# Entropy power inequalities for qudits

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arXiv:1503.0421

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# What is an entropy power inequality ?

- Inequalities between entropic quantities play a fundamental role in classical & quantum information theory

- Shannon's **entropy power inequality** is one such inequality (EPI) [1948]

- It has wide-ranging applications in:

information theory, probability theory, mathematical physics

- It deals with independent, continuous random variables.

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# Sums of Continuous Random Variables

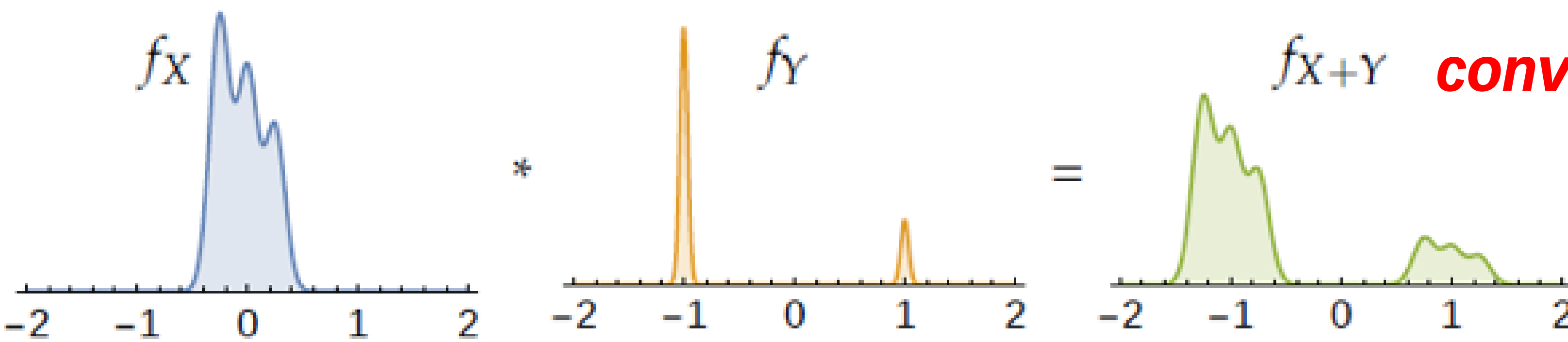
$X$ : random variable (r.v.) on $\mathbb{R}^d$ with p.d.f. $f_X$

$X$ takes values $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} f_X(\mathbf{x}) d\mathbf{x} = 1$$

If $X, Y$ independent r.v.s on $\mathbb{R}^d$ with p.d.f.s $f_X, f_Y$,

then $X + Y$ is a r.v. with p.d.f. $f_{X+Y} = f_X * f_Y$




***convolution***

addition rule : $(X, Y) \mapsto X + Y$

# Entropy Power

For a continuous random variable (r.v) $X$ on $\mathbb{R}^d$ with p.d.f. $f_X$

**entropy power**

$$\boxed{v(X) := e^{2H(X)/d}}$$

$$\boxed{H(X) := -\int_{\mathbb{R}^d} f_X(x) \log f_X(x) dx}$$

**differential entropy**

*Note: usually entropy power is defined as* $v(X) := \frac{e^{2H(X)/d}}{2\pi e}$ *but here we ignore the denominator.*

# Entropy Power Inequality (EPI)

Proposed by **Shannon (1948):** for $X, Y$ independent r.v.s on $\mathbb{R}^d$ with p.d.f.s $f_X, f_Y$ :

$$\boxed{\begin{aligned} v(X+Y) &\geq v(X) + v(Y) \\ &= \text{ iff } X, Y \text{ are Gaussian r.v.s} \end{aligned}}$$

$$v(X) := e^{2H(X)/d}$$

$$\boxed{e^{2H(X+Y)/d} \geq e^{2H(X)/d} + e^{2H(Y)/d}.}$$

# Lieb (1978): EPI equivalently expressible as an inequality between differential entropies

For $X, Y$ independent r.v.s on $\mathbb{R}^d$, $\forall\, a \in [0,1]$ :

$$\boxed{H(\sqrt{a}X + \sqrt{1-a}Y) \geq aH(X) + (1-a)H(Y)}$$

$$f_{\sqrt{a}X+\sqrt{1-a}Y} = f_{\sqrt{a}X} * f_{\sqrt{1-a}Y}$$

This motivates the definition of a 'scaled' addition rule:

$$\boxed{X \boxplus_a Y := \sqrt{a}X + \sqrt{1-a}Y}$$

$$\boxed{H(X \boxplus_a Y) \geq aH(X) + (1-a)H(Y)}$$

**("concavity" of the differential entropy under $\boxplus_a$)**

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## Scaling property of entropy power

$$v(X) := e^{2H(X)/d}$$

$$\boxed{v(\sqrt{\alpha}X) = \alpha v(X) \quad \forall\ \alpha > 0}$$

This follows from the scaling property of p.d.f.s:

If $f_{cX}$ : p.d.f. of a r.v. $cX$ on $\mathbb{R}^d$

$$f_{cX}(\mathbf{x}) = \frac{1}{c^d} f_X(\mathbf{x}/c)$$

$$\Longrightarrow \boxed{H(cX) = H(X) + d \ln c} \text{.........(2)}$$

Set $c = \sqrt{\alpha}$ and substitute (2) in

$$v(\sqrt{\alpha}X) := e^{2H(\sqrt{\alpha}X)/d} = \alpha e^{2H(X)/d} = \alpha v(X)$$

