Non-local and temporal steering with joint measurability

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Non-joint measurability or measurement incompatibility

- Heisenberg's uncertainty relation and Bohr's notion of complementarity point towards quantum measurements, which can not be implemented simultaneously for some physical observables. The measurements are *incompatible* because they are not *jointly measurable.*

- At the first sight incompatibility of quantum measurements appears like an obstacle. However, it has been realized that only incompatible measurements enable the violation of a Bell inequality, steering inequality and other no-go theorems.

- In particular, incompatible measurements are essential non-classical resources for quantum information processing tasks such as quantum cryptography.

H. S. Karthik, A. R. Usha Devi and A. K. Rajagopal, Phys. Rev. A 91, 012115 (2015)

- **In the classical world physical observables commute with each other and they can all be jointly measured. But in the quantum scenario, measurement of observables, which do not commute are usually declared to be *incompatible* in the quantum scenario.**

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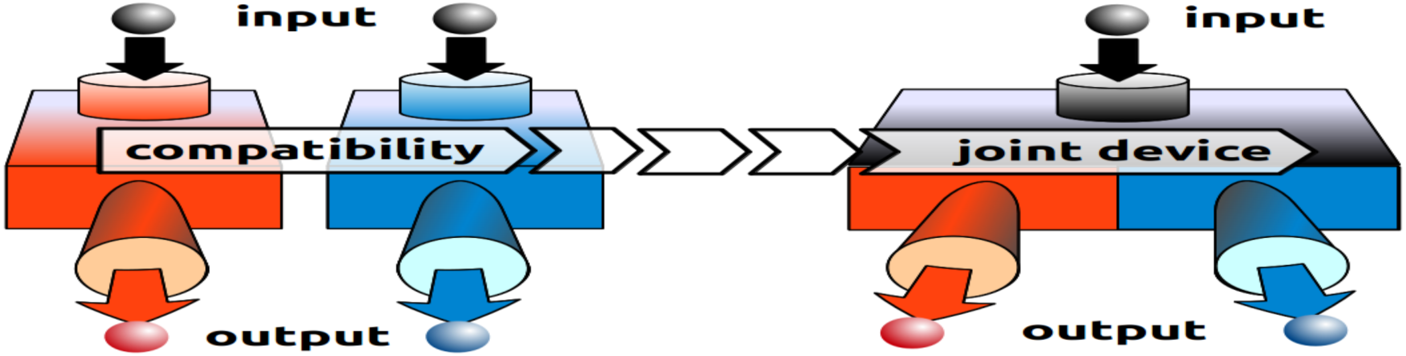

PV measurements of a pair of commuting observables

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Are joint unsharp measurements possible?

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

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Introduction of positive operator valued measures (POVMs) into physics:

- **1960s and 1970s – first by Ludwig and then by Davies, Helstrom, Holevo . . .**
- **A notion of joint measurement of noncommuting observables could be formulated in the Hilbert-space formalism of quantum mechanics.**

See: P. Busch, M. Grabowski, and P. Lahti, *Operational Quantum Physics*, 2nd ed. (Springer, Berlin, 1997)

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

- **The orthodox notion of *sharp* projective valued (PV) measurements of self adjoint observables gets broadened to include *unsharp* measurements of POV observables.**

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- **Generalized measurement framework: compatibility and commutativity are not synonymous notions.**

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

1. P. Busch, Phys. Rev. D **33**, 2253 (1986).
2. T. Heinosaari, D. Reitzner, and P. Stano, Found. Phys. **38**, 1133 (2008).
3. P. Busch, P. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement*, No. v. 2 in Environmental Engineering, Springer, 1996.
4. M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Phys. Rev. Lett. **103**, 230402 (2009).
5. M. Banik, Md. R. Gazi, S. Ghosh, and G. Kar, Phys. Rev. A **87**, 052125 (2013).
6. N. Stevens and P. Busch, Phys. Rev. A **89**, 022123 (2014)
7. M. T. Quintino, T. Vertesi, and N. Brunner, arXiv:1406.6976; Phys. Rev. Lett. **113**, 160402 (2014)
8. R. Uola, T. Moroder, and O. Gühne, arXiv:1407.2224; Phys. Rev. Lett. **113**, 160403 (2014).

Outline

- **Positive Operator Valued (POV) observables and joint unsharp measurements**
- **Compatible (jointly measurable) POVMs**
- **Non-local and time-like steering; joint measurability**
- **Moment matrix positivity and N-term correlation inequalities with joint measurability**

Positive Operator Valued (POV) observables and unsharp measurements

Def: POV observable $\mathbb{E}$ is a collection $\{E(x)\}$ of positive self-adjoint operators

$$0 \leq E(x) \leq 1$$

called *effects*.

The *effects* satisfy the condition

$$\sum_x E(x) = \mathbb{1}$$

($\mathbb{1}$ is the identity operator)

When a quantum system is prepared in the state ρ, measurement of the POV observable $\mathbb{E}$ gives an outcome x with probability

$$p(x) = \mathrm{Tr}[\rho\, E(x)]$$

Joint Measurability: Global Positive Operator Valued Measure (POVM)

Consider two POV observables

$$\mathbb{E}_i = \{E_i(x_i)\}, \quad i = 1, 2$$

These two POVMs are jointly measurable if there exists a *global* POVM

$$\mathbb{G} = \{G(x_1, x_2);\ 0 \leq G(\lambda) \leq \mathbb{1}, \sum_{\lambda} G(\lambda) = \mathbb{1}, \quad \lambda = \{x_1, x_2\}\}$$

such that the POV observables $\mathbb{E}_i$ can be realized as the marginals:

