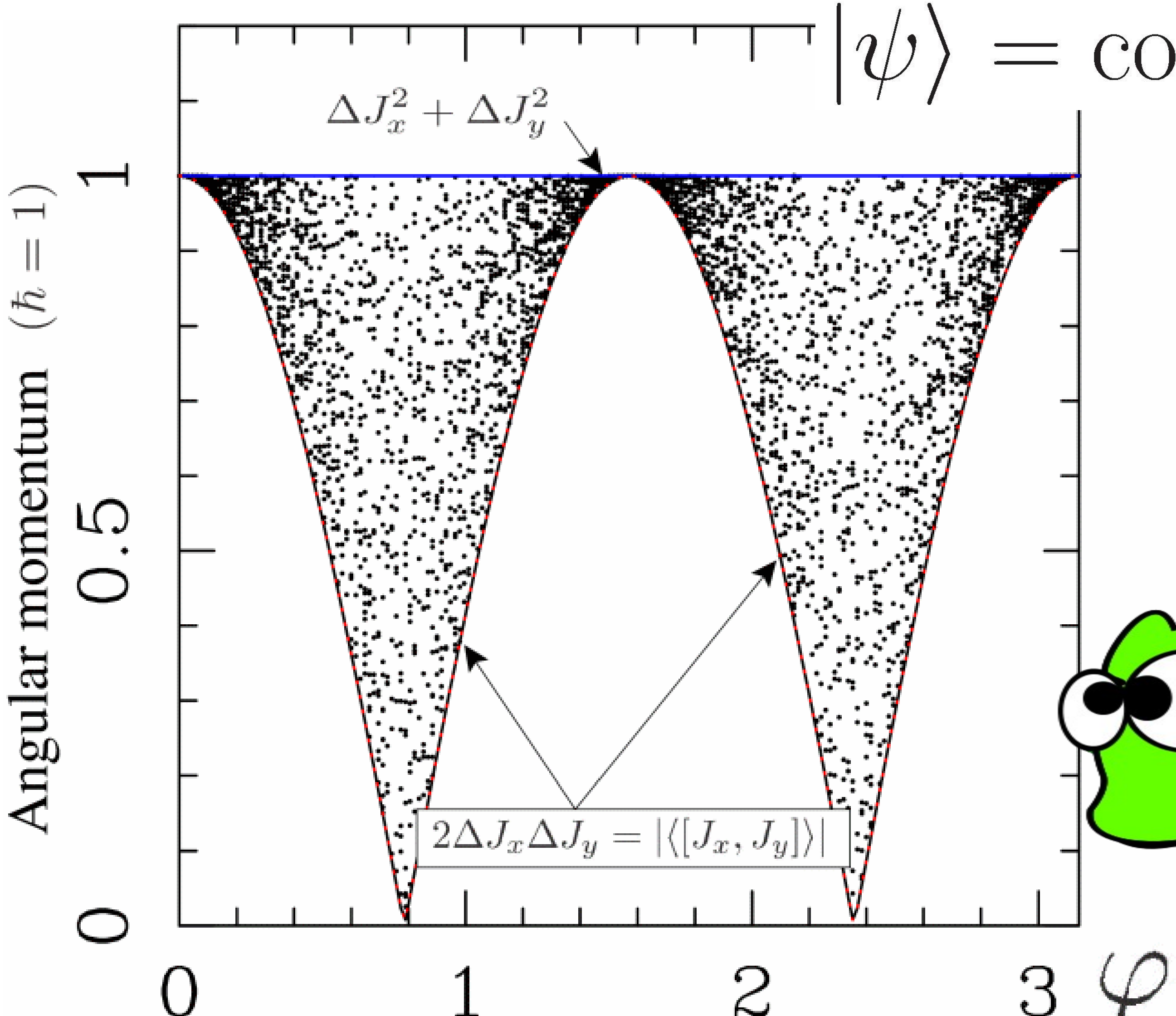
Minimizing over $|\psi^\perp\rangle$ → only HR (red)

Maximizing over $|\psi^\perp\rangle$ → ineq becomes eq

First bound: numerical test for random $|\psi^{\perp}\rangle$

$$\Delta A^2 + \Delta B^2 \geqslant \pm i\langle[A, B]\rangle + \left|\langle\psi|A \pm iB|\psi^{\perp}\rangle\right|^2$$

$$|\psi\rangle = \cos\varphi|0\rangle + \sin\varphi|1\rangle$$



$$A = \sigma_x/2$$

$$B = \sigma_y/2$$

First bound: proof

$$\Delta A^2 + \Delta B^2 \geqslant \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi^\perp \rangle \right|^2$$

application of Schwartz inequality

(like the Robertson)

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$$C \equiv A - \langle A\rangle;\ D \equiv B - \langle B\rangle$$

$$\|(C \mp iD)|\psi\rangle\|^2 = \Delta A^2 + \Delta B^2 \mp i\langle [A, B]\rangle$$

$$|\langle \psi | (A \pm iB) | \psi^\perp\rangle|^2 = |\langle \psi | A \pm iB - \langle A \pm iB\rangle | \psi^\perp\rangle|^2$$
$$= |\langle \psi | C \pm iD | \psi^\perp\rangle|^2 \leqslant \|(C \mp iD)|\psi\rangle\|^2$$

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proof by an anonymous referee
Masanao Ozawa
(original proof more complicated)

First bound: minimum uncert. st.

togliere questa slide: tutti gli stati sono minimum uncertainty states peri il primo bound: quando ottimizzo su |psi^perp>, la disuguaglianza diventa sempre una uguaglianza!

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Harmonic osc. $X = \frac{1}{\sqrt{2}}(a + a^\dagger), \ P = \frac{i}{\sqrt{2}}(a^\dagger - a)$

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MUS

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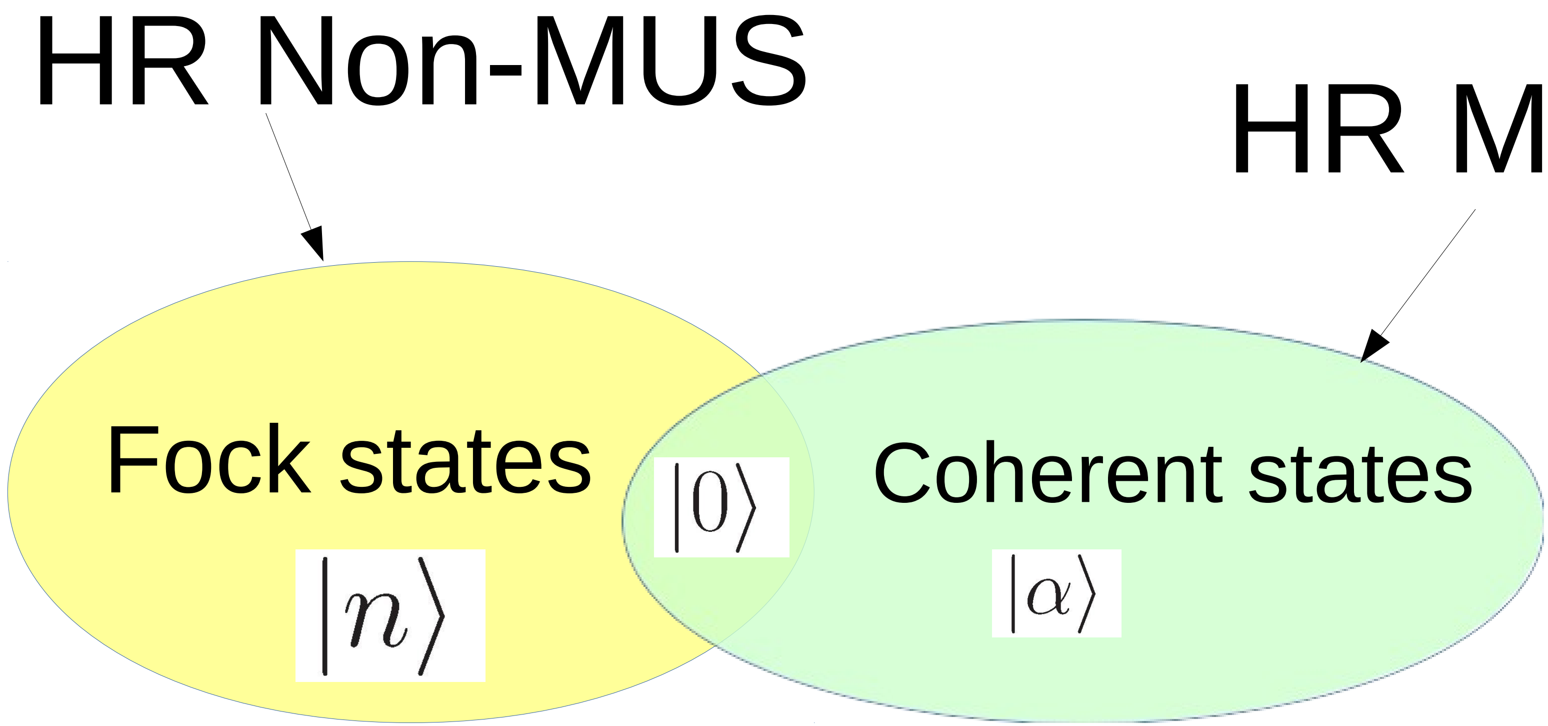
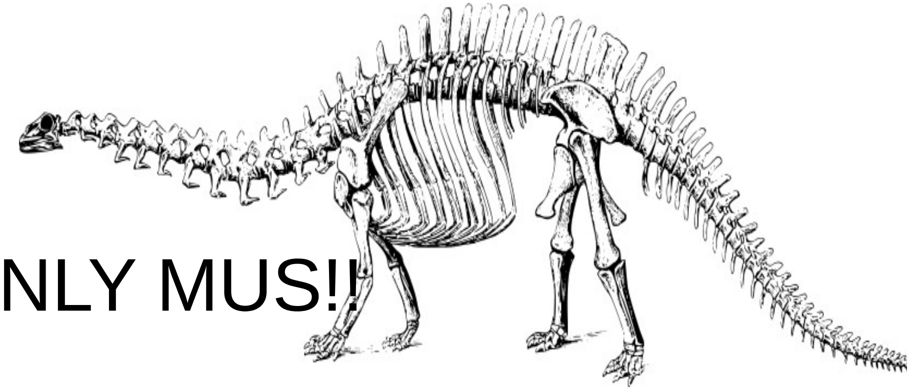
MUS

- Coherent states $|\alpha\rangle$

$$\Delta X^2 + \Delta P^2 = 1 \equiv \mathcal{L}$$



They're all MUS (for **first**, but not **second** bound)



These are just examples, not the ONLY MUS!!

