Exchangeable, stationary and entangled chains of Gaussian states

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A state $\rho$ in $\Gamma(\mathbb{C}^n)$ is called a $n$-mode Gaussian state if its Fourier transform $\hat{\rho}$ is given by

$$\hat{\rho}(x + iy) = \exp \left[ -i\sqrt{2}(l^T x - m^T y) - \begin{pmatrix} x \\ y \end{pmatrix}^T S \begin{pmatrix} x \\ y \end{pmatrix} \right] ,$$

(1)

for all $x, y \in \mathbb{R}^n$ where $l, m$ are elements of $\mathbb{R}^n$ and $S$ is a real $2n \times 2n$ symmetric matrix satisfying the matrix inequality

$$S + \frac{i}{2} J_{2n} \geq 0$$

(2)

with

$$J_{2n} = \bigoplus_{n\text{-copy}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

(3)
G-matrix

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Suppose $\Sigma = [[A_{ij}]], \quad i, j \in \{1, 2, \cdots\}$ is an infinite matrix where each $A_{ij}$ is a $2k \times 2k$ real matrix and $A_{ij}^T = A_{ji}$ for all $i, j$. For any finite subset $I = \{i_1 < i_2 < \cdots < i_n\} \subset \{1, 2, \cdots\}$, let $\Sigma(I) = [[A_{i_ri_s}]], r, s \in \{1, 2, \cdots\}$ be the $2kn \times 2kn$ matrix obtained from $\Sigma$ by restriction to its rows and columns numbered $i_1 < i_2 < \cdots < i_n$. 
G-matrix

A $2n \times 2n$ real symmetric positive matrix $S$ is said to be a \textit{G-matrix} if it satisfies the inequality

$$S + \frac{\imath}{2}J_{2n} \geq 0. \quad (4)$$

Suppose $\Sigma = [[A_{ij}]]$, $i, j \in \{1, 2, \cdots\}$ is an infinite matrix where each $A_{ij}$ is a $2k \times 2k$ real matrix and $A_{ij}^T = A_{ji}$ for all $i, j$. For any finite subset $I = \{i_1 < i_2 < \cdots < i_n\} \subset \{1, 2, \cdots\}$, let $\Sigma(I) = [[A_{i_r i_s}]]$, $r, s \in \{1, 2, \cdots\}$ be the $2kn \times 2kn$ matrix obtained from $\Sigma$ by restriction to its rows and columns numbered $i_1 < i_2 < \cdots < i_n$.

$\Sigma$ is said to be a \textit{G-chain of order $k$} if $\Sigma(I)$ is the covariance matrix of a $kn$-mode zero mean Gaussian state $\rho(I)$ in the boson Fock space $\Gamma(\mathbb{C}^{kn}) = \mathcal{H}_{i_1} \otimes \mathcal{H}_{i_2} \otimes \cdots \otimes \mathcal{H}_{i_n}$ where $\mathcal{H}_j$ denotes the $j$-th copy of the Hilbert space $\mathcal{H} = \Gamma(\mathbb{C}^k)$, $j = 1, 2, \cdots$. 
We say that $\Sigma$ is an *exchangeable G-chain* if it is a G-chain and there exist two $2k \times 2k$ matrices $A$, $B$ such that

$$A_{ij} = \begin{cases} 
B & \text{if } j > i, \\
A & \text{if } j = i, \\
B^T & \text{if } j < i.
\end{cases} \quad (5)$$

In such a case we write

$$\Sigma = \Sigma(A, B),$$
$$\Sigma(I) = \Sigma(I; A, B). \quad (6)$$
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$$\Sigma = \Sigma(A, B),$$

$$\Sigma(I) = \Sigma(I; A, B).$$

(6)

We say that a G-chain $\Sigma$ is *stationary* if there exist $2k \times 2k$ matrices $A, B_1, B_2, \cdots$ such that

$$A_{ij} = \begin{cases} A & \text{if } i = j, \\ B_{j-i} & \text{if } j > i, \\ B_{i-j}^T & \text{if } j < i \end{cases}$$

(7)

for all $i, j$. 
Theorem

Let \((A, B)\) be a pair of real \(2k \times 2k\) matrices. Then \(\Sigma(A, B)\) is a G-chain if and only if \(A\) and \(B\) are nonnegative definite and \(A - B\) is a G-matrix.
Theorem

