Quantum violation of various formulations of macrorealism and Leggett-Garg inequalities

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Outline

- Introduction
- Macrorealism and Leggett-Garg Inequalities(LGIs)
- LGIs for unsharp measurements
- Equivalent CHSH inequalities and Inequivalent LGIs
- On improved violation of LGIs using von Neumann rule
- Quantum violation of variants of LGIs upto algebraic maximum
- Summary

Relevant publications related to this talk

- S. Kumari and AKP, Phys. Rev. A 96, 042107 (2017)
- S. Kumari and AKP, EPL, 118, 50002 (2017).
- A. Kumari, Md. Qutubuddin and AKP, Phys. Rev. A 98, 042135 (2018).
- AKP, Md. Qutubuddin and S. Kumari, Phys. Rev. A, 98, 0621XX(2018).

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- We are certainly not concerned in which limit the mathematical structure of QM reduces to CM.

- How does the everyday world view about the nature of reality emerge from QM?

Introduction

- Schoedinger's question: When does a macroscopic system (an unlucky cat) stop existing as a superposition of states and become one (dead) or the other (alive)?

- Bohr never took the observer-induced collapse of the wave function seriously. So, the cat did not pose any riddle.

- Heisenberg proposed a bizarre 'cut' but remained silent about how such a 'cut' can be obtained within the very formalism of QM.

Introduction

- How fat is the cat?(Macroscopic quantum coherence)

  C_{60} molecule, 720 amu (Arndt et al., Nature, 2000)

  $C_{60}F_{48}$, 1632 amu (Hackermueller, et al.,PRL, 2003)

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Approach within QM:

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Realist approach:

Macrorealist model by Legget and Garg (1981): Analogues to Bell's approach

Macrorealism and Leggett-Garg Inequalities(LGIs)

Macrorealism and Legget-Garg inequalities (LGIs)

The notion of macrorealism consists of two main assumptions.

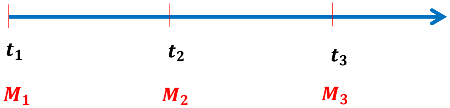
- *Macrorealism per se (MRps):* Macroscopic system which has available to it two or more macroscopic distinguishable ontic states can be found in one of those states at any instant of time.

- *Non-invasive measurability (NIM):* The ontic state of a macroscopic system can always be determined without affecting the state itself or its subsequent dynamics.

A. J. Leggett and A. Garg, Phys. Rev. Lett. 54, 857(1985).
A. J. Leggett, J. Phys. Condens. 14, R415 (2002).

Standard Leggett-Garg inequalities (SLGIs)

3-time LG scenario:



- Let physical observable $\hat{M}$ is measured at t_1, t_2 and $t_3 (t_3 > t_2 > t_1)$.

  MR: $$P(m_1, m_2) = \int P(m_1|\lambda) P(m_2|\lambda) \rho(\lambda) d\lambda$$

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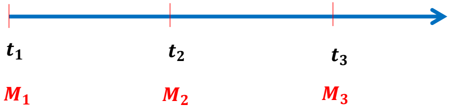

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  Using MR assumption, the following inequality can be derived,

  $$SLGI = \langle M_1 M_2 \rangle + \langle M_2 M_3 \rangle - \langle M_1 M_3 \rangle \leq 1$$

  Three more SLGIs can be proposed.

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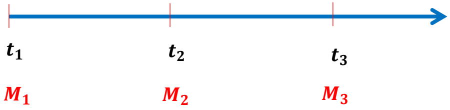

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- In QM: $$P_Q(m_1, m_2) = Tr[M_1^{m_1} \rho M_1^{m_1} M_2^{m_2}]$$

A. J. Leggett and A. Garg, Phys. Rev. Lett. 54, 857(1985).
A. J. Leggett, J. Phys. Condens. 14, R415 (2002).

Quantum violation standard LGIs

- Let the system is prepared in a state $\rho(t_1) = |\psi_{t_1}\rangle\langle\psi_{t_1}|$ at t_1, where

$$|\psi_{t_1}\rangle = \cos\theta|0\rangle + exp(i\phi)\sin\theta|1\rangle$$

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$$|\psi_{t_1}\rangle = \cos\theta|0\rangle + exp(i\phi)\sin\theta|1\rangle$$

- At t_1, we take $M_1 = \hat{\sigma_z}$ and Hamiltonian $\mathcal{H} = \omega\sigma_x$.

- $M_2 = \mathcal{U}_{12} M_1 \mathcal{U}_{12}^{\dagger}$ and $M_3 = \mathcal{U}_{13} M_1 \mathcal{U}_{13}^{\dagger} = \mathcal{U}_{23} M_2 \mathcal{U}_{23}^{\dagger}$.

