A necessary condition for local distinguishability of two-qudit maximally entangled states completely characterises that of the generalised Bell states for $d = 4$

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(*arXiv:1506.03667*)

Distinguishing quantum states

- If a quantum system is prepared in one of a **given** set of pairwise orthogonal states – but the **identity of these being undisclosed** – measurement in the basis consisiting of these states only can **reliably** distinguish these states, and thereby, identify them exactly.
- Obviously such a (projective) measurement is possible to implement in lab provided the lab has **full access** to the quantum system.
- What happens if the lab has access to **only** a subsystem of the original system?

Distinguishing quantum states (continued)

- For example, if the original quantum system consists of two subsystems A and B with A being in the possession of an experimenter (Alice, say) located in a lab far apart from the lab of the other experimenter (Bob, say), who is in possession of the subsystem B, will it be possible to identify the states shared *apriori* by A and B?
- It may be a **temptation** at this point to presume that such an effort of reliable distinguishability – by keeping the such systems far apart – is possible **at least** for pairwise orthogonal product states.
- Unfortunately, such a guess has been proved to be wrong, in general!

Distinguishing quantum states (continued)

- **Non-locality without entanglement:** There are sets of pairwise orthogonal product states – taken from $\mathcal{C}^d \otimes \mathcal{C}^d$ – which are not reliably distinguishable by performing measurements 'locally'.
- **Example:** $\mathcal{S} = \{|\Psi_1\rangle_{AB} = |11\rangle_{AB}, |\Psi_2\rangle_{AB} = |0\psi^+_{01}\rangle_{AB}, |\Psi_3\rangle_{AB} = |0\psi^-_{01}\rangle_{AB}, |\Psi_4\rangle_{AB} = |2\psi^+_{12}\rangle_{AB}, |\Psi_5\rangle_{AB} = |2\psi^-_{12}\rangle_{AB}, |\Psi_6\rangle_{AB} = |\psi^+_{12}0\rangle_{AB}, |\Psi_7\rangle_{AB} = |\psi^-_{12}0\rangle_{AB}, |\Psi_8\rangle_{AB} = |\psi^+_{01}2\rangle_{AB}, |\Psi_9\rangle_{AB} = |\psi^-_{01}2\rangle_{AB}\}$, where $|\psi^\pm_{ij}\rangle \equiv (1/\sqrt{2})(|i\rangle \pm |j\rangle)$.
- These states are not reliably distinguishable even if the experimenters are allowed to communicate classically (*e.g.*, via telephone calls) about their respective measurement outcomes, and thereby choose to select their respective future measurements accordingly – the so-called **LOCC** paradigm.

Distinguishing quantum states (continued)

- Thus it is **no wonder** that when Alice and Bob share one state from a given set of pairwise orthogonal joint states of A and B – which are generically entangled – the states can not be reliably distinguished (in general) by using **only** LOCC.
- Nevertheless, there are ensembles of pairwise orthogonal **pure** bi-partite states – shared by Alice and Bob – which can be reliably distinguished by using LOCC only.
- For example, **any given pair** of mutually orthogonal pure states of a bi-partite quantum system are reliably distinguishable by using LOCC only.
- Here we will discuss about reliable local distinguishability of pairwise orthogonal **maximally entangled states** of bi-partite quantum systems.

Outline

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Motivation behind LOCC discrimination

Motivation behind LOCC discrimination

- **Secret sharing:** Some secret massages have been encoded in terms of pairwise orthogonal states $\rho_{AB}^{(1)}$, $\rho_{AB}^{(2)}$, $\ldots$, $\rho_{AB}^{(n)}$ of a bi-partite system $A + B$, where two spatially separated parties Alice and Bob are in the possession of the subsystems A and B respectively. The parties can reveal the messages **only if** they come together and perform joint measurement on the total system to distinguish these states.

