Stronger uncertainty relations

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What I'm going to talk about

Heisenberg uncertainty relations do NOT convey fully the notion of complementarity
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\[ \Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle| \]

both terms can be zero, even if A and B are incompatible.
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both terms can be zero, even if A and B are incompatible.

e.g. if the state of the system is an eigenstate of A

\[ 0 \geq 0 \]
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Can we do better?
Heisenberg uncertainty relations do NOT convey fully the notion of complementarity

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both terms can be zero, even if A and B are incompatible

e.g. if the state of the system is an eigenstate of A

$$0 \geq 0$$

Can we do better? **YES!**
Reminder

Heisenberg-Robertson UR

\[ \Delta A^2 \Delta B^2 \geq \left| \frac{1}{2} \langle [A, B] \rangle \right|^2 \]

variances and expectations calculated on system state \[ |\psi\rangle \]
Reminder

Heisenberg-Robertson UR

$$\Delta A^2 \Delta B^2 \geq \left| \frac{1}{2} \langle [A, B] \rangle \right|^2$$

variances and expectations calculated on system state $$|\psi\rangle$$

Schroedinger UR

$$\Delta A^2 \Delta B^2 \geq \left| \frac{1}{2} \langle [A, B] \rangle \right|^2 + \left| \frac{1}{2} \langle \{A, B\}_+ \rangle - \langle A \rangle \langle B \rangle \right|^2$$
Reminder

Heisenberg-Robertson UR

$$\Delta A^2 \Delta B^2 \geq |\frac{1}{2} \langle [A, B] \rangle|^2$$

variances and expectations calculated on system state $|\psi\rangle$

Schroedinger UR

$$\Delta A^2 \Delta B^2 \geq |\frac{1}{2} \langle [A, B] \rangle|^2 + |\frac{1}{2} \langle \{A, B\}_+ \rangle - \langle A \rangle \langle B \rangle|^2$$

both trivial if $|\psi\rangle = \text{eigenstate of } A \text{ OR } B$
we need to give a bound to the sum

$$\Delta A^2 + \Delta B^2 \geq \ldots$$

(the product $\Delta A^2 \Delta B^2$ is trivially zero if one of the two is)
we need to give a bound to the sum

\[ \Delta A^2 + \Delta B^2 \geq \ldots \]

(the product \( \Delta A^2 \Delta B^2 \) is trivially zero if one of the two is)

simple try:

\[ (\Delta A - \Delta B)^2 \geq 0 \quad \Rightarrow \]

\[ \Delta A^2 + \Delta B^2 \geq 2\Delta A\Delta B \]
we need to give a bound to the \textit{sum} \[ \Delta A^2 + \Delta B^2 \geq \ldots \] (the product $\Delta A^2 \Delta B^2$ is trivially zero if one of the two is)

simple try:

\[ (\Delta A - \Delta B)^2 \geq 0 \implies \Delta A^2 + \Delta B^2 \geq 2\Delta A\Delta B \geq |\langle[A, B]\rangle| \]

Heisenberg
we need to give a bound to the sum

\[ \Delta A^2 + \Delta B^2 \geq \ldots \]

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nice, but...

Heisenberg
we need to give a bound to the \textbf{sum}
\[ \Delta A^2 + \Delta B^2 \geq \ldots \]

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simple try:
\[(\Delta A - \Delta B)^2 \geq 0 \Rightarrow \]
\[\Delta A^2 + \Delta B^2 \geq 2\Delta A\Delta B \geq |\langle [A, B] \rangle|\]

nice, but...

...useless!

Heisenberg
Main result:
two lower bounds
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two lower bounds

\[ \Delta A^2 + \Delta B^2 \geq L \]

nonzero except when \(|\psi\rangle\)

is a joint eigenstate of \(A\) and \(B\)
Main result:
two lower bounds

\[ \Delta A^2 + \Delta B^2 \geq \mathcal{L} \]

nonzero except when \( |\psi\rangle \) is a joint eigenstate of \( A \) and \( B \)

\[ \Delta A^2 + \Delta B^2 \geq \mathcal{M} \]
Main result:
two lower bounds

\[ \Delta A^2 + \Delta B^2 \geq \mathcal{L} \]