- $\mathcal{U}_{12} = e^{i\omega\sigma_x(t_2-t_1)}$ and $\mathcal{U}_{23} = e^{i\omega\sigma_x(t_3-t_2)}$. If one takes $\omega(t_2 - t_1) = \omega(t_3 - t_2) = g$,

$$SLG_Q = 2Cos(g) - Cos(2g)$$

$$SLGI_Q^{max} = 1.5 > 1 \text{ at } g = \pi/6.$$

Experimental tests of standard LGIs

- Electron spin

  Knee et al., Nature Comm.3, 606 (2012). (Negative result measurement)

- NV centre

  Waldherr et al., PRL 107, 090401 (2011) (assuming the stationarity of correlations)

  George et al., PNAS, 110, 3777(2013) (classically undetectable wavefunction collapse)

- NMR

  V. Athalye, S.S. Roy and T. S. Mahesh, PRL 107, 130402 (2011) .

- Photons

  Goggin et al., PNAS, 108, 1256(2011).(Weak measurement)

  Avella et al., Phys. Rev. A 96, 052123 (2017). (weak measurement)

- Cesium atom

  Robels et al., Phys. Rev. X, 5, 011003(2015). (Negative result measurement in quantum walks)

Quantum violation of LGIs for unsharp measurement

S. Kumari, **AKP**, Phys. Rev. A 96, 042107 (2017)

Quantum violation of LGIs for unsharp measurement

- We consider POVMs: $M^{\pm}(x) = \frac{\mathbb{I} \pm (x + \eta \vec{m}.\sigma)}{2}$; where x is biasedness parameter and η is sharpness parameter, $|x| + \eta \leq 1$ and $0 < \eta \leq 1$.
- At time t_1, $\vec{m} = \hat{z}$
- $\mathcal{U}_{12} = e^{i\omega\sigma_x(t_2 - t_1)}$ and $\mathcal{U}_{23} = e^{i\omega\sigma_x(t_3 - t_2)}$.

  We take $\omega(t_2 - t_1) = \omega(t_3 - t_2) = g$,
- POVMs at t_2, t_3: $M_2^{\pm}(x) = U_{12}^{\dagger} M_1^{\pm}(x) U_{12}^{\dagger}$, $M_3^{\pm}(x) = U_{23}^{\dagger} M_2^{\pm}(x) U_{23}^{\dagger}$.
- Pair-wise joint probability of different outcomes:

$$P(m_1, m_2) = Tr[U_{12} \sqrt{M_1^{m_1}} \rho(t_1) \sqrt{M_1^{m_1}} U_{12}^{\dagger} M_2^{m_2}]$$

S. Kumari, **AKP**, Phys. Rev. A 96, 042107 (2017)

Quantum violation of LGIs for unsharp measurements

$$|\psi_{t_1}\rangle = \cos\theta|0\rangle + exp(i\phi)\sin\theta|1\rangle$$

For spin-POVMs($x = 0$): $\mathrm{M}_1^{\pm} = \frac{I \pm \eta\sigma_z}{2}$

- Violation of SLGI: $\eta > 0.81$ (independent of the state).

For biased-POVMs($x = \eta - 1$): $\mathrm{M}_1^{+} = \eta\left(\frac{I + \sigma_z}{2}\right)$

- Quantum value of SLGI: $1 + \frac{\eta^2}{2}$ (for $\theta = \pi/3$, $\phi = \pi/2$, $g = 5\pi/6$)

Quantum violation of LGIs occurs for any non-zero value of unsharpness parameter.

Joint measurability and violation of LGI

Pairwise joint measurability condition for two different POVMs, $M^{\pm}(x, \vec{m})$ and $M^{\pm}(y, \vec{n})$.

$$(1 - F_x^2 - F_y^2)(1 - \frac{x^2}{F_x^2} - \frac{y^2}{F_y^2}) \leq (\vec{m}.\vec{n} - xy)^2$$

where $F_{x/y}$ are given by

$$F_{x/y} = \frac{\sqrt{(1 - x/y)^2 - m^2} + \sqrt{(1 + x/y)^2 - m^2}}{2};$$

For $x = 0$ and $y = 0$ we can obtain well-known joint measurability condition for the spin-POVMs is given by

$$||\vec{m} + \vec{n}|| + ||\vec{m} - \vec{n}|| \leq 2$$

P. Busch, Phys. Rev. D 33, 2253 (1986).
S. Yu, Nai-le Liu, Li-Li and C. H. Oh, Phys. Rev.A, 81, 062116 (2010).

Joint measurebility and violation of LGI

For Spin-POVM $(x = 0, y = 0)$**:**

The pairwise joint measurability condition for $M_1^{\pm}$ and $M_2^{\pm}$ (and $M_2^{\pm}$ and $M_3^{\pm}$) is

$$\eta \leq (\cos(g) + \sin(g))^{-1}$$

And for $M_1^{\pm}$ and $M_3^{\pm}$

$$\eta \leq (\cos 2(g) + \sin 2(g))^{-1}$$

- The pair-wise joint measurability condition is $\eta \leq 0.707$.
- But Wigner form of LGIs is violation for $\eta > 0.69$.

S. Kumari, **AKP**, Phys. Rev. A 96, 042107 (2017)

Joint Measurability and violation of LGI

For biased POVM:

The pairwise joint-measurability condition for $M_1^{\pm}$ and $M_2^{\pm}$ (and for $M_2^{\pm}$ and $M_3^{\pm}$) is

$$\eta \leq (1 + \cos(g))^{-1}$$

and for $M_2^{\pm}$ and $M_3^{\pm}$

$$\eta \leq (1 + \cos 2(g))^{-1}$$

- The pair-wise joint measurability condition is $\eta \leq 0.66$.
- But, standard LGI is violated for $0 < \eta \leq 1$.

Thus, pair-wise joint measurability has no role in LGI violation.