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Proof technique

Fix \(n\), and let \(I = \{1, 2, \cdots, n\}\). Set \(\Sigma_n(A, B) = \Sigma_n(I : A, B)\). Define \(N_n = \begin{bmatrix} x_{ij} \end{bmatrix}_{n \times n}\) where \(x_{ij} = 1\) if \(i > j\) and 0 otherwise. Let \(|\psi_n\rangle = n^{-\frac{1}{2}}[1, \cdots, 1]^T\) be the unit column vector of length \(n\).
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|\psi_n\rangle = n^{-\frac{1}{2}} [1, \cdots, 1]^T
\]
be the unit column vector of length \(n\).

\[
\Sigma_n(A, B) + \frac{\nu}{2} J_{2kn} = \left( A + \frac{\nu}{2} J_{2k} - \frac{1}{2} (B + B^T) \right) \otimes (I_n - |\psi_n\rangle \langle \psi_n|)
\]
\[
+ \left( A + \frac{\nu}{2} J_{2k} + \frac{1}{2} (n - 1)(B + B^T) \right) \otimes |\psi_n\rangle \langle \psi_n|
\]
\[
+ \frac{1}{2} (B - B^T) \otimes (N_n - N_n^T). (8)
\]
Multiplying both sides by $I_n \otimes |\psi_n\rangle\langle\psi_n|$ and take relative trace over the second component. Take relative trace over second component and relative trace over second component gives

$$\frac{1}{2}(B + B^T) \geq 0.$$
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$$\frac{1}{2} (B + B^T) \geq 0.$$ 

Similarly we get

$$A + \frac{\nu}{2} J_{2k} - \frac{1}{2} (B + B^T) \geq 0.$$
Multiplying both sides by $I_n \otimes |\psi_n\rangle\langle\psi_n|$ and take relative trace over the second component. Take relative trace over second component and relative trace over second component gives

$$\frac{1}{2}(B + B^T) \geq 0.$$ 

Similarly we get

$$A + \frac{\imath}{2}J_{2k} - \frac{1}{2}(B + B^T) \geq 0.$$ 

Consider complex unit vector $|\phi_n\rangle = \frac{1}{\sqrt{n}}[1, \omega, \omega^2, \cdots, \omega^{n-1}]^T$, where $\omega = e^{\frac{2\pi\imath}{n}}$. Now, multiplying both sides of (8) by $I_n \otimes |\phi_n\rangle\langle\phi_n|$ and tracing over the second Hilbert space we get the inequality

$$A + \frac{\imath}{2}J_{2k} - \frac{1}{2}(B + B^T) + \frac{\imath}{2} \frac{B - B^T}{\cot \frac{\pi}{n}} \geq 0.$$ 

for $n = 1, 2, \cdots$. Multiplying by $\tan \frac{\pi}{n}$ (for $n \geq 3$) and letting $n \to \infty$ we get the inequality

$$\frac{\imath}{2}(B - B^T) \geq 0.$$ 

Hence the result follows.
Corollary

In any exchangeable G-chain $\Sigma(A, B)$ of order $k$, for every finite set $I \subset \{1, 2, \cdots \}$, the underlying Gaussian state $\rho(I)$ is separable.
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Proof

Without loss of generality we may assume that $I = \{1, 2, \cdots , n\}$ for some $n$. Then the covariance matrix of $\rho(I)$ is equal to the $n \times n$ block matrix

$$
\begin{bmatrix}
A & B & \cdots & B \\
B & A & \cdots & B \\
\vdots & \vdots & \ddots & \vdots \\
B & B & \cdots & A
\end{bmatrix} = (A - B) \otimes I_n + B \otimes \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}.
$$

By previous theorem, $(A - B) \otimes I_n$ is the covariance matrix of an $n$-fold product Gaussian state and the second summand on the right hand side of the equation 8 above is a nonnegative definite matrix. Hence by Werner and Wolf’s theorem, $\rho(I)$ is separable.
Let $A$, $B$ be real $2k \times 2k$ symmetric matrices. For any fixed $j = 1, 2, \cdots$, denote by $\Delta^j(A, B)$ the infinite block matrix all of whose diagonal blocks are equal to $A$, $(n, n+j)$-th and $(n+j, n)$-th blocks are equal to $B$ for every $n$ and all the remaining blocks are zero matrices of order $2k \times 2k$. For example,

$$\Delta^1(A, B) = \begin{bmatrix} A & B & 0 & 0 & 0 & \cdots \\ B & A & B & 0 & 0 & \cdots \\ 0 & B & A & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$ 

Denote by $\Delta^j_n(A, B)$ the $2kn \times 2kn$ matrix obtained by $\Delta^j(A, B)$ by restriction to the first $n$ row and column blocks. For example,

$$\Delta^1_2(A, B) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \quad \Delta^2_3(A, B) = \begin{bmatrix} A & 0 & B \\ 0 & A & 0 \\ B & 0 & A \end{bmatrix}$$

and so on.