# Shannon's EPI in terms of the scaled addition rule

$$X \boxplus_a Y := \sqrt{a}X + \sqrt{1-a}Y$$

For $X, Y$ independent r.v.s on $\mathbb{R}^d$,

$$\boxed{v(X \boxplus_{1/2} Y) \geq \tfrac{1}{2} v(X) + \tfrac{1}{2} v(Y)}$$

Proof: since $a = 1/2$,

$$\begin{aligned} v(X \boxplus_{1/2} Y) &= v\left(\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y\right) \\ &= v\left(\frac{1}{\sqrt{2}}(X+Y)\right) \\ &= \tfrac{1}{2} v(X+Y) && [v(\sqrt{\alpha}X) = \alpha v(X)] \\ &\geq \tfrac{1}{2}\left(v(X) + v(Y)\right) && [v(X+Y) \geq v(X) + v(Y)] \end{aligned}$$

∎

# Summary : Classical EPI

For $X, Y$ independent r.v.s on $\mathbb{R}^d$, $\forall\ a \in [0,1]$

'scaled' addition rule: $$X \boxplus_a Y := \sqrt{a}X + \sqrt{1-a}Y$$

$$X \boxplus_a Y \begin{cases} X & \text{if} \quad a = 1 \\ Y & \text{if} \quad a = 0 \end{cases}$$

($\boxplus_a$ : interpolating map)

$$\boxed{H(X \boxplus_a Y) \geq aH(X) + (1-a)H(Y)}$$

$$\boxed{v(X \boxplus_{1/2} Y) \geq \frac{1}{2}v(X) + \frac{1}{2}(Y)}$$

$$\boxed{e^{2H(X \boxplus_{1/2} Y)/d} \geq \frac{1}{2}e^{2H(X)/d} + \frac{1}{2}e^{2H(Y)/d}}$$

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# Timeline of proofs

$$\boxed{v(X \boxplus_{1/2} Y) \geq \tfrac{1}{2} v(X) + \tfrac{1}{2}(Y)}$$ ...(cEPI1)

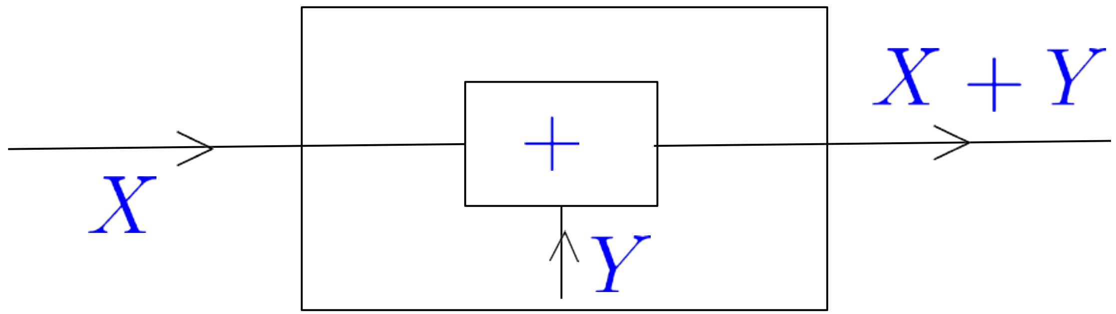
$$\boxed{H(X \boxplus_a Y) \geq aH(X) + (1-a)H(Y)}$$ ...(cEPI2)

- **1948** Shannon: proposed (cEPI1)
- **1959** Stam: proof of (cEPI1)
- **1965** Blachmann: simplied proof of (cEPI1)
- **1978** Lieb: proof of (cEPI2); (cEPI1) $\equiv$ (cEPI2)
- **1991** Dembo, Cover & Thomas: common proof of EPI & the Brunn-Minkowski inequality of convex geometry
- **2006** Verdu & Guo alternative proof of EPI
- ...... various other proofs, generalizations & applications

- Fisher info
- de Bruijn's identity

# Applications & generalizations of the EPI

- EPI proposed by Shannon as a means to bound the capacity of an additive noise channel:




- EPI has been used to find bounds on the capacity of Gaussian broadcast channel:

# Applications & generalizations of the EPI

- **Central Limit Theorem**: convergence in relative entropy

$X_1, X_2, \ldots,$ i.i.d. r.v.s; $\mathbb{E}(X_i) = \mu, \quad \mathrm{var} X_i = \sigma^2$

$Z_n := \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma},$ p.d.f. $f_n$

$\boxed{D(f_n \| \phi) \to 0 \text{ as } n \to \infty}$ $\phi$ : p.d.f. $N(0,1)$

- Many **generalizations** of EPI: e.g. monotonicity property [Artstein et. Al.]: $X_1, X_2, \ldots,$ i.i.d. continuous r.v.s;

$$\boxed{H\left(\frac{X_1+X_2+\ldots+X_n}{\sqrt{n}}\right) \geq H\left(\frac{X_1+X_2+\ldots+X_{n-1}}{\sqrt{n-1}}\right)}$$

# Quantum Setting [Koenig & Smith 2012]

Analogues of EPIs for continuous variable (CV) quantum systems (arising e.g. in quantum optics)

| Classical | Quantum |
|---|---|
| $X, Y$ on $\mathbb{R}^d$ | $\rho_X, \rho_Y$ quantum states describing bosonic modes of e.m. radiation (e.g. $n$-mode bosonic fields) |
| (differential entropy) | (von Neumann entropy) |
| $H(X)$ | $H(\rho_X) := -\mathrm{Tr}(\rho_X \log \rho_X)$ |

What is the analogue of 'scaled' addition rule $\boxplus_a$?

