Fault tolerant quantum metrology

Animesh Datta

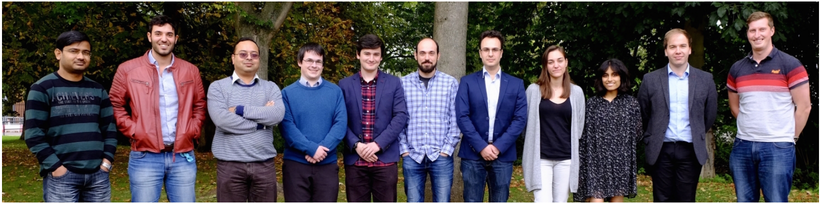
Department of Physics, University of Warwick, UK

Theodoros Kapourniotis

December 5, 2018

QIPA 2018, Allahabad

Us

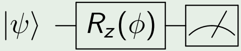


- Dominic Branford
- Samuele Ferracin
- Jamie Friel
- Evangelia Bisketzi
- Aiman Khan
- Andrew Jackson
- Theodoros Kapourniotis
- Max Marcus
- Francesco Albarelli

Enhancement using quantum probes

- Probe made of constituents qubits - quantum 2-level systems
- Using a probe $|\Psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, and

$$R_z(\phi) = e^{-i\phi Z}, \quad Z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

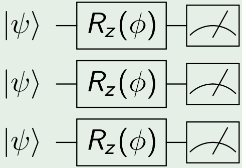


- $|\Psi\rangle = \frac{|0\rangle+e^{-i\phi}|1\rangle}{\sqrt{2}}$
- $M = X$
- $\langle M\rangle \sim \sin^2(\phi/2)$

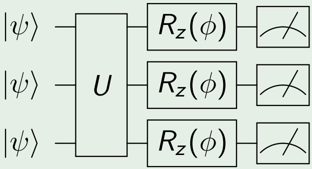
- $|\Psi\rangle = \frac{|00\rangle+e^{-2i\phi}|11\rangle}{\sqrt{2}}$
- $M = X \otimes X$
- $\langle M\rangle \sim \sin^2\phi$

Enhancement using quantum probes

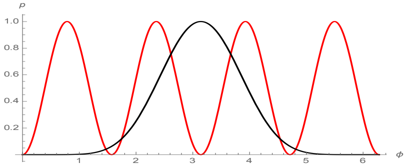
- Probe made of constituents, say N qubits, in a state $|\Psi\rangle$
- Field $R_z(\phi) = e^{-i\phi Z}$
- Measurement M



- $|\Psi\rangle = \left(\frac{|0\rangle + e^{-i\phi}|1\rangle}{\sqrt{2}}\right)^{\otimes N}$
- $\langle M\rangle \sim \left(\sin^2(\phi/2)\right)^N$



- $|\Psi\rangle = \frac{|00\cdots000\rangle + e^{-iN\phi}|11\cdots111\rangle}{\sqrt{2}}$
- $\langle M\rangle \sim \sin^2(N\phi/2)$



Estimation theory

- Precision: variance of a 'units-corrected' estimator $\tilde{\phi}$

$$\Delta\phi = \left\langle \left(\frac{\tilde{\phi}}{|\partial\tilde{\phi}/\partial\phi|} - \phi \right)^2 \right\rangle^{1/2}$$

- Cramér-Rao: Variance of estimator lower bounded

$$\Delta\phi \geq \frac{1}{\sqrt{\nu F_\phi}} \geq \frac{1}{\sqrt{\nu \mathcal{Q}_\phi}},$$

- F_ϕ is the classical Fisher information
- $\mathcal{Q}_\phi$ is the quantum Fisher information

Helstrom, Holevo, Braunstein, Caves, ...

Estimation theory

- Precision: variance of a 'units-corrected' estimator $\tilde{\phi}$

$$\Delta\phi = \left\langle \left(\frac{\tilde{\phi}}{|\partial\tilde{\phi}/\partial\phi|} - \phi \right)^2 \right\rangle^{1/2}$$

- Cramér-Rao: Variance of estimator lower bounded

$$\Delta\phi \geq \frac{1}{\sqrt{\nu F_\phi}} \geq \frac{1}{\sqrt{\nu \mathcal{Q}_\phi}},$$

- F_ϕ is the classical Fisher information
- $\mathcal{Q}_\phi$ is the quantum Fisher information

Helstrom, Holevo, Braunstein, Caves, ...