Motivation behind LOCC discrimination (continued)

- **Entanglement distillation:** If two far apart parties (Alice and Bob, say) share the state $|\psi^{(i)}\rangle_{A_1B_1}\langle\psi^{(i)}| \otimes \rho^{(i)}_{A_2B_2}$ with *apriori* probability p_i (for $i = 1, 2, \ldots, n$) – where Alice is in possession of both the subsystems A_1, A_2 while Bob is in possession of B_1, B_2, and $\rho^{(i)}_{A_2B_2}$'s are pairwise orthogonal – then, by reliably distinguishing these latter states using LOCC only, Alice and Bob will eventually come up with sharing the pure state $|\psi^{(i)}\rangle_{A_1B_1}$ in case the i-th state $\rho^{(i)}_{A_2B_2}$ has been identified during the discrimination process.

Brief historical background

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- Walgate et al. [PRL **85** (2000) pp. 4972] have shown that any **two** orthogonal states $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$ of a bi-partite quantum system – described by the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ – can be reliably distinguished by using LOCC. In fact this require only **one-way** LOCC.
- Bandyopadhyay and Walgate [JPA (2007); quant-ph/0612013] discussed about conclusive local discrimination of **any set of three linearly independent multi-partite pure states**.
- The general solution to the problem of reliable discrimination of an **arbitrary** set of linearly independent (or, even pairwise orthogonal) bi-partite or multi-partite states using only LOCC is still not known.

Brief historical background (continued)

- Because of symmetries of maximally entangled states (MES) of any given bi-partite system, it may be comparatively easier to look for conditions of reliable local distinguishability of pairwise orthogonal MES.
- In fact, it is not just a mathematical artifact: If we choose a state $|\Psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ **at random**, it is **most likely** that the state would turn out to be an MES [Lecture given on *Selected topics in Mathematical Physics: Quantum Information Theory* by Thomas Gläßle in Dec., 2013].

Brief historical background (continued)

- Fan [PRL **92** (2004) 177905] showed that when d is a **prime number** and l is a positive integer such that $l(l-1) \leq 2d$, then any l number of mutually orthogonal **generalized Bell states** in $\mathbb{C}^d \otimes \mathbb{C}^d$ are perfectly distinguishable by LOCC.
- **Generalized Bell states:** $|\psi_{nm}^{(d)}\rangle \equiv (1/\sqrt{d}) \sum_{j=0}^{d-1} e^{2\pi ijn/d} |j\rangle \otimes |(j+m) \bmod d\rangle$ for $n, m = 0, 1, 2, \ldots, d-1$
- We showed earlier [Ghosh et al., PRA **70** (2004) 022304] that **no** set of k number of generalized Bell states in $\mathbb{C}^d \otimes \mathbb{C}^d$ can be perfectly distinguished by LOCC if $k > d$.
- More recently, Yu et al. [PRL **109** (2012) 020506] have shown that **no** set of k number of pairwise orthogonal MES (**not necessarily generalized Bell states**) in $\mathbb{C}^d \otimes \mathbb{C}^d$ can be perfectly distinguished by LOCC if $k > d$.

Brief historical background (continued)

- Moreover, we [PRA **70** (2004) 022304] provided examples of ensembles of d number of generalized Bell states in $\mathbb{C}^d \otimes \mathbb{C}^d$ for $d = 4, 5$ such that the states from each such ensemble are not perfectly distinguishable by a **particular type of one-way LOCC**, namely, the standard quantum teleportation scheme of Bennett et al. [PRL **70** (1993) 1895].
- Later, Bandyopadhyay et al. [New J. Phys. **13** (2011) 123013] provided examples of such ensembles of generalized Bell states for d = 4, 5 etc, the states from none of which are perfectly distinguishable by **one-way** LOCC only.
- For more such examples in higher dimensional systems (with generalized Bell states), see [Quantum Inf. Process. **13** (2014) 795].

Brief historical background (continued)

- In the context of general MES in $\mathbb{C}^d \otimes \mathbb{C}^d$ – as mentioned above – no k number of pairwise orthogonal MES are perfectly distinguishable by LOCC if $k > d$ [Nathanson, JMP **46** (2005) 062103], where an upper bound on locally accessible information [Badziag et al., PRL **91** (2003) 117901] was used to get the result.
- Moreoever, it has been shown in [Nathanson, JMP **46** (2005) 062103] that **any** three pairwise orthogonal MES in $\mathbb{C}^3 \otimes \mathbb{C}^3$ are reliably distinguishable by LOCC.