- Can be applied to pairs of uncorrelated quantum states

$$(\rho_X, \rho_Y) \mapsto \rho_X \boxplus_a \rho_Y$$

- Given by a linear CPTP map: **(quantum channel)** $\mathcal{E}_a$ :

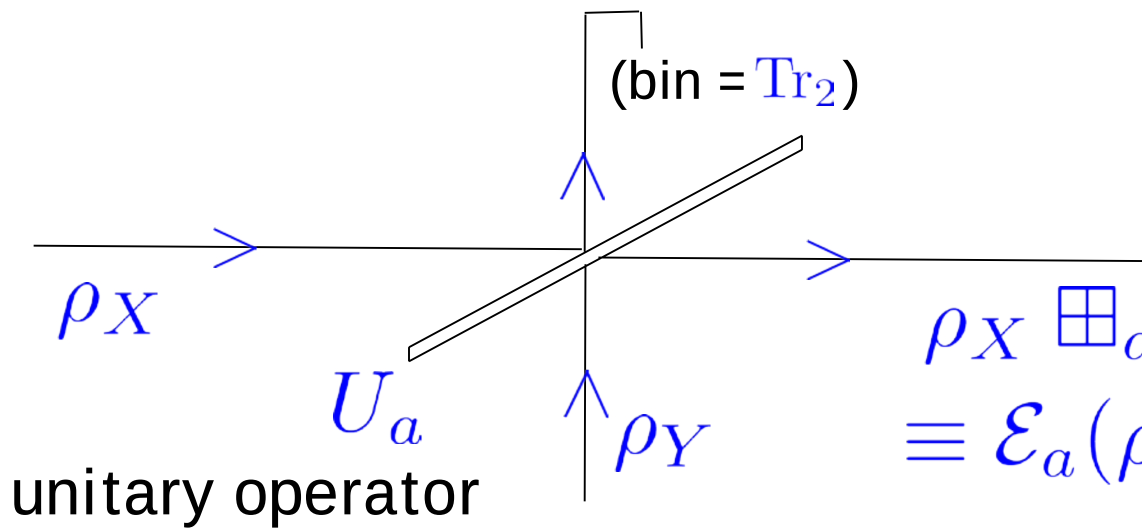
$$\mathcal{E}_a(\rho_X \otimes \rho_Y) := \rho_X \boxplus_a \rho_Y$$

# Quantum Addition Rule $\boxplus_a$ [Koenig & Smith]

Since the q-m analogue of **additive Gaussian noise** is modelled by the mixing of 2 bosonic modes at a beamsplitter;

$a \in [0,1]$ : transmissivity of a beamsplitter

$\rho_X, \rho_Y$ : states corresponding to its input modes

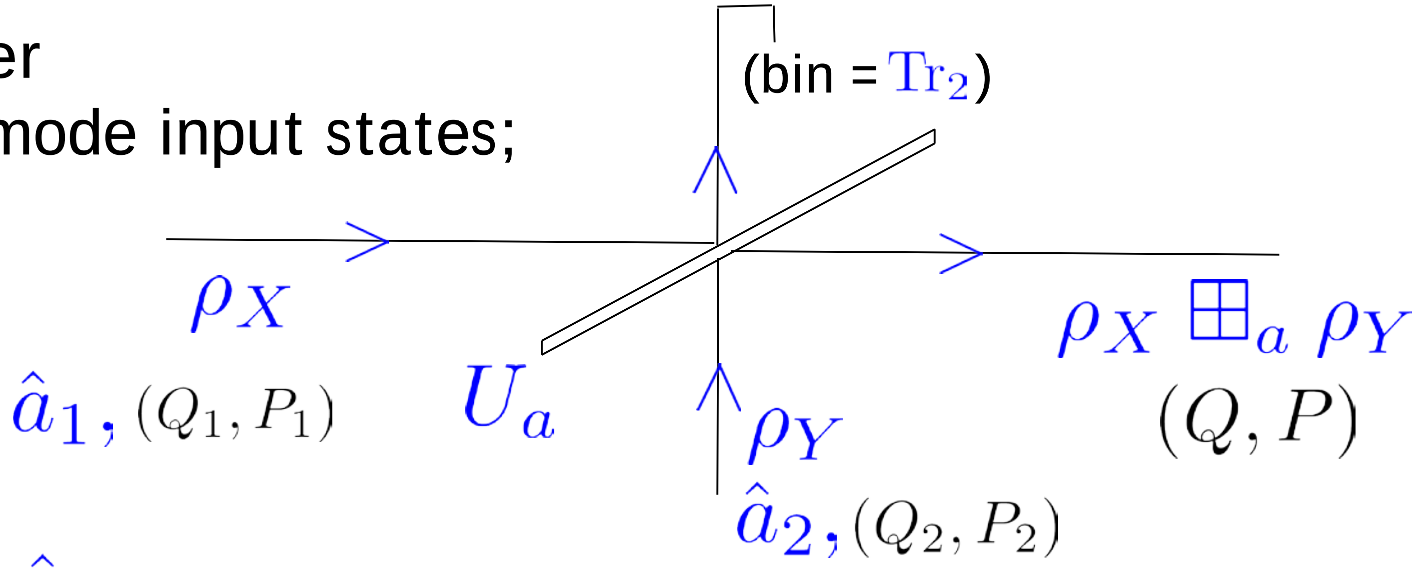



$$\rho_X \boxplus_a \rho_Y \equiv \mathcal{E}_a(\rho_X \otimes \rho_Y)$$

$$\mathcal{E}_a(\rho_X \otimes \rho_Y) = \mathrm{Tr}_2[U_a(\rho_X \otimes \rho_Y)U_a^\dagger]$$

# Motivation behind this choice of $\boxplus_a$

Consider
single mode input states;




$\hat{a}_i$ : annihilation ops.

