Bell-Type Operators as a Measure of Entanglement

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Outline

Introduction

- Bell-type inequalities have played a major role in unravelling the mysteries and structure of the quantum mechanics formalism.
- In 1964, Bell obtained an inequality to demonstrate the incompatibility between local realism and quantum mechanics. It was done for a singlet state.
- After more than 25 years, in 1991, Gisin showed that any pure entangled state of a bipartite system violates a Bell's inequality, more accurately CHSH inequality.
- Maximum value of CHSH operator of this inequality in quantum mechanics can be $2\sqrt{2}$ (Tsirelson's bound, 1980). The local-realistic value is 2.
- This also establishes a relationship between entanglement and nonlocality.

Introduction

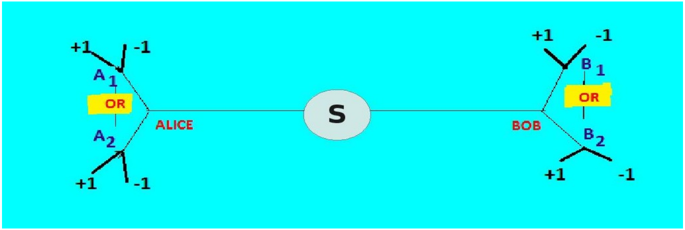
- The situation about mixed state is not clear. There are entangled states which don't violate standard CHSH inequality. Prototype example is Werner state. This leads to the phenomenon of hidden nonlocality.
- In literature, there are attempts to show that all entangled states are nonlocal. For example, Gisin (1996), Buscemi (2012), and Masanes et al (2008).
- So there may still be a relationship between nonlocality and entanglement.
- Question, then, is - what is the nature of this relationship ?
- Even for pure states, the violation of CHSH inequality depends on the observables that we choose to measure. Violation by a less entagled state in one setting, can be larger than by a more entangled state in a different setting.

Introduction

- We will first discuss CHSH inequality for two-qubit pure states and see its relation with a measure of entanglement, namely concurrence.
- Then we will go to our real goal of obtaining a relationship for a bipartite qudit state.
- The relationship that we get is between the value of the Bell-type operator for a state and its entanglement. So by measuring this operator, we can experimentally find the entanglement of a state; or compare the entanglement of two or more states.
- Whatever we have obtained, one should be able to do far better. That is for future.
- I should mention that this work has been done in collaboration with a student *Chandan Datta* and a PDF *Sujit Choudhury*.

Outline

CHSH Enequality



The CHSH inequality (John Clauser, Michael Horne, Abner Shimony, and Richard Holt, 1969) is given in terms of the following combination of the observables,

$$I_{CHSH} = A_1\ B_1 + A_1\ B_2 + A_2\ B_1 - A_2\ B_2$$

In a local-realistic theory,

$$\langle I_{CHSH} \rangle \leq 2$$

CHSH Inequality

- Let us now consider the measuremnet settings:

$$A_1 = \sigma_x, \qquad A_2 = \sigma_y,$$
$$B_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y), \quad B_2 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y).$$

- We take the nonmaximally entangled state as,

$$|\psi_0\rangle = c_0|00\rangle + c_1|11\rangle.$$

- Note that the state is in σ_z basis, while A_1 and A_2 are in the perpendicular directions.
- Then, we find,

$$\langle\psi_0|I_{CHSH}|\psi_0\rangle = 4\sqrt{2}c_0c_1 = 2\sqrt{2}\mathbb{C}.$$

Here $\mathbb{C}$ is the concurrence of the state.

CHSH Inequality

- The state we have used is in the σ_z basis. We will always get this result if the measurement settings are perpendicular to the state basis vector direction.
- An arbitrary two-qubit state after Schmidt decomposition can alway be written as

$$|\psi_n\rangle = c_0|\hat{n}_+\hat{n}_+\rangle + c_1|\hat{n}_-\hat{n}_-\rangle.$$

- We choose the measurement settings in the following way

$$A_1 = \hat{m}_1 \cdot \vec{\sigma}, \qquad A_2 = \hat{m}_2 \cdot \vec{\sigma},$$
$$B_1 = \tfrac{1}{\sqrt{2}}(\hat{m}_1 \cdot \vec{\sigma} + \hat{m}_2 \cdot \vec{\sigma}), \quad B_2 = \frac{1}{\sqrt{2}}(\hat{m}_1 \cdot \vec{\sigma} - \hat{m}_2 \cdot \vec{\sigma}).$$

Here $\hat{n}$, $\hat{m}_1$ and $\hat{m}_2$ are the unit vectors perpendicular to each other.