S. Kumari, **AKP**, Phys. Rev. A 96, 042107 (2017)

Inequivalent LGIs

Swati Kumari and AKP, EPL, 118, 50002 (2017).

Equivalent Bell-CHSH inequalities

- Aurther Fine showed that for a two-party, two measurements per party having two outcomes of each measurement, the only relevant Bell's inequality is the CHSH form.
- Any other form, such as, Wiger and CH forms of inequalities reduce to the CHSH inequality.

Equivalent Bell-CHSH inequalities

- Aurther Fine showed that for a two-party, two measurements per party having two outcomes of each measurement, the only relevant Bell's inequality is the CHSH form.
- Any other form, such as, Wiger and CH forms of inequalities reduce to the CHSH inequality.
- SLGIs are often considered to be the analogus to the CHSH inequalities.
- We showed that Wigner and CH form of LGIs are stronger than standard LGIs.

A. Fine, Phys. Rev. Lett., 48, 291,(1982).

Wigner form of LGIs

- The satisfaction of MR implies the existence of joint probabilities $P(m_1, m_2, m_3)$. The marginals can then be written as

$$P(m_2, m_3) = \sum_{m_1} P(m_1, m_2, m_3)$$

where $m_1, m_2, m_3 = \pm 1$.

Using similar pair-wise joint probabilities, 24 Wigner form of LGIs can be derived are the following;

$$P(m_2, m_3) - P(-m_1, m_2) - P(m_1, m_3) \leq 0$$

$$P(m_1, m_3) - P(m_1, -m_2) - P(m_2, m_3) \leq 0$$

$$P(m_1, m_2) - P(m_2, -m_3) - P(m_1, m_3) \leq 0$$

D. Saha, et al. Phys. Rev. A, 91, 032117 (2015).

Wigner LGI Vs standard LGI

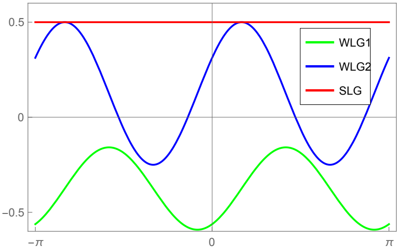
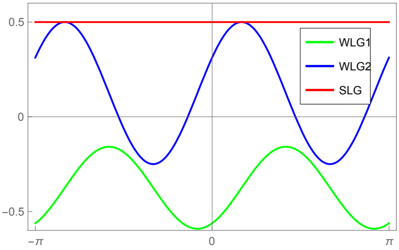


Figure : We plot $SLG - 1$ and two WLGIs against θ by taking $g = \pi/6$.

Wigner LGI Vs standard LGI



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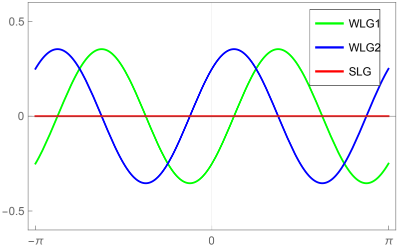

Figure : We plot $SLG - 1$ and two WLGIs against θ by taking $g = \pi/4$.

Clauser-Horne (CH) form of LGIs

In a macrorealistic theory, single marginal statistics for the measurement of an observable, say for M_2, is $P(m_2) = \sum_{m_1 m_2 = \pm} P(m_1, m_2, m_3)$.

By combining single and pair-wise probabilities, we can derive 24 inequalities are the following;

$$P(m_1, m_2) + P(m_2, m_3) - P(m_1, m_3) - P(m_2) \leq 0$$

$$P(m_1, m_3) + P(m_2, m_3) - P(m_1, m_2) - P(m_3) \leq 0$$

$$P(m_1, m_3) + P(m_1, m_2) - P(m_2, m_3) - P(m_1) \leq 0$$

We call them CH form of LGIs.

S. Kumari and **AKP**, EPL, 50002, 118 (2017).

Joint probabilities in QM

Three-time probability in terms of correlation functions:

$$P_{123}(m_1, m_2, m_3) = (1/8)(1 + m_1\langle M_1\rangle + m_2\langle M_2^{(1)}\rangle + m_3\langle M_3^{(12)}\rangle$$
$$+m_1, m_2\langle M_1 M_2\rangle + m_2, m_3\langle M_2 M_3^{(1)}\rangle + m_1, m_3\langle M_1 M_3^{(2)}\rangle + m_1, m_2 m_3 D)$$

The pair-wise probabilities are given by

$$\begin{aligned} P_{13}(m_1, m_3) &= \frac{(1 + m_1\langle M_1\rangle + m_3\langle M_3^{(1)}\rangle + m_1 m_3\langle M_1 M_3\rangle)}{4} \\ P_{23}(m_2, m_3) &= \frac{(1 + m_2\langle M_2\rangle + m_3\langle M_3^{(2)}\rangle + m_2 m_3\langle M_2 M_3\rangle)}{4} \\ P_{12}(m_1, m_2) &= \frac{(1 + m_1\langle M_1\rangle + m_2\langle M_2^{(1)}\rangle + m_1 m_2\langle M_1 M_2\rangle)}{4} \\ P(m_i) &= \frac{(1 + m_i\langle M_i\rangle)}{2} \qquad i = 1, 2, 3 \end{aligned}$$

S. Kumari and **AKP**, EPL, 50002, 118 (2017).

WLGIs and CHLGIs are stronger than SLGIs

Using pair-wise and single probabilities, 24 Wigner LGIs can be written as

$$|\langle M_2 \rangle - \langle M_2^{(1)} \rangle| + |\langle M_3^{(2)} \rangle - \langle M_3^{(1)} \rangle| + SLG_Q \leq 1$$

where $SLG_Q = m_1 m_2 \langle M_1 M_2 \rangle + m_2 m_3 \langle M_2 M_3 \rangle - m_1 m_3 \langle M_1 M_3 \rangle$.