$(Q_i, P_i)$ : corrs. pos. & mom. ops. in Heisenberg picture;

$$\boxed{Q_i = (\hat{a}_i + \hat{a}_i^\dagger)/\sqrt{2}\,; \qquad P_i = i(\hat{a}_i - \hat{a}_i^\dagger)/\sqrt{2}}$$

$$Q = \sqrt{a}Q_1 + \sqrt{1-a}Q_2\,;\; P = \sqrt{a}P_1 + \sqrt{1-a}P_2\,;$$

Mimics the classical scaled addition rule!

$$X \boxplus_a Y = \sqrt{a}X + \sqrt{1-a}Y$$

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# Analogues of EPIs for CV quantum systems

Koenig & Smith; For $n$-mode bosonic states, $\forall\, a \in [0,1]$

$$H(\rho_X \boxplus_a \rho_Y) \geq aH(\rho_X) + (1-a)H(\rho_Y) \quad \text{.......(q1)}$$

$$e^{H(\rho_X \boxplus_{1/2} \rho_Y)/n} \geq \frac{1}{2} e^{H(\rho_X)/n} + \frac{1}{2} e^{H(\rho_Y)/n} \quad \text{.......(q2)}$$

De Palma et. al; $\forall\, a \in [0,1]$

*entropy power for $n$-mode bosonic state*

$$e^{H(\rho_X \boxplus_a \rho_Y)/n} \geq a e^{H(\rho_X)/n} + (1-a) e^{H(\rho_Y)/n}$$

**Analogues - not generalizations!**

For $X, Y$ indep. r.v.s on $\mathbb{R}^d$

$$H(X \boxplus_a Y) \geq aH(X) + (1-a)H(Y) \quad \text{.......(c1)}$$

$$e^{2H(X \boxplus_{1/2} Y)/d} \geq \frac{1}{2} e^{2H(X)/d} + \frac{1}{2} e^{2H(Y)/d} \quad \text{.......(c2)}$$

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# Our Aim: To establish analogues of EPIs for finite-dimensional quantum systems (qudits)

**Classical**

$X, Y$ on $\mathbb{R}^d$

(differential entropy)

$$H(X)$$

$\boxplus_a$ (scaled addition rule)

**Finite-dl Quantum**

$\rho, \sigma \in \mathcal{D}(\mathbb{C}^d)$

(von Neumann entropy)

$$H(\rho) := -\operatorname{Tr}(\rho \log \rho)$$

$\boxplus_a$ (qudit addition rule)

- Given by a linear CPTP map (or quantum channel) $\mathcal{E}_a$

$$\rho \boxplus_a \sigma = \mathcal{E}_a(\rho \otimes \sigma)$$

- Mimics the behaviour of a beamsplitter
- Based on a continuous version of the swap operation

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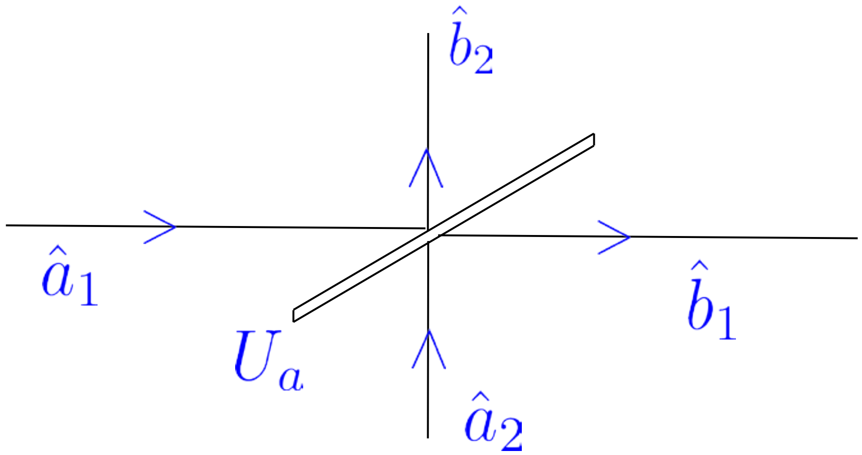
# To see how we obtain $\boxplus_a$ :

Let us look at a simple example of a beamsplitter




$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = U_a \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix};$$

scattering matrix

e.g.

$$U_a := \begin{pmatrix} \sqrt{a} & i\sqrt{1-a} \\ i\sqrt{1-a} & \sqrt{a} \end{pmatrix};$$

$$U_a = \sqrt{a}\, I + i\sqrt{1-a}\, \sigma_x$$

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_x \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} \hat{a}_2 \\ \hat{a}_1 \end{pmatrix}$$

$U_1 = I, \quad U_0 = i\sigma_x$ : swaps the 2 input modes (upto a phase)