Second bound:

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this bound is $\neq 0$ if $|\psi\rangle$ is not an eigenstate of *A*+*B*

Second bound: example

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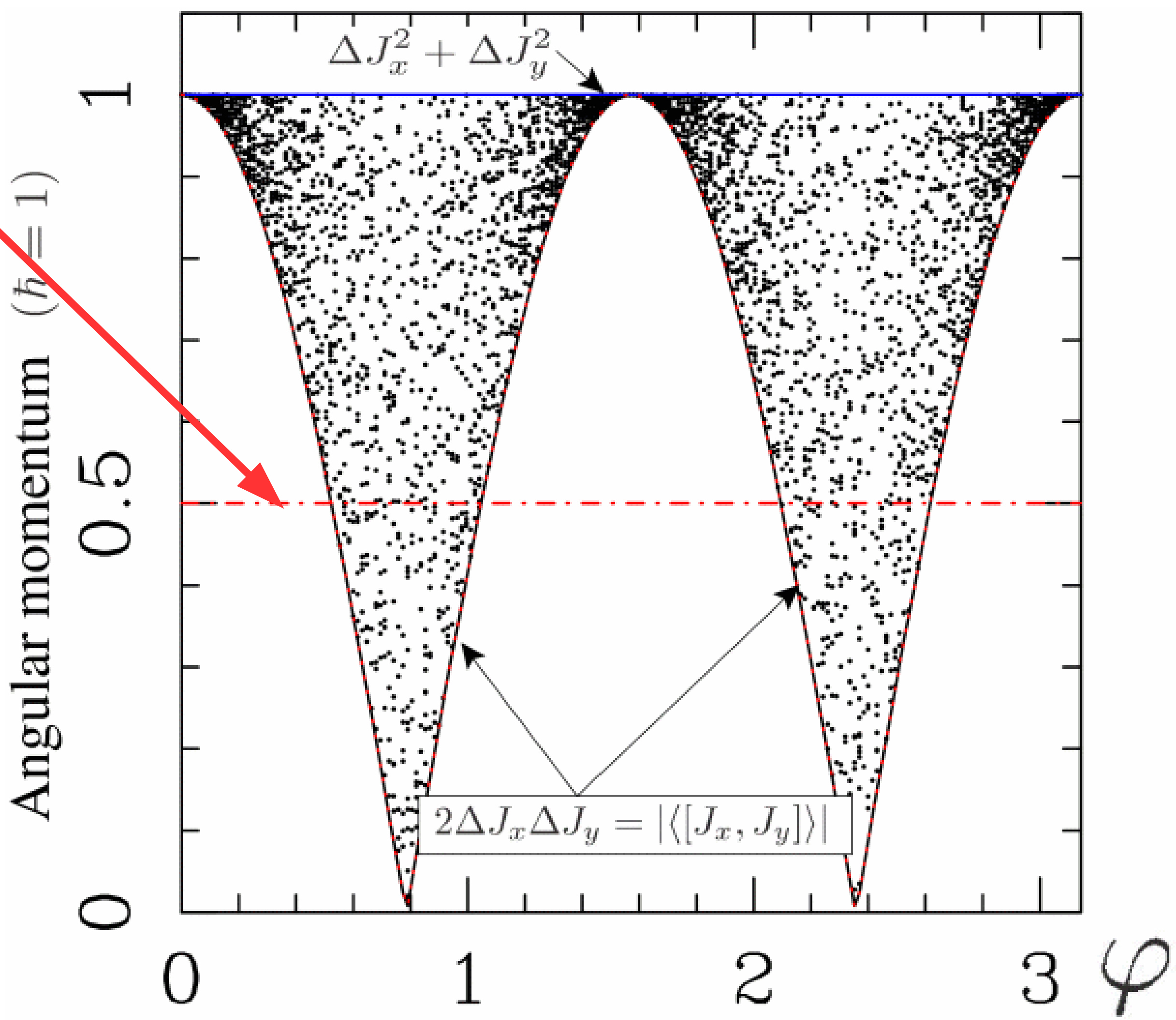

Second bound: proof

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application of the parallelogram ineq.

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$$2\Delta A^2 + 2\Delta B^2 = \|(C+D)|\psi\rangle\|^2 + \|(C-D)|\psi\rangle\|^2$$

$$\Delta(A+B) = \|(C+D)|\psi\rangle\|, \; \Delta(A-B) = \|(C-D)|\psi\rangle\|$$

$$\begin{aligned}\Delta A^2 + \Delta B^2 &= \tfrac{1}{2}[\Delta(A+B)^2 + \Delta(A-B)^2] \\ &\geqslant \tfrac{1}{2}\Delta(A+B)^2,\end{aligned}$$

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$$\Delta A^2 + \Delta B^2 \geqslant \frac{1}{2}\Delta S^2$$

$$\boxed{\min_{|\psi\rangle} \max_{|\psi^\perp(\psi)\rangle} \pm i\langle[A,B]\rangle + \left|\langle\psi|A \pm iB|\psi^\perp\rangle\right|^2}$$

Using the same techniques, we can

tighten the Heis-Rob bound!

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tighten the Heis-Rob bound!

$$\Delta A \Delta B \geqslant \pm\frac{i}{2}\langle[A,B]\rangle \Big/ \left(1 - \frac{1}{2}\left|\langle\psi|\frac{A}{\Delta A} \pm i\frac{B}{\Delta B}|\psi^{\perp}\rangle\right|^{2}\right)$$

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Heisenberg uncertainty

two

mu

paper w

u pri

tradeoff: A p ces a

disturbance

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In this talk: unc

RELATIONS!

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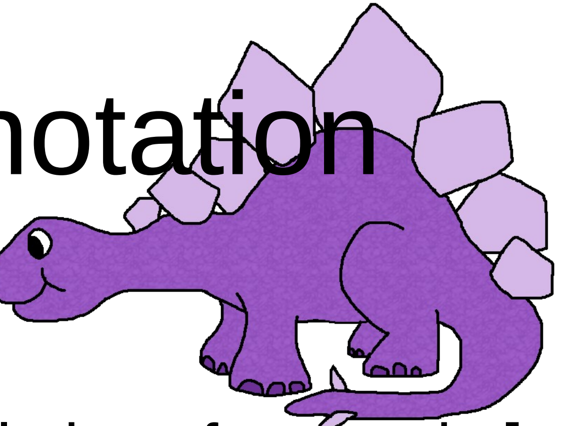
Asher Peres introduced this notation



<u>uncertainty relation</u>:

“The only correct interpretation of [the uncertainty relations for x and p] is the following: If the **same preparation** procedure is repeated many times, and is followed either by a measurement of x, or by a measurement of p, the various results obtained for x and for p have standard deviations, Δx and Δp, whose product cannot be less than ħ/2. **There never is any question here that a measurement of x ‘disturbs’ the value of p and vice versa, as sometimes claimed**. These measurements are indeed incompatible, but they are performed on different particles (all of which were identically prepared) and therefore these measurements cannot disturb each other in any way. The uncertainty relation [...] only reflects the intrinsic randomness of the outcomes of quantum tests.”

(he didn't want to talk about unc principle)

try looking up “uncertain principle” in his book.

What did I say?!?

Lorenzo Maccone
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stronger uncertainty relations for variances with nontrivial lower bound.

$$\Delta A^2 + \Delta B^2 \geqslant \pm i\langle[A,B]\rangle + \left|\langle\psi|A \pm iB|\psi^\perp\rangle\right|^2$$

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Entanglement and Complementarity

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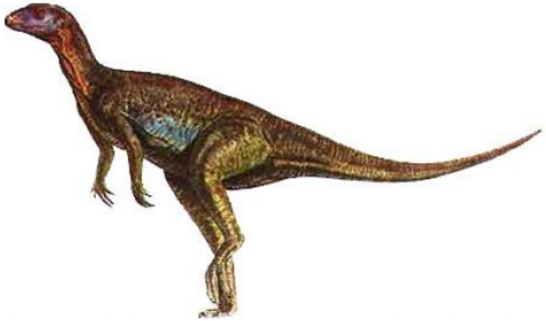
What I'm going to talk about

We always say that entangled states are more correlated... WHAT DOES IT MEAN exactly?



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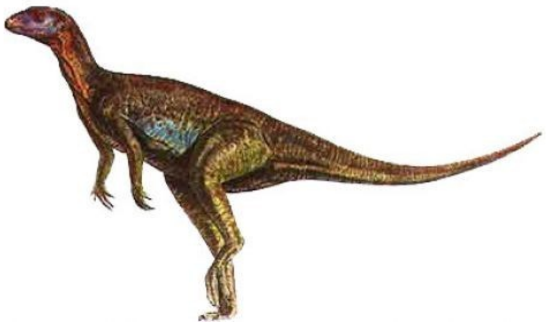
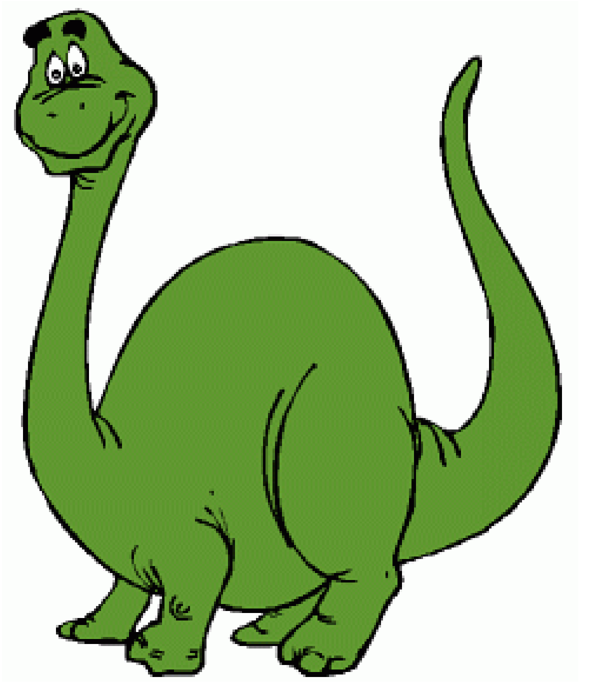
they have more correlations among complementary observables than separable ones



Usual approaches to study entanglement



- Non locality
- Negative partial transpose
- Bell inequality violations
- Enhanced precision in measurements
- etc.

Here: we use correlations among two (or more) COMPLEMENTARY PROPERTIES

different way to think about entanglement, as correlations among complementary properties

Remember: Complementary properties.