nonzero except when \( |\psi\rangle \) is a joint eigenstate of \( A \) and \( B \)

\[ \Delta A^2 + \Delta B^2 \geq \mathcal{M} \]

\[ \Rightarrow \Delta A^2 + \Delta B^2 \geq \max(\mathcal{L}, \mathcal{M}) \]
SUM of variances

\[ \Delta A^2 + \Delta B^2 \geq \ldots \Rightarrow \]

dimensional regularization needed if \( A \) and \( B \) have different units
SUM of variances

$$\Delta A^2 + \Delta B^2 \geq \ldots \implies$$

dimensional regularization needed if $A$ and $B$ have different units

$$\frac{\Delta A^2}{a} + \frac{\Delta B^2}{b} \geq \ldots$$
First bound:

$$\Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi\downarrow \rangle \right|^2$$
First bound:

\[ \Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi^\perp \rangle \right|^2 \]

choose sign so that this is positive (it's real)
First bound:

$$\Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi^\perp \rangle \right|^2$$

choose sign so that this is positive (it's real)

$$| \psi^\perp \rangle$$ arbitrary state orthogonal to the system state $$| \psi \rangle$$
First bound:

\[ \Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB |\psi^- \rangle \right|^2 \]

choose sign so that this is positive (it's real)

\[ |\psi^- \rangle \text{ arbitrary state orthogonal to the system state } |\psi \rangle \]

constructive procedure to choose \[ |\psi^- \rangle \], or just choose a random one
First bound:

$$\Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + |\langle \psi | A \pm i B | \psi \rangle|^2$$

Green: Heisenb-Robertson UR
First bound:

\[ \Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi^\perp \rangle \right|^2 \]

**Green**: Heisenberg-Robertson UR

**Red**: new part, always \( \neq 0 \) except if \( |\psi\rangle \) is joint eigenstate of \( A \) and \( B \)
First bound:

\[ \Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi \rangle \right|^2 \]

**Green:** Heisenb-Robertson UR

**Red:** new part, always \( \neq 0 \) except if \( |\psi\rangle \) is joint eigenstate of \( A \) and \( B \)

Minimizing over \( |\psi\rangle \) \(\rightarrow\) only HR (red)

Maximizing over \( |\psi\rangle \) \(\rightarrow\) ineq becomes eq
First bound: numerical test for random $|\psi^\perp\rangle$

$$\Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi^\perp \rangle \right|^2$$

$$|\psi\rangle = \cos \varphi |0\rangle + \sin \varphi |1\rangle$$

$$A = \sigma_x / 2$$
$$B = \sigma_y / 2$$
First bound: proof

$$\Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi^\perp \rangle \right|^2$$

application of Schwartz inequality (like the Robertson)
First bound: proof

\[ \Delta A^2 + \Delta B^2 \geq \pm i\langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi^\perp \rangle \right|^2 \]

application of Schwartz inequality (like the Robertson)

\[ C \equiv A - \langle A \rangle; \quad D \equiv B - \langle B \rangle \]

\[ \| (C \mp iD) | \psi \rangle \|^2 = \Delta A^2 + \Delta B^2 \mp i\langle [A, B] \rangle \]

\[ \left| \langle \psi | (A \pm iB) | \psi^\perp \rangle \right|^2 = \left| \langle \psi | A \pm iB - \langle A \pm iB | \psi^\perp \rangle \right|^2 \]

\[ = \left| \langle \psi | C \mp iD | \psi^\perp \rangle \right|^2 \leq \| (C \mp iD) | \psi \rangle \|^2 \]
First bound: proof

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\[ = \left| \langle \psi | C \pm iD | \psi^\perp \rangle \right|^2 \leq \| (C \mp iD) | \psi \rangle \|^2 \]

proof by an anonymous referee

Masanao Ozawa

(original proof more complicated)
First bound: minimum uncertain. st.

togliere questa slide: tutti gli stati sono minimum uncertainty states per il primo bound: quando ottimizzo su $|\psi^\perp>$, la disuguaglianza diventa sempre una uguaglianza!
First bound: minimum uncert. st.