- The effect of these above operators on the state is

$$\hat{m}_1 \cdot \vec{\sigma}|\hat{n}_+\rangle = -|\hat{n}_-\rangle, \quad \hat{m}_1 \cdot \vec{\sigma}|\hat{n}_-\rangle = -|\hat{n}_+\rangle,$$
$$\hat{m}_2 \cdot \vec{\sigma}|\hat{n}_+\rangle = -i|\hat{n}_-\rangle, \quad \hat{m}_2 \cdot \vec{\sigma}|\hat{n}_-\rangle = +i|\hat{n}_+\rangle.$$

CHSH Inequality

- Now we have to obtain the expectation value of the CHSH operator in the state $|\psi_n\rangle$. We get same as before

$$\langle\psi_n|I_S|\psi_n\rangle = 2\sqrt{2}\mathbb{C}.$$

- Since concurrence is a measure of entanglement, we find that there is relation between an entanglement measure and the value of CHSH operator for any pure two-qubit state. Of course, these measurement settings have a flaw. Some of the entangled state don't violate CHSH inequality.
- Advantage of these settings is that the value of operator is zero for product states, and non-zero for entangled states.
- So, in this setting, CHSH operator can act as a witness to the entanglement as well as measure it.

CHSH Inequality

- Let us now consider another measurement settings:

$$\begin{aligned} A_1 &= \sigma_z, \qquad & A_2 &= \sigma_x \\ B_1 &= \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x) \quad & B_2 &= \tfrac{1}{\sqrt{2}}(\sigma_z - \sigma_x). \end{aligned} \tag{1}$$

- For these settings we find,

$$\langle \psi_0 | I_{CHSH} | \psi_0 \rangle = \sqrt{2}(1 + \mathbb{C}).$$

  Here $\mathbb{C}$ is the concurrence of the state.
- We again see a relation, though a different one. We can again measure the entanglement. However, here the expectation value is higher, so violation is higher. But still some entangled states don't violate the inequality.
- Notice the difference in the settings. In the first case, the direction of the measurements were perpendicular to the "direction" of the state. In this case, one measurement "direction" is parallel.

CHSH Inequality

- Let us again consider a general two-qubit state,

$$|\psi_n\rangle = c_0|\hat{n}_+\hat{n}_+\rangle + c_1|\hat{n}_-\hat{n}_-\rangle.$$

- We choose the measurement settings in the following way

$$A_1 = \hat{n}\cdot\vec{\sigma}, \qquad A_2 = \hat{m}\cdot\vec{\sigma},$$
$$B_1 = \frac{1}{\sqrt{2}}(\hat{n}\cdot\vec{\sigma} + \hat{m}\cdot\vec{\sigma}), \quad B_2 = \frac{1}{\sqrt{2}}(\hat{n}\cdot\vec{\sigma} - \hat{m}\cdot\vec{\sigma}).$$

- We again find,

$$\langle\psi_0|I_{CHSH}|\psi_0\rangle = \sqrt{2}(1+\mathbb{C}).$$

  Here $\mathbb{C}$ is the concurrence of the state.
- We see that we have higher values and some entangled states do not violate CHSH inequality. But still there is a relation which can be used to measure entanglement.

