Exchangeable, stationary and entangled chains of Gaussian states

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Gaussian state

A state ρ in $\Gamma(\mathbb{C}^n)$ is called a n-mode Gaussian state if its Fourier transform $\hat{\rho}$ is given by

$$\hat{\rho}(\boldsymbol{x} + \imath \boldsymbol{y}) = \exp\left[-\imath\sqrt{2}(\boldsymbol{l}^T\boldsymbol{x} - \boldsymbol{m}^T\boldsymbol{y}) - \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}^T S \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right], \tag{1}$$

for all $\boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^n$ where $\boldsymbol{l}$, $\boldsymbol{m}$ are elements of $\mathbb{R}^n$ and S is a real $2n \times 2n$ symmetric matrix satisfying the matrix inequality

$$S + \frac{\imath}{2} J_{2n} \geq 0 \tag{2}$$

with

$$J_{2n} = \bigoplus_{n\text{-copy}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3}$$

G-matrix

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Suppose $\Sigma = [[A_{ij}]], \quad i, j \in \{1, 2, \cdots\}$ is an infinite matrix where each A_{ij} is a $2k \times 2k$ real matrix and $A_{ij}^T = A_{ji}$ for all i, j. For any finite subset $I = \{i_1 < i_2 < \cdots < i_n\} \subset \{1, 2, \cdots\}$, let $\Sigma(I) = [[A_{i_r i_s}]], r, s \in \{1, 2, \cdots\}$ be the $2kn \times 2kn$ matrix obtained from Σ by restriction to its rows and columns numbered $i_1 < i_2 < \cdots < i_n$.

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Σ is said to be a *G-chain of order k* if $\Sigma(I)$ is the covariance matrix of a kn-mode zero mean Gaussian state $\rho(I)$ in the boson Fock space $\Gamma(\mathbb{C}^{kn}) = \mathcal{H}_{i_1} \otimes \mathcal{H}_{i_2} \otimes \cdots \otimes \mathcal{H}_{i_n}$ where $\mathcal{H}_j$ denotes the j-th copy of the Hilbert space $\mathcal{H} = \Gamma(\mathbb{C}^k), j = 1, 2, \cdots$.

We say that Σ is an *exchangeable G-chain* if it is a G-chain and there exist two $2k \times 2k$ matrices $A,\ B$ such that

$$A_{ij} = \begin{cases} B & \text{if} \quad j > i, \\ A & \text{if} \quad j = i, \\ B^T & \text{if} \quad j < i. \end{cases} \tag{5}$$

In such a case we write

$$\begin{aligned} \Sigma &= \Sigma(A, B), \\ \Sigma(I) &= \Sigma(I; A, B). \end{aligned} \tag{6}$$

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We say that a G-chain Σ is *stationary* if there exist $2k \times 2k$ matrices $A,\ B_1,\ B_2, \cdots$ such that

$$A_{ij} = \begin{cases} A & \text{if} \quad i = j, \\ B_{j-i} & \text{if} \quad j > i, \\ B_{i-j}^T & \text{if} \quad j < i \end{cases} \tag{7}$$

for all $i,\ j$.

Theorem

Let (A, B) be a pair of real $2k \times 2k$ matrices. Then $\Sigma(A, B)$ is a G-chain if and only if A and B are nonnegative definite and $A - B$ is a G-matrix.

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Proof technique

Fix n, and Let $I = \{1, 2, \cdots, n\}$. Set $\Sigma_n(A, B) = \Sigma_n(I : A, B)$. Define $N_n = [[x_{ij}]]_{n \times n}$ where $x_{ij} = 1$ if $i > j$ and 0 otherwise. Let $|\psi_n\rangle = n^{-\frac{1}{2}}[1, \cdots, 1]^T$ be the unit column vector of length n.