$\forall a \in (0,1), U_a$ partially swaps the 2 input modes

This intuition → partial swap operator for 2 qudits → $\boxplus_a$

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# Obtaining a qudit addition rule $\boxplus_a$ :

$\rho, \sigma \in \mathcal{D}(\mathbb{C}^d)$; $\{|i\rangle\}_{i=1}^d$ orthonormal basis of $\mathbb{C}^d$

## Swap operator

$$S := \sum_{i,j=1}^d |i\rangle\langle j| \otimes |j\rangle\langle i| \qquad S = S^\dagger; S^2 = I$$

$$\forall\, |ij\rangle = |i\rangle \otimes |j\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d, \; S|ij\rangle = |ji\rangle$$

$[U_a = \sqrt{a}\, I + i\sqrt{1-a}\, \sigma_x]$

## Unitary operator

$$U_a := \sqrt{a}\, I + i\sqrt{1-a} S$$

$U_1 = I,$ $\quad U_0 = iS$ : swaps the qudits under conjugation

$U_1(\rho \otimes \sigma)U_1^\dagger = \rho \otimes \sigma;$ $\quad U_0(\rho \otimes \sigma)U_0^\dagger = \sigma \otimes \rho$

For $a \in (0,1)$, $U_a$ partially swaps the 2 qudits

$\Longrightarrow$ **partial swap channel** $\mathcal{E}_a$ & hence $\boxplus_a$

# Partial swap channel $\mathcal{E}_a$ ➡ Qudit addition rule $\boxplus_a$

$$\mathcal{E}_a : \mathcal{D}(\mathbb{C}^d \otimes \mathbb{C}^d) \to \mathcal{D}(\mathbb{C}^d)$$

$$\forall\, \rho_{12} \in \mathcal{D}(\mathbb{C}^d \otimes \mathbb{C}^d),$$

$$\boxed{\mathcal{E}_a(\rho_{12}) := \mathrm{Tr}_2(U_a \rho_{12} U_a^\dagger)\,;}$$

$$U_a := \sqrt{a}\, I + i\sqrt{1-a}S$$

In particular, if

$$\rho_{12} = \rho \otimes \sigma,\ \ \rho, \sigma \in \mathcal{D}(\mathbb{C}^d),$$

$$\boxed{\mathcal{E}_a(\rho \otimes \sigma) \equiv \rho \boxplus_a \sigma = \mathrm{Tr}_2\, U_a(\rho \otimes \sigma) U_a^\dagger\,;}$$

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# Calculations

$$U_a := \sqrt{a}\, I + i\sqrt{1-a} S$$

$$\mathcal{E}_a(\rho \otimes \sigma) \equiv \rho \boxplus_a \sigma = \mathrm{Tr}_2\, U_a(\rho \otimes \sigma) U_a^\dagger\,;$$

$$= \mathrm{Tr}_2 \left[ (\sqrt{a} I + i\sqrt{1-a} S)(\rho \otimes \sigma)(\sqrt{a} I - i\sqrt{1-a} S) \right]$$

$$= a\, \mathrm{Tr}_2(\rho \otimes \sigma) + (1-a)\, \mathrm{Tr}_2(\sigma \otimes \rho) \qquad [S(\rho \otimes \sigma) S = \sigma \otimes \rho]$$

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# Calculations

$$U_a := \sqrt{a}\, I + i\sqrt{1-a}\, S$$

$$\mathcal{E}_a(\rho \otimes \sigma) \equiv \rho \boxplus_a \sigma = \mathrm{Tr}_2\, U_a(\rho \otimes \sigma) U_a^\dagger\, ;$$

$$= \mathrm{Tr}_2 \left[(\sqrt{a} I + i\sqrt{1-a} S)(\rho \otimes \sigma)(\sqrt{a} I - i\sqrt{1-a} S)\right]$$

$$= a\, \mathrm{Tr}_2(\rho \otimes \sigma) + (1-a)\, \mathrm{Tr}_2(\sigma \otimes \rho)$$
$$-i\sqrt{a(1-a)}\, \mathrm{Tr}_2 \left[(\rho \otimes \sigma) S - S(\rho \otimes \sigma)\right]$$

$$= a\rho + (1-a)\sigma \; - i\sqrt{a(1-a)}\, \mathrm{Tr}_2 \left[(\rho \otimes \sigma) S - S(\rho \otimes \sigma)\right]$$

$$= \rho\sigma - \sigma\rho$$

$$\boxed{\mathcal{E}_a(\rho \otimes \sigma) = a\rho + (1-a)\sigma - i\sqrt{a(1-a)}[\rho, \sigma]}$$

# Partial swap channel $\mathcal{E}_a$ ➡ Qudit addition rule $\boxplus_a$

$$\mathcal{E}_a(\rho \otimes \sigma) \equiv \rho \boxplus_a \sigma = \mathrm{Tr}_2\, U_a(\rho \otimes \sigma) U_a^\dagger \, ;$$

**Partial swap channel** $\forall\ a \in [0, 1]$

$$\boxed{\mathcal{E}_a(\rho \otimes \sigma) = a\rho + (1-a)\sigma - i\sqrt{a(1-a)}[\rho, \sigma]}$$

$\rho$ $(a = 1)$

$\sigma$ $(a = 0)$

combines the 2 qudit states in a non-trivial manner, which mimics the action of a beamsplitter