Brief historical background (continued)

- Yu et al. [PRL **109** (2012) 020506] provided the **first** example of a set of four pairwise orthogonal **ququad-ququad** states (namely, MES of the form $(1/2)\sum_{i,j=0}^{1} |ij\rangle_{A_0 B_0} \otimes \sigma_\alpha |i\rangle_{A_1} \otimes \sigma_\beta |j\rangle_{B_1}$ where $\sigma_\alpha, \sigma_\beta \in \{I_{2\times 2}, \sigma_x, \sigma_y, \sigma_z\}$) which are not perfectly distinguisable by LOCC.
- In fact, they have shown that these four states are distinguisable by **no PPT POVM** - an operation more general than LOCC.
- Imperfect distinguishability of these states was later discussed by Cosentino [PRA **87** (2013) 012321].

Brief historical background (continued)

- Until recently, a general consensus - based on the available examples - was that if pairwise orthogonal states from any given set are at all perfectly distinguishable, one-way LOCC would be enough.
- But in a recent work, Nathanson [PRA **88** (2013) 062316] gave an example of three pairwise orthogonal MES in $\mathbb{C}^4 \otimes \mathbb{C}^4$ which are although **not** perfectly distinguishable by one-way LOCC, are so by **two-way** LOCC.
- Is this a generic feature that whenever the states from a set of k $(\leq d)$ number of pairwise orthogonal MES in $\mathbb{C}^d \otimes \mathbb{C}^d$ are not perfectly distinguishable by one-way LOCC, they should be distinguishable by two-way LOCC in most of the cases?

Brief historical background (continued)

- Even though the answer **may be in the affirmative in general**, it is not true for the case of four pairwise orthogonal ququad-ququad MES.
- In fact, Tian et al. [PRA **91** (2015) 052314] have recently shown that there is **only one class** - upto local unitary - of four pairwise orthogonal ququad-ququad MES states where the states are **not** distinguishable perfectly by LOCC.
- This is the same example taken by Yu et al.above to show the perfect indistinguishability of the states even with PPT POVMs.

Brief historical background (continued)

- Here we show, in the case of generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, that such a scenario (that most of the ensembles of pairwise orthogonal MES are perfectly distinguishable by two-way LOCC, whenever they are indistinguishable by one-way LOCC) is **not true**.

Bound on locally accessible information

Bound on locally accessible information

- **Holevo's bound:** When the (classical) information of a random variable X – distributed as $\mathrm{Prob}(X = x) = p_x$ for $x \in \Lambda$– is encoded in terms of the ensemble $\mathcal{E} \equiv \{p_x, \rho_x\}$ of states of a d dim. quantum system S, by performing most general measurement $\{E_y : y \in \Lambda'\}$ on the system S, the maximum amount of (classical) information about X that can be extracted (called as **accessible information** I^{acc}) is **upper bounded** by the Holevo's quantity: $\chi(\mathcal{E}) \equiv S(\sum_{x \in \Lambda} p_x \rho_x) - \sum_{x \in \Lambda} p_x S(\rho_x)$.
- The bound $\chi(\mathcal{E})$ on I^{acc} can be shown to be achievable in the asymptotic limit of many copies of the ensemble.

Bound on locally accessible information (continued)

- **Bound on locally accessible information:** When the (classical) information of a random variable X – distributed as $\mathrm{Prob}(X = x) = p_x$ for $x \in \Lambda$– is encoded in terms of the ensemble $\mathcal{E} \equiv \{p_x, \rho_{AB}^{(x)}\}$ of states of a bi-partite quantum system $A + B$ described by the Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$, by most general LOCC operation, what is the maximum amount of (classical) information about X that can be extracted (called as **locally accessible information** I_{LOCC}^{acc})?
- Can one find out a **tight bound** on I_{LOCC}^{acc} – even in the asymptotic form?