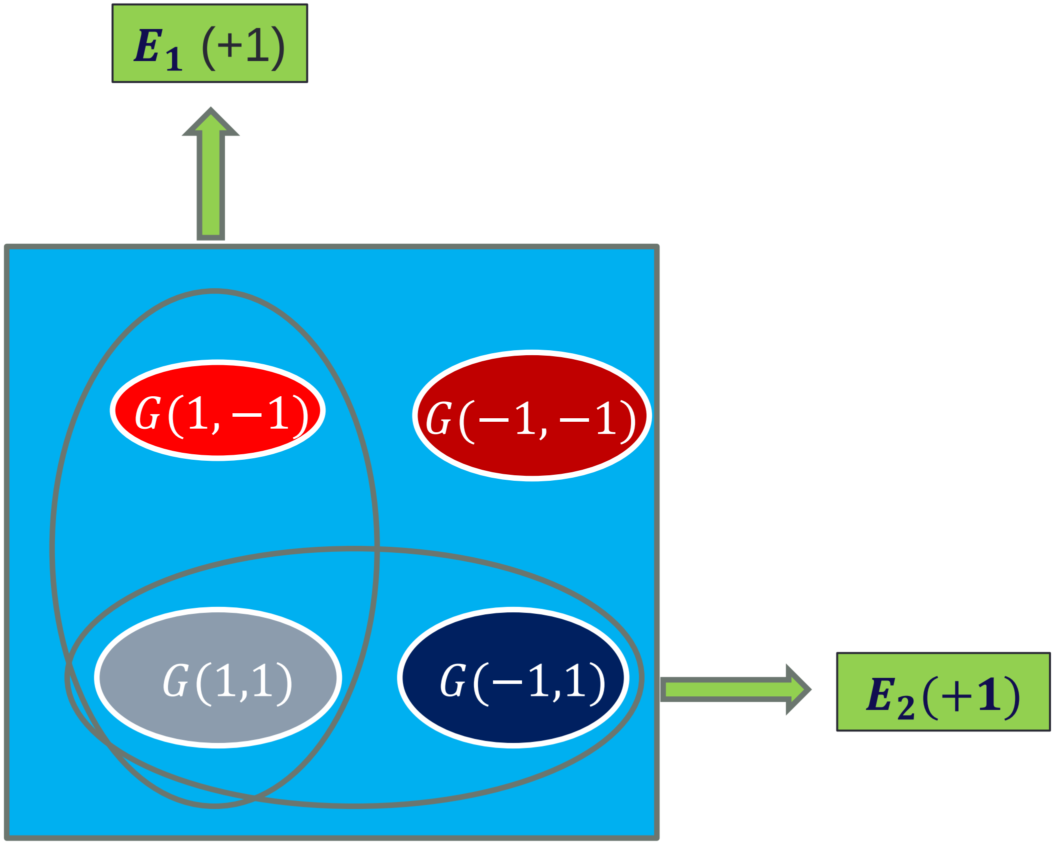
$$E_1(x_1) = \sum_{x_2} G(x_1, x_2), \quad E_2(x_2) = \sum_{x_1} G(x_1, x_2)$$

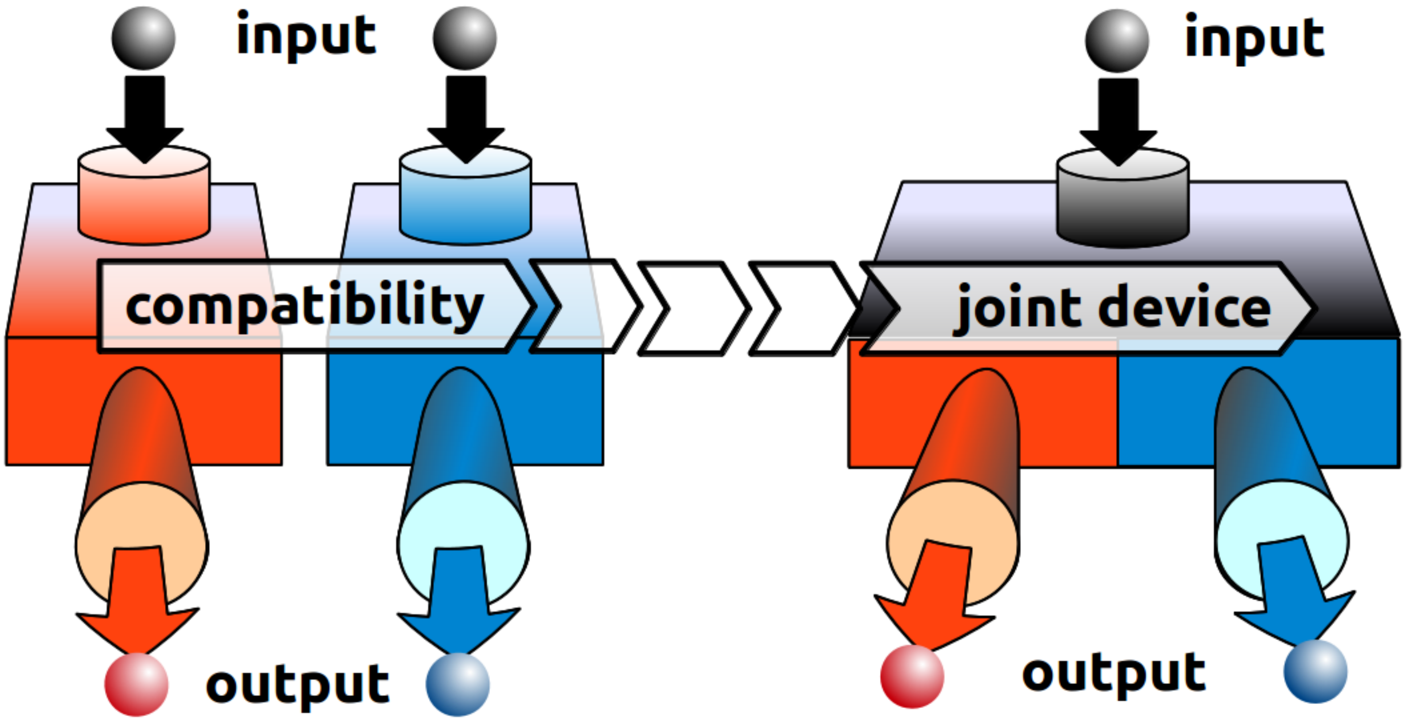
In general, if the effects $E_i(x_i)$ can be expressed as

$$E_i(x_i) = \sum_{\lambda} p(x_i|i, \lambda)\, G(\lambda) \quad \forall \ \ i$$

where $\sum_{x_i} p(x_i|i, \lambda) = 1$, the fuzzy observables $\mathbb{E}_i$ are jointly measurable.