- Quantum metrology is mostly about bounds on the variance

We will provide exact variances

Paedagogical examples

GHZ (Greenberger-Horne-Zeilinger) states

$$\frac{|00\cdots000\rangle + |11\cdots111\rangle}{\sqrt{2}} \qquad \mathcal{Q}_\phi = N^2 \qquad \Delta\phi \sim \frac{1}{N}$$

Bollinger *et al.*, PRA, **54**, R4649, (1996)

N00N states

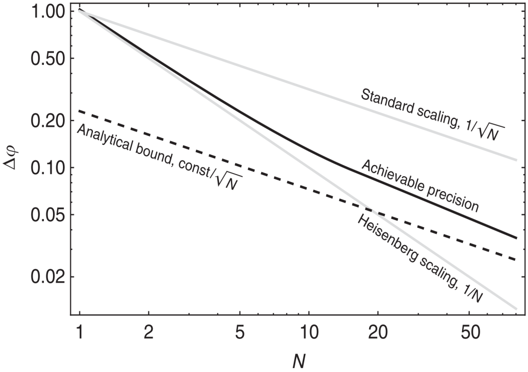
$$\frac{|N,0\rangle + |0,N\rangle}{\sqrt{2}} \qquad \mathcal{Q}_\phi = N^2 \qquad \Delta\phi \sim \frac{1}{N}$$

Kok *et al.*, Phys. Rev. A **65**, 052104 (2002)

Classical scaling

$$\mathcal{Q}_\phi \sim N \qquad \Delta\phi \sim \frac{1}{\sqrt{N}}$$

Noisy quantum metrology



Demkowicz-Dobrzanski/Kolodynski/Guta, Nat. Comms. 3, 1063, (2012)

- ✗ No quantum scaling with noise
- ✗ Quantum scaling lost no matter how small the noise
- ✗ Quantum-enhanced scaling is impossible!

Error corrected quantum metrology

- Attempts to recover quantum-enhanced scaling

Preskill, quant-ph:0010098

- Assume specific forms of noise (and all others to be absent)

W. Dur *et al.*, PRL, 112, 080801 (2014) | D. A. Herrera-Marti *et al.*, PRL, 115, 200501 (2015)

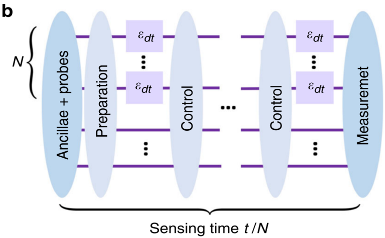
- Assume short sensing times to commute noise to the end

E. M. Kessler *et al.*, PRL, 112, 150802 (2014) | X.-M. Lu *et al.*, Nat. Comms. 6, 7282 (2015)

- Assume instantaneous, perfect correction & control operations

G. Arrad *et al.*, PRL, 112, 150801 (2014) | Demkowicz-Dobrzanski *et al.* PRX, 7, 1, (2017)

Zhou *et al.*, Nat. Comms. 9, 78, (2018)



- Error correction helps iff generator $\notin$ span of noise

Our results

- Separate noise into two types
  - beyond our control: associated with the field ($R_z(\phi)$)
  - under our control: devices (prepare/measure probes & ancillae)
- Introduce noise thresholds
- ✔✔ Show better devices counter more noise beyond our control
- Give actual variances (not bounds and scalings)
- Retrieve more information with local, full-rank Pauli noise
- No assumption on time-scales

arXiv.org > quant-ph > arXiv:1807.04267

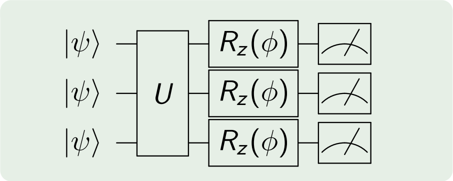
Quantum Physics

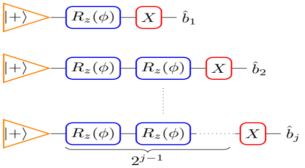
Fault-tolerant quantum metrology

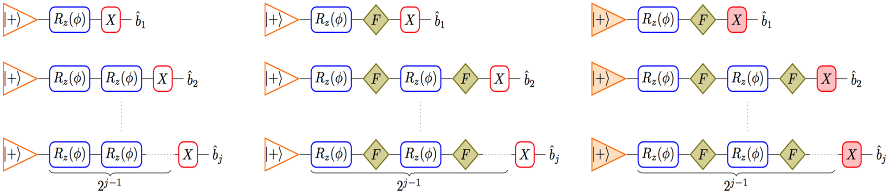
Theodoros Kapourniotis, Animesh Datta

(Submitted on 11 Jul 2018)

