Measurement incompatibility and Schrödinger-EPR steering in a class of probabilistic theories

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QIPA2015
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[J. Math. Phys. **56**, 052101 (2015)]

MOTIVATION...

- Steering is one of the characteristic features of bipartite quantum system
- Measurement incompatibility is another non-classical feature of quantum theory
- Recently it has been shown that these two distinct non-classical features have connections [Phys. Rev. Lett. 113, 160402 (2014); Phys. Rev. Lett. 113, 160403 (2014)]
- The concept of steering has been generalized for more general abstract tensor product theories
- The notion of measurement incompatibility can be extended for general probability theories
- The connection between steering and measurement incompatibility holds in a border class of tensor product theories rather than just quantum theory.

PLAN OF THIS TALK...

- CONVEX OPERATIONAL THEORIES
  - State space
  - Observable
  - State space for composite system
  - Marginal and conditional states
- CONCEPT OF STEERING
- CONCEPT OF JOINT MEASUREMENT
- CONNECTION BETWEEN MEASUREMENT INCOMPATIBILITY AND STEERING

♣ The framework was initially introduced in the 1960s. Due to the emphasis on the convex structure of the set of states and the use of operations to model state transformations it gets the name.

- State Space
  - State space for a system S is denoted by $\Omega_S \subset V_S$
  - Ω_S is convex, i.e., $\forall\ \omega_1, \omega_2 \in \Omega_S$
    $$\Omega_S \ni C_{\omega_1,\omega_2} := \{p\omega_1 + (1-p)\omega_2 \mid 0 \leq p \leq 1\}$$
  - The extremal points of $\Omega_S \Rightarrow$ pure states
    others are called mixed states
  - If the number of extremal points is finite $\Rightarrow$ polytopes
    simplex $\Rightarrow$ mixed states have unique decomposition

- **Observable**
  - **Set of affine functionals on $\Omega \Rightarrow \mathcal{A}(\Omega)$**
  - $\mathcal{A}(\Omega)$ **is a ordered linear space, ordering given point-wise, i.e.**
    $$\mathcal{A}(\Omega) \ni f \geq 0, \; if \; f(\omega) \geq 0 \; \forall \; \omega \in \Omega$$
  - **The unit functional** $\Rightarrow u \mid u(\omega) = 1 \; \forall \; \omega \in \Omega$
  - **The set of effects on** $\Omega \Rightarrow \mathcal{E}(\Omega) \subset \mathcal{A}(\Omega)$
    $$\mathcal{E}(\Omega) := \{e \in \mathcal{A}(\Omega) \mid 0 \leq e(\omega) \leq 1, \; \forall \; \omega \in \Omega\}$$
  - **An observable $\mathcal{O}$ is function from an outcome set $\mathcal{K}$ into $\mathcal{E}(\Omega)$ satisfying normalization, i.e.**
    $$\mathcal{O} : \mathcal{K} \mapsto \mathcal{E}(\Omega) \mid \sum_{k \in \mathcal{K}} e^k = u$$
  - **An operational theory must assign a rule to calculate the outcome probability**
    $$p(e^k|\omega) \equiv e^k(\omega) : \Omega \times \mathcal{A}(\Omega) \mapsto [0, 1]$$
  - **Convex combination of effects are again a valid effect, i.e. $\mathcal{E}(\Omega)$ is an convex set in V^***

- State space for composite system
  - Suppose systems A and B have state spaces Ω_A and Ω_B
  - The joint system AB will have its own state space, Ω_{AB}, which is convex by definition
  - Assumptions:
    - a joint state defines a joint probability for each pair of effects (e^A, e^B)
    - these joint probabilities respect the no-signaling principle
    - if the joint probabilities for all pairs of effects (e^A, e^B) are specified, then the joint state is specified (local tomography)
  - Ω_{AB} is a convex set in $V_{AB} = V_A \otimes V_B$
  - Furthermore, it must lie between two extremes, the maximal tensor products $\Omega_A \otimes_{max} \Omega_B$ and the minimal tensor products $\Omega_A \otimes_{min} \Omega_B$

- **Marginal and conditional states**
  - **Consider the effects e_A^k and e_B^l on the state $\omega_{AB} \in \Omega_{AB}$. The marginal probability**
    $p(e_A^k, e_B^l|\omega_{AB}) = \sum_l p(e_A^k, e_B^l|\omega_{AB}) = \sum_l [e_A^k \otimes e_B^l](\omega_{AB})$
    $$= e_A^k \otimes \sum_l [e_B^l](\omega_{AB}) = [e_A^k \otimes u_B](\omega_{AB}) = e_A^k(\omega_A^{u_B})$$
    $\Rightarrow \omega_A^{u_B} \in \Omega_A$ **is the marginal state of the system A, given the composite state Ω_{AB}**
  - **The conditional probability is given by**
    $p(e_A^k|e_B^l, \omega_{AB}) = \frac{p(e_A^k, e_B^l|\omega_{AB})}{p(e_B^l|\omega_{AB})} = \frac{[e_A^k \otimes e_B^l](\omega_{AB})}{e_B^l(\omega_B^{u_A})}$
    $$= e_A^k \left(\frac{\tilde{\omega}_A^{e_B^l}}{e_B^l(\omega_B^{u_A})} \right) = e_A^k(\omega_A^{e_B^l})$$
    $\Rightarrow \omega_A^{e_B^l}$ **is the (normalized) conditional state of Alice given the effect e_B^l of Bob**