Wigner LGIs are stronger than standard LGIs.

WLGIs and CHLGIs are stronger than SLGIs

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Wigner LGIs are stronger than standard LGIs.

Similarly, corresponding to 24 CH form of LGIs, we get

$$|\langle M_2\rangle - \langle M_2^{(1)}\rangle| + |\langle M_3\rangle - \langle M_3^{(1)}\rangle| + |\langle M_3\rangle - \langle M_3^{(2)}\rangle| + SLG_Q \leq 1$$

CH form of LGIs are stronger than Wigner form of LGIs.

S. Kumari and **AKP**, EPL, 50002, 118 (2017).

On the violation of L$\ddot{u}$der bound of LGIs.

A. Kumari, Md. Qutubuddin, and **AKP**, Phys. Rev. A 98, 042135 (2018).

Lüder bound of LGI and its violation

- Q violation of CHSH inequality is constraint by Cir'elsen bound.
- By analogy, the violation of LGIs is restricted by Lüders bound.
- It is shown that violation of LGIs can exceed the Lüders bound, if degeneracy breaking von Neumann measurement rule is invoked.

C. Budroni and C. Emary, Phys. Rev. Lett. 113, 050401 (2014).

- Experimental test:

H. Katiyar, A. Brodutch, D. Lu and R. LaflammeH. New J. Phys. 19, 023033 (2017).

K. Wang et al., Opt. Express 25, 31462 (2017)

- We examine the relevance of violation of Lüders bound in LG test.

A. Kumari, Md. Qutubuddin, and **AKP**, Phys. Rev. A 98, 042135 (2018).

L$\ddot{u}$der and von Neumann projection rule

Let an observable $\hat{A}$ has discrete eigenvalues $a_1, a_2, a_3...$ having degeneracies $x_1, x_2, x_3...$ respectively.

Consider $P_n^i = |\phi_n^i\rangle\langle\phi_n^i|$ is the projection operator corresponding to n^{th} eigenvalue of $\hat{A}$ and ρ_0 is density matrix of the system.

Reduced density matrix using L$\ddot{u}$ders rule:

$$\rho_l = \sum_n P_n \rho_0 P_n \text{ where } P_n = \sum_{i=1}^{x_n} |\phi_n^i\rangle\langle\phi_n^i|.$$

Reduced density matrix using von Neumann rule:

$$\rho_v = \sum_{n,i} P_n^i \rho_0 P_n^i \text{ where } P_n^i = |\phi_n^i\rangle\langle\phi_n^i|.$$

G. C. Hegerfeldt and R. Sala Mayato, Phy Lett. A, 375 (36), 3167-3170, (2011).
A. K. Pan, K. Mandal, Int J Theor Phys, 55, 3472-3478 (2016).
A. Kumari, Md. Qutubuddin, and **AKP**, Phys. Rev. A 98, 042135 (2018).

Lüder and von Neumann projection rule: An example

Let $A = |3\rangle\langle 3| + |2\rangle\langle 2| - |1\rangle\langle 1| \equiv P_+ - P_-$

Then, $P_+ = |3\rangle\langle 3| + |2\rangle\langle 2| \equiv P_+^1 + P_+^2$ and $P_- = |1\rangle\langle 1|$.

State reduction using Lüders rule:

$$\rho_+ = P_+\rho P_+ = \big(|3\rangle\langle 3| + |2\rangle\langle 2|\big)\rho\big(|3\rangle\langle 3| + |2\rangle\langle 2|\big)$$

State reduction using von Neumann rule:

$$\rho_+ = P_+^1\rho P_+^1 + P_+^2\rho P_+^2 = |3\rangle\langle 3|\rho|3\rangle\langle 3| + |2\rangle\langle 2\rho|2\rangle\langle 2|$$

Since $P_+ = |3\rangle\langle 3| + |2\rangle\langle 2| \equiv |3'\rangle\langle 3'| + |2'\rangle\langle 2'|$

where $|2'\rangle = \xi|2\rangle + \sqrt{1-\xi^2}|3\rangle$ and $|3'\rangle = \sqrt{1-\xi^2}|2\rangle - \xi|3\rangle)$

A. Kumari, Md. Qutubuddin, and **AKP**, Phys. Rev. A 98, 042135 (2018).

Sequential correlation for L$\ddot{u}$der and von Neumann rule

Let for a qutrit system two dichotomic observables

$$\hat{A} = A_+^1 + A_+^2 - A_- \text{ and } \hat{B} = B_+^1 + B_+^2 - B_-.$$

Then, sequential correlation between $\hat{A}$ and $\hat{B}$, using L$\ddot{u}$ders rule is

$$\langle \hat{A}\hat{B} \rangle_{seq}^{l} = \frac{1}{2}(Tr[\rho\{A, B\}])$$

But, using von Neumann rule, one gets

$$\langle \hat{A}\hat{B} \rangle_{seq}^{v} = \langle \hat{A}\hat{B} \rangle_{seq}^{l} - Tr[(A_+^1 \rho A_+^2 + A_+^2 \rho A_+^1)\hat{B}]$$

The quantity $Tr[(A_+^1 \rho A_+^2 + A_+^2 \rho A_+^1)\hat{B}]$ may depend on the choice of basis and responsible for the violation of L$\ddot{u}$ders bound.