Harmonic osc. \[ X = \frac{1}{\sqrt{2}} (a + a^\dagger), \quad P = \frac{i}{\sqrt{2}} (a^\dagger - a) \]
First bound: minimum uncert. st.

Harmonic osc. \[ X = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad P = \frac{i}{\sqrt{2}}(a^\dagger - a) \]

\[ \Delta X^2 + \Delta P^2 \geq 1 + 2|\langle \psi |a^\dagger|\psi^\perp \rangle|^2 \]
First bound: minimum uncert. st.

**Harmonic osc.** \( X = \frac{1}{\sqrt{2}}(a + a^\dagger), \ P = \frac{i}{\sqrt{2}}(a^\dagger - a) \)

\[
\Delta X^2 + \Delta P^2 \geq 1 + 2|\langle \psi |a^\dagger |\psi^\perp \rangle|^2
\]

- Fock states \( |n\rangle \)

\[
\Delta X^2 + \Delta P^2 = (2n + 1) \equiv \mathcal{L}
\]
First bound: minimum uncertainty state.

Harmonic osc. \[ X = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad P = \frac{i}{\sqrt{2}}(a^\dagger - a) \]

\[ \Delta X^2 + \Delta P^2 \geq 1 + 2|\langle \psi | a^\dagger | \psi^{\perp} \rangle|^2 \]

- Fock states \( |n\rangle \)

\[ \Delta X^2 + \Delta P^2 = (2n + 1) \equiv \mathcal{L} \]

MUS
First bound: minimum uncert. st.

Harmonic osc. \( X = \frac{1}{\sqrt{2}}(a + a^\dagger), \ P = \frac{i}{\sqrt{2}}(a^\dagger - a) \)

\[ \Delta X^2 + \Delta P^2 \geq 1 + 2|\langle \psi | a^\dagger | \psi \rangle|^2 \]

· Fock states \( |n\rangle \)

\[ \Delta X^2 + \Delta P^2 = (2n + 1) \equiv L \]

· Coherent states \( |\alpha\rangle \)

\[ \Delta X^2 + \Delta P^2 = 1 \equiv L \]
They're all MUS (for first, but not second bound)

These are just examples, not the ONLY MUS!!
Second bound:

\[ \Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2 \]

\[ S = A + B \]
Second bound:

\[ \Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2 \]

\[
S = A + B \implies \\
\Delta S^2 = \langle (A + B)^2 \rangle - \langle A + B \rangle^2
\]
Second bound:

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2$$

$$S = A + B \implies \Delta S^2 = \langle (A + B)^2 \rangle - \langle A + B \rangle^2$$

This bound is $\neq 0$ if $|\psi\rangle$ is not an eigenstate of $A+B$
Second bound: example

\[ \Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2 \]
Second bound: proof

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2$$

application of the parallelogram ineq.
Second bound: proof

\[ \Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2 \]

application of the parallelogram ineq.

\[ C \equiv A - \langle A \rangle; \quad D \equiv B - \langle B \rangle \]

\[ 2\Delta A^2 + 2\Delta B^2 = \| (C + D) \psi \|^2 + \| (C - D) \psi \|^2 \]

\[ \Delta (A + B) = \| (C + D) \psi \|, \quad \Delta (A - B) = \| (C - D) \psi \| \]

\[ \Delta A^2 + \Delta B^2 \geq \frac{1}{2} [\Delta (A + B)^2 + \Delta (A - B)^2] \geq \frac{1}{2} \Delta (A + B)^2, \]
State dependence.

\[ \Delta A^2 + \Delta B^2 \geq \max(\mathcal{L}, \mathcal{M}) \]

both our bounds are state-dependent.
State dependence.

\[ \Delta A^2 + \Delta B^2 \geq \max(\mathcal{L}, \mathcal{M}) \]

both our bounds are state-dependent.

\[ \mathcal{L} = \mathcal{L}(|\psi\rangle) \quad \mathcal{M} = \mathcal{M}(|\psi\rangle) \]
State dependence.

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get a state-independent bound from them?
State dependence.

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the second becomes trivial

minimizing over $|\psi\rangle$ (just take an eigenstate of $A+B$)

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2$$
State dependence.