Our first result gives a necessary and sufficient condition for $\Delta^j(A, B)$ to be a G-chain.
Theorem

Let $A, B$ be a pair of $2k \times 2k$ real symmetric matrices. In order that $\Delta^j(A, B)$ may be a G-chain of order $k$ it is necessary and sufficient that $A + tB$ is a G-matrix for every $t \in [-2, 2]$. 

Proof:

Denote by $L_j^n$ the upper triangular matrix whose $(j+1)$-th upper diagonal entries are all equal to 1 and all the remaining entries are zero. Thus $L_j^n$ is defined for $1 \leq j \leq n-1$. Then $\Delta^j(A, B) = A \otimes I_n + B \otimes (L_j^n + (L_j^n)^T)$.

(9)

Consider the spectral decomposition of the $n \times n$ symmetric matrix $L_j^n + (L_j^n)^T$:

$L_j^n + (L_j^n)^T = \sum_{r=1}^{n} \lambda_{nr} \langle \psi_{nr} \rangle \psi_{nr}$

(10)

where $\{\lambda_{nr} : r = 1, 2, \ldots, n\}$ are the eigenvalues and $\{\langle \psi_{nr} \rangle : r = 1, 2, \ldots, n\}$ are the corresponding orthonormal basis of eigenvectors for $L_j^n + (L_j^n)^T$. Since each $L_j^n$ is a matrix with operator norm equal to unity and therefore $L_j^n + (L_j^n)^T$ has operator norm not exceeding 2.
Theorem

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Proof: Denote by $L^j_n$ the upper triangular matrix whose $(j + 1)$-th upper diagonal entries are all equal to 1 and all the remaining entries are zero. Thus $L^j_n$ is defined for $1 \leq j \leq n - 1$. Then

$$\Delta^j_n(A, B) = A \otimes I_n + B \otimes (L^j_n + (L^j_n)^T).$$

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\[ |\lambda_{nr}| \leq 2, \quad 1 \leq r \leq n, \quad n = 1, 2, \cdots. \]  
\hspace{1cm} (11)

Equations (9)–(10) imply

\[ \Delta^j_n(A, B) + \frac{\nu}{2} J_{2kn} = \sum_{r=1}^{n} \left( A + \lambda_{nr}B + \frac{\nu}{2} J_{2k} \right) \otimes |\psi_{nr}\rangle\langle\psi_{nr}|. \]  
\hspace{1cm} (12)

Thus \( \Delta^j_n(A, B) \) is a G-matrix if and only if \( A + \lambda_{nr}B \) is a G-matrix for each \( r \).
Equations (9)–(10) imply

\[
\Delta_j^i(A, B) + \frac{\imath}{2} J_{2kn} = \sum_{r=1}^{n} \left( A + \lambda_{nr} B + \frac{\imath}{2} J_{2k} \right) \otimes |\psi_{nr}\rangle\langle\psi_{nr}|. \tag{12}
\]

Thus \(\Delta_j^i(A, B)\) is a G-matrix if and only if \(A + \lambda_{nr} B\) is a G-matrix for each \(r\). To prove necessity, we appeal to the theorem of Kac, Murdock and Szegö [J. Ration. Mech. Anal. 2, 767–800, 1953]. Consider the probability distribution

\[
\mu_n = \frac{1}{n} \sum_{r=1}^{n} \delta_{\lambda_{nr}}
\]

where \(\lambda_{nr}, r = 1, 2, \cdots, n\) are as in (10). The left hand side of (10) is a Toeplitz matrix of order \(n\) for each \(n\). KMS theorem implies that the sequence \(\{\mu_n\}\) converges weakly as \(n \to \infty\) to the probability measure \(Lh^{-1}\) where \(L\) is the Lebesgue measure in the unit interval and \(h(t) = 2 \cos 2\pi jt, t \in [0, 1]\). This, in particular, implies that \(\{\lambda_{nr} : r = 1, 2, \cdots, n, n = 1, 2, \cdots \}\) is dense in the interval \([-2, 2]\). The proof of necessity is now complete.
**Corollary**