- **Think of G as a common measurement device with four LEDs (corresponding to four outcomes); two of the LEDs correspond to the measurement outcome $+1$ for the binary POVMs E_1 and similarly for E_2.**





Unsharp measurements of a pair of compatible POVMs

Courtesy: Heinosari et al., arXiv: 1511.07548 v1

Fuzzy measurements of noisy qubit observables

Positive operator valued *fuzzy* spin observables:

Unsharp x-spin $\longrightarrow$ $\{E_x(+1), E_x(-1)\}$
Unsharp z-spin $\longrightarrow$ $\{E_z(+1), E_z(-1)\}$

$$E_x(\pm 1) = \frac{1}{2}[\mathbb{1} \pm \eta\, \sigma_x]$$
$$E_z(\pm 1) = \frac{1}{2}[\mathbb{1} \pm \eta\, \sigma_z]$$

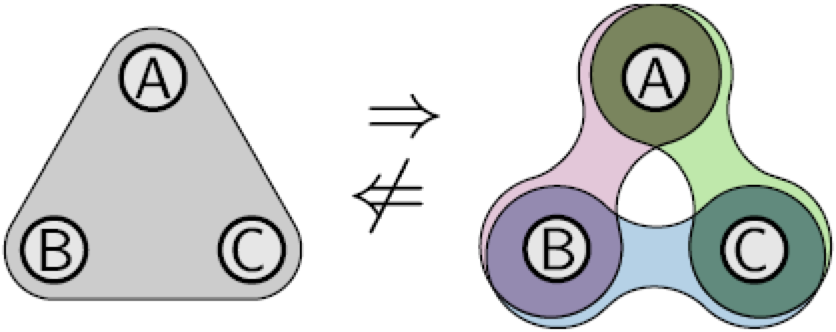
$$0 \leq \eta \leq 1$$

$\eta \longrightarrow$ sharpness parameter

- **PV measurements $\longrightarrow \eta = 1 \Rightarrow$ sharp measurement**
- **Joint measurability of σ_x, σ_z requires $\eta \leq \frac{1}{\sqrt{2}}$**
- **Global POVM for pairwise joint measurabililty:**

$$G(x, z) = \frac{1}{4}\left[\mathbb{1} + \frac{x}{\sqrt{2}}\sigma_x + \frac{z}{\sqrt{2}}\sigma_z\right], \quad x, z = \pm 1.$$

- **Joint measurability of three orthogonal spin components σ_x, σ_y, σ_z implies $\eta \leq \frac{1}{\sqrt{3}}$**
- **Three orthogonal qubit orientations are pairwise measurable but not tripplewise measurable iff $\frac{1}{\sqrt{3}} \leq \eta \leq \frac{1}{\sqrt{2}}$.**



Existence of a global observable for three observables A, B and C implies that their exist joint observables for each of the possible pairs $\{A, B\}$, $\{A, C\}$, $\{B, C\}$ but the converse need not be true for unsharp observables

See: P. Busch, Phys. Rev. D 33, 2253 (1986),
T. Heinosaari, D. Reitzner, and P. Stano, Found. Phys. 38, 1133 (2008).

Necessary condition for joint measurability of N dichotomic POVMs with qubit orientations $\hat{n}_k,\ k = 1, 2, \ldots, N$

$$\eta \leq \frac{1}{N} \max_{x_1, x_2, \ldots, x_N} \left| \vec{m}_{x_1, x_2, \ldots, x_N} \right|$$

$$\vec{m}_{x_1, x_2, \ldots, x_N} = \sum_{k=1}^{N} x_k \, \hat{n}_k; \quad x_k = \pm 1$$

See: Ravi Kunjwal and Sibasish Ghosh, Phys. Rev. A 89, 042118 (2014)
Y. C. Liang, R. W. Spekkens, and H. M. Wiseman, Phys. Rep. 506, 1 (2011).

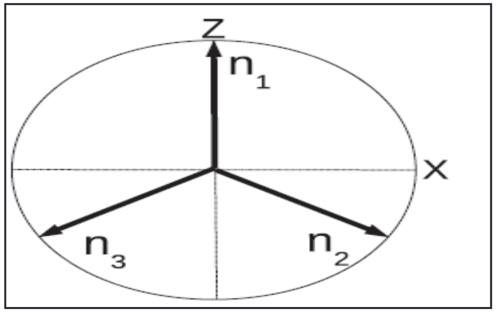
- In the example of three orthogonal orientations $\hat{n}_1,\ \hat{n_2}, \hat{n}_3;\ \hat{n}_1\cdot\hat{n}_2 = 0 = \hat{n}_2\cdot\hat{n}_3 = \hat{n}_1\cdot\hat{n}_3$, we find that

$$\begin{aligned} \eta &\le \frac{1}{3}\max_{x_1,x_2,x_3=\pm 1}|(\hat{n}_1\,x_1+\hat{n}_2\,x_2+\hat{n}_3\,x_3)| \\ &= \frac{1}{3}\times\sqrt{3} \\ i.e.,\ \eta &\le \frac{1}{\sqrt{3}}. \end{aligned}$$

- For trine axes $\hat{n}_1,\ \hat{n_2}, \hat{n_3},\ \hat{n}_1\cdot\hat{n_2} = \hat{n}_2\cdot\hat{n_3} = -\hat{n}_1\cdot\hat{n_3} = \cos(\pi/3)$, we obtain the compatiblity condition

$$\begin{aligned} \eta &\le \frac{1}{3}\max_{x_1,x_2,x_3=\pm 1}|(\hat{n}_1\,x_1+\hat{n}_2\,x_2+\hat{n}_3\,x_3)| \\ &= \frac{1}{3}\max_{x_1,x_2,x_3=\pm 1}\sqrt{3+2\,\cos(\pi/3)\,(x_1\,x_2+x_2\,x_3-x_1\,x_3)} \\ i.e.,\ \eta &\le \frac{2}{3} \end{aligned}$$

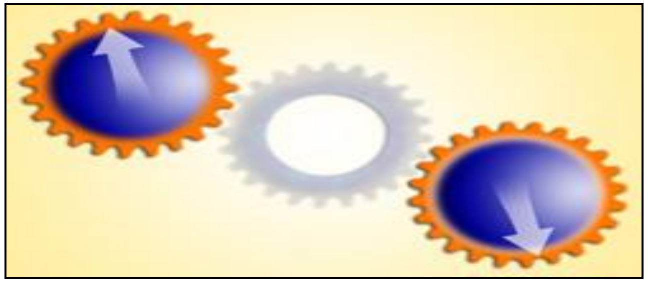
for joint measurability of the POVMs $\{E_k(x_k) = \frac{1}{2}(I+\eta\,x_k\,\vec{\sigma}\cdot\hat{n}_k), k = 1,2,3\}$.



