EFFICIENCY OF QUANTUM CONTROLLED NON-MARKOVIAN THERMALIZATION

Victor Mukherjee
Weizmann Institute of Science, Rehovot, Israel


**MOTIVATION**

- *Open question* – Can we exploit the memory effects of NM dynamics to control a system more effectively?

Our focus: Using *information back-flow* from non-Markovian bath to a generic system to our advantage - controlling the relaxation dynamics of the system

Application → Cooling a quantum system in the minimum possible time
MAIN CHALLENGES

- Naïve inclusion of arbitrary time dependent unitary control might give us unphysical results, eg. negative probabilities.

- Non-Markovian baths are in general finite size baths - Unitary control applied to a system influences the bath non-trivially.

---

OUTLINE

- Markovian and non-Markovian dynamics

- Controlling a quantum system in presence of a Markovian (M) bath

- Controlling a quantum system in presence of a non-Markovian (NM) bath
  
  A necessary condition to be satisfied so that the effectiveness of optimal control is enhanced in presence of a NM bath subject to suitable unitary controls.

- Conclusion
Quantum system in presence of a dissipative bath

System in contact with a dissipative bath at temperature $\beta_f$

$$\rho_i = \rho(t = 0) = \begin{pmatrix} \alpha & \zeta \\ \zeta^* & 1 - \alpha \end{pmatrix}$$

$$\rho_{fp} = \begin{pmatrix} e^{-\beta_f} & 0 \\ 1 + e^{-\beta_f} & 0 \end{pmatrix}$$

Fixed point

Aim: We want to reach $\rho_{fp}$ in the shortest possible time

$\downarrow$ Quantum Speed Limit
\[ \dot{\rho}(t) = \gamma_t L(\rho(t)) \]

\[ \gamma_t > 0 \quad \forall t \quad \rightarrow \quad M \]

\[ \gamma_t < 0 \quad \text{For some time interval} \quad \rightarrow \quad \text{NM} \]

\[ \int_0^t \gamma_\tau d\tau \geq 0 \quad \rightarrow \quad \text{Ensures CP} \]

\[ d(t) = \| \rho(t) - \rho_{fp} \| \]

\[ \frac{\partial d(t)}{\partial t} > 0 \quad \rightarrow \quad \text{Related to measure of non-Markovianity} \]

Optimal protocols to minimize the time to relax to the fixed point

Controlling a single qubit in presence of a Markovian bath
\[ \rho = -i[H_s(t), \rho] + \gamma_0 L(\rho) \]

\[ L(\rho) = \sum_a \left[ L_a \rho L_a^+ - \frac{1}{2} \left( L_a^+ L_a \rho + \rho L_a^+ L_a \right) \right] \]

\[ (L_1)_{AD} = \sigma^+, \ (L_2)_{AD} = e^{\beta/2} \sigma^-, \ \gamma_0 > 0 \]

\[ L(\rho_{fp}) = 0 \rightarrow \text{Fixed point} \]

\[ \rho = \frac{1}{2} \left( I + \vec{\sigma} \cdot \vec{r} \right) \rightarrow \text{Bloch Sphere} \]

\[ P = Tr(\rho^2) = \frac{1}{2} \left( 1 + r^2 \right) \rightarrow \text{purity} \]

\[ \nu_{AD}(r, \theta) = \frac{\partial P(r)}{\partial t} \rightarrow \text{Maximize / minimize this} \]

Error tolerance $\varepsilon$
Time taken to reach the fixed point without optimal control

Worst case scenario without optimal control

Worst case scenario with optimal control

Gain while heating $\sim 70$ !!

Time taken to reach the fixed point
With infinite optimal control

$= 2$

Control in presence of a non-Markovian bath
\[ \dot{\rho} = -i[H_s, \rho] + \gamma_t L(\rho) \]
$\dot{\rho} = -i[H_s, \rho] + \gamma_t L(\rho)$

Let us restrict unitary controls at beginning and end of dissipative dynamics

$\rho(t) = U_{fin} \bullet D \bullet U_{in} \rho(0)$

That is all we need for system in presence of a Markovian Bath
\[ \dot{\rho} = \gamma_t L(\rho) \]

\[ L(\rho(T_F)) = L(\rho_{ fp}) = 0, \quad \dot{\rho}(t \to T_F^+) \neq 0 \]
\[ \dot{\rho} = \gamma_t L(\rho) \]

\[ L(\rho_f) = 0, \quad \rho \neq 0 \implies |\gamma_t(t \to T_F)| \to \infty \]

\[ T_{QSL} = T_F \quad \text{Depends on} \]