A. Kumari, Md. Qutubuddin, and **AKP**, Phys. Rev. A 98, 042135 (2018).

Violation of Lüders bound of LGIs

Lüders bound of standard LGIs:

$$SLG = \langle M_1 M_2 \rangle + \langle M_2 M_3 \rangle - \langle M_1 M_3 \rangle \leq 1$$

For any arbitrary dimensional system having dichotomic outcomes $SLG_Q^{opt} = 1.5$.

This is the Lüders bound of 3-time LGIs.

- Using von Neumann rule, Budroni and Emary showed that $SLG_Q = 1.75$ for a qutrit system.
- SLG_Q can even approach algebraic maximum 3 in the asymptotic limit of the dimension of system.

C. Budroni and C. Emary, Phys. Rev. Lett. 113, 050401 (2014).

Violation of Lüders bound of realist inequalities

Let M_1, M_2 and M_3 be mutually commuting dichotomic observables.

$$\beta_{13} = \langle M_1 M_2 \rangle + \langle M_2 M_3 \rangle - \langle M_1 M_3 \rangle \leq 1$$

$$\beta_{23} = \langle M_1 M_2 \rangle - \langle M_2 M_3 \rangle + \langle M_1 M_3 \rangle \leq 1$$

$$\beta_{12} = -\langle M_1 M_2 \rangle + \langle M_2 M_3 \rangle + \langle M_1 M_3 \rangle \leq 1$$

β_{31}, β_{23} and β_{12} are not violated by QM if Lüders rule is used.

$P_Q(M_1, M_2, M_3)$ exists whose marginal provides all pair-wise joint probabilities satisfied by QM.

Using von Neumann rule we showed that $(\beta_{13})_Q, (\beta_{23})_Q, (\beta_{12})_Q > 1$.

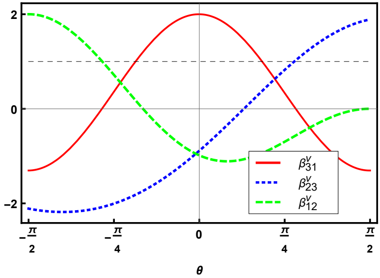
A. Kumari, Md. Qutubuddin, and **AKP**, Phys. Rev. A 98, 042135 (2018).

A specific example for qutrit system

Let the initial state $|\psi\rangle = (\sin(\theta), \cos(\theta), 0)^T$.

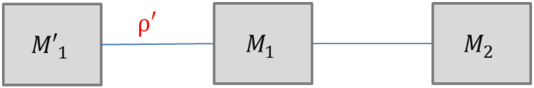
$\hat{M}_i = I - 2|\alpha_i\rangle\langle\alpha_i|$ with $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$ where $i, j = 1, 2, 3$.

$|\alpha_1\rangle = (-1, 0, 1)^T/\sqrt{2}$, $|\alpha_2\rangle = (1, 0, 1)^T/\sqrt{2}$ and $|\alpha_3\rangle = (0, 1, 0)^T$.



A. Kumari, Md. Qutubuddin, and **AKP**, Phys. Rev. A 98, 042135 (2018).

What the violation of L$\ddot{u}$ders bound means?



$[M'_1, M_1] = 0$ $\qquad$ $[M'_1, M_2] \neq 0$

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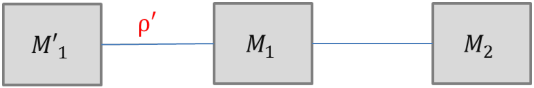

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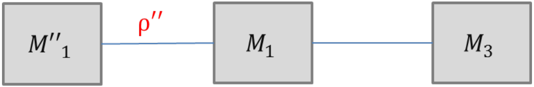
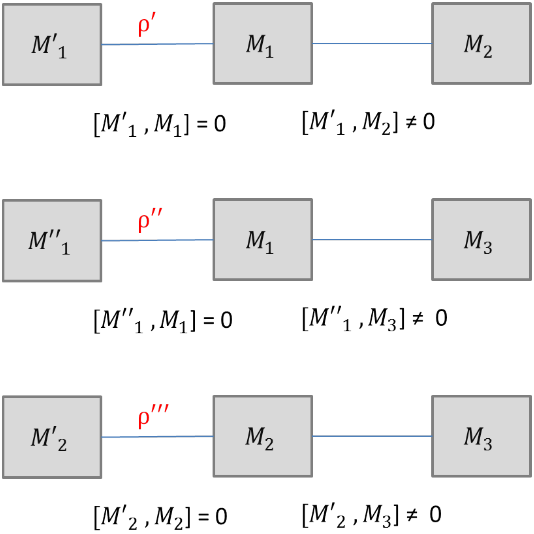


$[M''_1, M_1] = 0 \qquad [M''_1, M_3] \neq 0$

What the violation of Lüders bound means?