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minimizing over \( |\psi\rangle \) (just take an eigenstate of \( A+B \))

the first?

\[ \Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2 \]
State dependence.

\[ \Delta A^2 + \Delta B^2 \geq \max(\mathcal{L}, \mathcal{M}) \]

both our bounds are state-dependent.

\[ \mathcal{L} = \mathcal{L}(|\psi\rangle) \quad \mathcal{M} = \mathcal{M}(|\psi\rangle) \]

get a state-independent bound from them?

the second becomes trivial minimizing over \( |\psi\rangle \) (just take an eigenstate of \( A+B \))

the first?

\[ \min \max \left( \pm i\langle[A, B]\rangle \pm \langle \psi | A \pm i B | \psi^\perp \rangle \right)^2 \]
Using the same techniques, we can tighten the Heis-Rob bound!

\[ \Delta A \Delta B \geq \pm \frac{i}{2} \langle [A, B] \rangle \]
Using the same techniques, we can tighten the Heis-Rob bound!

\[\Delta A \Delta B \geq \pm \frac{i}{2} \langle [A, B] \rangle \left/ \left(1 - \frac{1}{2} \langle \psi \left| \frac{A}{\Delta A} \right| \psi \rangle \pm i \frac{B}{\Delta B} \langle \psi \left| \psi \right| \right) \right. \]
Heisenberg uncertainty relation vs. principle

two very different notions!!!!!!!!!!!!!!!!!!!!!
Heisenberg uncertainty relation vs. principle

two very different notions!!!!!!!!!!!!!!!!!!!!!!!

much confusion in the literature, originating from Heisenberg's paper where he uses them interchangeably (criticized by Bohr)
Heisenberg uncertainty relation vs. principle

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_u principle_: measurement-disturbance tradeoff. A precise measure induces a disturbance
Heisenberg uncertainty relation vs. principle

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**u principle**: measurement-disturbance tradeoff. A precise measure induces a disturbance ➔ **WRONG** interpretation!
Heisenberg uncertainty relation vs. principle

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much confusion in the literature, originating from Heisenberg's paper where he uses them interchangeably (criticized by Bohr)

**u principle**: measurement-disturbance tradeoff. A precise measure induces a disturbance

WRONG interpretation!

**u relation**: refers only to the preparation of the state. Can't prepare a state with sharp values for incompatible observables.
Heisenberg uncertainty relation vs. principle: two very different notions!

- **Heisenberg uncertainty relation**: refers only to the preparation of the state. Can't prepare a state with sharp values for incompatible observables.

- **Principle**: much confusion in the literature, originating from Heisenberg's paper where he uses them interchangeably (criticized by Bohr)

Measurement-disturbance tradeoff. A precise measure induces a disturbance!
Asher Peres introduced this notation

uncertainty relation:

“The only correct interpretation of [the uncertainty relations for \( x \) and \( p \)] is the following: If the same preparation procedure is repeated many times, and is followed either by a measurement of \( x \), or by a measurement of \( p \), the various results obtained for \( x \) and for \( p \) have standard deviations, \( \Delta x \) and \( \Delta p \), whose product cannot be less than \( \hbar/2 \). There never is any question here that a measurement of \( x \) ‘disturbs’ the value of \( p \) and vice versa, as sometimes claimed. These measurements are indeed incompatible, but they are performed on different particles (all of which were identically prepared) and therefore these measurements cannot disturb each other in any way. The uncertainty relation […] only reflects the intrinsic randomness of the outcomes of quantum tests.”

(he didn't want to talk about unc principle)

try looking up “uncertain principle” in his book.
stronger uncertainty relations for variances with nontrivial lower bound.

\[
\Delta A^2 + \Delta B^2 \geq \pm i \langle [A, B] \rangle + \left| \langle \psi | A \pm iB | \psi^\perp \rangle \right|^2
\]

\[
\Delta A^2 + \Delta B^2 \geq \frac{1}{2} \Delta S^2
\]

\[
S = A + B
\]

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Entanglement and Complementarity

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We always say that entangled states are more correlated... WHAT DOES IT MEAN exactly?
What I'm going to talk about

We always say that entangled states are more correlated... WHAT DOES IT MEAN exactly?

they have more correlations among complementary observables than separable ones
Usual approaches to study entanglement

- Non locality
- Negative partial transpose
- Bell inequality violations
- Enhanced precision in measurements
- etc.
Here: we use correlations among two (or more) COMPLEMENTARY PROPERTIES

different way to think about entanglement, as correlations among complementary properties
Remember: Complementary properties.
Remember: Complementary properties.