Fault tolerance (FT) quantum metrology







(a) Protocol Ia (b) Protocol Ib (c) Protocol Ic

- Ia: Noise everywhere, FT nowhere
- Ib: Noiseless/noisy devices + field noise, FT for field only
- Ic: Noise everywhere, FT everywhere

Fault tolerant quantum metrology

Since $\phi \in \mathbb{R}$, we cannot have

- ✗ a stabiliser code that is transversal for $R_z(\phi) = e^{-i\phi Z}$

Jochym-O'Connor/Kubica/Yoder, PRX 8, 021047 (2018)

Digital phase estimation

$$\phi = 2\pi \times 0.b_0 b_1 b_2 \ldots = b_0 \pi + b_1 \frac{\pi}{2} + b_2 \frac{\pi}{4} + \ldots$$

$$\text{If } T_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{2\pi}{2^n}} \end{pmatrix}, \; R_z(\phi) = T_1^{b_0} T_2^{b_1} T_3^{b_2} \ldots$$

Since ϕ is unknown, we cannot use

- ✗ gate synthesis to acquire a FT gate set
- ✗ magic state distillation
- ✗ state twirling to diagonalise noise in the magic state basis

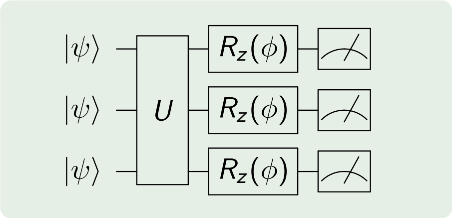
Performance of FT quantum metrology depends on

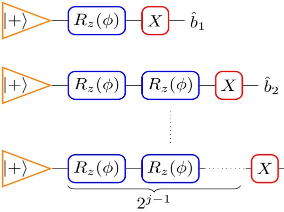
- Noise model

$$\mathcal{E}(\rho) = (1-p)\rho + p(p_x X\rho X + p_y XZ\rho ZX + p_z Z\rho Z),$$

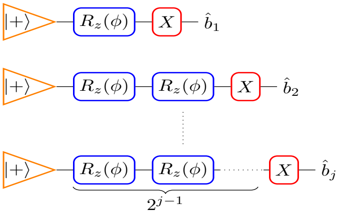
$$0 \le p, p_x, p_y, p_z \le 1, \quad p_x + p_y + p_z = 1$$

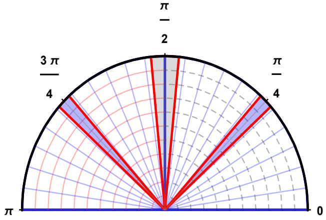
  - Local, full-rank noise that applies everywhere
- Error correcting code
- Estimator (encapsulates the protocol)





Bit-wise estimator





- Set $j = 1$
- p_1: Prob. of obtaining $+1$
- $\widehat{p}_1$: Estimate of obtaining $+1$
- $\mathrm{prob}(|\widehat{p}_1 - p_1| \leq \delta) \geq 1 - e^{-2M\delta^2}$
- $\delta = \left|\cos^2(\frac{\pi}{4}) - \cos^2(\frac{\pi}{4} - \frac{\gamma}{2})\right| = |\sin\gamma|/2$
- If $0 \leq \phi < \frac{\pi}{2} - \gamma$,

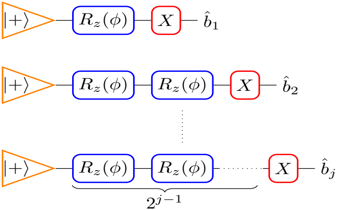
$$\mathrm{prob}(0 \leq \widehat{\phi} < \frac{\pi}{2}) \geq 1 - e^{-2M\delta^2}$$

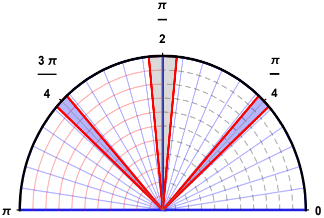
$$\Rightarrow \mathrm{prob}(\widehat{b}_1 = b_1 = 0) \geq 1 - e^{-2M\delta^2}$$

Rudolph/Grover, PRL. 91, 217905, (2003)

Ji/Wang/Duan/Feng/Ying, IEEE Trans. Inf. Th. 54, 5172 (2008)