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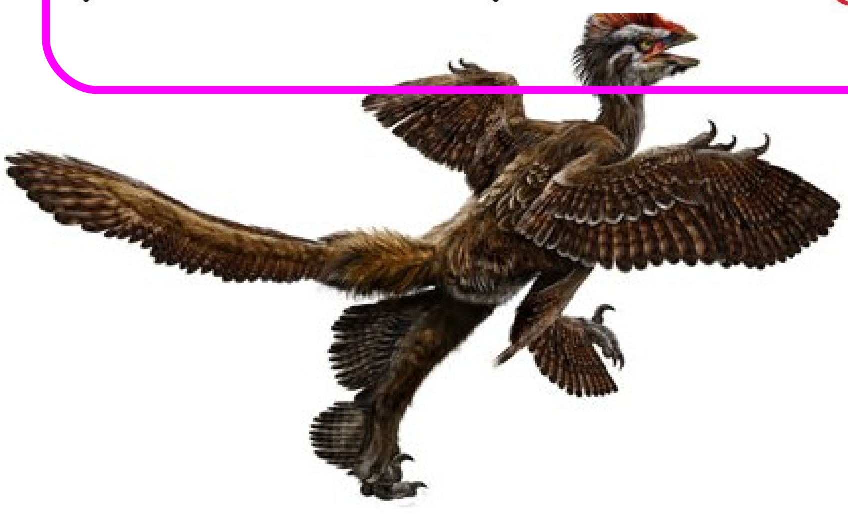
Two observables: the knowledge of one gives no knowledge of the other

$$A = \sum_a f(a) |a\rangle\langle a|$$

$$C = \sum_c g(c) |c\rangle\langle c|$$

$$\boxed{\left| \langle a | c \rangle \right|^2 = \frac{1}{d}}$$



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$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|++\rangle + |--\rangle}{\sqrt{2}}$$

Maximally entangled state: perfect correlation BOTH on 0/1 and on +/-

$$|\pm\rangle \equiv \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$$

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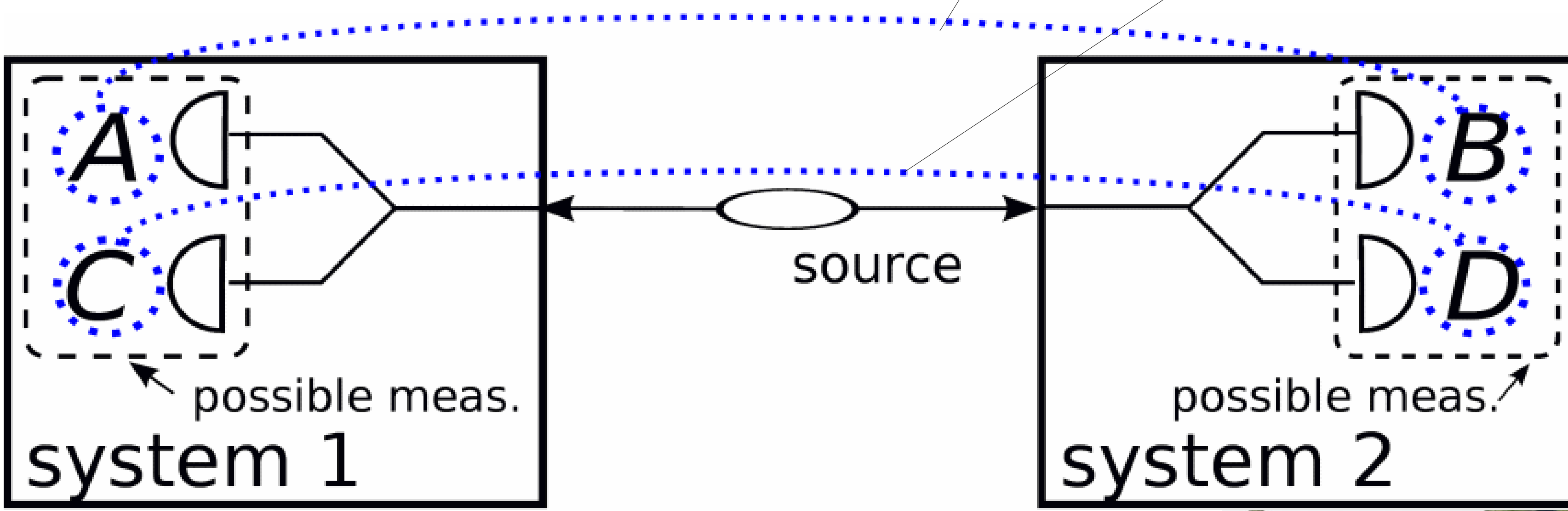
Maximally entangled state: perfect correlation BOTH on 0/1 and on +/-

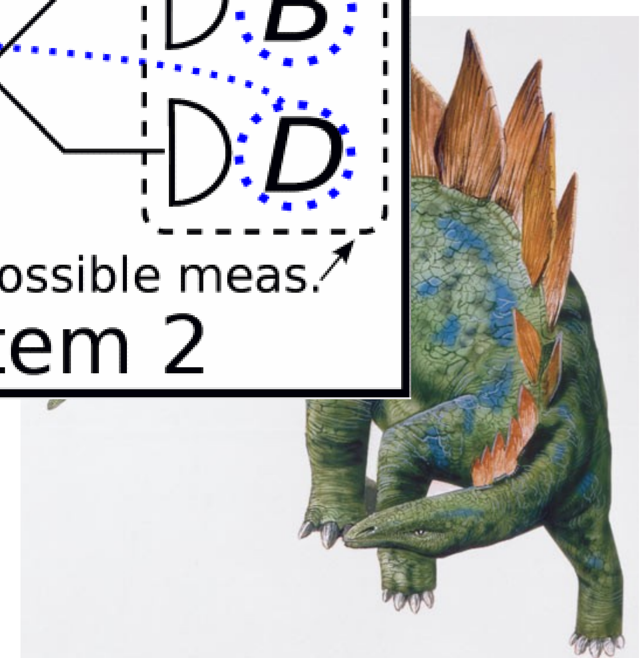
$$(|00\rangle\langle 00| + |11\rangle\langle 11|)/2 =$$
$$(|+\rangle\langle +| + |-\rangle\langle -|)/2 \otimes (|+\rangle\langle +| + |-\rangle\langle -|)/2$$

separable state: perfect correlation for 0/1, no correlation for +/-

Simple experiment

- On system 1 measure either *A* or *C*
- On system 2 measure either *B* or *D*
- Calculate correlations *A-B* and *C-D*





How to measure correlation?

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- Mutual information

$$I_{AB} = H(A) + H(B) - H(A,B)$$

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$|\mathcal{C}_{AB}| = 1 \Rightarrow$
perfect correlation
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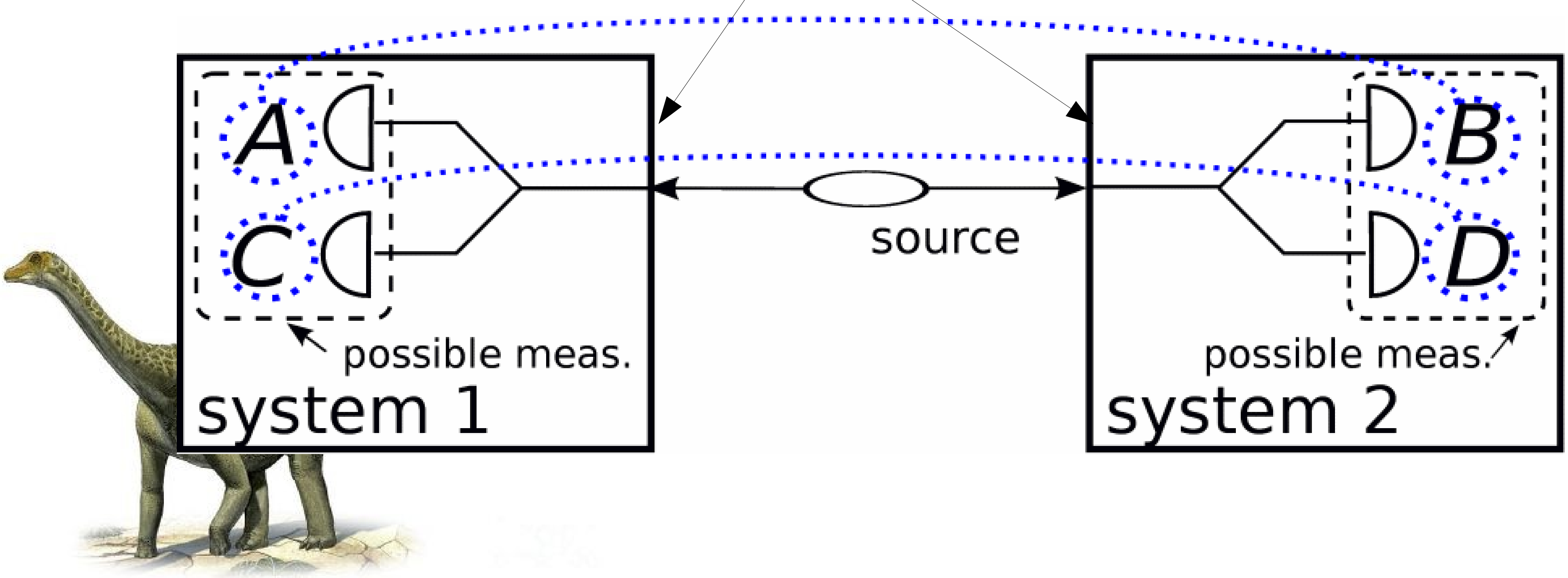
Use these to measure correlations among