Two observables: the knowledge of one gives no knowledge of the other

\[ A = \sum_a f(a) |a\rangle \langle a| \]
\[ C = \sum_c g(c) |c\rangle \langle c| \]

\[ \left| \langle a | c \rangle \right|^2 = \frac{1}{d} \]
simplest example:
simplest example:

\[
\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|++\rangle + |--\rangle}{\sqrt{2}}
\]

Maximally entangled state: perfect correlation BOTH on 0/1 and on +/-
simplest example:

\[
\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|++\rangle + |--\rangle}{\sqrt{2}}
\]

Maximally entangled state: perfect correlation BOTH on 0/1 and on +/-?

\[
(\langle 00|00\rangle + \langle 11|11\rangle)/2
\]
simplest example:

\[
\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|++\rangle + |--\rangle}{\sqrt{2}}
\]

Maximally entangled state: perfect correlation BOTH on 0/1 and on +/-.

\[
\frac{(|00\rangle\langle00| + |11\rangle\langle11|)}{2} = \frac{(|+\rangle\langle+| + |\ -\rangle\langle\ -\|)/2 \otimes (|+\rangle\langle+| + |\ -\rangle\langle\ -\|)/2}{2}
\]

Separable state: perfect correlation for 0/1, no correlation for +/-.
Simple experiment

- On system 1 measure either $A$ or $C$
- On system 2 measure either $B$ or $D$
- Calculate correlations $A-B$ and $C-D$
How to measure correlation?
How to measure correlation?

- Mutual information

\[ I_{AB} = H(A) + H(B) - H(A, B) \]
How to measure correlation?

- Mutual information
  \[ I_{AB} = H(A) + H(B) - H(A, B) \]

- Pearson correlation coefficient
  \[ C_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B} \]
  \[ |C_{AB}| = 1 \Rightarrow \text{perfect correlation or anticorrelation} \]
Use these to measure correlations among 2 complementary properties of 2 systems.

\[ A \otimes B \quad \text{complem to} \quad C \otimes D \]
Some results…
Start with mutual information

\[ I_{AB} = H(A) + H(B) - H(A, B) \]

“total” correlation given by the sum

\[ I_{AB} + I_{CD} \]
The system state is \textbf{maximally entangled} iff perfect correlation on both $A-B$ and $C-D$.
The system state is **maximally entangled** iff perfect correlation on **both** $A-B$ and $C-D$

\[ I_{AB} + I_{CD} = 2 \log d \]

(for some observ $ABCD$)

\[ \iff \left| \Psi_{12} \right\rangle \text{ maximally entangled} \]
The system state is maximally entangled iff perfect correlation on both A-B and C-D.

\[ I_{AB} \leq \log_2 d \]

\[ I_{CD} \leq \log_2 d \]

\[ I_{AB} + I_{CD} = 2 \log_2 d \]

(for some observ A B C D)

\[ \iff \left| \Psi_{12} \right\rangle \text{ maximally entangled} \]
The system state is **entangled** if correlations on **both** \(A-B\) and \(C-D\) are large enough.
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough

\[ I_{AB} + I_{CD} > \log d \]

\[ \rho_{12} \text{ ent} \]
The system state is entangled if correlations on both $A-B$ and $C-D$ are large enough.

\[ I_{AB} \leq \log_2 d \]
\[ I_{CD} \leq \log_2 d \]

\[ I_{AB} + I_{CD} > \log d \]

\[ \rho_{12} \text{ ent} \]
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

Can the bound be made **tighter**?

$$I_{AB} + I_{CD} > \log d \quad \Rightarrow \quad \rho_{12}^{\text{ent}}$$
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

$\rho_{12}^{\text{ent}}$ \[ I_{AB} + I_{CD} > \log d \]

Can the bound be made **tighter**?