CHSH Inequality

- Let us now consider a third set of measurement settings:

$$A_1 = \sigma_z, \qquad A_2 = \sigma_x, \tag{2}$$
$$B_1 = cos(\eta)\ \sigma_z + sin(\eta)\ \sigma_x, \quad B_2 = cos(\eta)\ \sigma_z - sin(\eta)\ \sigma_x.$$

  Here $cos(\eta) = \frac{1}{\sqrt{1+\mathbb{C}^2}}$.
- For a nonmaximally entangled state

$$|\psi_0\rangle = c_0|00\rangle + c_1|11\rangle$$

  we find,

$$\langle\psi_0|I_{CHSH}|\psi_0\rangle = 2\sqrt{1+\mathbb{C}^2}.$$

  Here $\mathbb{C}$ is the concurrence of the state.
- These settings are the best. For any entangled state there is a violation of CHSH inequality. These settings have been optimized for each state to give maximum possible value. (Horodecki[3], 1995)

CHSH Inequality

- Again we consider the general two-qubit state.

$$|\psi_n\rangle = c_0|\hat{n}_+\hat{n}_+\rangle + c_1|\hat{n}_-\hat{n}_-\rangle$$

- We choose the measurement settings in the following way

$$\begin{aligned} A_1 &= \hat{n}\cdot\vec{\sigma}, & A_2 &= \hat{m}\cdot\vec{\sigma}, \\ B_1 &= \hat{n}\cdot\vec{\sigma}\, cos(\eta) + \hat{m}\cdot\vec{\sigma}\, sin(\eta), & B_2 &= \hat{n}\cdot\vec{\sigma}\, cos(\eta) - \hat{m}\cdot\vec{\sigma}\, sin(\eta). \end{aligned}$$

  Here $\hat{n}$ and $\hat{m}$ are the unit vectors perpendicular to each other.
- We again get

$$\langle\psi_n|I_{CHSH}|\psi_n\rangle = 2\sqrt{1+\mathbb{C}^2}.$$

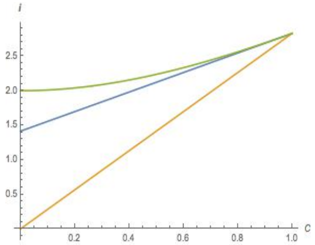
- Though these setting give the optimized value and largest violation, but there is a flaw. You have to know the state in advance for these settings.
- So if we wish to find how entangled an unknown state is, we should be using earlier settings.

CHSH Inequality

- The question may be asked what about the most general state-independent settings ? One can show that there is a relation,

$$\langle\psi_0|I_{CHSH}|\psi_0\rangle = A + B\,\mathbb{C}.$$

  Here A and B would depend on measurement setting angles.
- Here is a plot to show the value of CHSH operator for the three different settings:



CHSH Inequality

- We had written above the CHSH inequality in terms of correlation functions. It is possible to rewrite this inequality in terms of joint probabilities:

$$I_{CHSH} = P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1).$$

- In this expression $P(A = B + k)$, more generally, stands for

$$P(A = B + k) = \sum_{j=0}^{d-1} P(A = j + k \bmod d, B = j).$$

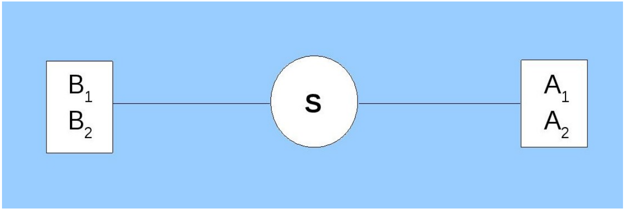
For qubits $d = 2$. P(A=j, B=k) is a joint probability of obtaining $A = j$ and $B = k$ on measuring A and B.

- This form of CHSH inequality was generalized to qudits and is known as CGLMP inequality. (D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, 2002)

Outline

Bell-type Inequalities and Qudits

- Qudits, i.e., systems with *d*-dimensional Hilbert space, can play important role in quantum information processing. It would also be interesting to investigate their nonlocality structure.
- Here is the typical set-up to observe Bell-type inequality violation



- Gisin (1991) and Gisin & Peres (1992) examined CHSH inequality with dichotomic observables for a system of two qudits. Gisin theorem also holds for two-qudit pure states.