A. Kumari, Md. Qutubuddin, and **AKP**, Phys. Rev. A 98, 042135 (2018).

What the violation of Lüders bound means?

Standard LG expression:

$$SLG = \langle M_1 M_2 \rangle + \langle M_2 M_3 \rangle - \langle M_1 M_3 \rangle$$

$$\langle M_i M_j \rangle = \sum_{m_i, m_j} m_i m_j P(m_i, m_j); \quad i, j = 1, 2, 3 \;\; with \;\; j > i$$

What the violation of L$\ddot{u}$ders bound means?

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$P(m_1, m_2) = \int P(m_1|\lambda')P(m_2|\lambda')\rho(\lambda')d\lambda'$

$P(m_1, m_3) = \int P(m_1|\lambda'')P(m_3|\lambda'')\rho(\lambda'')d\lambda''$

$P(m_2, m_3) = \int P(m_2|\lambda''')P(m_3|\lambda''')\rho(\lambda''')d\lambda'''$

$$SLG \leq 3$$

Quantum violation of variants of LGIs

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219

Variants of LGIs

- Due to the sequential nature of correlation involved in LG test, there is a flexibility to propose new inequalities, even in 3-time LG test.

- By apparently keeping the assumption of MRps and NIM same, we propose interesting variants of standard LGIs for 3-time measurement scenario.

- Quantum violation of variants of LGIs is larger than the standard case, even for qubit system.

- Variants of LGIs is more robust to unsharpness compared to standard LGIs in unsharp measurement scenario.

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219.

Variants of LGIs for three-time measurements

- Considering a three-time correlation function $\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle$, a two-time function $\langle \hat{M}_i \hat{M}_j \rangle$ and $\langle \hat{M}_k \rangle$, we propose the inequality

$$K_3^3 = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle + \langle \hat{M}_i \hat{M}_j \rangle - \langle \hat{M}_k \rangle \leq 1$$

  where $i, j, k = 1, 2, 3$ with $j > i$.

  We call those inequalities as variants of LGIs.

- Let the system state to be $\rho_0(t_1) = |\psi_0\rangle\langle\psi_0|$ at t_1 is

$$|\psi_0\rangle = \cos\theta|0\rangle + exp(i\phi)\sin\theta|1\rangle$$

$M_1 = \sigma_z$ and M_2, M_3 are as taken earlier.

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219.

Quantum violation of variants of LGIs

Standard LGI: $K_3 = \langle \hat{M}_1 \hat{M}_2 \rangle + \langle \hat{M}_2 \hat{M}_3 \rangle - \langle \hat{M}_1 \hat{M}_3 \rangle \leq 1.$

- The QM expression of K_3 is given by

$$(K_3)_Q = 2\cos 2g - \cos 4g$$

$$(K_3)_Q^{max} = 1.5 \text{ for } g = \pi/6.$$

- The QM expression of K_3^3 is given by

$$\begin{aligned}(K_3^3)_Q &= \cos 2g(1 + 4\sin^2 g \cos 2\theta) + 2\sin^2 g \cos 2\theta \\ &- \sin 4g \sin 2\theta \sin \phi\end{aligned}$$

$$(K_3^3)_Q^{max} = 2, \text{ for } g = 0.41,\ \theta = 2.66 \text{ and } \phi = \pi/2.$$

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219.

Leggett-Garg Inequalities for n-time measurements

- If the measurement of M is performed n times,

$$SGL_n = \langle \hat{M}_1 \hat{M}_2 \rangle + ... + \langle \hat{M}_{n-1} \hat{M}_n \rangle - \langle \hat{M}_1 \hat{M}_n \rangle$$

If n is odd, $-n \leq K_n \leq n-2$ for $n \geq 3$ and

if n is even, $-(n-2) \leq K_n \leq n-2$ for $n \geq 4$.

- For a two-level system, the maximum quantum value $(SGL_n)_Q^{max} = n \cos \frac{\pi}{n}$. For $n = 3$, $(SGL_3)_Q^{max} = 1.5$.

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219.

Variants of LGIs for 4-time measurements

- If $n = 4$, we can formulate a variant of LGI is given by

$$K_4^3 = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \hat{M}_4 \rangle + \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle - \langle \hat{M}_4 \rangle \leq 1$$

- Interestingly, for $n = 4$, another variant can be proposed as

$$\hat{L}_4^3 = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle + \langle \hat{M}_2 \hat{M}_3 \hat{M}_4 \rangle - \langle \hat{M}_1 \hat{M}_4 \rangle \leq 1$$

By generalizing for n-time measurement scenario, we propose

$$K_n^3 = \langle \hat{M}_1 \hat{M}_2 ... \hat{M}_n \rangle + \langle \hat{M}_1 \hat{M}_2 ... \hat{M}_{n-1} \rangle - \langle \hat{M}_n \rangle \leq 1$$

$$\hat{L}_n^3 = \langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \hat{M}_4 ... \hat{M}_{n-1} \rangle + \langle \hat{M}_2 \hat{M}_3 ... \hat{M}_n \rangle - \langle \hat{M}_1 \hat{M}_n \rangle \leq 1$$

where $\langle \hat{M}_1 ... \hat{M}_n \rangle = \sum_{m_1,...,m_n} m_1 ... m_n P(M_1^{m_1}, ..., M_n^{m_n})$.