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$$\begin{aligned}\Sigma_n(A, B) + \frac{\imath}{2}J_{2kn} &= \left(A + \frac{\imath}{2}J_{2k} - \frac{1}{2}(B + B^T)\right) \otimes (I_n - |\psi_n\rangle\langle\psi_n|) \\ &\quad + \left(A + \frac{\imath}{2}J_{2k} + \frac{1}{2}(n-1)(B + B^T)\right) \otimes |\psi_n\rangle\langle\psi_n| \\ &\quad + \frac{1}{2}(B - B^T) \otimes (N_n - N_n^T). \qquad (8)\end{aligned}$$

Multiplying both sides by $I_n \otimes |\psi_n\rangle\langle\psi_n|$ and take relative trace over the second component. Take relative trace over second component and relative trace over second component gives

$$\frac{1}{2}(B + B^T) \geq 0.$$

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Consider complex unit vector $|\phi_n\rangle = \frac{1}{\sqrt{n}}[1, \omega, \omega^2, \cdots, \omega^{n-1}]^T$, where $\omega = e^{\frac{2\pi\imath}{n}}$. Now, multiplying both sides of (8) by $I_n \otimes |\phi_n\rangle\langle\phi_n|$ and tracing over the second Hilbert space we get the inequality

$$A + \frac{\imath}{2}J_{2k} - \frac{1}{2}(B + B^T) + \imath\frac{B - B^T}{2}\cot\frac{\pi}{n} \geq 0.$$

for $n = 1, 2, \cdots$. Multiplying by $\tan\frac{\pi}{n}$ (for $n \geq 3$) and letting $n \to \infty$ we get the inequality

$$\frac{\imath}{2}(B - B^T) \geq 0.$$

Hence the result follows.

Corollary

In any exchangeable G-chain $\Sigma(A, B)$ of order k, for every finite set $I \subset \{1, 2, \cdots\}$, the underlying Gaussian state $\rho(I)$ is separable.

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Proof

Without loss of generality we may assume that $I = \{1, 2, \cdots, n\}$ for some n. Then the covariance matrix of $\rho(I)$ is equal to the $n \times n$ block matrix

$$\begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \cdots & A \end{bmatrix} = (A - B) \otimes I_n + B \otimes \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

By previous theorem, $(A - B) \otimes I_n$ is the covariance matrix of an n-fold product Gaussian state and the second summand on the right hand side of the equation 8 above is a nonnegative definite matrix. Hence by Werner and Wolf's theorem, $\rho(I)$ is separable.

Let A, B be real $2k \times 2k$ symmetric matrices. For any fixed $j = 1, 2, \cdots$, denote by $\Delta^j(A, B)$ the infinite block matrix all of whose diagonal blocks are equal to A, $(n, n+j)$-th and $(n+j, n)$-th blocks are equal to B for every n and all the remaining blocks are zero matrices of order $2k \times 2k$. For example,

$$\Delta^1(A, B) = \begin{bmatrix} A & B & 0 & 0 & 0 & \cdots \\ B & A & B & 0 & 0 & \cdots \\ 0 & B & A & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Denote by $\Delta^j_n(A, B)$ the $2kn \times 2kn$ matrix obtained by $\Delta^j(A, B)$ by restriction to the first n row and column blocks. For example,

$$\Delta^1_2(A, B) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \Delta^2_3(A, B) = \begin{bmatrix} A & 0 & B \\ 0 & A & 0 \\ B & 0 & A \end{bmatrix}$$

and so on.
Our first result gives a necessary and sufficient condition for $\Delta^j(A, B)$ to be a G-chain.

Theorem

Let A, B be a pair of $2k \times 2k$ real symmetric matrices. In order that $\Delta^j(A, B)$ may be a G-chain of order k it is necessary and sufficient that $A + tB$ is a G-matrix for every $t \in [-2, 2]$.

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Proof: Denote by L_n^j the upper triangular matrix whose $(j+1)$-th upper diagonal entries are all equal to 1 and all the remaining entries are zero. Thus L_n^j is defined for $1 \leq j \leq n-1$. Then

$$\Delta_n^j(A, B) = A \otimes I_n + B \otimes (L_n^j + (L_n^j)^T). \tag{9}$$

Consider the spectral decomposition of the $n \times n$ symmetric matrix $L_n^j + (L_n^j)^T$:

$$L_n^j + (L_n^j)^T = \sum_{r=1}^{n} \lambda_{nr} \langle \psi_{nr} \rangle \, \psi_{nr} \tag{10}$$

where $\{\lambda_{nr} : r = 1, 2, \cdots, n\}$ are the eigenvalues and $\{|\psi_{nr}\rangle : r = 1, 2, \cdots, n\}$ are the corresponding orthonormal basis of eigenvectors for $L_n^j + (L_n^j)^T$. Since each L_n^j is a matrix with operator norm equal to unity and therefore $L_n^j + (L_n^j)^T$ has operator norm not exceeding 2.