# Main Results: EPIs for qudits

$$\forall\, \rho, \sigma \in \mathcal{D}(\mathbb{C}^d),\ \forall\, a \in [0,1],$$

- $$H(\rho \boxplus_a \sigma) \geq aH(\rho) + (1-a)H(\sigma)$$ .......(qd1)
- $$e^{H(\rho \boxplus_a \sigma)/d} \geq a e^{H(\rho)/d} + (1-a) e^{H(\sigma)/d}$$ .......(qd2)

i.e. analogues of EPIs for $n$-mode bosonic states

$$H(\rho_X \boxplus_a \rho_Y) \geq aH(\rho_X) + (1-a)H(\rho_Y)$$

$$e^{H(\rho_X \boxplus_a \rho_Y)/n} \geq a e^{H(\rho_X)/n} + (1-a) e^{H(\rho_Y)/n}$$

- **and much more!**

To state our full result, recall the definition of majorization

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# Majorization

$$\vec{r}, \vec{s} \in \mathbb{R}^d$$

$$\vec{r} = (r_1, r_2, \ldots, r_d) \;\; ; \qquad \vec{s} = (s_1, s_2, \ldots, s_d)$$

$$r_1^\downarrow \geq r_2^\downarrow \geq \ldots \geq r_d^\downarrow \quad ; \quad s_1^\downarrow \geq s_2^\downarrow \geq \ldots \geq s_d^\downarrow$$

$\vec{r} \prec \vec{s}$ ; $\vec{r}$ is majorized by $\vec{s}$

if $\sum_{i=1}^{k} r_i^\downarrow \leq \sum_{i=1}^{k} s_i^\downarrow \quad (\forall\ 1 \leq k < d)$

& $\sum_{i=1}^{d} r_i^\downarrow = \sum_{i=1}^{d} s_i^\downarrow$

To state our full result, we define the following class of functions

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**Class of functions:** $\mathcal{F} := \{f : \mathcal{D}(\mathbb{C}^d) \to \mathbb{R}\}$

**1. Concave :** For $\rho, \sigma \in \mathcal{D}(\mathbb{C}^d), \forall\, a \in [0,1]$:

$$f(a\rho + (1-a)\sigma) \geq a f(\rho) + (1-a) f(\sigma).$$

**2. Symmetric** : $f(\rho)$ depends **only on** the eigenvalues of $\rho$ and is symmetric in them.

**e.g. of a function** $f \in \mathcal{F}$ :

$$f(\rho) = H(\rho) := -\operatorname{Tr}(\rho \log \rho)$$ von Neumann entropy

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**2. Symmetric** : $f(\rho)$ depends **only on** the eigenvalues of $\rho$

and is symmetric in them.

**1 & 2 imply that:** $f(\rho)$ is Schur-concave:

Let $\lambda(\rho) = (r_1, r_2, \ldots, r_d)$
$\lambda(\sigma) = (s_1, s_2, \ldots, s_d)$ vectors of eigenvalues of $\rho, \sigma$

**Schur Concavity** $\boxed{\lambda(\rho) \prec \lambda(\sigma) \quad \Longrightarrow \quad f(\rho) \geq f(\sigma)}$

# Main Result: analogues of EPIs for qudits

**Theorem 1:** $\forall f \in \mathcal{F},\ a \in [0,1]$,

$$f(\rho \boxplus_a \sigma) \geq a f(\rho) + (1-a) f(\sigma)$$

- Since the von Neumann entropy $H(\rho) \in \mathcal{F}$,

**Theorem 1** $\Longrightarrow$ $H(\rho \boxplus_a \sigma) \geq a H(\rho) + (1-a) H(\sigma)$

- **What other functions belong to the class $\mathcal{F}$ ?**

e.g. Renyi entropy:

$$H_\alpha(\rho) = \frac{1}{\alpha - 1} \operatorname{Tr} \rho^\alpha; \quad \alpha > 0\,; \alpha \neq 1.$$

# Further examples of functions in the class $\mathcal{F}$

For an $n$-mode bosonic state $\rho$ :

- **entropy power**: $e^{H(\rho)/n}$

In analogy, for a qudit $\rho \in \mathcal{D}(\mathbb{C}^d)$ define

**Entropy power for a qudit :** $\forall\, c \geq 0,\ \rho \in \mathcal{D}(\mathbb{C}^d),$

$$E_c(\rho) := e^{cH(\rho)}$$

**Result:** $E_c(\rho) \in \mathcal{F}, \quad \forall\, 0 \le c \le 1/(\log d)^2$

Note: functions in $\mathcal{F}$,

1. concave
2. symmetric ✓

$E_c(\rho) = e^{cH(\rho)}$ : depends only on $H(\rho)$

**Theorem 2:** $E_c(\rho)$ is concave $\quad \forall\, 0 \le c \le 1/(\log d)^2$

**Theorem 1:** $\forall\, f \in \mathcal{F},\; f(\rho \boxplus_a \sigma) \ge a f(\rho) + (1-a) f(\sigma)$

**Theorem 1** $\Longrightarrow$

$$E_c(\rho \boxplus_a \sigma) \ge a E_c(\rho) + (1-a) E_c(\sigma)$$

$$\forall\, 0 \le c \le 1/(\log d)^2$$

**Theorem 1:** $\forall\, f \in \mathcal{F},\; f(\rho \boxplus_a \sigma) \geq a f(\rho) + (1-a) f(\sigma)$

## Key ingredient of the proof of Theorem 1

**Theorem 3:** $\forall\, \rho, \sigma \in \mathcal{D}(\mathbb{C}^d),\, \forall\, a \in [0,1],$

$$\lambda(\rho \boxplus_a \sigma) \prec a\lambda(\rho) + (1-a)\lambda(\sigma)$$

## Proof of Theorem 1, given Theorem 3:

Define diagonal matrices $\tilde{\rho} := \mathrm{diag}(\lambda(\rho)); \tilde{\sigma} := \mathrm{diag}(\lambda(\sigma))$ with their **entries** arranged in non-increasing order.