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219.

Correlation formula for n-time sequential measurements

For $n = 2$: $\langle \hat{M}_1 \hat{M}_2 \rangle_{seq} = \frac{1}{2} Tr \left[\rho \left\{ \hat{M}_1, \hat{M}_2 \right\} \right]$

The correlation function for 3-time measurement:

$$
\begin{aligned}
\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle_{seq} &= \sum_{m_1, m_2, m_3 = \pm 1} m_1 m_2 m_3 P(m_1, m_2, m_3) \\
&= \sum_{m_1, m_2, m_3 = \pm 1} m_1 m_2 m_3 \, Tr[\Pi_{M_2}^{m_2} \Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1} \Pi_{M_2}^{m_2} \Pi_{M_3}^{m_3}] \\
&= \sum_{m_1, m_2 = \pm 1} m_1 m_2 \, Tr[\Pi_{M_2}^{m_2} \Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1} \Pi_{M_2}^{m_2} \Pi_{M_3}^{+}] \\
&- \sum_{m_1, m_2 = \pm 1} m_1 m_2 \, Tr[\Pi_{M_2}^{m_2} \Pi_{M_1}^{m_1} \rho \Pi_{M_1}^{m_1} \Pi_{M_2}^{m_2} \Pi_{M_3}^{-}]
\end{aligned}
$$

Correlation formula for n-time sequential measurenets

Using $\hat{M}_3 = \Pi^{+}_{M_3} - \Pi^{-}_{M_3}$ and putting the value of $m_2 = \pm 1$, we have

$$\begin{aligned}
\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle_{seq} &= \sum_{m_1=\pm 1} m_1 \, Tr[(\Pi^{+}_{M_2} \Pi^{m_1}_{M_1} \rho \Pi^{m_1}_{M_1} \Pi^{+}_{M_2}).\hat{M}_3] \\
&- \sum_{m_1=\pm 1} m_1 \, Tr[(\Pi^{-}_{M_2} \Pi^{m_1}_{M_1} \rho \Pi^{m_1}_{M_1} \Pi^{-}_{M_2}).\hat{M}_3] \\
&= \frac{1}{2} \sum_{m_1=\pm 1} m_1 \, Tr\left[(\Pi^{m_1}_{M_1} \rho \Pi^{m_1}_{M_1}).\left\{\hat{M}_2, \hat{M}_3\right\}\right]
\end{aligned}$$

Finally, $\langle \hat{M}_1 \hat{M}_2 \hat{M}_3 \rangle_{seq} = \frac{1}{4} Tr\left[\rho \left\{\hat{M}_1, \left\{\hat{M}_2, \hat{M}_3\right\}\right\}\right]$

For the case of n-time measurements, we derive

$$\langle \hat{M}_1 \hat{M}_2 .... \hat{M}_n \rangle_{seq} = \frac{1}{2^{n-1}} Tr\left[\rho \left\{\hat{M}_1, \left\{\hat{M}_2, ..., \left\{\hat{M}_{n-2}, \left\{\hat{M}_{n-1}, \hat{M}_n\right\}\right\}\right\}\right\}\right]$$

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219.

For the qubit state $|\psi_0\rangle = \cos\theta|0\rangle + exp(i\phi)\sin\theta|1\rangle$,

$$\begin{aligned}(K^3_{n_{even}})_Q &= (\cos 2g)^{\frac{n}{2}} + (\cos 2g)^{\frac{n}{2}-1} - \big(\cos 2(n-1)g \\ &\quad \cos 2\theta + \sin 2(n-1)g \sin 2\theta \sin\phi\big)\end{aligned}$$

For odd n,

$$\begin{aligned}(K^3_{n_{odd}})_Q &= (\cos 2g)^{\frac{n-1}{2}} \cos 2\theta + (\cos 2g)^{\frac{n-1}{2}} - \big(\cos 2(n-1)g \\ &\quad \cos 2\theta + \sin 2(n-1)g \sin 2\theta \sin\phi\big)\end{aligned}$$

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QM violation of variants of LGIs upto algebraic maximum

By considering $g = \frac{\pi}{2n}$, take the form

$$(K^3_{n_{even}})_Q = \left(\cos\frac{\pi}{n}\right)^{\frac{n}{2}} + \left[\left(\cos\frac{\pi}{n}\right)^{(\frac{n}{2}-1)} + \cos\frac{\pi}{n}\right]\cos 2\theta - \sin\frac{\pi}{n}\sin 2\theta\sin\phi$$

$$(K^3_{n_{odd}})_Q = \left[\left(\cos\frac{\pi}{n}\right)^{\frac{n-1}{2}} + \cos\frac{\pi}{n}\cos 2\theta\right] + \left(\cos\frac{\pi}{n}\right)^{(\frac{n-1}{2})} - \sin\frac{\pi}{n}\sin 2\theta\sin\phi$$

In the large n limit, both of them reduces to

$$(K^3_{n_{even}})_Q = (K^3_{n_{odd}})_Q \approx 1 + 2\cos 2\theta$$

When $\theta \approx 0$, $(K^3_{n_{even}})_Q = (K^3_{n_{odd}})_Q \approx 3$.

Quantum violation approaches algebraic maximum even for qubit system.

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219.