$$|\lambda_{nr}| \leq 2, \quad 1 \leq r \leq n, \;\; n = 1, 2, \cdots . \tag{11}$$

Equations (9)–(10) imply

$$\Delta_n^j(A, B) + \frac{\imath}{2} J_{2kn} = \sum_{r=1}^{n} \left(A + \lambda_{nr} B + \frac{\imath}{2} J_{2k} \right) \otimes |\psi_{nr}\rangle\langle\psi_{nr}| \, . \tag{12}$$

Thus $\Delta_n^j(A, B)$ is a G-matrix if and only if $A + \lambda_{nr} B$ is a G-matrix for each r.

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Thus $\Delta_n^j(A, B)$ is a G-matrix if and only if $A + \lambda_{nr} B$ is a G-matrix for each r. To prove necessity, we appeal to the theorem of Kac, Murdock and Szegö [J. Ration. Mech. Anal. 2, 767–800, 1953]. Consider the probability distribution

$$\mu_n = \frac{1}{n} \sum_{r=1}^{n} \delta_{\lambda_{nr}}$$

where $\lambda_{nr}, \; r = 1, 2, \cdots, n$ are as in (10). The left hand side of (10) is a Toeplitz matrix of order n for each n. KMS theorem implies that the sequence $\{\mu_n\}$ converges weakly as $n \to \infty$ to the probability measure Lh^{-1} where L is the Lebesgue measure in the unit interval and $h(t) = 2 \cos 2\pi j t, \; t \in [0, 1]$. This, in particular, implies that $\{\lambda_{nr} : \; r = 1, 2, \cdots, n, \; n = 1, 2, \cdots\}$ is dense in the interval $[-2, 2]$. The proof of necessity is now complete.

Corollary

Let A, B_1, B_2, $\cdots$ be real $2k \times 2k$ symmetric matrices satisfying the condition that $A + tB_j$ is a G-matrix for every $j = 1, 2, \cdots$ and $t \in [-2, 2]$. Suppose $p_1, p_2, \cdots$, is a probability distribution on the set $\{1, 2, \cdots\}$. Then the block Toeplitz matrix

$$\Sigma(A; p_1B_1, p_2B_2, \cdots) = \begin{bmatrix} A & p_1B_1 & p_2B_2 & \cdots & \cdots \\ p_1B_1 & A & p_1B_1 & p_2B_2 & \cdots \\ p_2B_2 & p_1B_1 & A & p_1B_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

is a stationary G-chain.

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$$\Sigma(A; p_1B_1, p_2B_2, \cdots) = \sum_{j=1}^{\infty} p_j \Delta^j(A, B_j)$$

and each $\Delta^j(A, B_j)$ is a G-chain.

Entropy rate of stationary G-chain

Suppose $\Sigma = \Sigma(A, B_1, B_2, \cdots)$ is a stationary G-chain. For any G-matrix C denote by $S(C)$ the von Neumann entropy of a Gaussian state ρ with covariance matrix C. Let

$$\begin{aligned} \Sigma_n &= \Sigma(\{1, 2, \cdots, n\}), \\ S_n &= S(\Sigma_n). \end{aligned}$$

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Proposition

The sequences $\{S_n - S_{n-1}\}$, $\left\{\frac{1}{n} S_n\right\}$ monotonically decrease to the same limit $\bar{S} \geq 0$ as $n \to \infty$. Furthermore, $S_n \geq S_{n-1}$ for all n.