**Theorem 3** $\implies$

$$\begin{aligned} \lambda(\rho \boxplus_a \sigma) &\prec a\lambda(\tilde{\rho}) + (1-a)\lambda(\tilde{\sigma}) \\ &= \lambda(a\tilde{\rho}) + \lambda((1-a)\tilde{\sigma}) \\ &= \lambda(a\tilde{\rho} + (1-a)\tilde{\sigma}) \end{aligned}$$

**Theorem 1:** $\forall\, f \in \mathcal{F},\; f(\rho \boxplus_a \sigma) \geq a f(\rho) + (1-a) f(\sigma)$

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**Theorem 3** $\Longrightarrow$

$$\forall\, f \in \mathcal{F},\ \ f(\rho \boxplus_a \sigma) \geq f(a\tilde{\rho} + (1-a)\tilde{\sigma}) \quad \text{Schur-concavity}$$

$$\geq a f(\tilde{\rho}) + (1-a) f(\tilde{\sigma}) \quad \text{concavity}$$

$$= a f(\rho) + (1-a) f(\sigma) \quad \text{symmetric} \quad \blacksquare$$

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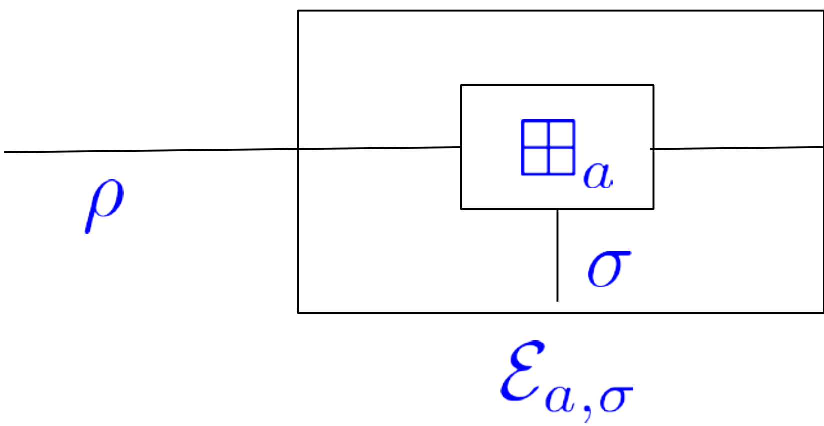
# An application of the EPIs

**Partial swap channel:** $\mathcal{E}_a(\rho \otimes \sigma) \equiv \rho \boxplus_a \sigma$

$$\mathcal{E}_a(\rho \otimes \sigma) = a\rho + (1-a)\sigma - i\sqrt{a(1-a)}[\rho, \sigma]$$

Now consider $\sigma$ to be fixed $\forall\, \rho \in \mathcal{D}(\mathbb{C}^d)$

➡ **Additive noise channel:** $\mathcal{E}_{a,\sigma}(\rho) = \mathcal{E}_a(\rho \otimes \sigma)$




a (family of) quantum channels which depends on the parameter $a$ & the state $\sigma$

**e.g**. If $\sigma = I/d$

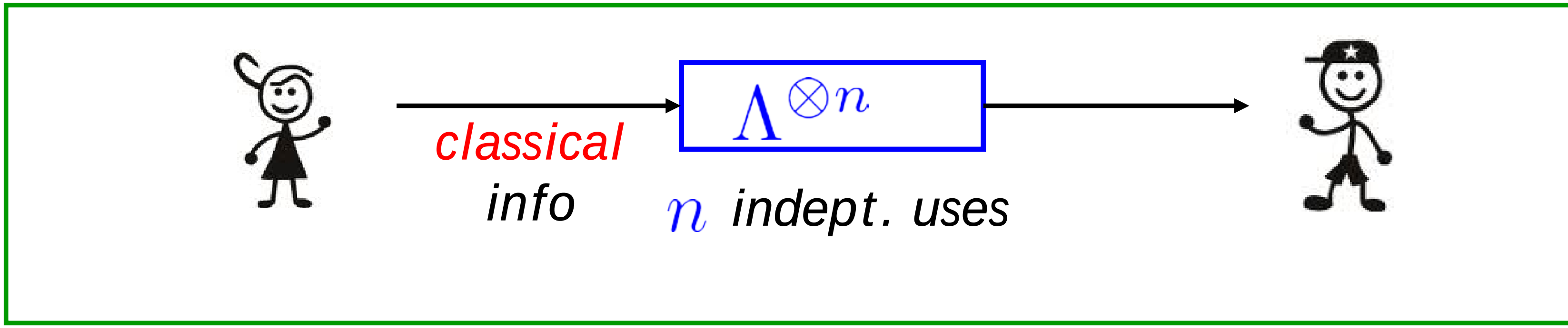
$$[\rho, I/d] = 0$$

$$\boxed{\mathcal{E}_{a,\sigma}(\rho) = a\rho + (1-a)I/d}$$

depolarizing channel

# *Sending classical info through the quantum channel*

$$\Lambda \equiv \mathcal{E}_{a,\sigma}$$




$x \rightarrow$ $E_n$ (encoding) $\xrightarrow{\rho_x^{(n)}}$ input $\rightarrow$ $\Lambda^{\otimes n}$ $\xrightarrow{\Lambda^{\otimes n}(\rho_x^{(n)})}$ channel output $\rightarrow$ $D_n$ (decoding) $\rightarrow x'$