Non-local steering

EPR Steering

The ability to non-locally alter the states of one part of a composite system by performing measurements on another space-like separated part.

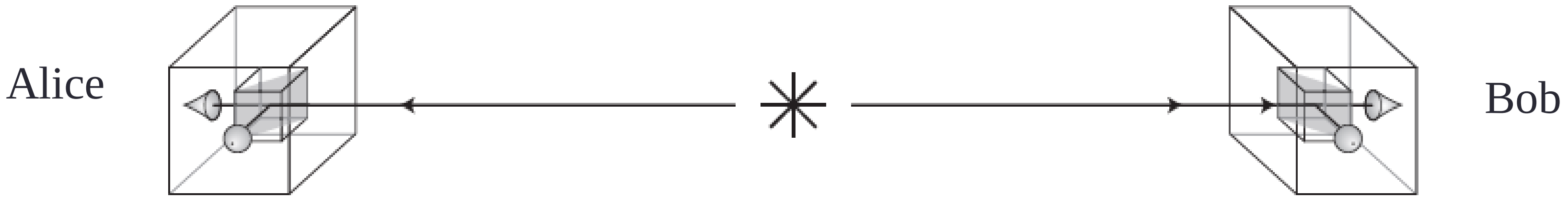


Non-local steering

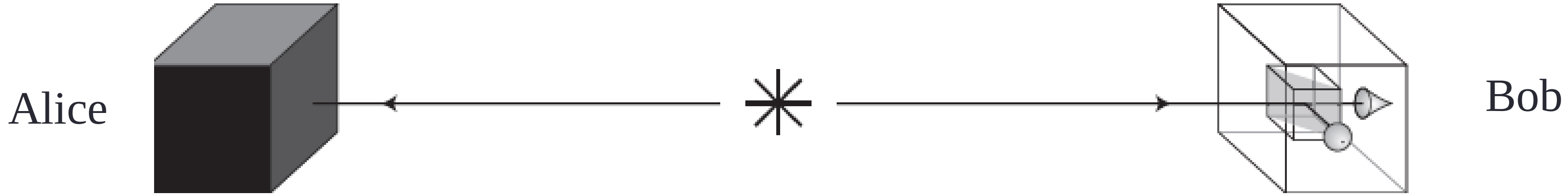
(E. Schrodinger, Proc. Camb. Phil. Soc. 31, 555563 (1935))

- Alice, who holds a source of photons, tries to convince a sceptical Bob that they share an entangled state.
- To convince Bob, Alice claims that after having sent the photon to Bob, she can steer its state from a distance.
- If the photons are actually entangled, quantum theory predicts that by performing different measurements on her photon, Alice can remotely prepare different states for Bobs photon (the prepared state depends on Alice's measurement outcome).
- But how can Bob make sure that Alice is not cheating – for instance by sending him an uncorrelated photon and only *pretending* to make a measurement on her side?
- Bob asks Alice to perform different measurements in each experimental run and evaluates *steering inequality* – which is *always* obeyed when Alice is not trustworthy i.e., she cannot steer Bob's state by her remote measurements.

(See: N. Brunner, News and views, nature physics, 6, 842 (2010))



Trusted devices



Bob cannot trust Alice! Verifies if a 'steering inequality' is violated.

EPR Steering

- **It could be that Alice is not honest; she does not prepare a composite state ρ_{AB} at all; but she chooses the states ρ_λ with probability $g(\lambda)$ from a chosen ensemble and sends it to Bob.**

- **When Bob asks Alice to perform measurement of the observable X_k, she could merely communicate a fake outcome x_k to have occurred with the probability $p(x_k|k) = \sum_\lambda g(\lambda)\, p(x_k|k,\lambda)$. Bob would then be able to verify that the assemblage has Local Hidden State form.**

EPR Steering

Assemblage:

$$\rho_{x_k|k} = \mathrm{Tr}_A \left[\rho_{AB}\, E_k(x_k) \otimes I\right]$$

Local Hidden State Form:

$$\rho_{x_k|k} \;=\; \sum_{\lambda} g(\lambda)\, p(x_k|k,\lambda)\, \rho_{\lambda},$$

EPR Steering and Joint Measurability

- **Suppose that Alice is honest. She indeed prepares a steerable, entangled state ρ_{AB}, part of which she sends to Bob.**
- **But when Alice's measurements $\{E_k(x_k)\}$ are compatible, i.e., there exists a global POVM $G(\lambda)$ such that**

$$E_k(x_k) = \sum_{\lambda} p(x_k|k,\lambda)\, G(\lambda),$$

Bob's assemblage takes the form:

$$\begin{aligned}\rho_{x_k|k} &= \mathrm{Tr}_A\left[\rho_{AB}\, E_k(x_k) \otimes I\right] \\ &= \sum_{\lambda} g(\lambda)\, p(x_k|k,\lambda)\, \rho_\lambda,\end{aligned}$$

which is in the Local Hidden State (LHS) form – irrespective of *steerability* of the entangled state ρ_{AB} shared between Alice and Bob.