Consider Gaussian systems P, Q, R such that $PQR = \rho(\{1, 2, \cdots, n+1\})$, $Q = \rho(\{2, \cdots, n\})$, $PQ = \rho(\{1, 2, \cdots, n\})$ and $QR = \rho(\{2, \cdots, n+1\})$. Using stationarity $S(\rho(PQR)) = S_{n+1}$, $S(\rho(PQ)) = S_n$, $S(\rho(QR)) = S_n$, $S(\rho(Q)) = S_{n-1}$. By the strong subadditivity,

$$S_{n+1} + S_{n-1} \leq 2S_n$$

or

$$S_{n+1} - S_n \leq S_n - S_{n-1}.$$

Since

$$\frac{S_n}{n} = \frac{(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_1 - S_0)}{n}$$

where S_0 is defined to be zero, it follows that $\frac{S_n}{n}$ decreases monotonically to a limit $\bar{S} \geq 0$. This also implies that $S_n - S_{n-1}$ cannot decrease to $-\infty$ and hence $S_n - S_{n-1}$ also decreases monotonically to $\bar{S}$. This also shows that $S_n \geq S_{n-1}$ for all n.

Entropy rate

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Proof: Using the fact that $S(C \oplus D) = S(C) + S(D)$, we get from

$$\Sigma_n(A, B) \quad = \quad (A - B) \otimes (I_n - |\psi_n\rangle\langle\psi_n|) + (A + (n-1)B) \otimes |\psi_n\rangle\langle\psi_n|$$

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ρ^A be the mean zero Gaussian state with covariance matrix A. $\boldsymbol{\xi} = \boldsymbol{\xi}_1 \oplus \boldsymbol{\xi}_2$, $W(\boldsymbol{\xi})$ is Weyl or displacement operator at $\boldsymbol{\xi}_1 + \imath\boldsymbol{\xi}_2$ and $\phi(\boldsymbol{\xi})$ is the Gaussian density function with mean zero and covariance matrix $(n-1)B$. Then

$$\rho^{A+(n-1)B} = \int_{\mathbb{R}^{2k}} W(\boldsymbol{\xi})\rho^A W(\boldsymbol{\xi})^\dagger \phi(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi}$$

Using concavity of von Neumann entropy, we get

$$\begin{aligned} S(A + (n-1)B) &= S(\rho^{A+(n-1)B}) \\ &\leq \int S(A)\phi(\boldsymbol{\xi})\ \mathrm{d}\boldsymbol{\xi} + H(\phi) \end{aligned} \tag{14}$$

where $H(\phi)$ is the Shannon differential entropy of the density function ϕ.

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$$H(\phi) = k \log 2\pi e + \frac{1}{2} \log \det[(n-1)B] \tag{15}$$

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it follows from (13)–(15) that

$$\begin{aligned} \left| \frac{S_n}{n} - \frac{n-1}{n} S(A-B) \right| &\leq \frac{S(A)}{n} + \frac{k}{n} \log 2\pi e + \frac{1}{2n} \log (n-1)^{2k} \det B \\ &\leq \frac{1}{n} \left[S(A) + k \log 2\pi e + \frac{1}{2} \log \det B \right] + \frac{k}{n} \log(n-1 \end{aligned}$$

Take $n \to \infty$ to get the result.

Theorem

Let $p_1, p_2, \cdots$ be a probability distribution over $\{1, 2, 3, \cdots\}$, and let A and B be $2k \times 2k$ symmetric real matrices satisfying the condition that $A + tB$ is a G-matrix for every $t \in [-2, 2]$. Let Σ be the stationary G-chain defined by the infinite block Toeplitz matrix

$$\Sigma = \begin{bmatrix} A & p_1B & p_2B & \cdots & \cdots \\ p_1B & A & p_1B & p_2B & \cdots \\ p_2B & p_1B & A & p_1B & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Then the entropy rate of Σ is given by

$$\bar{S}(\Sigma) = \int_0^1 S(A + h(s)B) \, \mathrm{d}s$$

where $h(s) = 2\sum_{j=1}^{\infty} p_j \cos 2\pi js, \quad s \in [0, 1]$.