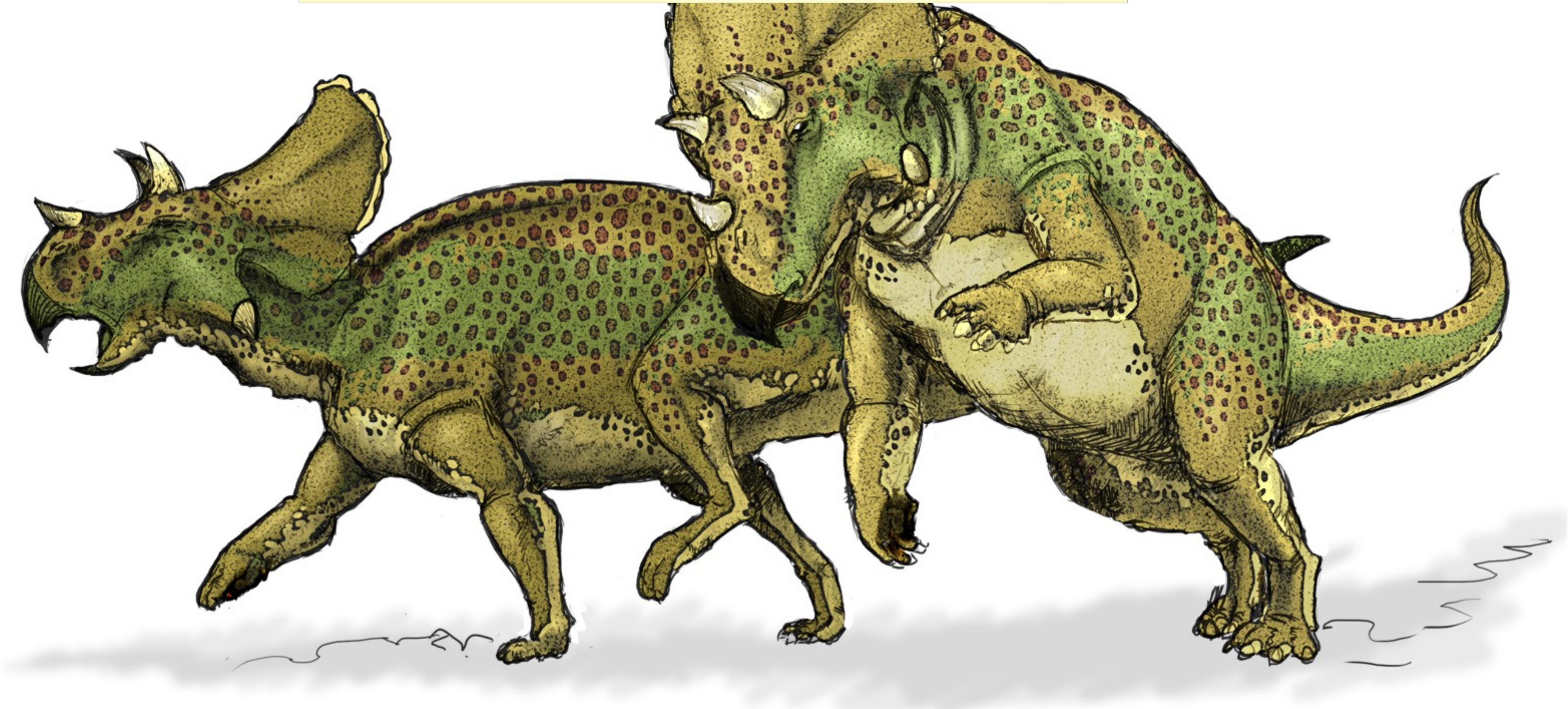
2 complementary properties

$A \otimes B$ $\xleftrightarrow{\text{complem to}}$ $C \otimes D$

of 2 systems



Some results...

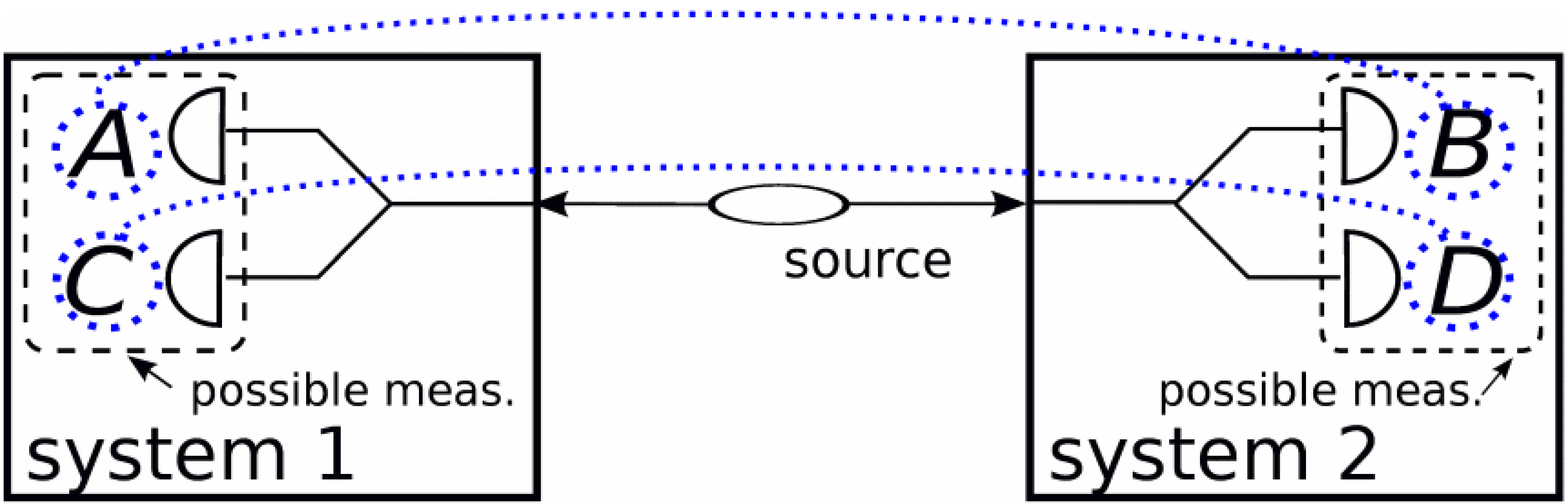


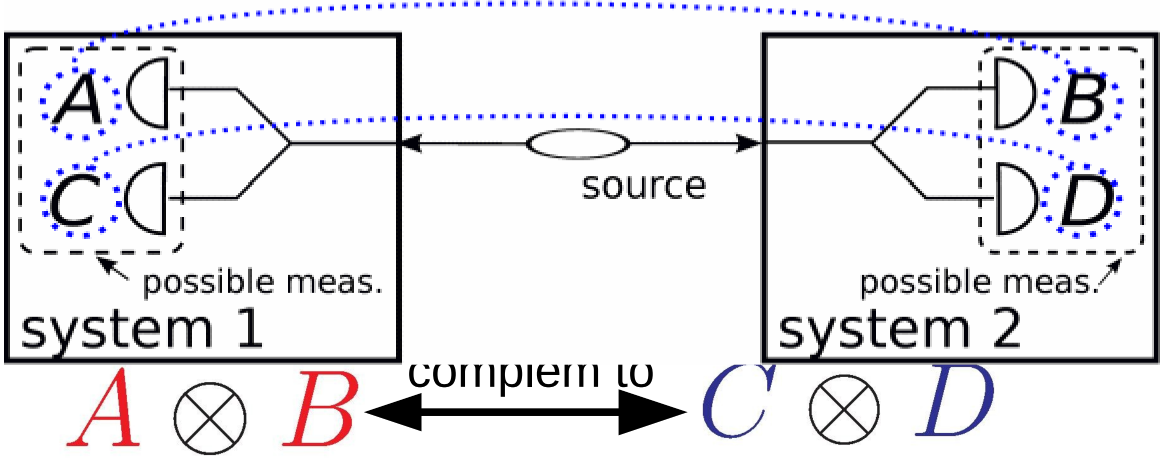
Start with mutual information

$$I_{AB} = H(A) + H(B) - H(A,B)$$

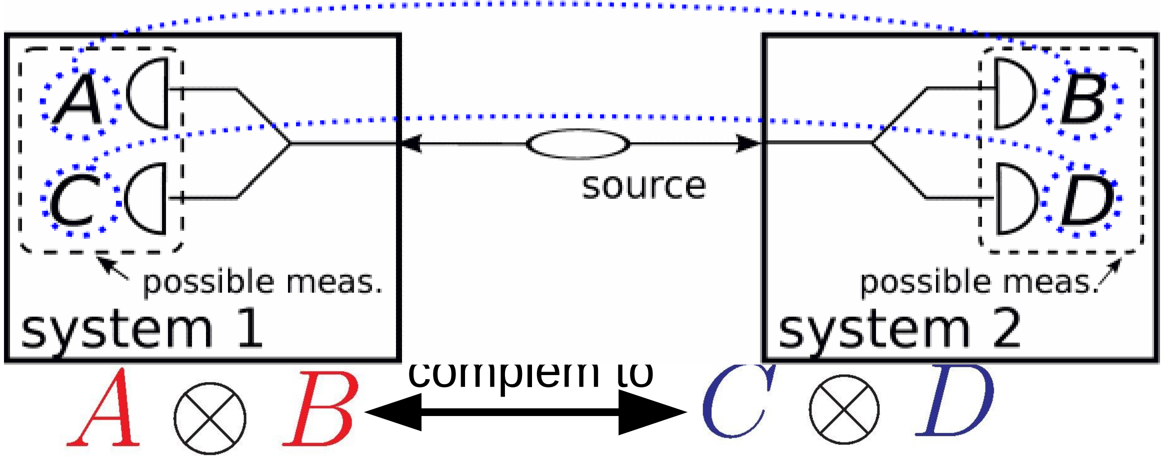
"total" correlation given by the sum

$$I_{AB} + I_{CD}$$





The system state is **maximally entangled** iff perfect correlation on **both** *A*-*B* and *C*-*D*

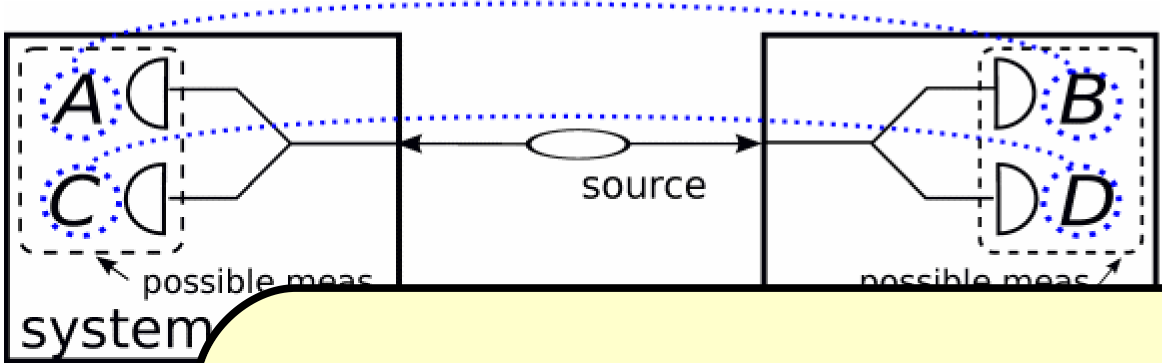


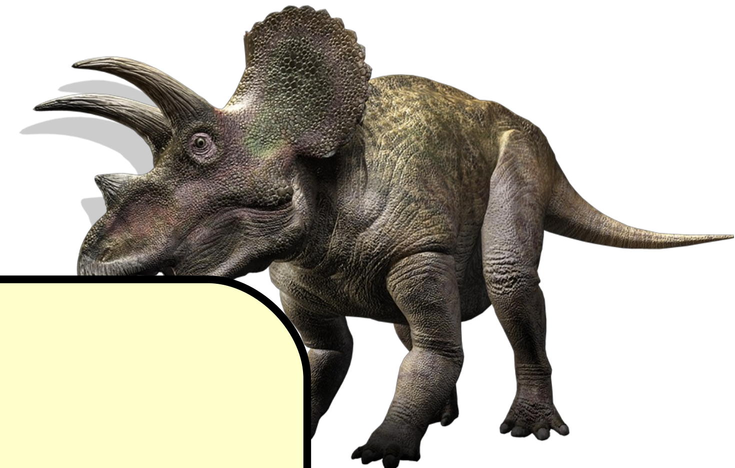
The system state is maximally entangled iff perfect correlation on both *A*-*B* and *C*-*D*

$$I_{AB} + I_{CD} = 2 \log d$$

(for some observ *ABCD*)