Bell-type Inequalities and Qudits

- It is natural to examine inequalities, where the observables can take d different values - like $0, 1, 2, ..., d-1$. One can also measure more than 2 observables on each qudit.
- Our focus will be on inequalities with two observables with d values on each side.
- One early development in this direction was the introduction of CGLMP inequality. (D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, 2002). This inequality has its own advantages and disadvantages.
- Subsequently, many more inequalities for two or more qudit systems have been proposed. We will pick one such inequality, which was one of many that were proposed by W. Son, J. Lee and M. S. Kim (2006). It is called SLK inequality.
- We will obtain a relation between an entanglement measure and the expectation value of SLK operator in a particular set of observation settings. For these settings no such relation exists for CGLMP operator.

Outline

CGLMP Inequality

- The CGLMP inequality is a generalization of CHSH inequality. It is a specific generalization in terms of joint probability distributions:

$$
\begin{aligned}
I_d \quad = \quad & \sum_{k=0}^{[\frac{d}{2}]-1} (1 - \frac{2k}{(d-1)})[(P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) \\
& + P(A_2 = B_2 + k) + P(B_2 = A_1 + k)) \\
& - (P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) \\
& + P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1))] \qquad (3)
\end{aligned}
$$

- This generalization was obtained by first trying to find an optimum expression for $d = 3$ and $d = 4$.
- There are other ways to write it, as we will see later.
- The maximum local-realistic value for this is 2 and maximum possible value is 4.

CGLMP Inequality

- This has been more popular qudit inequality. It has been tested experimentally also. (Dada et al, 2011)
- However this inequality has one drawbeck. A nonmaximally entangled state violates it more than a maximally entangled state. Following table from Acin, Durt, Gisin, and Latorre (2002) illustrates this.

Dimension	Violation for $\|\Psi\rangle$	Maximal violation (for $\|\Psi_{mv}\rangle$)	Difference (%)
3	2.8729	2.9149	1.4591
4	2.8962	2.9727	2.6398
5	2.9105	3.0157	3.6133
6	2.9202	3.0497	4.4345
7	2.9272	3.0776	5.1411
8	2.9324	3.1013	5.7588

- Given this, it would appear unlikely that a relation where a relation like that for CHSH inequality may exist for CGLMP inequality.

Outline

SLK Inequality

- In the case of SLK (Son-Lee-Kim) inequality, two far separated observers Alice and Bob, can independently choose one of the two observables denoted by A_1, A_2 for Alice and B_1, B_2 for Bob. Measurement outcomes of the observables are elements of the set, $V = \{1, \omega, \cdots, \omega^{d-1}\}$, where $\omega = \exp(2\pi i/d)$.
- In a variant of SLK inequality, the SLK function, I_{SLK}, is given by

$$
\begin{aligned}
I_{SLK} \quad &= \quad \frac{1}{\sqrt{2}} \sum_{n=1}^{d-1} \left(\omega^{-n/4} C^n_{1,1} + \omega^{-3n/4} C^n_{2,1} \right. \\
&\quad \left. + \omega^{n/4} C^n_{1,2} + \omega^{-n/4} C^n_{2,2} \right) + c.c.,
\end{aligned}
$$

$\omega = exp(2\pi i/d)$, c.c. is for complex conjugate.

SLK Inequality

- For $d = 2$, it reduces to CHSH inequality. It has been shown that maximum local-realistic value is

$$\frac{1}{\sqrt{2}}[3\ cot(\frac{\pi}{4d}) - cot(\frac{3\pi}{4d})] - 2\sqrt{2}$$

- The correlation function can be written as

$$\begin{aligned} C^n_{a,b} &= \int d\lambda \rho(\lambda)(A^*_a(\lambda)\ B_b(\lambda))^n \\ &= \sum_{k,l=0}^{d-1} \omega^{n(l-k)} P(A_a = k, B_b = l) \\ &= \sum_{\alpha} \omega^{n\alpha} P(A_a = B_b + \alpha \bmod d) \end{aligned}$$

where $P(A_a = k, B_b = l)$ represents the probability that Alice and Bob get ω^k and ω^l on measurement.