*If* $x' \neq x$ *then error.*

***IF*** *inputs are restricted to be product states, i.e.*

$$\rho_x^{(n)} = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$$

*Then, product-state classical capacity:*

$C_{prod}(\Lambda)$ : *max. number of bits that can be sent per use of the channel s.t.* $p_{err}^{(n)} \to 0$ *as* $n \to \infty$

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It follows from the HSW Theorem that

$$\boxed{C_{prod}(\Lambda) \leq \log d - \min_{\rho} H(\Lambda(\rho))} \qquad (\Lambda \equiv \mathcal{E}_{a,\sigma})$$

$$H(\Lambda(\rho)) \equiv H(\mathcal{E}_{a,\sigma}(\rho)) = H(\rho \boxplus_a \sigma)$$

**EPI:** $$\boxed{H(\rho \boxplus_a \sigma) \geq aH(\rho) + (1-a)H(\sigma)}$$

$$C_{prod}(\Lambda) \leq \log d - a \underbrace{\min_{\rho} H(\rho)}_{=0} - (1-a)H(\sigma)$$

$$\boxed{C_{prod}(\Lambda) \leq \log d - (1-a)H(\sigma)}$$

upper bound on the product-state classical capacity!

# Entropy Photon Number Inequality (EPnI)

- another generalization of the classical EPI to the CV quantum setting

Thermal state of a bosonic mode with annihilation op. $\hat{a}$,

$$\rho_{th} = \sum_{i=0}^{\infty} \frac{N^i}{(N+1)^{i+1}} |i\rangle\langle i|$$

average photon number of the state $\rho_{th}$ $\quad N := \mathrm{Tr}\left(\rho_{th}\, \hat{a}^\dagger a\right)$

Its von Neumann entropy :

$$\boxed{H(\rho_{th}) = g(N)}, \qquad g(x) := (1+x)\log(1+x) - x\log x,$$

$$\boxed{N = g^{-1}(H(\rho_{th}))}$$

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For an $n$-mode bosonic state:

$$\boxed{N = g^{-1}(H(\rho_{th})/n)}$$

**Conjecture: [Guha et al.]**

$$\boxed{N(\rho \boxplus_a \sigma) \geq aN(\rho) + (1-a)N(\sigma)}$$

- has important implications
- proved only for Gaussian states.

## Entropy photon number for qudits

$$\forall\ \rho \in \mathcal{D}(\mathbb{C}^d), \quad N(\rho) := g^{-1}(H(\rho)/d)$$

where $g(x) = -x \log x + (1+x)\log(1+x)$

## Entropy photon number inequality for qudits

$$N(\rho \boxplus_a \sigma) \geq aN(\rho) + (1-a)N(\sigma)$$

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# Summary & Open Questions

- Analogues of EPIs for qudits under a qudit addition rule

$$\rho \boxplus_a \sigma \equiv \mathcal{E}_a(\rho \otimes \sigma) = a\rho + (1-a)\sigma - i\sqrt{a(1-a)}[\rho, \sigma]$$

(partial swap channel)

**Theorem 1:** $\forall\, f \in \mathcal{F},\ a \in [0,1],$

$$f(\rho \boxplus_a \sigma) \geq a f(\rho) + (1-a) f(\sigma)$$

$$H(\rho \boxplus_a \sigma) \geq aH(\rho) + (1-a)H(\sigma)$$

$$e^{H(\rho \boxplus_a \sigma)/d} \geq a e^{H(\rho)/d} + (1-a) e^{H(\sigma)/d}$$

$$N(\rho \boxplus_a \sigma) \geq aN(\rho) + (1-a)N(\sigma)$$

**Theorem 2:** $\forall\, \rho, \sigma \in \mathcal{D}(\mathbb{C}^d),\ \forall\, a \in [0,1],$

$$\lambda(\rho \boxplus_a \sigma) \prec a\lambda(\rho) + (1-a)\lambda(\sigma)$$

# Open Questions

- Can our proof of the qudit analogue of the entropy photon number inequality (EPnI) be generalized to establish EPnI for the bosonic case ?

- Is the partial swap channel $\mathcal{E}_a$ (& hence the $\boxplus_a$ that we define) the unique channel resulting in an interpolation between the input states & resulting in a non-trivial EPI?

Mixing: $$\tilde{\mathcal{E}}_a(\rho \otimes \sigma) \equiv \rho \tilde{\boxplus}_a \sigma := a\rho + (1-a)\sigma$$

$$H(\rho \tilde{\boxplus}_a \sigma) \geq aH(\rho) + (1-a)H(\sigma)$$

$$H(a\rho + (1-a)\sigma) \geq aH(\rho) + (1-a)H(\sigma)$$ concavity!

- Is it possible to generalize $\boxplus_a$ for combining more than two states ? Multi-mode generalization of our EPI?

- EPI for conditional entropies ?

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# Thank you!