Bit-wise estimator





Protocol Ia
For $j = 1, \dots, t$

1. Repeat M times:
   (i) Prepare $|+\rangle$.
   (ii) Interrogate field 2^{j-1} times.
   (iii) Measure X.
2. Calculate $\widehat{p}_j$ as the fraction of the $+1$ measurement outcomes out of M. If $\widehat{b}_{j-1} = 0$ set $\widehat{\phi}_j = \cos^{-1}(2\widehat{p}_j - 1)$ in $[0, \pi]$, or else in $[\pi, 2\pi]$. If
   (i) $\widehat{b}_{j-1}\pi \le \widehat{\phi}_j < \widehat{b}_{j-1}\pi + (\pi/2 - \gamma)$, set $\widehat{b}_j = 0$.
   (ii) $\widehat{b}_{j-1}\pi + (\pi/2 + \gamma) \le \widehat{\phi}_j \le \widehat{b}_{j-1}\pi + \pi$, set $\widehat{b}_j = 1$.
   Otherwise output estimate up to bit $j-1$ and exit.
3. If $j \ne t$ increase j by one and go to step 1, otherwise exit and output

$$\widehat{\phi} = \widehat{b}_1 \frac{\pi}{2} + \widehat{b}_2 \frac{\pi}{4} + \dots + \widehat{b}_t \frac{\pi}{2^t}$$

- Convergences if outputs the first t bits with confidence ϵ

Rudolph/Grover, PRL. 91, 217905, (2003) | Ji/Wang/Duan/Feng/Ying, IEEE Trans. Inf. Th. 54, 5172 (2008)

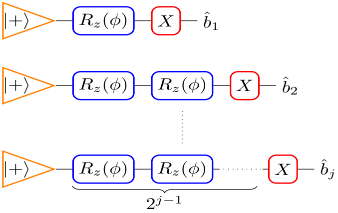
Ia: No fault tolerance

- Local, full-rank noise

$$\mathcal{E}(\rho) = (1-p)\rho + p(p_x X\rho X + p_y XZ\rho ZX + p_z Z\rho Z),$$

- For $\widehat{b_j}$, failure probability $1-(1-p)^{2^{j-1}}$
- Protocol Ia converges if $p < p_{\rm th}$ which is the solution of

$$1-(1-p)^{2^{j-1}} = |\sin\gamma|/2.$$



Kapourniotis/AD, arXiv:1807.04267

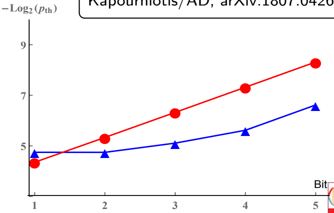
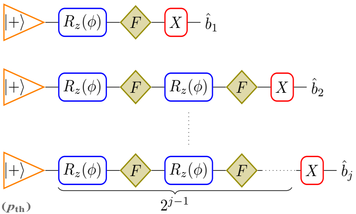


Figure: Protocol Ia, $\gamma = \pi/32$. (Later!)

Ib: Fault tolerance for noise beyond our control



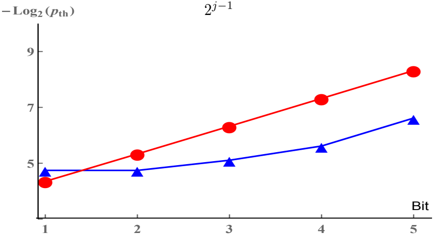


Figure: Protocol Ia, Protocol Ib. $\gamma = \frac{\pi}{32}$.

(assume noiseless devices)

Protocol Ib

For $j = 1, \ldots, t$

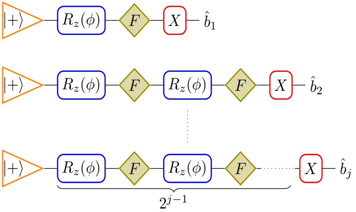
1. Repeat M times
   (i) Prepare probe $|+\rangle$. Set $k = 1$.
   (ii) Prepare ancilla $|0\rangle$. Apply CNOT between probe and ancilla. Encode probe by QRM$(1, j+2)$.
   (iii) Interrogate field transversally with probe. Apply error detection on probe. Restart (i) if syndrome measurements reject.
   (iv) Teleport by measuring probe in logical X and adapting Pauli frame accordingly (See Fig. (3)).
   (v) If $k < 2^{j-1}$, increase k by one, use ancilla as new probe and return to (ii).
   (vi) Measure X.
2. Step 2 of Protocol Ia with γ replaced by γ'.
3. If $j \neq t$ increase j by one and go to step 1, otherwise exit and output