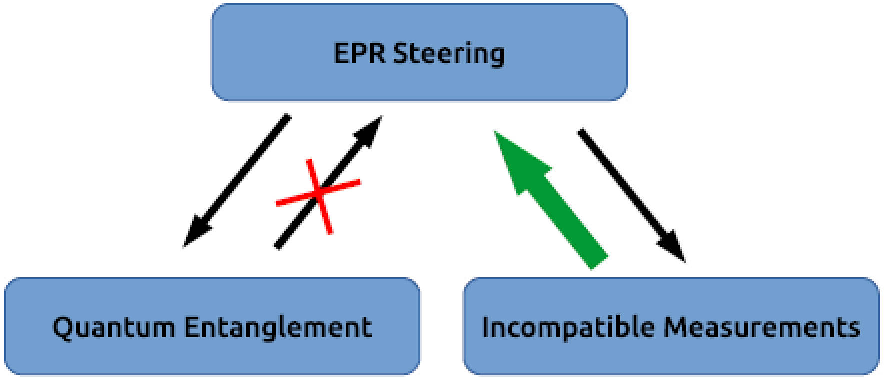
- **In other words, the assemblage can be classically simulated using the local hidden states ρ_λ.**

Thus, if Alice employs measurements of compatible POVMs at her end, Bob's assemblage would take the LHS form even when they share an entangled state. To witness steering phenomena, it is necessary that Alice employs incompatible POVMs.

Converse statement viz., "*a set of measurements are incompatible if they are useful to demonstrate EPR steering phenomena*" has also been established.

M. T. Quintino, T. Vértesi, and N. Brunner, arXiv:1406.6976; Phys. Rev. Lett. 113, 160402 (2014)
R. Uola, T. Moroder, and O. Gühne, arXiv:1407.2224; Phys. Rev. Lett. 113, 160403 (2014)

Steering and Non-joint Measurability are synonymous



Steering implies both entanglement and incompatible measurements

M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014)
R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014)

Separation of local and nonlocal resources: Time analogue of steering

Incompatible measurements are *local* resources whereas, space-like separated bipartite entangled states are *non-local* resources. But the connection between EPR steerability and measurment incompatibility leads to mixing of both local and non-local resources. To understand incompatibility, can one formulate a resource theory without relying on non-locality?

M. F. Pussey, JOSA B, **32**, A56 (2015); T. Heinosaari et. al., arXiv:1504.05768); H. S. Karthik et. al., JOSA B, **32**, A34 (2015).

Time-like analogue of steering in a single quantum system??

Time-like analogue of steering

- **A system, prepared in a quantum state $\rho = \rho(0)$, evolves under the Hamiltonian evolution $U(t) = e^{-i\,H\,t/\hbar}$.**

- **Schrodinger picture:**

$$\rho \to \rho(t_k) = \rho_k = U(t_k)\,\rho\,U^{\dagger}(t_k)$$

.

- **Heinsenberg picture: The physical observables undergo dynamical evolution as**

$$X(0) \to X(t_k) = X_k = U^{\dagger}(t_k)\,X(0)\,U(t_k)$$

.

- **The observable X_k (X at different time instants t_k) do not commute in general. And hence their *sharp* PV measurements are incompatible.**

Time-like analogue of steering

- **Alice measures the time-separated observables X_k using the fuzzy POVMs $E_k(x_k)$.**
- **Bob's task is to verify if Alice has given him a genuine set of post measured assemblage $\{\rho_{x_k|k}\}$ or not.**
- **Bob may choose to measure the observables X_l at a later time, $l \geq k$ on the assemblage $\rho_{x_k|k}$ and record the conditional probabilities $\mathcal{P}(x_l|x_k)$ of his outcomes x_l (given that Alice had obtained an outcome x_k in her measurements of of the observable X_k); he then explores if the *temporal correlations* of the observables X_k, X_l violate any *temporal steering inequality*.**

Time-like analogue of steering

- **Correlations in the measurement outcomes of time-separated observables in a single system mimic the non-local correlations in a spatially separated entangled system.**

- **The Leggett-Garg inequality (A. J. Leggett and A. Garg, Phys. Rev. Lett. 54, 857 (1985)) – the *temporal* Bell inequality – involves linear combinations of correlations $\langle X_k X_l \rangle$ between temporally separated observables, measured sequentially.**

- **Temporal steering inequalities could be formulated by considering measurements of observables at different time instants in a single quantum system, as a natural temporal counterpart of EPR non-local steering inequalities (Chen et. al., Phys. Rev. A 89, 032112 (2014)).**

Time-like analogue of steering

- **Suppose Alice performs measurement of a global POVM $\mathbb{G} = \{G(\lambda)\}$ corresponding to compatible measurements of X_k.**

- **After her measurement, the post measured states have the form:**

$$\rho_\lambda = \sqrt{G(\lambda)}\,\rho\,\sqrt{G(\lambda)}/g(\lambda)$$

where $g(\lambda) = \mathrm{Tr}\,[\rho\, G(\lambda)]$ is the probability of outcome λ.

- **It is possible for Alice to classically post process the measurement data of the global POVM $\mathbb{G} = \{G(\lambda)\}$ to obtain the probabilities of outcomes $p(x_k|k)$ of measurement of any compatible POVMs $\mathbb{E}_k$ to have resulted in an outcome x_k as,**

$$\begin{aligned} p(x_k|k) &= \mathrm{Tr}\,[\rho\, E_k(x_k)] \\ &= \sum_\lambda p(x_k|k,\lambda)\,\mathrm{Tr}[\rho\, G(\lambda)] \\ &= \sum_\lambda p(x_k|k,\lambda)\, g(\lambda). \end{aligned}$$

Time-like analogue of steering

- **More specifically, Alice could discern the results of measurements of *compatible* POVMs $\mathbb{E}_k = \{E_k(x_k)\}$ via measurement of a global POVM $\{G(\lambda)\}$ and then using the decomposition $E_k(x_k) = \sum_\lambda p(x_k|k,\lambda)\, G(\lambda)$.**

- **After Alice announces her measurement results $\{x_k,\, p(x_k|k)\}$ of $E_k(x_k)$ and hands over the post measured set of states, Bob detects that his assemblage $\{\rho(x_k|k)\}$ is of the HS form $\rho(x_k|k) = \sum_\lambda g(\lambda)\, p(x_k|k,\lambda)\, \rho_\lambda$. Thus, Bob concludes that there is no temporal steering.**