Bound on locally accessible information (continued)

- Badziag et al. [PRL **91** (2003) 117901] provided an upper bound on I_{LOCC}^{acc}: $I_{LOCC}^{acc} \leq S(\sum_{x \in \Lambda} p_x \rho_A^{(x)}) + S(\sum_{x \in \Lambda} p_x \rho_B^{(x)}) - \max\{\sum_{x \in \Lambda} p_x S(\rho_A^{(x)}),\ \sum_{x \in \Lambda} p_x S(\rho_B^{(x)})\}$.
- Here $\rho_A^{(x)} \equiv \mathrm{Tr}_B \rho_{AB}^{(x)}$ and $\rho_B^{(x)} \equiv \mathrm{Tr}_A \rho_{AB}^{(x)}$ for all x.
- The above bound on I_{LOCC}^{acc} has been made tighter, but the asymptotic attainability is still not known.
- Here we will use Badziag et al.'s bound on I_{LOCC}^{acc}.

Necessary condition for LOCC discrimination of pairwise orthogonal MES in $\mathbb{C}^d \otimes \mathbb{C}^d$

Necessary condition for LOCC discrimination of pairwise orthogonal MES in $\mathbb{C}^d \otimes \mathbb{C}^d$

- For any given set $\{|\psi_i\rangle_{AB} : i = 1, 2, \ldots, m\}$ of pairwise orthogonal MES in $\mathbb{C}^d \otimes \mathbb{C}^d$ (where $m \leq d$), Badziag et al.'s bound turns out to be:
$I^{acc}_{LOCC} \leq S((1/m)\sum_{i=1}^{m} \mathrm{Tr}_B|\psi_i\rangle_{AB}\langle\psi_i|) + S((1/m)\sum_{i=1}^{m} \mathrm{Tr}_A|\psi_i\rangle_{AB}\langle\psi_i|) - \max \{(1/m)\sum_{i=1}^{m} S(\mathrm{Tr}_A|\psi_i\rangle_{AB}\langle\psi_i|), (1/m)\sum_{i=1}^{m} S(\mathrm{Tr}_B|\psi_i\rangle_{AB}\langle\psi_i|)\} = S((1/d)I_{d\times d}) + S((1/d)I_{d\times d}) - S((1/d)I_{d\times d}) = \log_2 d$.
- The constraint $I^{acc}_{LOCC} \leq \log_2 d$ will now be used to find out constratint on the kind of measurements the first party (Alice, say) is supposed to perform to distinguish the states $|\psi_i\rangle$.

Necessary condition for LOCC discrimination ....

- Assume that $\{K_\alpha^\dagger K_\alpha : \alpha = 1, 2, \ldots, n\}$ be a POVM Alice is going to perform on A. Thus here $\sum_{\alpha=1}^{n} K_\alpha^\dagger K_\alpha = I_{d \times d}$.
- The action of the α-th POVM effect on $|\psi_i\rangle_{AB}$ gives rise to the state $|\psi_{i,\alpha}\rangle = (K_\alpha \otimes I_{d\times d})|\psi_i\rangle_{AB} / \sqrt{\langle\psi_i|(K_\alpha^\dagger K_\alpha \otimes I_{d\times d})|\psi_i\rangle}$.
- Condition for orthogonality of post-measurement joint states require: $\langle\psi_i|(K_\alpha^\dagger \otimes I)(K_\alpha \otimes I)|\psi_j\rangle = \delta_{ij}$ for $i \neq j$.
- A necessary condition for local distinguishability of $m = d$ no. of pairwise orthogonal MES in $\mathbb{C}^d \otimes \mathbb{C}^d$ is that the average reduced density matrix of B – after Alice's measurement – has to be maximally mixed, together with the orthogonality condition for post-measurement states.

Necessary condition for LOCC discrimination ....

- **Lemma:** *If Alice starts the measurement protocol to distinguish m MES $\{|\psi_i\rangle : i = 1, 2, \ldots, m\}$ by LOCC, the post-measurement reduced states (PMRS) on her side are completely indistinguisgable.*
- **Proof:** Reduced states of Alice before measurement: $\rho_i^{(A)} = (1/d) I_{d \times d}$, which gets transformed into (after Alice's measurement): $\rho_{i,\alpha}^{(A)}$, proportional to $K_\alpha K_\alpha^\dagger$ <u>for all i</u> when α-th effect 'clicks' in Alice's measurement.Thus, after Alice's measurement, the PMRS on her side are all same, and hence, completely indistinguishable.