SLK Inequality

- In terms of probabilities, it can be written as

$$
\begin{aligned}
I_{SLK} &= \sum_{\alpha=0}^{d-1} f(\alpha)[P(A_1 = B_1 + \alpha) \\
&\quad + P(B_1 = A_2 + \alpha + 1) + P(A_2 = B_2 + \alpha) \\
&\quad + P(B_2 = A_1 + \alpha)],
\end{aligned}
$$

  where sums inside the probabilities are modulo d sums, and

$$f(\alpha) = \frac{1}{\sqrt{2}}\Big(\cot[\frac{\pi}{d}(\alpha + \frac{1}{4})] - 1\Big).$$

- CGLMP inequality can also be written in the same form as above, except,

$$f_{CGLMP}(\alpha) = 1 - \frac{2\,\alpha}{d-1}.$$

- Using identity $\sum_{k=0}^{d-1} \cot(\frac{4k+1}{4d})\pi = d$, we get $\sum_{\alpha=0}^{d-1} f(\alpha) = 0$.

SLK Inequality

- One good feature of this inequality is that it is maximally violated by maximally entangled state.
- For maximally entangled state

$$I_{SLK} = 2\sqrt{2}(d - 1)$$

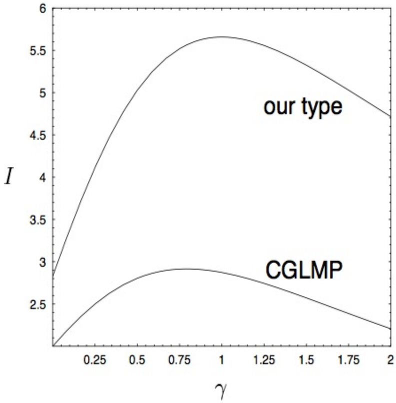
- For a two-qutrit state

$$|\varphi\rangle = N(|00\rangle + \gamma\,|11\rangle + |22\rangle),$$

  the difference between the CGLMP and SLK inequality can be seen easily. CGLMP function has maximum value for this state when $\gamma = 0.7923$. The value is 2.9149. This is same as given in the table on an earlier slide. For SLK function, the maximum value of $4\sqrt{2}$ is obtained for $\gamma = 1$.

SLK Inequality

- Following plot from Lee, Ryu, and Lee (2006) illustrates this.



- We clearly see that SLK inequality is maximally violated for maximally entangled two-qutrit state.

SLK Inequality

- We now calculate the value of the Bell-SLK function for an arbitrary pure two-qudit state $|\psi\rangle = \sum_i c_i |ii\rangle$ and for the measurement settings originally given by Durt, Kaszlikowski and Zukowski (2001). The nondegenerate eigenvectors of the operators $\hat{A}_a$, $a = 1, 2$, and $\hat{B}_b$, $b = 1, 2$, are respectively

$$|k\rangle_{A,a} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^{(k+\delta_a)j} |j\rangle,$$

$$|l\rangle_{B,b} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^{(-l+\epsilon_b)j} |j\rangle,$$

where $\delta_1 = 0, \delta_2 = 1/2, \epsilon_1 = 1/4$ and $\epsilon_2 = -1/4$.

- Putting probabilities together, we get

$$I_{SLK} = \frac{4}{d} \sum_{\alpha=0}^{d-1} f(\alpha) \sum_{p,q=0}^{d-1} c_p c_q \omega^{(\alpha+1/4)(p-q)}.$$

SLK Inequality

- We have to now compute the expression. We will need to find some sums.
- We can now rewrite

$$
\begin{aligned}
I_{SLK} &= \frac{4}{d}\sum_{\alpha=0}^{d-1} f(\alpha) \sum_{\substack{p\neq q\\ p>q}} 2c_p c_q \cos\left(\frac{2\pi}{d}(\alpha+\frac{1}{4})(p-q)\right) \\
&= \frac{4}{d}\sum_{\alpha=0}^{d-1} \frac{1}{\sqrt{2}}\left(\cot[\frac{\pi}{d}(\alpha+\frac{1}{4})] - 1\right) \\
&\quad \sum_{\substack{p\neq q\\ p>q}} 2c_p c_q \cos\left(\frac{2\pi}{d}(\alpha+\frac{1}{4})(p-q)\right).
\end{aligned}
$$

- In particular, we need to find

$$
\sum_{\alpha=0}^{d-1} \cos\left(\frac{2\pi m}{d}(\alpha+\frac{1}{4})\right) \cot\left(\frac{\pi}{d}(\alpha+\frac{1}{4})\right), \tag{4}
$$

where we have replaced $p-q$ by m (an integer).