Let $A$, $B_1$, $B_2$, $\cdots$ be real $2k \times 2k$ symmetric matrices satisfying the condition that $A + tB_j$ is a G-matrix for every $j = 1, 2, \cdots$ and $t \in [-2, 2]$. Suppose $p_1, p_2, \cdots$, is a probability distribution on the set $\{1, 2, \cdots\}$. Then the block Toeplitz matrix

$$
\Sigma(A; p_1B_1, p_2B_2, \cdots) = \begin{bmatrix}
A & p_1B_1 & p_2B_2 & \cdots & \cdots \\
p_1B_1 & A & p_1B_1 & p_2B_2 & \cdots \\
p_2B_2 & p_1B_1 & A & p_1B_1 & \cdots \\
: & : & : & : & \ddots \\
\end{bmatrix}
$$

is a stationary G-chain.
Corollary

Let $A$, $B_1$, $B_2$, $\cdots$ be real $2k \times 2k$ symmetric matrices satisfying the condition that $A + tB_j$ is a G-matrix for every $j = 1, 2, \cdots$ and $t \in [-2, 2]$. Suppose $p_1, p_2, \cdots$, is a probability distribution on the set $\{1, 2, \cdots\}$. Then the block Toeplitz matrix

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p_2B_2 & p_1B_1 & A & p_1B_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}$$

is a stationary G-chain.

$$\Sigma(A; p_1B_1, p_2B_2, \cdots) = \sum_{j=1}^{\infty} p_j \Delta^j(A, B_j)$$

and each $\Delta^j(A, B_j)$ is a G-chain.
Entropy rate of stationary G-chain

Suppose \( \Sigma = \Sigma(A, B_1, B_2, \cdots) \) is a stationary G-chain. For any G-matrix \( C \) denote by \( S(C) \) the von Neumann entropy of a Gaussian state \( \rho \) with covariance matrix \( C \). Let

\[
\Sigma_n = \Sigma(\{1, 2, \cdots, n\}), \\
S_n = S(\Sigma_n).
\]
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\Sigma_n = \Sigma(\{1, 2, \cdots, n\}), \\
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\]

**Proposition**

The sequences $\{S_n - S_{n-1}\}$, $\{\frac{1}{n}S_n\}$ monotonically decrease to the same limit $\bar{S} \geq 0$ as $n \to \infty$. Furthermore, $S_n \geq S_{n-1}$ for all $n$. 
Consider Gaussian systems $P, Q, R$ such that $PQR = \rho(\{1, 2, \cdots, n + 1\})$, $Q = \rho(\{2, \cdots, n\})$, $PQ = \rho(\{1, 2, \cdots, n\})$ and $QR = \rho(\{2, \cdots, n + 1\})$. Using stationarity $S(\rho(PQR)) = S_{n+1}$, $S(\rho(PQ)) = S_n$, $S(\rho(QR)) = S_n$, $S(\rho(Q)) = S_{n-1}$. By the strong subadditivity,

$$S_{n+1} + S_{n-1} \leq 2S_n$$

or

$$S_{n+1} - S_n \leq S_n - S_{n-1}.$$  

Since

$$\frac{S_n}{n} = \frac{(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_1 - S_0)}{n}$$

where $S_0$ is defined to be zero, it follows that $\frac{S_n}{n}$ decreases monotonically to a limit $\bar{S} \geq 0$. This also implies that $S_n - S_{n-1}$ cannot decrease to $-\infty$ and hence $S_n - S_{n-1}$ also decreases monotonically to $\bar{S}$. This also shows that $S_n \geq S_{n-1}$ for all $n$. 
Entropy rate

We denote the limit $\bar{S}$ by $\bar{S}(\Sigma)$ and call it the *entropy rate* of the stationary G-chain $\Sigma$. 

Theorem

Let $\Sigma = \Sigma(A, B)$ be an exchangeable G-chain. Then $\bar{S}(\Sigma) = S(A-B)$. 

Proof:

Using the fact that $S(C \oplus D) = S(C) + S(D)$, we get from $\Sigma_n(A, B) = (A-B) \otimes (I_n - |\psi_n\rangle\langle\psi_n|) + (A + (n-1)B) \otimes |\psi_n\rangle\langle\psi_n|$. 