Time-like analogue of steering

- **Conversely, non-jointly measurable (incompatible) POVMs are sufficient to demonstrate time-like steering.**
- **Consider a completely random state $\rho = \mathbb{1}/d$ and a set of POVMs $\{\mathbb{E}_k\}$ for the measurements of the observables $\{X_k\}$. The post measured assemblage $\{\rho_{x_k|k}\}$ is characterized by its elements,**

$$\begin{aligned} \rho_{x_k|k} &= \sqrt{E_k(x_k)}\,\rho\,\sqrt{E_k(x_k)} \\ &= \frac{1}{d}\,E_k(x_k). \end{aligned}$$

- **It is thus possible to express the elements $E_k(x_k)$ of the POVM in terms of the assemblage $\{\rho_{x_k|k}\}$** as,

$$E_k(x_k) = d\,\rho_{x_k|k}$$

Time-like analogue of steering

- **If there is no temporal steering, then the assemblage $\{\rho_{x_k|k}\}$ is described by a Hidden State form leading to**

$$\begin{aligned} E_k(x_k) &= d\sum_{\lambda} g(\lambda)\, p(x_k|k,\lambda)\rho_\lambda \\ &= \sum_{\lambda} p(x_k|k,\lambda)G(\lambda) \end{aligned}$$

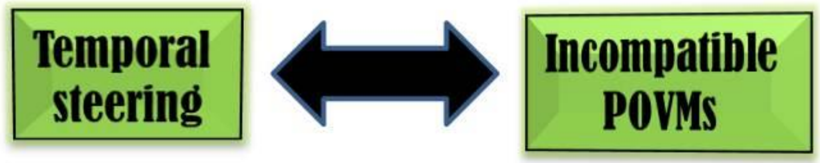
where $G(\lambda) = d\; g(\lambda)\,\rho_\lambda$.

- **This is essentially the joint measurability (compatibility) condition for POVMs $\{E_k(x)\}$.**

- **A set of POVMs is said to be non-jointly measurable if and only if it is useful for demonstrating temporal steering.**

Time-like analogue of steering

This result highlights that one does not require a steerable, space-like separated entangled state (non-local resource) to determine if a given set of measurements is compatible or not; it suffices to detect time-like analogue of non-steerability in a single quantum system itself



- **Time-like analogue of steering phenomena in a single quantum system (percieved via a hidden state assemblage structure – analogous to the LHS model for spatially separated systems) $\Rightarrow$ falsification of hidden state model.**

- **Connection between incompatibility of quantum measurements and temporal steering phenomena: A set of measurements are incompatible if and only if they can be used to demonstrate temporal steering in any quantum state.**

- **Resource theory of incompatible measurements? M. F. Pussey, JOSA B, 32, A56 (2015); T. Heinosaari et. al., arXiv:1504.05768); H. S. Karthik et. al., JOSA B, 32, A34 (2015).**

Moment matrix positivity

- Probabilities of measurement outcomes arising in the quantum framework turn out to be different from those arising in the traditional classical statistical scenario.
- This has invoked a wide range of debates on the quantum-classical worldviews of nature.
- Investigations by Bell, Kochen-Specker, Leggett-Garg tied the puzzling quantum features in terms of no-go theorems (CHSH inequality, Leggett-Garg Inequality..).
- Proofs of these no-go theorems essentially point towards the non-existence of a joint probability distribution for the outcomes of all possible measurements performed on a quantum system.

Classical Moment Problem

➡ Addresses the issue of finding a probability distribution given a set of moments.

It brings forth the fact that

A given sequence of real numbers qualifies to be moment sequence of a legitimate probability distribution if and only if the associated moment matrix is positive.

Existence of joint probability distribution ⟺ **Moment matrix is positive**

J.A Sholat and J.D. Tamarkin, *The problem of moments,* AMS (1943)

N.J. Akhiezer, *The Classical Moment Problem*, Hofuer Publishing Co., (1965)

Classical Moment problem

Moment Inversion

↓

Are the probabilities moment determinate?

Moment Matrix Positivity

↓

Is the given sequence of “moments” admit *legitimate* probabilities?

H. S. Karthik, H. Katiyar, A. Shukla, T. S. Mahesh, A. R. Usha Devi and A. K. Rajagopal, Phys. Rev. A 87, 052118 (2013).

Chained correlation inequality

- **Consider N classical random variables X_k with outcomes $x_k = \pm 1$.**
- **Construct 4×4 moment matrices $M_k = \langle \xi_k \, \xi_k^T \rangle$ containing only pairwise moments of a set of three random variables each.**

$$\xi_k = \begin{pmatrix} 1 \\ x_1 \, x_k \\ x_k \, x_{k+1} \\ x_1 \, x_{k+1} \end{pmatrix}, \; k = 2, 3, \ldots N-1$$

and $\langle \cdot \rangle$ denotes expectation value.

Chained correlation inequality ...

The 4×4 moment matrix M_k has the form:

$$M_k = \begin{pmatrix} 1 & \langle X_1 \, X_k \rangle & \langle X_k \, X_{k+1} \rangle & \langle X_1 \, X_{k+1} \rangle \\ \langle X_1 \, X_k \rangle & 1 & \langle X_1 \, X_{k+1} \rangle & \langle X_k \, X_{k+1} \rangle \\ \langle X_k \, X_{k+1} \rangle & \langle X_1 \, X_{k+1} \rangle & 1 & \langle X_1 \, X_k \rangle \\ \langle X_1 \, X_{k+1} \rangle & \langle X_k \, X_{k+1} \rangle & \langle X_1 \, X_k \rangle & 1 \end{pmatrix}.$$

Chained correlation inequality . . .

In the classical probability setting, the moment matrix is real, symmetric and positive semidefinite by construction.

Chained correlation inequality . . .

- **The eigenvalues** $\mu_i^{(k)}; i = 1, 2, 3, 4$ **of the moment matrix:**

$$\begin{aligned}
\mu_1^{(k)} &= 1 + \langle X_1\, X_k\rangle - \langle X_k\, X_{k+1}\rangle - \langle X_1\, X_{k+1}\rangle \\
\mu_2^{(k)} &= 1 - \langle X_1\, X_k\rangle + \langle X_k\, X_{k+1}\rangle - \langle X_1\, X_{k+1}\rangle \\
\mu_3^{(k)} &= 1 - \langle X_1\, X_k\rangle - \langle X_k\, X_{k+1}\rangle + \langle X_1\, X_{k+1}\rangle \\
\mu_4^{(k)} &= 1 + \langle X_1\, X_k\rangle + \langle X_k\, X_{k+1}\rangle + \langle X_1\, X_{k+1}\rangle.
\end{aligned}$$

- **Positivity of the moment matrix implies that the eigenvalues** $\mu_i^{(k)}$ **are positive.**

Chained correlation inequality ...