Independent of any unitary transformation of \( \rho \)

**Unitary control is ineffective for Class A NM dynamics.**
Class A (divergence): *System reaches fixed point before information backflow* ×
Class B (no divergence): *Information backflow before system reaches fixed point* ✔

**Necessary Condition**: We need class B NM dynamics (black curve) for unitary control to be effective
SPECIFIC EXAMPLE: TWO LEVEL SYSTEM

\[ \dot{\rho} = \gamma_t L(\rho) \]
\[ \gamma_t = \frac{2\lambda \gamma_0 \sinh\left(\frac{td}{2}\right)}{d \cosh\left(\frac{td}{2}\right) + \lambda \sinh\left(\frac{td}{2}\right)} \]
\[ d = \sqrt{\lambda^2 - 2\gamma_0 \lambda} \]

\( \lambda^{-1} \rightarrow \) time scale of bath; \( \gamma_0^{-1} \rightarrow \) time scale of system

\( \lambda / \gamma_0 \rightarrow \infty \rightarrow \) Markovian \( \rightarrow \gamma_t \approx \gamma_0 > 0 \)

\( \lambda / \gamma_0 \rightarrow 0 \rightarrow \) Non-Markovian \( \rightarrow \gamma_t \propto \tan\left(\sqrt{\frac{\lambda \gamma_0}{2}} t\right) \)


Quantum speed up ratio for cooling

\[ R = \frac{T_{\text{free}}}{T_{QSL}} \geq 1 \rightarrow \text{Advantage one gains by optimal control} \]

\[ R_{M,A}^{\text{cool}} \rightarrow 2; \quad R_{NM,A}^{\text{cool}} \rightarrow 1 \]

\[ R_A^M \geq R_A^{NM} \]
CLASS B NON-MARKOVIAN DYNAMICS

\[
\frac{R_M^A}{R_{NM}^A} \geq 1
\]

\[\rho = \gamma_t L(\rho) ; \quad L(\rho_{fp}) = 0\]

\[\gamma_t = \exp(-\xi t)\cos(\Omega t)\]

\(\Omega = 0 \rightarrow M\)

\(\Omega \neq 0 \rightarrow NM\)

Exploit NM effects to cool faster

\[
\rho(t) = U_{fin} \bullet D \bullet U_{in} \rho(0)
\]

EXPLOITING CLASS B NON-MARKOVIAN DYNAMICS

\[ \rho(t) = U_{\text{fin}} \bullet D_2 \bullet \widetilde{U} \bullet D_1 \bullet U_{\text{in}} \rho(0) \]

Follow the maxima of  \( v_{AD}(r, \theta) = \partial P(t) / \partial t \)
Main Result

We can determine if there is a possibility of efficiently controlling the relaxation of a generic system exhibiting non-Markovian dynamics just by looking at the above figure, without solving any complicated master equation !!

Independent of the details of the system, bath: dimensionality, Hamiltonian, explicit form of the dissipative bath, etc.

SUMMARY

- Class A NM dynamics associated with divergence → precludes unitary quantum control

- **Necessary Condition**: We need class B NM dynamics for unitary control to be effective

- Control protocols can be extended to $N$ level system by studying Casimir invariants $\sim Tr(\rho^j)$, $j = 2, 3, ..., N$

- Similar results obtained for $\dot{\rho}(t) = \sum_k \gamma_k(t) L_k(\rho(t))$

- What are the sufficient conditions?
Time Scales

\[ \rho = -i[H, \rho] + \gamma_t L(\rho) \]

\[ \gamma_t = \frac{2\lambda \gamma_0 \sinh \left( \frac{td}{2} \right)}{d \cosh \left( \frac{td}{2} \right) + \lambda \sinh \left( \frac{td}{2} \right)}; \quad d = \sqrt{\lambda^2 - 2\gamma_0 \lambda} \]

\[ \lambda / \gamma_0 \rightarrow \infty \quad \Rightarrow \quad \text{Markovian} \quad \Rightarrow \quad \gamma_t \approx \gamma_0 \]

\[ \lambda / \gamma_0 \rightarrow 0 \quad \Rightarrow \quad \text{Non-Markovian} \quad \Rightarrow \quad d \approx i\sqrt{2\gamma_0 \lambda} \]

\[ T_M \sim \frac{1}{\gamma_0} \quad \text{Decreases both for increasing Markovianity } (\lambda) \]

\[ T_{NM} \sim \frac{1}{\sqrt{\lambda \gamma_0}} \quad \text{as well as increasing non-Markovianity } (\gamma_0) \]