- **Let Alice and Bob share a bipartite state $\omega_{AB} \in \Omega_{AB}$.**
- **Alice's measurements $\Rightarrow \{e^{k_x}\}_x$; $x \rightarrow$ measurement choices, $k_x \rightarrow$ measurement outcomes**
- **Bob's sub normalized states $\{\tilde{\omega}^{e_B^{k_x}}\} \mid \tilde{\omega}^{e_B^{k_x}} = [e^{k_x} \otimes u_B](\omega_{AB})$;** $0 \leq u_B(\tilde{\omega}^{e_B^{k_x}}) \leq 1$; $\sum_x \tilde{\omega}^{e_B^{k_x}} = \omega_B^{u_A} = \sum_{x'} \tilde{\omega}^{e_B^{k_{x'}}}$
- **The Assemblages $\{\tilde{\omega}^{e_B^{k_x}}\}$ is unsteerable if** $\tilde{\omega}^{e_B^{k_x}} = \sum_\lambda \Gamma(\lambda) p(k|\lambda, x)\omega^\lambda, \ \forall \ k_x; \ \ \sum_\lambda \Gamma(\lambda) = 1, \sum_\lambda \omega^\lambda = \omega_B^{u_A}$
- **Definition:** A state ω_{AB} is called (**strongly**) steerable for its B marginal if (**all possible decompositions**) there exists at least one steerable decomposition of that marginal which Alice can remotely prepare by performing local measurement on her particle.

CONCEPT OF JOINT MEASUREMENT

- **A set of observables is said to be jointly measurable if all of them can be evaluated in a single measurement.**
- **Consider a set of m observables $\mathcal{O}_j = \{e^{k_j} \mid \sum_k e^{k_j} = u, \ \forall\, j\}$**
- **jointly measurable $\Rightarrow$ if there exists a measurement $\{e^{\vec{k}}\}$ with outcome $\vec{k} = [k_{j=1}, k_{j=2}, ..., k_{j=m}]$, where k_j gives the outcome of j^{th} measurement, i.e.**
  $e^{\vec{k}} \geq 0;\ \sum_{\vec{k}} e^{\vec{k}} = u;\ \sum_{\vec{k}\setminus k_j} e^{\vec{k}} = e^{k_j},\ \forall\, j$
- **If m observables are jointly measurable, then any subset of these m observables is also jointly measurable.**
- **However, the converse is not true in general, i.e., joint measurability of all possible proper subsets of these m observables does not necessarily imply that they are jointly measurable in all together.**

- **Theorem: The assemblages $\{\tilde{\omega}^{e_B^{k_x}}\}$, with $\tilde{\omega}^{e_B^{k_x}} = [e^{k_x} \otimes u_B](\omega_{AB})$, are un-steerable for any state $\omega_{AB} \in \Omega_{AB}$ if and only if the set of effects $\{e^{k_x}\} \subset \mathcal{E}(\Omega_A)$ is jointly measurable.**
- **Proof**
  - **"if part"**
    Let $\{e^{k_x}\}$ is jointly measurable, i.e.
    $e^{\vec{k}} \geq 0;\ \sum_{\vec{k}} e^{\vec{k}} = u;\ \sum_{\vec{k}\setminus k_j} e^{\vec{k}} = e^{k_j},\ \forall\ j$
    Define Alice's local variable to be $\lambda = \vec{k}$, distributed as
    $\Gamma(\lambda) = e^{\vec{k}}(\omega_B^{u_A})$
    Alice sends the local state $\omega^{\vec{k}} = [e^{\vec{k}} \otimes u_B](\omega_{AB})/\Gamma(\vec{k})$
    When asked by Bob to perform measurement x, Alice announces an outcome k according to $p(k|x, \vec{k}) = \delta_{k,k_x}$

- **Proof:**
  - **"only if part"**
    Consider an arbitrary state $\omega_{AB} \in \Omega_{AB}$. The assemblage resulting from a set of observables $\{e^{k_x}\}$ on state ω_{AB} is given by $\tilde{\omega}_B^{e^{k_x}} = [e^{k_x} \otimes u_B](\omega_{AB})$. Let $\{\tilde{\omega}_B^{e^{k_x}}\}$ unsteerable, i.e.
    $$\tilde{\omega}_B^{e^{k_x}} = \sum_\lambda \Gamma(\lambda) p(k|x,\lambda)\omega^\lambda$$
    Let e^λ be the effect on Alice's side such that
    $\omega^\lambda := \tilde{\omega}_B^{e^\lambda} = [e^\lambda \otimes u_B](\omega_{AB})$
    define the effect $e^{\vec{k}}$ as following
    $$e^{\vec{k}} := \sum_\lambda \Gamma(\lambda) e^\lambda \prod_x p(k_x|x,\lambda)$$
    $e^{\vec{k}}$ produces $\{e^{k_x}\}$ as marginals.

CONCLUSION

- **The concept of measurement incompatibility and the concept of steering can be extended in more general class of probability theories with quantum theory, a special example.**
- **we show that the connection between measurement incompatibility and steering as established in [Phys. Rev. Lett. 113, 160402 (2014)] and [Phys. Rev. Lett. 113, 160403 (2014)] also holds in a broader class of theories allowing strongly steerable state.**


Thank you!