For a set of $N-1$ **moment matrices** $M_2, M_3, \ldots, M_{N-1}$**, positivity condition** $\sum_{k=2,3,\ldots,N-1} \mu_i^{(k)} \geq 0$, **for the sum of eigenvalues** $\mu_i^{(k)}$**,** $i = 1,2,3,4$ **leads to four chained inequalities for pairwise moments:**

$$\sum_{k=2}^{N-1} \langle X_k\, X_{k+1}\rangle + \langle X_1\, X_N\rangle - \langle X_1\, X_2\rangle \leq N-2$$

$$2\sum_{k=1}^{N-2} \langle X_1\, X_k\rangle - \sum_{k=2}^{N-1}\langle X_k\, X_{k+1}\rangle + \langle X_1\, X_N\rangle - \langle X_1\, X_2\rangle \leq N-2$$

$$\sum_{i=1}^{N-1} \langle X_i\, X_{i+1}\rangle - \langle X_1\, X_N\rangle \leq N-2$$

$$-\sum_{k=2}^{N-1} \langle X_k\, X_{k+1}\rangle - 2\sum_{k=2}^{N-2}\langle X_1\, X_{k+1}\rangle - \langle X_1\, X_N\rangle + \langle X_1\, X_2\rangle \leq N-2.$$

Chained correlation inequality ...

Of the four inequalities we find the generalized N-term Leggett-Garg/non-contextual/Bell inequality:
(S. Wehner, Phys. Rev. A 73, 022110 (2006); C. Budroni, T. Moroder, M. Kleinmann, and O. Gühne, Phys. Rev. Lett. 111, 020403 (2013))

$$\mathcal{S}_N = \sum_{i=1}^{N-1} \langle X_i\, X_{i+1}\rangle - \langle X_1\, X_N\rangle \leq N-2.$$

- For $N=3$, we have $\langle X_1\, X_2\rangle + \langle X_2\, X_3\rangle - \langle X_1\, X_3\rangle \leq 1$ (3 term Leggett-Garg Inequality).
- For $N=5$, we have $\langle X_1\, X_2\rangle + \langle X_2\, X_3\rangle + \langle X_3\, X_4\rangle\langle X_4\, X_5\rangle - \langle X_1\, X_5\rangle \leq 3$ (5 term LGI)

Chained correlation inequality ...

- **We replace the classical random variables by a set of N dichotomic qubit observables**

$$X_k = \vec{\sigma} \cdot \hat{n}_k, k = 1, 2, \ldots, N$$

and the classical probability distribution by an arbitrary single qubit density matrix.

- **The pairwise moments**

$$\langle X_k \, X_l \rangle \equiv \langle X_k \, X_l \rangle_{\rm seq}$$

are obtained from sequential measurements of the observables – in the order in which they are written.

- **We obtain the chained inequality**

$$\mathcal{S}_N = \sum_{i=1}^{N-1} \langle X_i \, X_{i+1} \rangle_{\rm seq} - \langle X_1 \, X_N \rangle_{\rm seq} \leq N - 2.$$

Chained correlation inequality . . .

- **Budroni et. al (Phys. Rev. Lett. 111, 020403 (2013)) have evaluated the Tsirelsen Bound for the linear combination of the pair-wise correlations in the LHS of the chained N term inequality, when sequential sharp projective measurements are employed for suitably chosen orientations $\hat{n}_k$ for the qubit observables:**

$$\mathcal{S}_N^{(\mathrm{quantum})} \leq N\,\cos\left(\frac{\pi}{N}\right),$$

- **Thus the classical bound $N-2$ on the chained N term inequality can get violated in the quantum framework.**

PRL **111,** 020403 (2013) PHYSICAL REVIEW LETTERS week ending 12 JULY 2013

Bounding Temporal Quantum Correlations

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(Received 15 March 2013; published 10 July 2013)

Sequential measurements on a single particle play an important role in fundamental tests of quantum mechanics. We provide a general method to analyze temporal quantum correlations, which allows us to

Chained correlation inequality ...

- **The Tsirelson bound,** $N\,\cos\left(\frac{\pi}{N}\right)$ **can be reached, when the system is prepared in a maximally mixed state** $\rho = I/2$**; and sequential projective measurements of qubit observables** $\vec{\sigma}\cdot\hat{n}_k$**, with unit vectors** $\hat{n}_k$ **equally separated by an angle** π/N **in a plane, one obtains pairwise correlations** $\langle X_k\, X_{k+1}\rangle = \hat{n}_k\cdot\hat{n}_{k+1} = \cos\left(\frac{\pi}{N}\right)$ **and** $\langle X_1\, X_{k+1}\rangle = \hat{n}_1\cdot\hat{n}_N = -\cos\left(\frac{\pi}{N}\right)$ **leading to the the Tsirelsen bound** $N\cos\left(\frac{\pi}{N}\right)$**.**

- **Do we get violation of the inequality if generalized compatible (but unsharp) qubit POVMs are employed?**

- Do classical features emerge when one merely confines to measurements *compatible* unsharp observables? Is it possible to classify physical theories based on the *fuzziness* required for joint measurability?