**NO!!**
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

\[ I_{AB} + I_{CD} > \log d \]

\[ \rho_{12} \text{ ent} \]

Can the bound be made **tighter**?

**NO!!**

the **separable** state

\[ \frac{1}{2}(|00\rangle\langle00| + |11\rangle\langle11|) \]

saturates it: \[ I_{AB} + I_{CD} = \log d \]
The system state is **entangled** if correlations on **both** A-B and C-D are large enough.

Is the converse true?
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

Is the converse true? **NO!!**
The system state is **entangled** if correlations on both $A-B$ and $C-D$ are large enough.

is the converse true? NO!! NO!!

$$|\psi_\epsilon\rangle = \epsilon |00\rangle + \sqrt{1 - \epsilon^2} |11\rangle$$

is entangled but has negligible mutual info for $\epsilon \to 0$
Another measure of correlation...
Pearson correlation coefficient

\[ C_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B} \]

\[ |C_{AB}| = 1 \Rightarrow \text{perfect correlation or anticorrelation} \]
Pearson correlation coefficient

\[ \mathcal{C}_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B} \]

\[ |\mathcal{C}_{AB}| = 1 \Rightarrow \text{perfect correlation or anticorrelation} \]

it can be complex for quantum expectation values
Pearson correlation coefficient

\[ C_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B} \]

\[ |C_{AB}| = 1 \Rightarrow \]

perfect correlation or anticorrelation

It can be complex for quantum expectation values

... but its modulus is still \( \leq |1| : \)
Pearson correlation coefficient

\[ C_{AB} \equiv \frac{\langle AB \rangle - \langle A \rangle \langle B \rangle}{\sigma_A \sigma_B} \]

| | \[ |C_{AB}| = 1 \Rightarrow \text{perfect correlation or anticorrelation} \]

it can be complex for quantum expectation values

not a problem for us: \( A \) and \( B \) commute, so it's REAL

\[ A \otimes B = A \otimes 1 + 1 \otimes B \]
Total correlation: again use the sum

$$|C_{AB}| + |C_{CD}|$$
The system state is **maximally entangled** iff perfect correlation on both $A-B$ and $C-D$. 
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True also using Pearson! (for linear observables: Pearson measures only linear correl)
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\[ |C_{AB}| \leq 1 \]

\[ |C_{CD}| \leq 1 \]

\[ |C_{AB}| + |C_{CD}| = 2 \] (for some observ $ABCD$)

\[ \iff \left| \Psi_{12} \right\rangle \] maximally entangled
The system state is entangled if correlations on both \( A-B \) and \( C-D \) are large enough?
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\[ |C_{AB}| + |C_{CD}| > 1 \implies \rho_{12} \text{ ent} \]

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Separable state \[
\frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}
\]

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Conjecture: $|C_{AB}| + |C_{CD}| > 1 \Rightarrow \text{state is ent.}$

Again, the inequality is tight:

separable state $\frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$

$|C_{AB}| + |C_{CD}| = 1$

(perfect correl on one basis, no correl on the complem)
Is the Pearson correlation weaker than the mutual info?
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Is the Pearson correlation weaker than the mutual info?

**NO!**

\[ |\psi_\epsilon\rangle = \epsilon |00\rangle + \sqrt{1 - \epsilon^2} |11\rangle \]

Has negligible mutual info for \( \epsilon \to 0 \)
Is the Pearson correlation weaker than the mutual info?

**NO!**

\[ |\psi_\epsilon\rangle = \epsilon |00\rangle + \sqrt{1 - \epsilon^2} |11\rangle \]

Has negligible mutual info for \( \epsilon \rightarrow 0 \) but **Pearson correlation always >1!**
Simple criterion for entanglement detection!!

Just measure two complementary properties. Are the correlations greater than perfect correlation on one?

⇒ The state is entangled!

Simple to measure and simple to optimize.

Unfortunately: not very effective in finding entanglement in random states.
What did I say?!?

- Entanglement as correlation among complementary observables

- Using different measures of correlation:
  - Mutual info
  - Pearson correlation

- Some theorems and some conjectures
The most correlated states are entangled but ent states are not the most correlated.

Correlations on complementary prop. help understanding entanglement.

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