Necessary condition for LOCC discrimination ....

- **Theorem:** *The von Neumann entropy of the average PMRS on Bob's side – after Alice's measurement – has to be at least* $\log_2 m$ *bits for the states to be perfectly distinguishable by LOCC.*
- **Proof:** The Holevo-like upper bound for the locally accessible information of the set of states $\{|\psi_{i,\alpha}\rangle\}_{i=1}^{m}$ is given by:
  $I_{acc}^{LOCC} \leq S\left(\rho_\alpha^{(A)}\right) + S\left(\rho_\alpha^{(B)}\right)$
  $- \max\left\{(1/m)\sum_{i=1}^{m} S\left(\rho_{i,\alpha}^{(X)}\right) \; : X = A, B\right\}$
  Spectrum of $\rho_{i,\alpha}^{(A)}$ and $\rho_{i,\alpha}^{(B)}$ are same – as the joint post-measurement state (namely, $|\psi_{i,\alpha}\rangle$) is pure.
  Also $\rho_{i,\alpha}^{(A)} = \rho_\alpha^{(A)}$ for all i. This imples that $I_{acc}^{LOCC} \leq S(\rho_\alpha^{(B)})$.
  Since we need to perfectly distinguish m states, we must have: $S(\rho_\alpha^{(B)}) \geq \log_2 m$.

Necessary condition for LOCC discrimination ....

- **Forms of $|\psi_i\rangle$:** With respect to the standard ONB $\{|j\rangle_A\}_{j=1}^{d}$, we have: $|\psi_i\rangle = (1/\sqrt{d})\sum_{j=1}^{d} |j\rangle_A \otimes |b_j^{(i)}\rangle_B$ where $\{|b_j^{(i)}\rangle_B\}_{j=1}^{d}$ is an ONB for Bob's system for all i.
- The i-th PMRS on Bob's side is then given by:
  $\rho_{i,\alpha}^{(B)} = [1/(\mathrm{Tr}(K_\alpha^\dagger K_\alpha))]\sum_{j,k=1}^{d} \langle j|K_\alpha^\dagger K_\alpha|k\rangle |b_k^{(i)}\rangle\langle b_j^{(i)}| =$
  $U_i^{(B)}[K_\alpha^T K_\alpha^* / (\mathrm{Tr}(K_\alpha^T K_\alpha^*))] U_i^{(B)\dagger}$
  $U_i^{(B)}$'s are $d \times d$ unitaries such that $U_i^{(B)}|j\rangle_B = |b_j^{(i)}\rangle_B$ for $j = 1, 2, \ldots, d$ with each $i = 1, 2, \ldots, m$.
- Bob's average PMRS:
  $\rho_\alpha^{(B)} = (1/m)\sum_{i=1}^{m} U_i^{(B)}[K_\alpha^T K_\alpha^* / (\mathrm{Tr}(K_\alpha^T K_\alpha^*))] U_i^{(B)\dagger}$
  which should satisfy the Theorem – giving constraint on K_α.
- Orthogonality preservation is subsumed by this constraint.

Necessary condition for LOCC discrimination ....

- **Corollary:** *If $m = d$ in the Theorem, then the average PMRS on Bob's side has to be maximally mixed.*
- **Proof:** For $m = d$, $\log_2 d$ bits of information can be extracted by perfectly distinguishing the states.
  The maximum value $S(\rho_\alpha^{(B)})$ can take is $\log_2 d$ – which happens only when $\rho_\alpha^{(B)}$ is a maximally mixed state.
- Thus we have here:
  $(1/d) \sum_{i=1}^{d} U_i^{(B)} [K_\alpha^T K_\alpha^* / (\mathrm{Tr}(K_\alpha^T K_\alpha^*))] {U_i^{(B)}}^\dagger = (1/d) I_{d \times d}$.

Necessary condition for LOCC discrimination ....