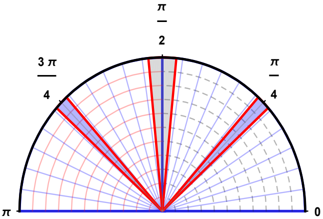
$$\widehat{\phi} = \widehat{b}_1 \frac{\pi}{2} + \widehat{b}_2 \frac{\pi}{4} + \ldots + \widehat{b}_t \frac{\pi}{2^t}$$

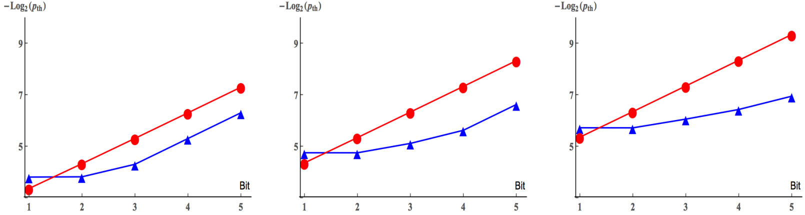
- Threshold is solution

$$1 - (1 - p_{\mathrm{err}}^X)^{2^{j-1}} (1 - p_{\mathrm{err}}^Z)^{2^{j-1}} = |\sin(\gamma')|/2$$

Ib: Fault tolerance for noise beyond our control





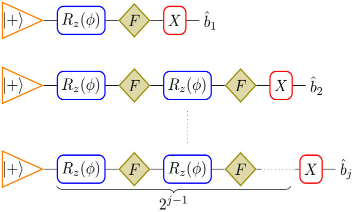


(a) $\gamma = \pi/16$ (b) $\gamma = \pi/32$ (c) $\gamma = \pi/64$

Figure: Protocol Ia, Protocol Ib (assume noiseless devices).

Kapourniotis/AD, arXiv:1807.04267

Ib: Fault tolerance for noise beyond our control



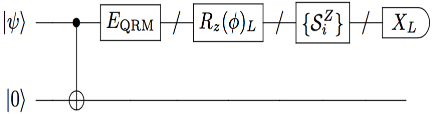


Figure: For $k = 1, |\psi\rangle = |+\rangle$, else output of $k-1$. $\{\mathcal{S}_i^Z\}$: Z stabliliser measurements, X_L: logical X syndromes. Only $R_z(\phi)_L$ is noisy.

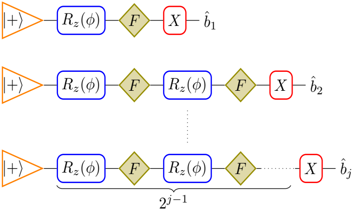
(assume noiseless devices)

Protocol Ib
For $j = 1, \ldots, t$

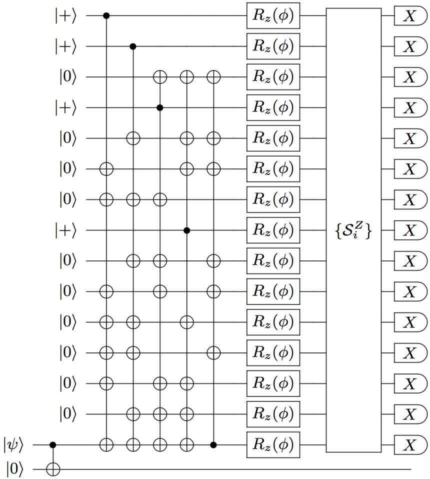
1. Repeat M times
   (i) Prepare probe $|+\rangle$. Set $k = 1$.
   (ii) Prepare ancilla $|0\rangle$. Apply CNOT between probe and ancilla. Encode probe by QRM$(1, j+2)$.
   (iii) Interrogate field transversally with probe. Apply error detection on probe. Restart (i) if syndrome measurements reject.
   (iv) Teleport by measuring probe in logical X and adapting Pauli frame accordingly (See Fig. (3)).
   (v) If $k < 2^{j-1}$, increase k by one, use ancilla as new probe and return to (ii).
   (vi) Measure X.
2. Step 2 of Protocol Ia with γ replaced by γ'.
3. If $j \neq t$ increase j by one and go to step 1, otherwise exit and output

$$\widehat{\phi} = \widehat{b}_1 \frac{\pi}{2} + \widehat{b}_2 \frac{\pi}{4} + \ldots + \widehat{b}_t \frac{\pi}{2^t}$$

- Error detection, NOT correction



(assume noiseless devices)



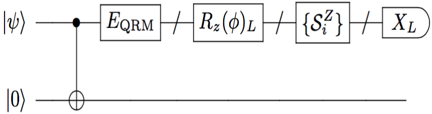
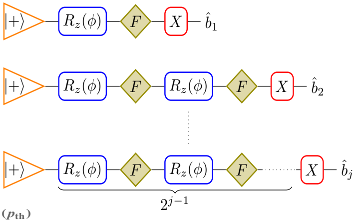


Figure: FT application of transversal $R_z(\phi)$ using QRM(1,4). Only $R_z(\phi)$ is noisy.