$$\Leftrightarrow \quad |\Psi_{12}\rangle \text{ maximally entangled}$$





A ⊗

$$I_{AB} \leqslant \log_2 d$$

$$I_{CD} \leqslant \log_2 d$$

T… …mally e… …lation on **both** *A*-*B* and *C*-*D*

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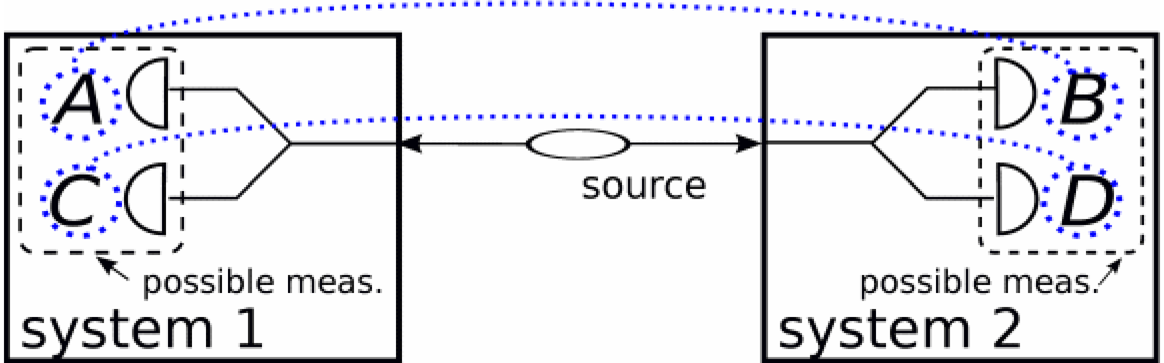
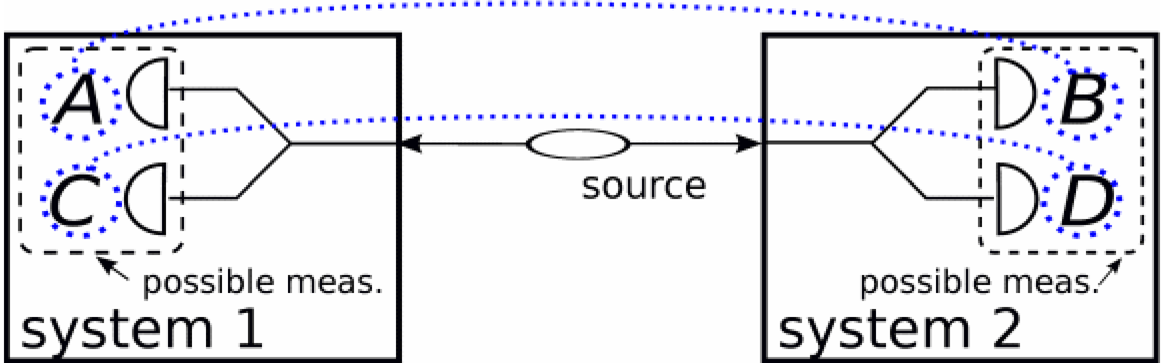


$$A \otimes B \xleftrightarrow{\text{complem to}} C \otimes D$$

The system state is **entangled** if correlations on **both** *A*-*B* and *C*-*D* are large enough



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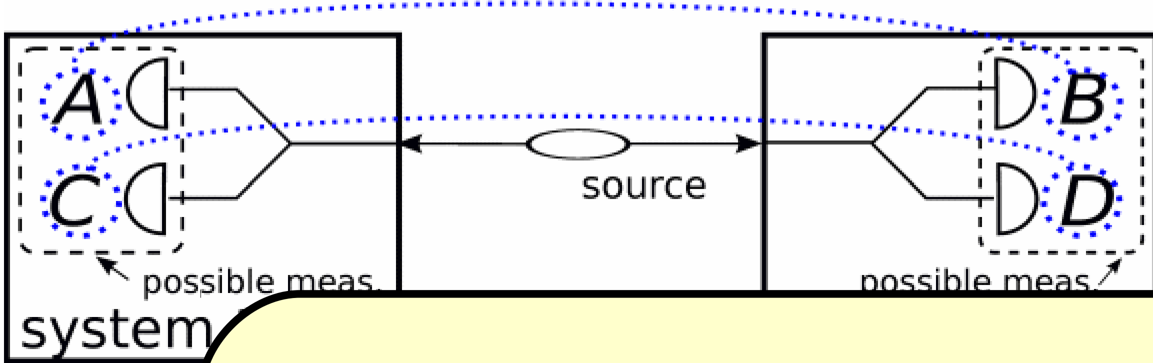
ρ_{12} ent



$A \otimes$

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T… **…gled** if
co… …d *C*-*D*
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Can the bound be made **tighter**?

The system state is **entangled** if correlations on **both** *A-B* and *C-D* are large enough

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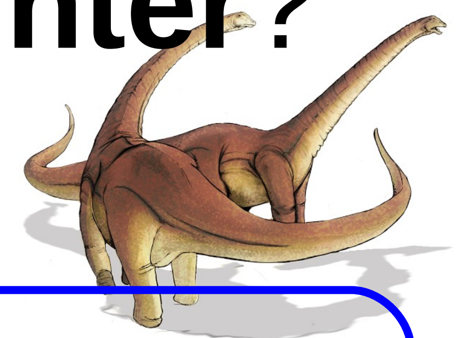


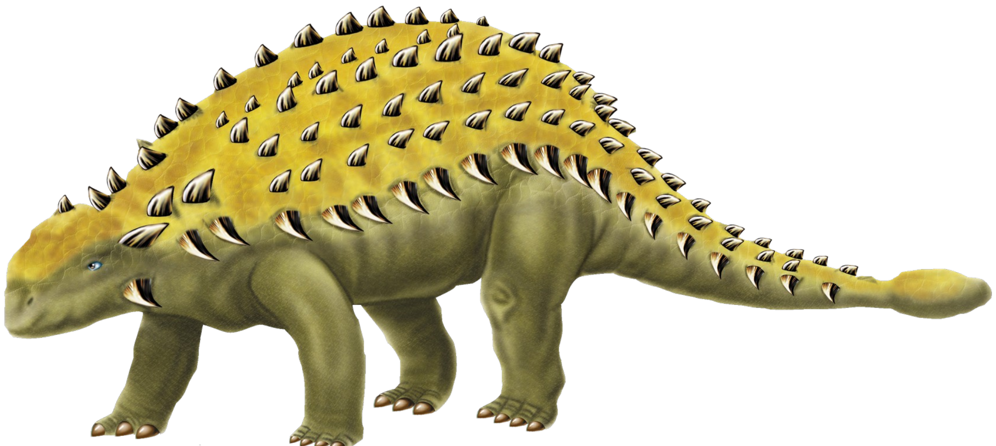
the **separable** state

$$\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

saturates it: $I_{AB} + I_{CD} = \log d$

The systen state is **entangled** if correlations on **both** *A*-*B* and *C*-*D* are large enough

is the converse true?



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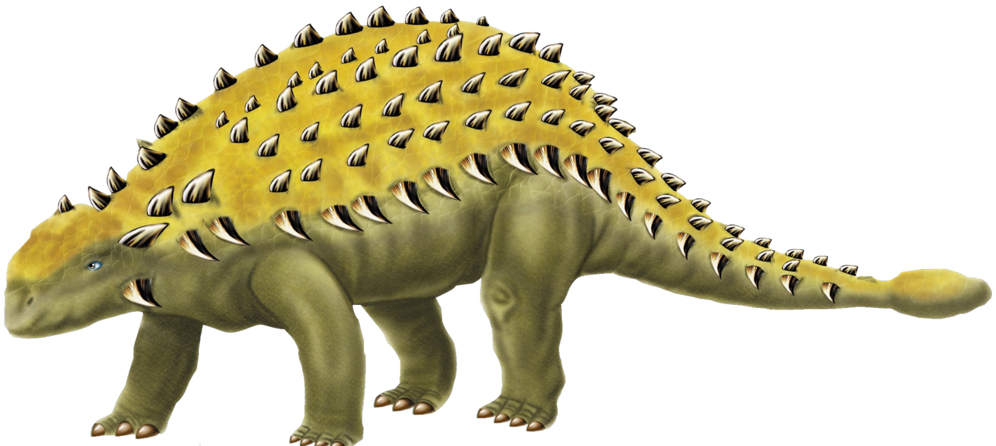

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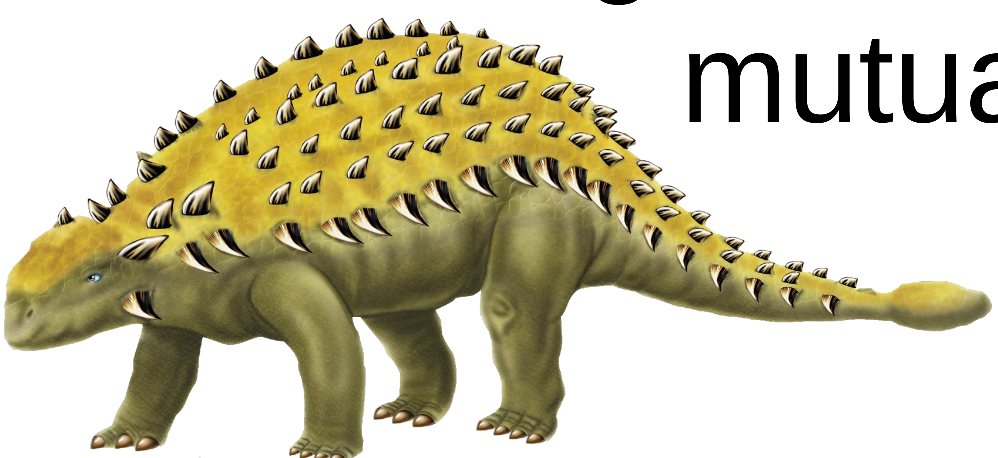
is the converse true?