SLK Inequality

- This sum could not be found in various handbooks or googling. However, it turns out that we can compute these sums using method of residues.
- To compute the sum, we have proved two lemmas.
- With a and k being positive integer such that $a < k$, and $0 < b < 1$,

$$\sum_{j=0}^{k-1} \cos \frac{2\pi aj}{k} \cot\left(\frac{\pi j}{k} + \pi b\right) = k \cos\left[b(2a-k)\pi\right] \operatorname{cosec}(bk\pi).$$

- With a and k being positive integer such that $a < k$, and $0 < b < 1$.

$$\sum_{j=0}^{k-1} \sin \frac{2\pi aj}{k} \cot\left(\frac{\pi j}{k} + \pi b\right) = -k \sin\left[b(2a-k)\pi\right] \operatorname{cosec}(bk\pi).$$

SLK Inequality

- Using these sums, we obtain

$$\sum_{\alpha=0}^{d-1} \cos\left(\frac{2\pi m}{d}(\alpha + \frac{1}{4})\right) \cot\left(\frac{\pi}{d}(\alpha + \frac{1}{4})\right) = d.$$

- We note that the value of this sum is independent of m. This is most important. If there were dependence on m, then we would not have been able to write SLK function in terms of concurrence. In the case of CGLMP funnction, the sum that appears is not independent of m. Therefore, such a relation dows not exist.
- Putting everything together

$$I_{SLK} = 4\sqrt{2} \sum_{\substack{p \neq q \\ p > q}} c_p c_q.$$

- This sum is proportional to the concurrence of the state.

SLK Inequality

- The concurrence, $\mathbb{C}$, for a two-qudit pure state is defined as

$$\mathbb{C} = \sum_{\substack{p \neq q \\ p > q}} c_p c_q \frac{2}{d-1}.$$

- This is a generalization of the concurrence for a system of two qubits. Using this we finally get

$$I_{SLK} = 2\sqrt{2}(d-1)\mathbb{C}.$$

- Note that for $d = 2$, it reduces to the expression that we had obtained for the CHSH operator, for the first measurement settings. Actually it is not surprising, since DKZ measurement settings reduce to our first measurement settings.
- This result establishes a relation between nonlocality and entanglement. For product states this function is zero. So it is an entanglement witness also.

SLK Inequality

- So given an unknown state, we can find its entanglement, by measuring this function. Given a set of states, we can also find which state is more entangled
- On the negative side, for some entangled states, this inequality is not violated.
- We may also like to find a relation that is analog of the third measurement setting for qubits. Then one can relate the nonlocality, as reflected in the violation of the inequality, with the entanglement.
- For this one has to find appropriate state dependent settings. For each state, the settings would depend on its entanglement.

Outline

Conclusions

- As we saw, CHSH operator can be used to measure the entanglement of a pure qubit state with several different settings. For mixed states, such relations are still to be established, as we have the phenomenon of hidden nonlocality.
- In the case of qudits, CGLMP inequality is violated more by a nonmaximally entangled state than by a maximally entangled state. So it may not be useful to find a relation between nonlocality and entanglement measure.
- However, we show that SLK function can be used to characterize the entanglement. However, it would be better to find an inequality, or settings where the violation and entanglement are related.
- In the case of multiqubit states, it is far from clear if one can find such relationships. For this, one has to find a way to characterize a state's entanglement.

Advertisement

- In the end, let me advertise a school-cum-conference that we will be organizing at the Institute of Physics, Bhubaneswar.

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  It is 5th in the series that started in 2008. There will be a five-day school followed by a four-day conference. Students are encouraged to register and others are invited to the conference.