- Error detection, NOT correction

Ib: Fault tolerance for noise beyond our control



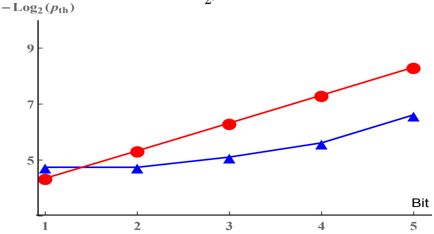


Figure: Protocol Ia, Protocol Ib. $\gamma = \frac{\pi}{32}$.

(assume noiseless devices)

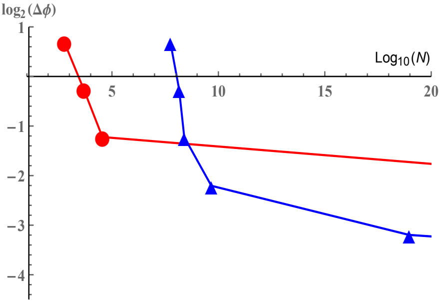


Figure: Ia, Ib. $\gamma = \pi/32, p = 0.63\%$

- Threshold is solution to

$$1 - (1 - p_{\mathrm{err}}^X)^{2^{j-1}} (1 - p_{\mathrm{err}}^Z)^{2^{j-1}} = |\sin(\gamma')|/2$$

Quantum Reed-Muller codes (very briefly)

- Quantum stabiliser codes

Steane, IEEE Trans. Inf. Th. 45, 1701 (1999)

- QRM$(1, n+1)$ transversal for $T_j = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{2\pi}{2^j}} \end{pmatrix}$, for $j \leq n$
- Transversality allows logical operation on $2^{n+1} - 1$ qubits ✗
- Operate as error detecting code for full-rank noise in T_n gates

---

- QRM$(1, n+1)$ not transversal for T_j for $j > n$

$$\gamma' = \gamma - 2\arctan\left(\frac{\sin(2^{j+1}\gamma)}{(2^{j+2}-1) + \cos(2^{j+1}\gamma)}\right) \sim \gamma - O(2^{-j})$$

Noisy devices

- Noisy non-transversal encoding
- Noisy syndrome measurements
- Assume device noise independent of ϕ

$$\mathcal{E}(\rho) = (1 - p')\rho + p'(p'_x X\rho X + p'_y XZ\rho ZX + p'_z Z\rho Z),$$

  in addition of noise beyond our control

- Threshold equation changes
  from

$$1 - (1 - p_{\text{err}}^X)^{2^{j-1}}(1 - p_{\text{err}}^Z)^{2^{j-1}} = \frac{|\sin(\gamma')|}{2}$$

  to

$$1 - (1 - p'^X_{\text{err}})^{2^{j-1}}(1 - p'^Z_{\text{err}})^{2^{j-1}}(1 - p')^{3\times 2^{j-1}+2} = \frac{|\sin(\gamma')|}{2}$$

Kapourniotis/AD, arXiv:1807.04267

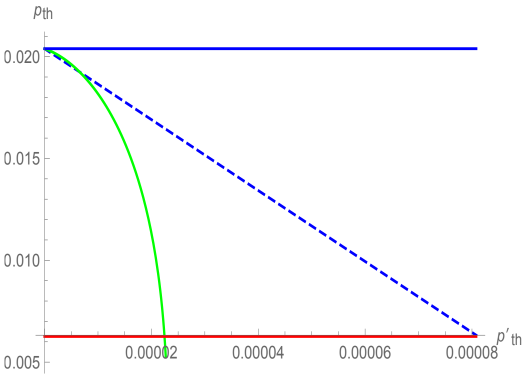


Figure: $j = 4, \gamma = \pi/32$. Protocol Ia.
Ib with device noise (dashed).
Ib without device noise (solid).
(Later!)