$S_n = S(\Sigma_n(A, B)) = (n-1)S(A-B) + S(A + (n-1)B)$. (13)
**Entropy rate**

We denote the limit $\tilde{S}$ by $\tilde{S}(\Sigma)$ and call it the *entropy rate* of the stationary $G$-chain $\Sigma$.

**Theorem**

*Let* $\Sigma = \Sigma(A, B)$ *be an exchangeable* $G$-*chain. Then* $\tilde{S}(\Sigma) = S(A - B)$.*
Entropy rate

We denote the limit $\tilde{S}$ by $\tilde{S}(\Sigma)$ and call it the *entropy rate* of the stationary G-chain $\Sigma$.

Theorem

*Let $\Sigma = \Sigma(A, B)$ be an exchangeable G-chain. Then $\tilde{S}(\Sigma) = S(A - B)$.***

**Proof:** Using the fact that $S(C \oplus D) = S(C) + S(D)$, we get from

$$
\Sigma_n(A, B) = (A - B) \otimes (I_n - |\psi_n\rangle\langle\psi_n|) + (A + (n - 1)B) \otimes |\psi_n\rangle\langle\psi_n|
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Entropy rate

We denote the limit $\bar{S}$ by $\bar{S}(\Sigma)$ and call it the entropy rate of the stationary G-chain $\Sigma$.

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$$

$$
S_n = S(\Sigma_n(A, B)) = (n - 1)S(A - B) + S(A + (n - 1)B). \quad (13)
$$

$\rho^A$ be the mean zero Gaussian state with covariance matrix $A$. $\xi = \xi_1 \oplus \xi_2$, $W(\xi)$ is Weyl or displacement operator at $\xi_1 + i\xi_2$ and $\phi(\xi)$ is the Gaussian density function with mean zero and covariance matrix $(n - 1)B$. Then

$$
\rho^{A+(n-1)B} = \int_{\mathbb{R}^{2k}} W(\xi) \rho^A W(\xi) \phi(\xi) \, d\xi
$$
Using concavity of von Neumann entropy, we get

\[ S(A + (n - 1)B) = S(\rho^{A+(n-1)B}) \leq \int S(A)\phi(\xi) \, d\xi + H(\phi) \]  

(14)

where \( H(\phi) \) is the Shannon differential entropy of the density function \( \phi \).
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\[ H(\phi) = k \log 2\pi e + \frac{1}{2} \log \det[(n - 1)B] \]  

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it follows from (13)–(15) that

\[ \left| \frac{S_n}{n} - \frac{n - 1}{n}S(A - B) \right| \leq \frac{S(A)}{n} + \frac{k}{n} \log 2\pi e + \frac{1}{2n} \log(n - 1)^{2k} \det B \]

\[ \leq \frac{1}{n} \left[ S(A) + k \log 2\pi e + \frac{1}{2} \log \det B \right] + \frac{k}{n} \log(n - 1) \]

Take \( n \to \infty \) to get the result.
Theorem

Let $p_1, p_2, \cdots$ be a probability distribution over $\{1, 2, 3, \cdots\}$, and let $A$ and $B$ be $2k \times 2k$ symmetric real matrices satisfying the condition that $A + tB$ is a $G$-matrix for every $t \in [-2, 2]$. Let $\Sigma$ be the stationary $G$-chain defined by the infinite block Toeplitz matrix

$$
\Sigma = \begin{bmatrix}
A & p_1B & p_2B & \cdots & \cdots \\
p_1B & A & p_1B & p_2B & \cdots \\
p_2B & p_1B & A & p_1B & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}.
$$

Then the entropy rate of $\Sigma$ is given by

$$
\bar{S}(\Sigma) = \int_0^1 S(A + h(s)B) \, ds
$$

where $h(s) = 2 \sum_{j=1}^{\infty} p_j \cos 2\pi js$, $s \in [0, 1]$. 

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Gaussian chains

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Express $\Sigma_n$ as

$$\Sigma_n = A \otimes I_n + B \otimes T_n(p)$$

where

$$T_n(p) = \begin{bmatrix}
0 & p_1 & p_2 & \cdots & p_{n-1} \\
p_1 & 0 & p_1 & \cdots & p_{n-2} \\
p_2 & p_1 & 0 & \cdots & p_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{n-1} & p_{n-2} & p_{n-3} & \cdots & 0
\end{bmatrix}.$$ 