QM violation of variants of LGIs upto algebraic maximum

Other variant of LGIs:

$$(L^3_{n_{even}})_Q = (\cos 2g)^{\frac{n}{2}-1}\cos 2\theta + (\cos 2g)^{\frac{n}{2}-1}\big(\cos 2g\cos 2\theta + \sin 2g\big) - \cos 2(n-1)g$$

$$(L^3_{n_{odd}})_Q = (\cos 2g)^{\frac{n-1}{2}} + (\cos 2g)^{\frac{n-1}{2}} - \cos 2(n-1)g$$

If $g = \frac{\pi}{2n}$ and n is very large, $(L^3_{n_{even}})_Q = (L^3_{n_{even}})_Q = 3$.

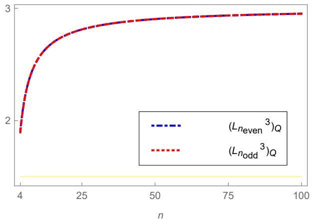


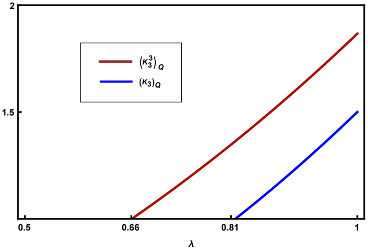
Figure : $(L^3_{n_{odd}})_Q$ and $(L^3_{n_{even}})_Q$ are plotted against n by taking $\theta = 0$.

Variants of LGIs for Unsharp Measurement

- Let at t_1, the POVMs is of the form $M_1^{\pm} = \frac{\mathbb{I} \pm \eta \sigma_z}{2}$.
- The quantum mechanical expression of K_3^3 and K_3 are given by

$$(K_3)_Q = \eta^2 (2\cos 2g - \cos 4g)$$

$$(K_3^3)_Q = \eta\big(\eta \cos 2g(\eta \cos 2\theta + 1) - \sin 4g \sin 2\theta \sin\phi - \cos 4g \cos 2\theta\big)$$



$(K_3)_Q$ and $(K_3^3)_Q$ are plotted against η. For $\eta \in (0.66, 0.81)$, where K_3^3 is violated, but K_3 does not.

AKP, Md. Qutubuddin, S. Kumari, Phys. Rev. A, 98, 06XXXX(2018); arXiv:1806.01219.

Variants of LGIs for Unsharp Measurement

- For the n-time unsharp measurement scenario, quantum expression of $(K_n)_Q$, $(K^3_{n_{even}})_Q$ and $(K^3_{n_{odd}})_Q$ respectively are given by,

$$(K_n)_Q = \eta^2 n(\cos\frac{\pi}{n})$$

$$\begin{aligned}(K^3_{n_{even}})_Q &= \eta^n\Big(\cos\frac{\pi}{n}\Big)^{\frac{n}{2}} + \eta^{n-1}\Big(\cos\frac{\pi}{n}\Big)^{\frac{n}{2}-1}\cos 2\theta \\ &+ \eta(\cos\frac{\pi}{n}\cos 2\theta - \sin\frac{\pi}{n}\sin 2\theta\sin\phi)\end{aligned}$$

and

$$\begin{aligned}(K^3_{n_{odd}})_Q &= \eta^n\Big(\cos\frac{\pi}{n}\Big)^{\frac{n-1}{2}}\cos 2\theta + \eta^{n-1}\Big(\cos\frac{\pi}{n}\Big)^{\frac{n-1}{2}} \\ &+ \eta(\cos\frac{\pi}{n}\cos 2\theta - \sin\frac{\pi}{n}\sin 2\theta\sin\phi)\end{aligned}$$

where $g = \pi/2n$.

Variants of LGIs for Unsharp Measurement

- Variants of LGIs is shown to be more robust to unsharpness than standard LGIs for the any value of n.

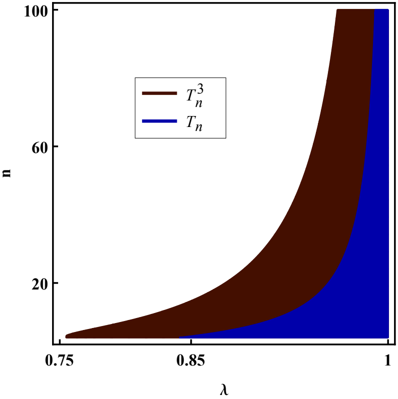


Figure : $T_n = (K_n)_Q - (K_n)$ and $T_n^3 = (K_n^3)_Q - (K_n^3)$ are plotted against η for $\theta = 0$.

- The violation of the inequality K_n^3 is obtained for a range of $\eta \in [0.75, 0.81]$, where no violation of K_n occur.

Summary and Conclusions

- We studied the violation of LGIs in unsharp measurement scenario and found that joint measurability of POVMs has no role in the violation of standard LGIs.
- We have shown that in 3-time LG scenario, there exist inequalities (Wigner and CH forms)which are not only inequivalent to standard LGIs but also stronger than SLGIs. This feature is in contrast to the CHSH inequalities.
- We argued the violation of L*ü*ders bound of LGIs cannot be consideres as the violation of LGIs in its usual sense.
- We proposed variants of LGIs, for 3-time and n-time measurement scenario where n is arbitrary. When n is sufficiently large, QM violation of variants of LGIs approaches algebraic maximum, even for a qubit system.

Thank You