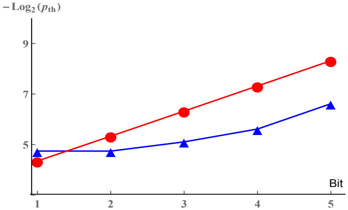
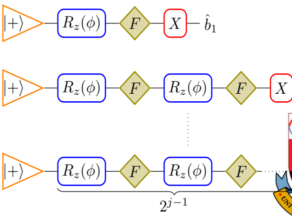
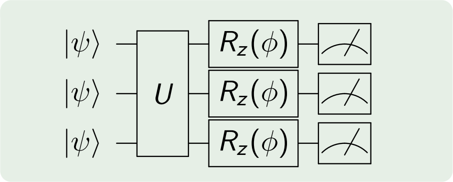
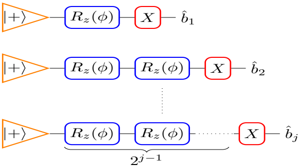


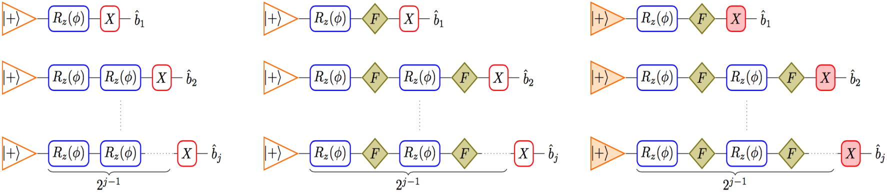
Figure: $\gamma = \pi/32$, $p = 0.63\%$



Fault tolerance (FT) quantum metrology







(a) Protocol Ia (b) Protocol Ib (c) Protocol Ic

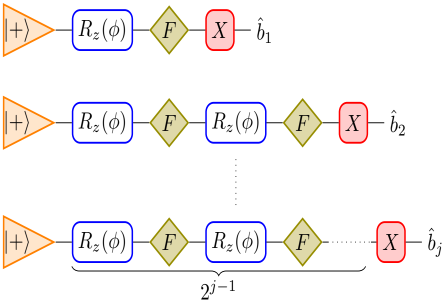
Ia: Noise everywhere, FT nowhere

Ib: Noiseless/noisy devices + field noise, FT for field only

Ic: Noise everywhere, FT everywhere

Ic: Fault tolerance everywhere

- Local full rank noise everywhere



Protocol Ic

For $j = 1, \ldots, t$

1. Repeat M times
   (i) Prepare $|+_L\rangle$ using FT procedure employing the Steane code and switch to QRM$(1, j+2)$. Set $k = 1$.
   (ii) Prepare ancilla $|0_L\rangle$ using FT procedure employing QRM$(1, j+2)$. Apply transversal FT CNOT between probe and ancilla.
   (iii) Interrogate field transversally with probe. Apply error detection on probe. Restart (i) if syndrome measurements reject.
   (iv) Teleport by measuring probe in logical X and adapting Pauli frame accordingly (See Fig. (9) in Appendix F 2).
   (v) If $k < 2^{j-1}$, increase k by one, use ancilla as new probe and return to (ii).
   (vi) FT measurement of logical X.
2. Step 2 and 3 of Protocol Ib.

- Local full rank noise everywhere

Protocol Ib
For $j = 1, \ldots, t$

1. Repeat M times
   (i) Prepare probe $|+\rangle$. Set $k = 1$.
   (ii) Prepare ancilla $|0\rangle$. Apply CNOT between probe and ancilla. Encode probe by QRM$(1, j+2)$.
   (iii) Interrogate field transversally with probe. Apply error detection on probe. Restart (i) if syndrome measurements reject.
   (iv) Teleport by measuring probe in logical X and adapting Pauli frame accordingly (See Fig. (3)).
   (v) If $k < 2^{j-1}$, increase k by one, use ancilla as new probe and return to (ii).
   (vi) Measure X.
2. Step 2 of Protocol Ia with γ replaced by γ'.
3. If $j \neq t$ increase j by one and go to step 1, otherwise exit and output

$$\widehat{\phi} = \widehat{b}_1 \frac{\pi}{2} + \widehat{b}_2 \frac{\pi}{4} + \ldots + \widehat{b}_t \frac{\pi}{2^t}$$

Protocol Ic
For $j = 1, \ldots, t$

1. Repeat M times
   (i) Prepare $|+_L\rangle$ using FT procedure employing the Steane code and switch to QRM$(1, j+2)$. Set $k = 1$.
   (ii) Prepare ancilla $|0_L\rangle$ using FT procedure employing QRM$(1, j + 2)$. Apply transversal FT CNOT between probe and ancilla.
   (iii) Interrogate field transversally with probe. Apply error detection on probe. Restart (i) if syndrome measurements reject.
   (iv) Teleport by measuring probe in logical X and adapting Pauli frame accordingly (See Fig. (9) in Appendix F 2).
   (v) If $k < 2^{j-1}$, increase k by one, use ancilla as new probe and return to (ii).
   (vi) FT measurement of logical X.
2. Step 2 and 3 of Protocol Ib.