Let $\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nn}$ be the eigenvalues of $T_n(p)$ and let $|\psi_{n1}\rangle, |\psi_{n2}\rangle, \cdots, |\psi_{nn}\rangle$ the corresponding eigenvectors constituting an orthonormal basis for $\mathbb{R}^n$ so that

$$\Sigma_n = \sum_{j=1}^{n} (A + \lambda_{nj}B) \otimes |\psi_{nj}\rangle\langle\psi_{nj}|$$

$$\frac{1}{n}S(\Sigma_n) = \frac{1}{n} \sum_{j=1}^{n} S(A + \lambda_{nj}B) = \int S(A + sB) \, d\mu_n(s),$$

where $\mu_n$ is the probability measure defined by $\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta \lambda_{nj}$. 
By KMZ theorem $\mu_n$ converges weakly as $n \to \infty$ to the distribution $Lh^{-1}$ where $L$ denotes the Lebesgue measure in $[0, 1]$ and

$$h(s) = 2 \sum_{j=1}^{\infty} p_j \cos 2\pi js.$$ 

Note that $\|T_n(p)\| \leq 2$ and the eigenvalues $\lambda_{nj}$ lie in the interval $[-2, 2]$. Furthermore, the symplectic spectrum of $A + sB$ is a continuous function of $s$ and hence the entropy $S(A + sB)$ is a continuous function of $s$ in $[-2, 2]$. Thus

$$\lim_{n \to \infty} \frac{1}{n} S(\Sigma_n) = \int_{-2}^{2} S(A + sB)Lh^{-1}(\,ds)$$

$$= \int_{0}^{1} S(A + h(s)B) \, ds.$$
A = \lambda I_2

B_j = B = b \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad j = 1, 2, \cdots ,

\text{where } \lambda \text{ and } b \text{ are positive scalars with } \lambda > \frac{1}{2}. \text{ We start with two elementary lemmas. Let}

\Sigma = \begin{bmatrix}
\lambda I_2 & p_1B & p_2B & \cdots & \cdots \\
p_1B & \lambda I_2 & p_1B & p_2B & \cdots \\
p_2B & p_1B & \lambda I_2 & p_1B & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}  \quad (16)
Entanglement property

\[ A = \lambda I_2 \]
\[ B_j = B = b \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad j = 1, 2, \cdots, \]

where \( \lambda \) and \( b \) are positive scalars with \( \lambda > \frac{1}{2} \). We start with two elementary lemmas. Let

\[
\Sigma = \begin{bmatrix}
\lambda I_2 & p_1 B & p_2 B & \cdots & \cdots \\
p_1 B & \lambda I_2 & p_1 B & p_2 B & \cdots \\
p_2 B & p_1 B & \lambda I_2 & p_1 B & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\]

(16)

**Lemma**

The infinite block matrix \( \Sigma \) in (16) is a stationary G-chain of order one if \( b < \frac{1}{2} \left( \lambda^2 - \frac{1}{4} \right)^{\frac{1}{2}} \).
Lemma

Let $\lambda > \frac{1}{2}$, $c > 0$. Then the matrix

$$
\Gamma = \begin{bmatrix}
\lambda & 0 & c & 0 \\
0 & \lambda & 0 & -c \\
c & 0 & \lambda & 0 \\
0 & -c & 0 & \lambda \\
\end{bmatrix}
$$

is the covariance matrix of an entangled 2-mode Gaussian state if

$$
\lambda - \frac{1}{2} < c < \left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}.
$$
**Lemma**

Let $\lambda > \frac{1}{2}$, $c > 0$. Then the matrix

$$\Gamma = \begin{bmatrix} \lambda & 0 & c & 0 \\ 0 & \lambda & 0 & -c \\ c & 0 & \lambda & 0 \\ 0 & -c & 0 & \lambda \end{bmatrix}$$

is the covariance matrix of an entangled 2-mode Gaussian state if

$$\lambda - \frac{1}{2} < c < \left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}.$$

**Proposition**

Let $\frac{1}{2} < \lambda < \frac{5}{6}$, $\lambda - \frac{1}{2} < b < \sqrt{\lambda^2 - \frac{1}{4}}$. Suppose $p_j b > \lambda - \frac{1}{2}$ for some $j$. Then the 2-mode Gaussian state $\rho(\{1,j\})$ determined by the stationary G-chain $\Sigma$ defined by (16) is entangled.
Open problem

Find out what happens if the covariance matrix is a general block Toeplitz matrix.
Open problem
Find out what happens if the covariance matrix is a general block Toeplitz matrix.

Thank you!!!