Express Σ_n as

$$\Sigma_n = A \otimes I_n + B \otimes T_n(\boldsymbol{p})$$

$$T_n(\boldsymbol{p}) = \begin{bmatrix} 0 & p_1 & p_2 & \cdots & p_{n-1} \\ p_1 & 0 & p_1 & \cdots & p_{n-2} \\ p_2 & p_1 & 0 & \cdots & p_{n-3} \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdots & 0 \end{bmatrix}.$$

Let $\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nn}$ be the eigenvalues of $T_n(\boldsymbol{p})$ and let $|\psi_{n1}\rangle, |\psi_{n2}\rangle, \cdots, |\psi_{nn}\rangle$ the corresponding eigenvectors constituting an orthonormal basis for $\mathbb{R}^n$ so that

$$\begin{aligned} \Sigma_n &= \sum_{j=1}^{n} (A + \lambda_{nj} B) \otimes |\psi_{nj}\rangle\langle\psi_{nj}| \\ \frac{1}{n} S(\Sigma_n) &= \frac{1}{n} \sum_{j=1}^{n} S(A + \lambda_{nj} B) = \int S(A + sB)\, \mathrm{d}\mu_n(s), \end{aligned}$$

where μ_n is the probability measure defined by $\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_{nj}}$.

By KMZ theorem μ_n converges weakly as $n \to \infty$ to the distribution Lh^{-1} where L denotes the Lebesgue measure in $[0, 1]$ and

$$h(s) = 2 \sum_{j=1}^{\infty} p_j \cos 2\pi js.$$

Note that $\|T_n(\boldsymbol{p})\| \leq 2$ and the eigenvalues λ_{nj} lie in the interval $[-2, 2]$. Furthermore, the symplectic spectrum of $A + sB$ is a continuous function of s and hence the entropy $S(A + sB)$ is a continuous function of s in $[-2, 2]$. Thus

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} S(\Sigma_n) &= \int_{-2}^{2} S(A + sB) Lh^{-1}(\,\mathrm{d}s) \\ &= \int_{0}^{1} S(A + h(s)B)\ \mathrm{d}s. \end{aligned}$$

$$\begin{aligned} A &= \lambda I_2 \\ B_j &= B = b\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad j = 1, 2, \cdots, \end{aligned}$$

where λ and b are positive scalars with $\lambda > \frac{1}{2}$. We start with two elementary lemmas. Let

$$\Sigma = \begin{bmatrix} \lambda I_2 & p_1 B & p_2 B & \cdots & \cdots \\ p_1 B & \lambda I_2 & p_1 B & p_2 B & \cdots \\ p_2 B & p_1 B & \lambda I_2 & p_1 B & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \tag{16}$$

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where λ and b are positive scalars with $\lambda > \frac{1}{2}$. We start with two elementary lemmas. Let

$$
\Sigma = \begin{bmatrix} \lambda I_2 & p_1 B & p_2 B & \cdots & \cdots \\ p_1 B & \lambda I_2 & p_1 B & p_2 B & \cdots \\ p_2 B & p_1 B & \lambda I_2 & p_1 B & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \tag{16}
$$

Lemma

The infinite block matrix Σ in (16) is a stationary G-chain of order one if $b < \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}$.

Lemma

Let $\lambda > \frac{1}{2},\ c > 0$. Then the matrix

$$\Gamma = \begin{bmatrix} \lambda & 0 & c & 0 \\ 0 & \lambda & 0 & -c \\ c & 0 & \lambda & 0 \\ 0 & -c & 0 & \lambda \end{bmatrix}$$

is the covariance matrix of an entangled 2-mode Gaussian state if

$$\lambda - \frac{1}{2} < c < \left(\lambda^2 - \frac{1}{4}\right)^{\frac{1}{2}}.$$

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Proposition

Let $\frac{1}{2} < \lambda < \frac{5}{6}$, $\lambda - \frac{1}{2} < b < \sqrt{\lambda^2 - \frac{1}{4}}$. Suppose $p_j b > \lambda - \frac{1}{2}$ for some j. Then the 2-mode Gaussian state $\rho(\{1, j\})$ determined by the stationary G-chain Σ defined by (16) is entangled.

Open problem

Find out what happens if the covariance matrix is a general block Toeplitz matrix.

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Thank you!!!

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