For a generic system in presence of a generic \( L \)
\[
\lim_{\varepsilon \to 0} R_{M}^{\text{cool}} = \frac{T_{\text{free},M}}{T_{\text{cool},M}} \rightarrow 2
\]

\[
R_{NM}^{\text{cool}} = \frac{T_{\text{free},NM}}{T_{\text{cool},NM}} \rightarrow 1
\]

\[
\lim_{\varepsilon \to 0} R_{M}^{\text{heat}} = \frac{T_{\text{free},M}}{T_{\text{heat},M}} \approx 2 \frac{\ln \varepsilon}{\ln \left( \frac{r_{fp} + r_i}{2r_{fp}} \right)} \rightarrow \infty
\]

\[
R_{NM}^{\text{heat}} \approx \frac{\pi / 2}{\cos^{-1} \left( \left( \frac{2r}{r_i + r_{fp}} \right)^{1/2(e^{\beta} + 1)} \right)} \rightarrow \text{finite}
\]

\[
R_{M} \geq R_{NM}
\]

**Optimal control works better in the Markovian limit**
**DAMPED JAYNES–CUMMINGS MODEL**

\[
H = H_S + H_B + H_I
\]

\[
H_S = \omega_0 \sigma^+ \sigma^-
\]

\[
H_B = \sum_k \omega_k b_k^+ b_k
\]

\[
H_I = \sigma^+ \otimes B + \sigma^- \otimes B^+
\]

\[
B = \sum_k g_k b_k
\]

\[
\tilde{g}(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \omega)^2 + \lambda^2}
\]

Markovian / non-Markovian dynamics determined by form of \( g_k \)

---

TIME LOCAL NON-MARKOVIAN MASTER EQUATION

\[ \dot{\rho}(t) = L(t - t_0)\rho(t) \]
\[ \rho(t) = \Lambda(t, t_0)\rho(t_0) \]
\[ \Lambda(t, t_0) = \exp \left[ \int_{t_0}^{t-t_0} L(\tau) d\tau \right] \]
\[ L(\tau) = \frac{d}{d\tau} \Lambda(\tau)\Lambda^{-1}(\tau) \]

Figure 2. (a) Parametric plot showing variation of time $T_{QSL}^{\text{cool}}$ of reaching the fixed point with $\lambda$ and $\gamma_0$ for $\beta = 2, r_i = 0.5$ and $\epsilon = 0.01$. The Markovian (M) and non-Markovian (NM) regions are separated by the blue line on the $\lambda - \gamma_0$ plane. (b) Plot showing variation of quantum speed up ratio $R^A$ with $\lambda$ and $\gamma_0$ for $\beta = 2, \tau_{ii} = 0.3, r_{ji} = 0, \tau_{ji} = 0.4$ and $\epsilon = 0.01$. $R^A$ saturates to $R^A_M \approx 2$ ($R^A_{NM} \approx 1$) in the extreme M (NM) limit.
FREEZING OF A QUBIT – STOPPING DECOHERENCE FOR INFINITE TIME

\[ \dot{\rho} = -i[H_s, \rho] + \gamma_0 L(\rho) \rightarrow \dot{r} \equiv \partial P / \partial t = 0 \]
\[ T_{\text{con}}^{\text{cool}}(\vec{r}_i, \varepsilon) = \frac{1}{\gamma_0(e^\beta + 1)} \ln \left[ \frac{(r_{fp} - r_i)}{\varepsilon} \right] \]

\[ T_{\text{con}}^{\text{heat}}(\vec{r}_i, \varepsilon) = \frac{1}{\gamma_0(e^\beta + 1)} \ln \left[ \frac{r_{fp} + r_i}{2r_{fp} + \varepsilon} \right] \]

\[ \rho = \frac{1}{2} (I + \vec{r} \cdot \vec{\sigma}) = \frac{1}{2} \begin{pmatrix}
1 + r_z & r_x - ir_y \\
1 + ir_y & r_z
\end{pmatrix} \begin{pmatrix}
1 + r_z & r_x - ir_y \\
r_x + ir_y & 1 - r_z
\end{pmatrix} \]