- **Observation:** After imposing the last condition on the effects $K_\alpha^\dagger K_\alpha$ of Alice's POVM, if the allowed POVM turns out to be trivial – that is, if all the corresponding effects are multiples of $I_{d\times d}$ – the given set of pairwise orthogonal MES $\{|\psi_i\rangle\}_{i=1}^d$ fails to satisfy the necessary condition that the von Neumann entropy of the average PMRS of Bob must be $\log_2 d$.
- If the allowed POVM is non-trivial, the given states may or may not be perfectly distinguishable by LOCC.
- It is this observation that is used in our work to sort out the locally *indintinguishable* classes of d pairwise orthogonal MES in $\mathbb{C}^d \otimes \mathbb{C}^d$.

Classification of generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ under local unitaries

Classification of generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ under local unitaries

- There are ${}^{16}C_4 = 1820$ distinct sets of four pairwise orthogonal generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$.
- Under the action of local unitaries, they form 122 **inequivalent** classes – determined through elaborate calculations!.
- So, local (in)distinguishability of states for any representative set from such an 'equivalent' class will confirm the same for all other member sets of the class.
- And local (in)distinguishability of states from sets of one class **does not predict** anything about the same for states taken from sets of any other class 'inequivalent' to the former.

Locally indistinguishable classes of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$

Locally indistinguishable classes of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$

- It can be shown that exactly 39 out of the 122 inequivalent classes violate the necessary condition.
- Hence, states from **none** of the representative sets from any one of these 37 classes are reliably distinguishable.
- For example, the generalized Bell states $|\psi_{00}^{(4)}\rangle$, $|\psi_{11}^{(4)}\rangle$, $|\psi_{31}^{(4)}\rangle$, and $|\psi_{32}^{(4)}\rangle$ are not reliably distinguishable by LOCC.

On local distinguishability of classes of four generalized Bell states in $\mathcal{C}^4 \otimes \mathcal{C}^4$ satisfying the necessary condition

On local distinguishability of classes of four generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ satisfying the necessary condition

- What can be said about local distinguishability of rest of the 122 - 39 $=$ 87 inequivalent classes?
- The states for each set chosen from any of these classes do satisfy the necessary condition.
- Surprisingly, we figured out that states from each set from any one of these 87 classes **can be** reliably distinguishable by LOCC.
- In fact, one-way LOCC with only rank-one projective measurements are enough here.
- For example, the generalized Bell states $|\psi^{(4)}_{00}\rangle$, $|\psi^{(4)}_{02}\rangle$, $|\psi^{(4)}_{20}\rangle$, and $|\psi^{(4)}_{22}\rangle$ are reliably distinguishable by LOCC.

Discussion

Discussion

- Based on an upper bound on I_{acc}^{LOCC}, we formulated a necessary condition for the perfect distinguishability of any set of pairwise orthogonal MES by LOCC.
- This necessary condition genuinely decreses the complexity of the local distinguishability problem: in the case of generalized Bell states for $d = 4$, this condition turns out to be sufficient also.
- We tested our necessary condition for the set of four ququad-ququad states of Yu et al. [PRL **109**, 020506 (2012)] and found to be locally indistinguishable. And the other classes satisfy the necessary condition
- Thus – with the support of Tian et al.'s result [PRA **91**, 052314 (2015)] – our necessary condition turns out to be sufficient also for the ququad-ququad states.

Discussion (continued)

- Note that the classification scheme for local distinguishability of ququad-ququad states is *very specific* to these states only.
- It will be interesting to figure out: for what *other* kind of MES, our necessary condition turns out to be sufficient also.
- It may be interesting to figure out whether any one (or, more than one) of the locally indistinguishable classes of generalized Bell states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ remains indistinguishable by PPT POVMs also.
- It may also be interesting to look for common structure of generalized Bell states in general d for their local (in)distinguishability – as our approach for $d > 4$ may turn out to be computationally challenging!

Reference: "Complete Analysis of Perfect Local Distinguishability of Ensemble of Four Generalized Bell States in $\mathbb{C}^4 \otimes \mathbb{C}^4$" by Tanmay Singal, Ramij Rahman, S.G., and Guruprasad Kar [arXiv:1506.03667 (quant-ph)]

THANK YOU!