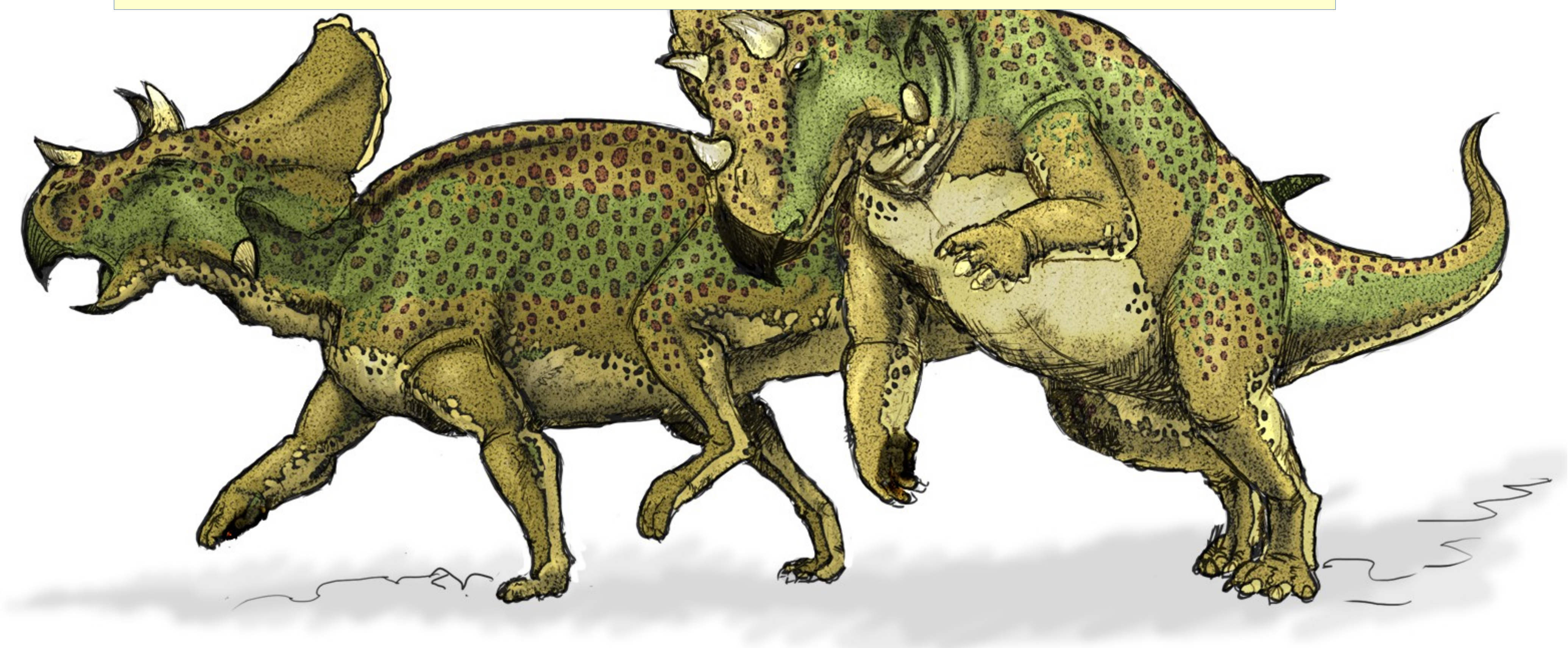
NO!!

$$|\psi_\epsilon\rangle = \epsilon|00\rangle + \sqrt{1-\epsilon^2}|11\rangle$$

is entangled but has negligible mutual info for $\epsilon \to 0$



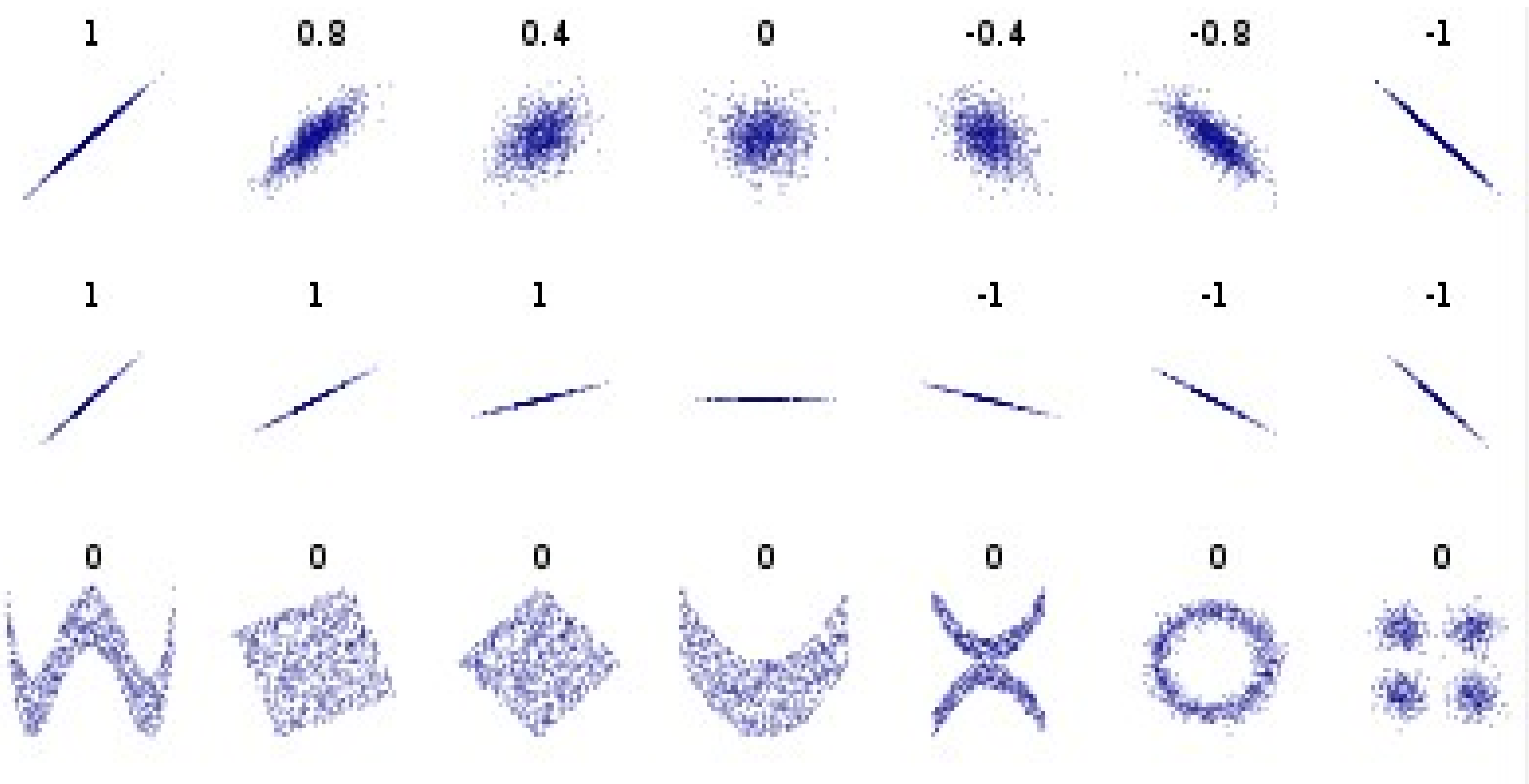
Another measure of correlation...



Pearson correlation coefficient

$$\mathcal{C}_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B}$$

$|\mathcal{C}_{AB}| = 1 \Rightarrow$ perfect correlation or anticorrelation



Pearson correlation coefficient

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it can be **complex** for quantum expectation values

... but its modulus is still $\leqslant |1|$:

Pearson correlation coefficient

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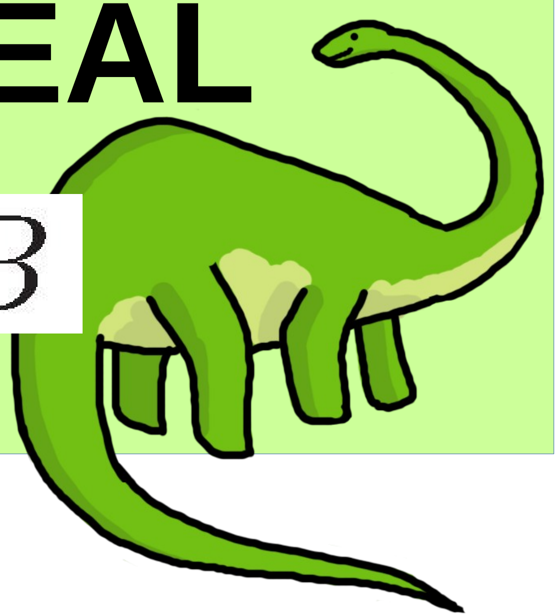
$|\mathcal{C}_{AB}| = 1 \Rightarrow$

perfect correlation or anticorrelation

it can be **complex** for quantum expectation values

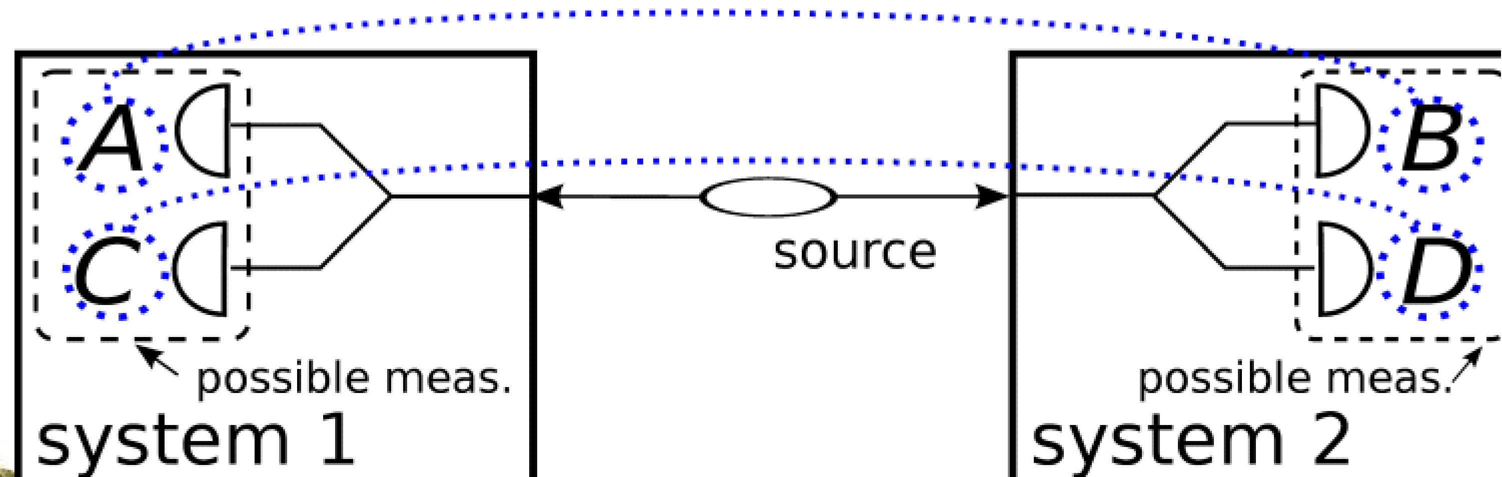
not a problem for us: *A* and *B* commute, so it's **REAL**