PHYSICAL REVIEW A **87**, 052125 (2013)

Degree of complementarity determines the nonlocality in quantum mechanics

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Bohr's complementarity principle is one of the central concepts in quantum mechanics which restricts joint measurement for certain observables. Of course, later development shows that joint measurement could be possible for such observables with the introduction of a certain degree of unsharpness or fuzziness in the

PHYSICAL REVIEW A **89**, 022123 (2014)

Steering, incompatibility, and Bell-inequality violations in a class of probabilistic theories

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We show that connections between a degree of incompatibility of pairs of observables and the strength of violations of Bell's inequality found in recent investigations can be extended to a general class of probabilistic physical models. It turns out that the property of universal uniform steering is sufficient for the saturation of a generalized Tsirelson bound, corresponding to maximal violations of Bell's inequality. It is also found that a limited form of steering is still available and sufficient for such saturation in some state spaces where universal uniform steering is not given. The techniques developed here are applied to the class of regular polygon state spaces, giving a strengthening of known results. However, we also find indications that the link between incompatibility and Bell violation may be more complex than originally envisaged.

$$|\langle \mathsf{A}_1\mathsf{B}_1\rangle_\eta + \langle \mathsf{A}_1\mathsf{B}_2\rangle_\eta + \langle \mathsf{A}_2\mathsf{B}_1\rangle_\eta - \langle \mathsf{A}_2\mathsf{B}_2\rangle_\eta| \leqslant 2.$$

$$\boxed{\lambda_{\text{opt}} = 1} \Longrightarrow \textbf{classical}$$

$$|\langle \mathsf{A}_1\mathsf{B}_1\rangle_\eta + \langle \mathsf{A}_1\mathsf{B}_2\rangle_\eta + \langle \mathsf{A}_2\mathsf{B}_1\rangle_\eta - \langle \mathsf{A}_2\mathsf{B}_2\rangle_\eta| \leqslant \frac{2}{\lambda_{\text{opt}}}$$

$$\boxed{\lambda_{\text{opt}} = \frac{1}{2}} \Longrightarrow \textbf{GPT}$$

Tsirelson bound of $2\sqrt{2}$

$$\boxed{\lambda_{\text{opt}} = \frac{1}{\sqrt{2}}} \Longrightarrow \textbf{quantum}$$

Chained correlation inequality ...

- **Using fuzzy qubit POVMs**

$$\{E_k(x_k) = \frac{1}{2}(I + \eta\, x_k\, \vec{\sigma}\cdot\hat{n}_k), k = 1, 2, \ldots N\}$$

with successive unit vectors $\hat{n}_k$ separated by π/N in a plane, we obtain

$$\langle X_i\, X_{i+1}\rangle_{POVM} = \eta\,\langle X_i\, X_{i+1}\rangle_{\rm sharp} = \eta\,\cos(\pi/N)$$

and

$$\langle X_1\, X_N\rangle_{POVM} = \eta\,\langle X_1\, X_N\rangle_{\rm sharp} == -\eta\,\cos(\pi/N)$$

.

Chained correlation inequality . . .

- **The POVMs**
$$\{E_k(x_k) = \frac{1}{2}(I + \eta\, x_k\, \vec{\sigma} \cdot \hat{n}_k)$$
are all jointly measurable/compatible if the *unsharpness* parameter is less than the optimal value $0 \leq \eta \leq \eta_{\rm opt}$.

- **A tabulation of the optimal values of unsharpness parameter for different values of** N**:**

N	$\eta_{\rm opt}$
2	$1/\sqrt{2}$
3	$2/3$
4	0.6533
5	0.647
7	0.642
$\vdots$	$\vdots$

Chained correlation inequality ...

- **For $N = 3$, the product of the optimal value of unsharpness parameter $\eta_{\rm opt} = 2/3$ and the Tsirelson Bound $3\cos(\pi/3) = 3/2$ is equal to the classical bound $N - 3 = 1$. Thus, the three term inequality is never violated when the POVMs are jointly measurable.**

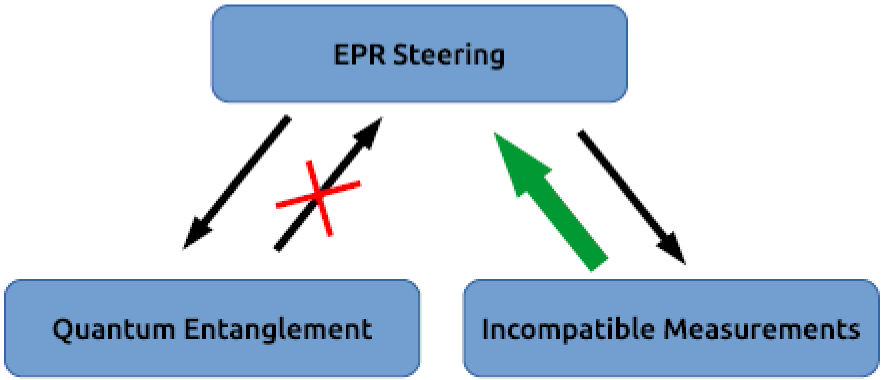
- **For $N > 3$ we identify that**

$$\eta_{\rm opt} \times N\cos(\pi/N) < N - 2,$$

confirming that in the joint measurability range $0 \le \eta \le \eta_{\rm opt}$ the chained N term inequality is *never* violated.

- **But even for a slightly larger domain $\eta > \eta_{\rm opt}$, where the POVMs are *incompatible*, the chained N term inequality is NOT violated – indicating that $\eta_{\rm opt}$ is only sufficient, but not necessary to obtain the classical bound.**

Steering and non-joint Measurability are synonymous



Steering implies both entanglement and incompatible measurements

M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014)
R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014)

But Bell non-locality and joint measurability not synonymous (except in the N=4 CHSH case).

N term time-like steering inequalities in single qubit system to identify that joint measurability is necessary and sufficient for classicality??

Connection between joint measurability and time-like steering in single system is discussed in

- **H. S. Karthik, J. Prabhu Tej, A. R. Usha Devi, and A. K. Rajagopal, J. Opt. Soc. Am. B. 32, A34 (2015)**
- **M. Pusey, J. Opt. Soc. Am. B. 32, A56 (2015).**

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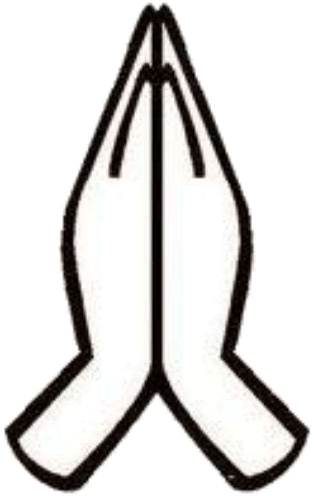


namaste