$$A \otimes B = A \otimes \mathbb{1} + \mathbb{1} \otimes B$$

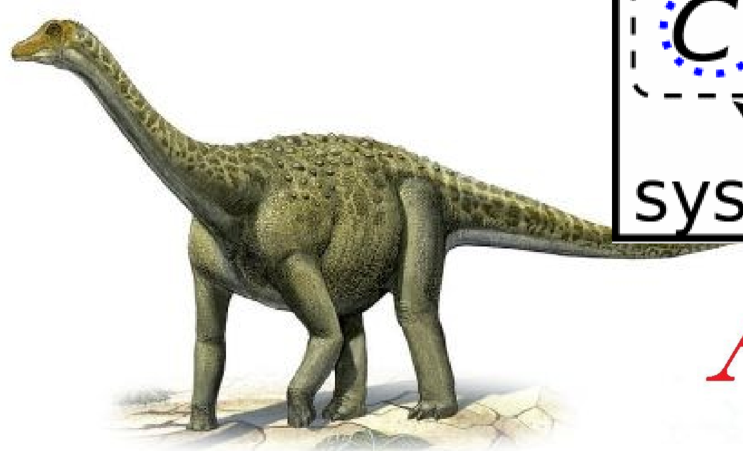


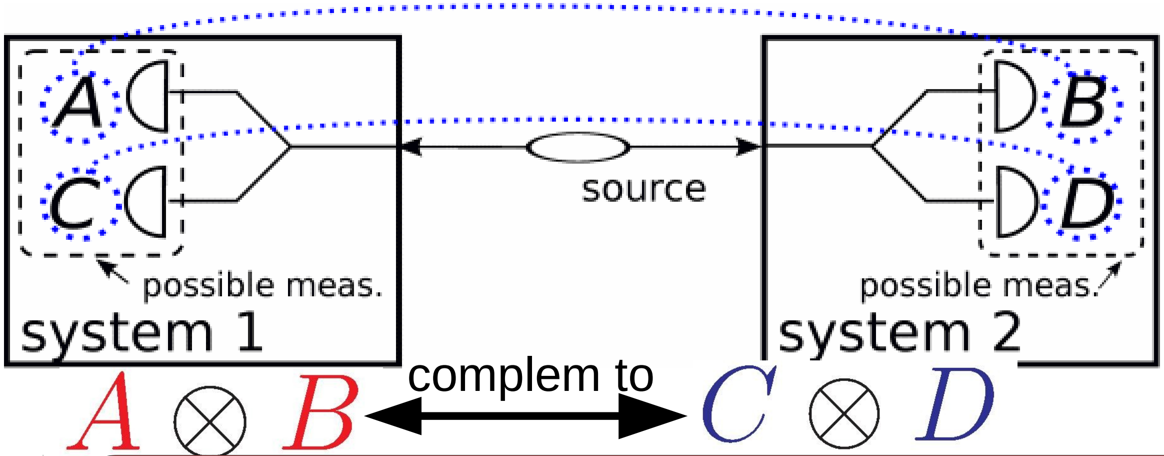
Total correlation: again use the sum

$$|C_{AB}| + |C_{CD}|$$

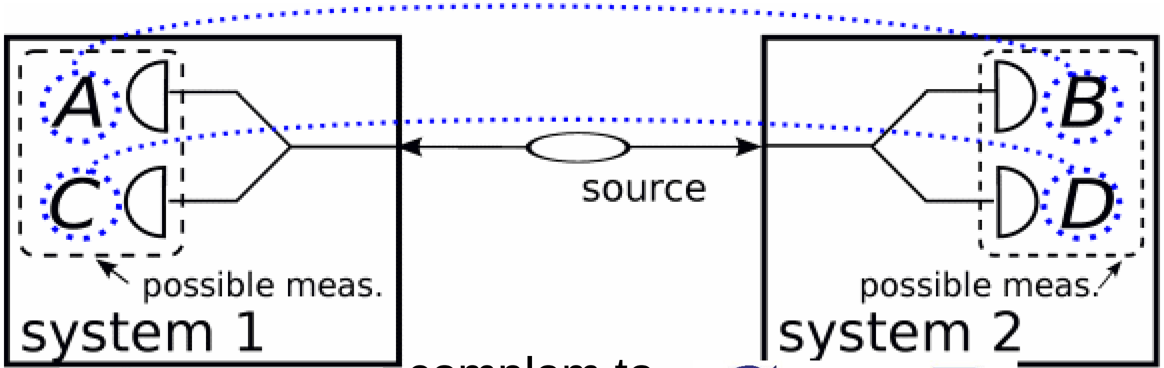


$$A \otimes B \xleftrightarrow{\text{complem to}} C \otimes D$$





The system state is **maximally entangled** iff perfect correlation on **both** *A*-*B* and *C*-*D*



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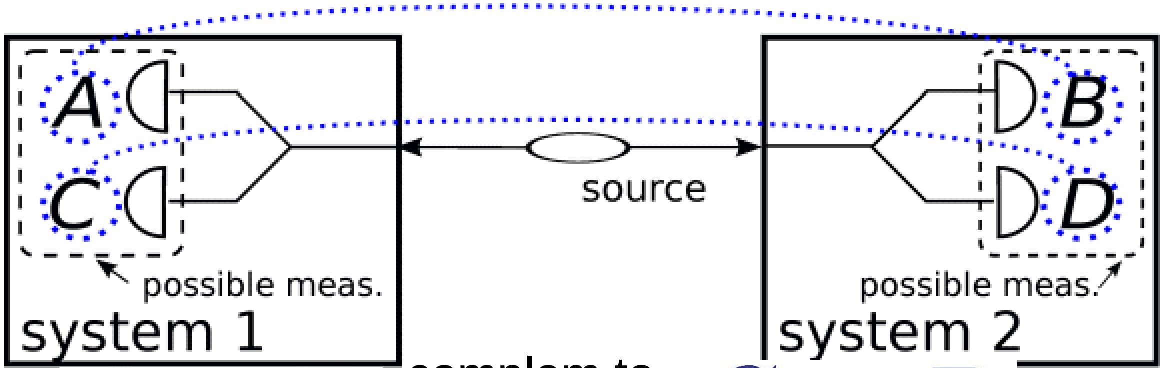
True also using Pearson! (for linear observables: Pearson measures only linear correl)



$A \otimes B \longleftrightarrow C \otimes D$

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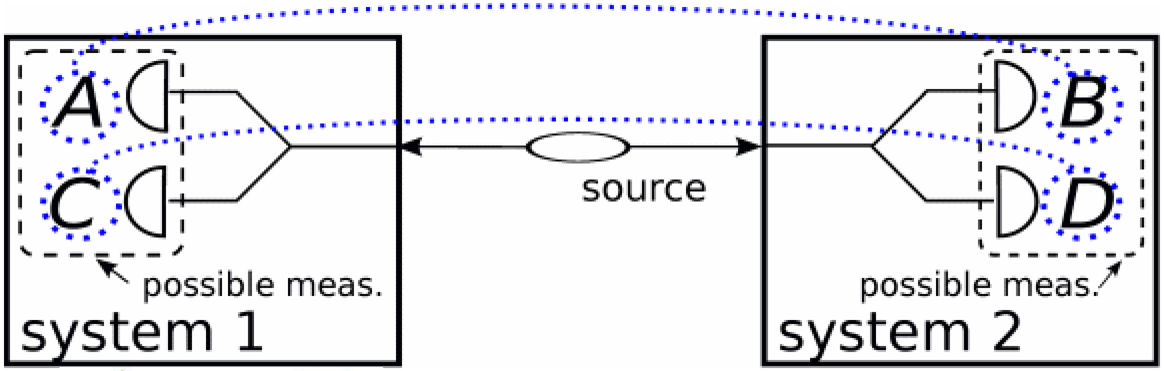
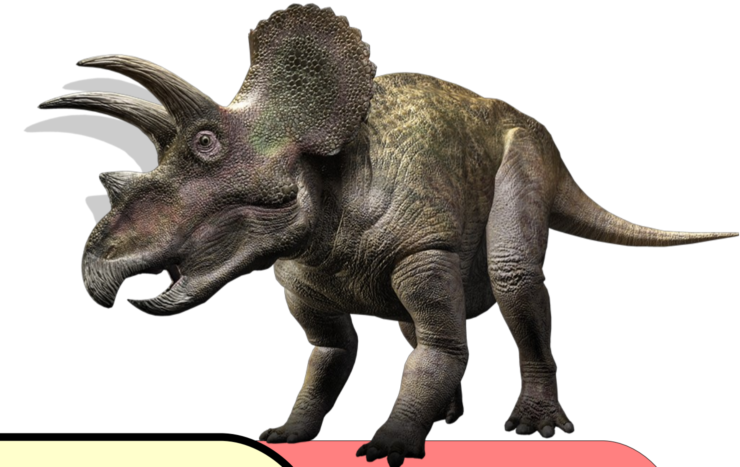


$A \otimes B$



T… **mally**

e… ation

$$|\mathcal{C}_{AB}| \leqslant 1$$

$$|\mathcal{C}_{CD}| \leqslant 1$$

Tru… **linear**

observables: Pearson measures only linear correl)

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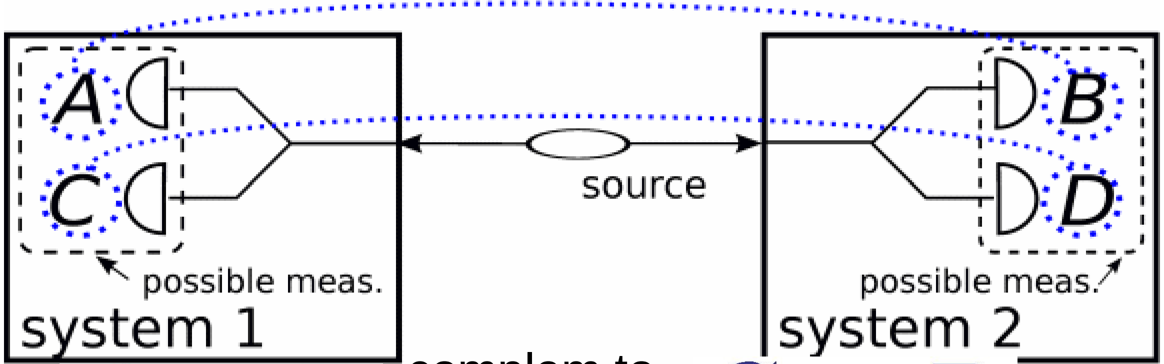
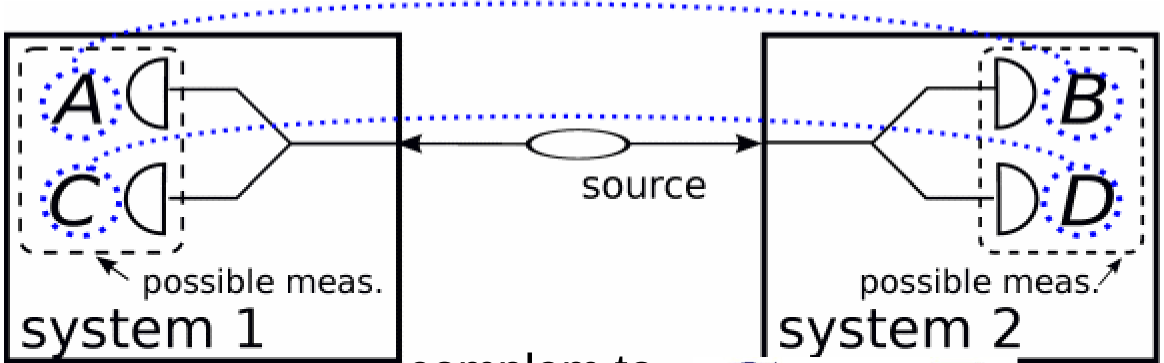


$A \otimes B$ ⟵ complem to ⟶ $C \otimes D$

The system state is **entangled** if correlations on **both** *A*-*B* and *C*-*D* are large enough?



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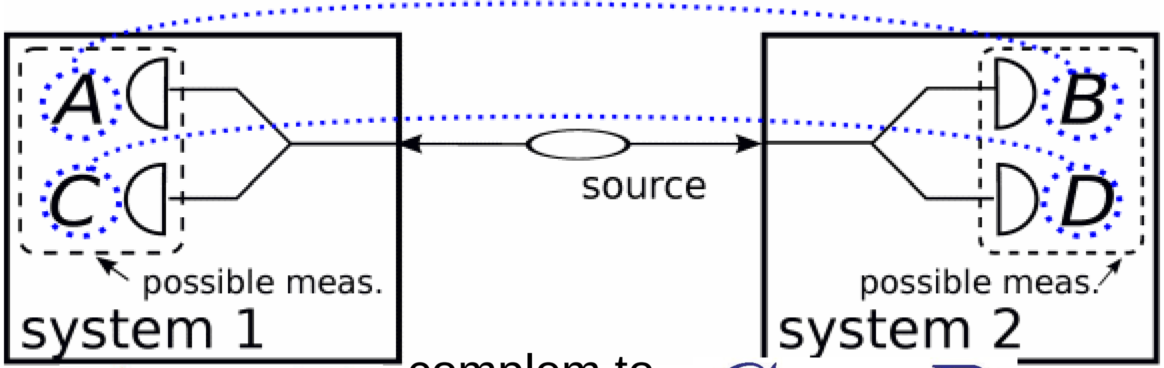
CONJECTURE: we don't know if it's true also using Pearson!



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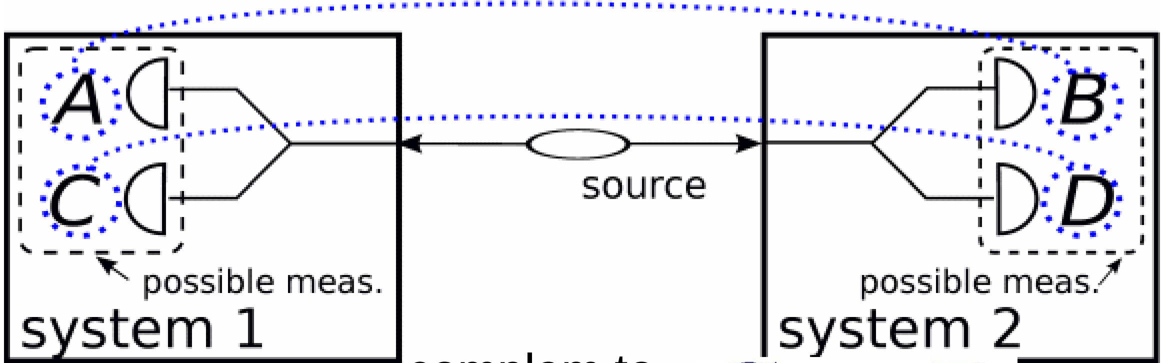

$A \otimes B \xleftrightarrow{\text{complem to}} C \otimes D$

Th… **…gled** if

co… …d *C*-*D*

$$|\mathcal{C}_{AB}| \leqslant 1$$

$$|\mathcal{C}_{CD}| \leqslant 1$$

co… …ow if

it's true also using Pearson!

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Conjecture: $|\mathcal{C}_{AB}| + |\mathcal{C}_{CD}| > 1 \;\Rightarrow$ state is ent.

Again, the inequality is tight:

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separable state $\dfrac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$

$$|\mathcal{C}_{AB}| + |\mathcal{C}_{CD}| = 1$$

Conjecture: $|\mathcal{C}_{AB}| + |\mathcal{C}_{CD}| > 1 \;\Rightarrow$ state is ent.

Again, the inequality is tight:

separable state $\frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$

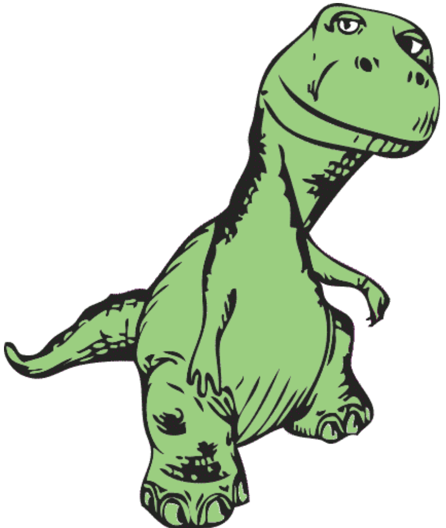
$$|\mathcal{C}_{AB}| + |\mathcal{C}_{CD}| = 1$$

(perfect correl on one basis,
no correl on the complem)

Is the Pearson correlation weaker than the mutual info?

only linear correlations

all correlations



Is the Pearson correlation weaker than the mutual info?

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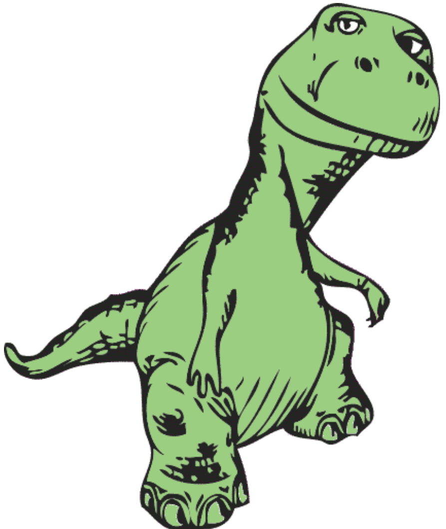
- NO!!



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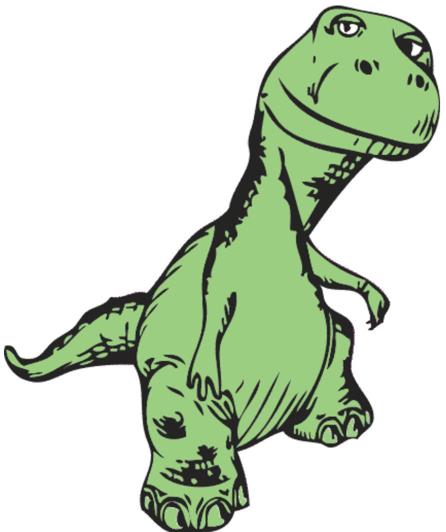
Has negligible mutual info for $\epsilon \to 0$



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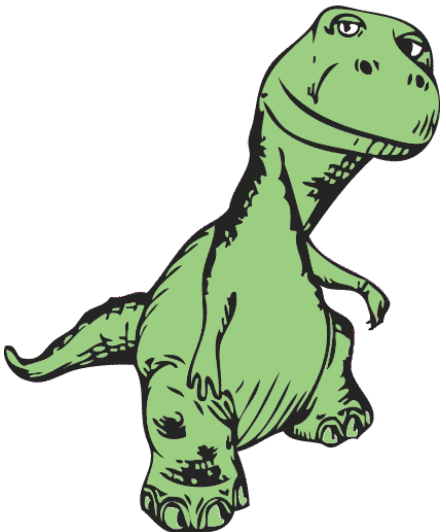
$$|\psi_\epsilon\rangle = \epsilon|00\rangle + \sqrt{1-\epsilon^2}|11\rangle$$

Has negligible mutual info for $\epsilon \to 0$

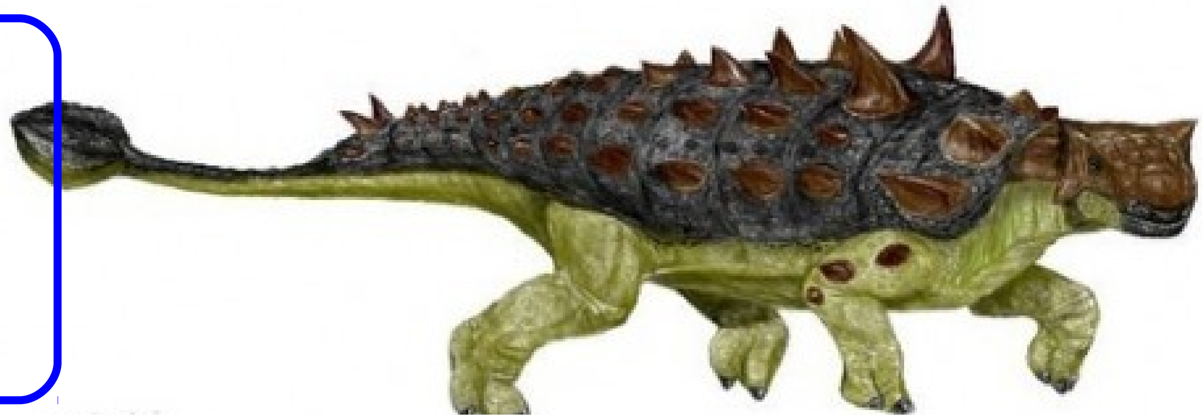
but Pearson correlation **always >1!**



Simple criterion for entanglement detection!!

Just measure two complementary properties. Are the correlations greater than perfect correlation on one?

$\Rightarrow$ The state is entangled!



Simple to measure and simple to optimize.

Unfortunately: not very effective in finding entanglement in random states

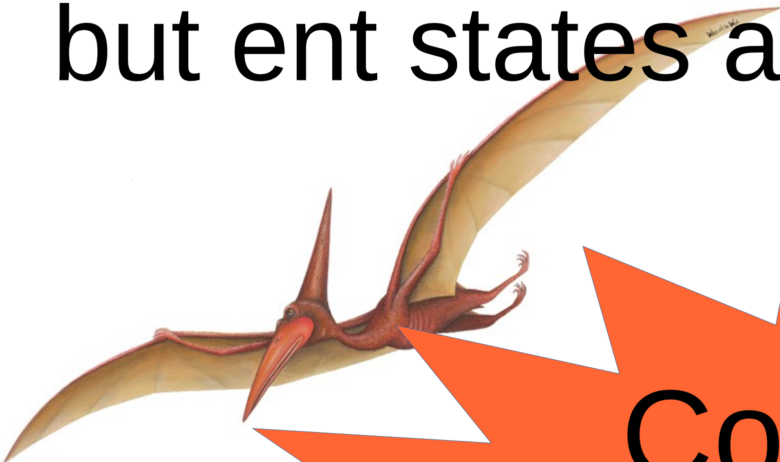
What did I say?!?

- Entanglement as correlation among complementary observables

- Using different measures of correlation:
  - Mutual info
  - Pearson correlation

- Some theorems and some conjectures

The most correlated states are entangled but ent states are not the most correlated



Correlations on complementary prop. help understanding entanglement

Lorenzo Maccone
maccone@unipv.it

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