- Switching between QRM$(1, n)$ and Steane code (QRM(1,3))

Why code switching?

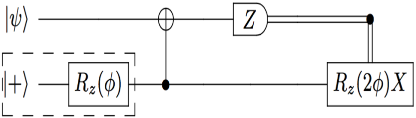


Figure: Gate distillation

$$R_z(\phi)XR_z^\dagger(\phi) \propto R_z(2\phi)X$$

- For b_k, $\sim R_z(2^k\phi)$
- Effective noise $\sim 2^k p$
- Loose FT advantage

Protocol Ic

For $j = 1, \ldots, t$

1. Repeat M times
   (i) Prepare $|+_L\rangle$ using FT procedure employing the Steane code and switch to QRM$(1, j{+}2)$. Set $k = 1$.
   (ii) Prepare ancilla $|0_L\rangle$ using FT procedure employing QRM$(1, j+2)$. Apply transversal FT CNOT between probe and ancilla.
   (iii) Interrogate field transversally with probe. Apply error detection on probe. Restart (i) if syndrome measurements reject.
   (iv) Teleport by measuring probe in logical X and adapting Pauli frame accordingly (See Fig. (9) in Appendix F 2).
   (v) If $k < 2^{j-1}$, increase k by one, use ancilla as new probe and return to (ii).
   (vi) FT measurement of logical X.
2. Step 2 and 3 of Protocol Ib.

- Switching between QRM$(1, n)$ and Steane code (QRM(1,3))
- Error correction in the device (still error detection in field)

$$1 - (1 - p_{\text{err}}''^X)^{2^{j-1}}(1 - p_{\text{err}}''^Z)^{2^{j-1}}(1 - p_{\text{EC}})^{3\times 2^{j-1}+j+1} = \frac{|\sin\gamma'|}{2}$$

Fault tolerance everywhere

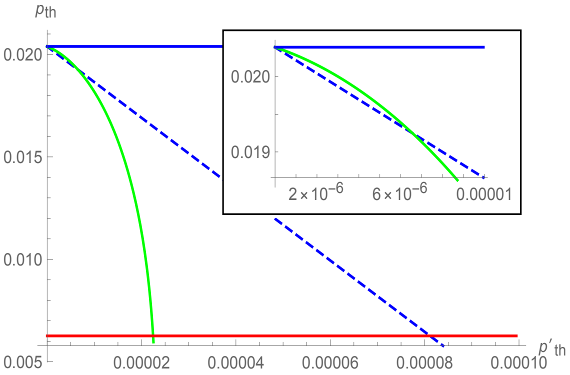


Figure: $j = 4, \gamma = \pi/32$. Protocol Ia. Ib with device noise (dashed). Ib without device noise (solid). Protocol Ic

- ✓ Better devices counter noise beyond our control
- Small improvements!!!

So,

Fault tolerant quantum metrology can ...

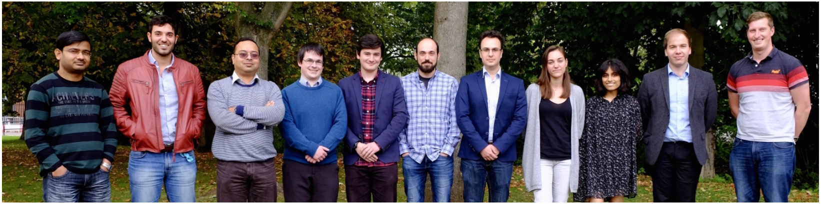
- protect from noise everywhere (fairly loose assumptions)
- estimate higher bits, giving more precision
- counter noise beyond our control with better devices ✓

Much room for improvements. We need ...

- codes with better rates (QRM$(1, n)$ has rate $\sim 2^n$ ✗)
- estimator and code optimisation in tandem
- better understanding of fault tolerance on unknown gates

Kapourniotis/AD, arXiv:1807.04267

Us



- Dominic Branford
- Samuele Ferracin
- Jamie Friel
- Evangelia Bisketzi
- Aiman Khan
- Andrew Jackson
- Theodoros Kapourniotis
- Max Marcus
- Francesco Albarelli

Thank you!

www.warwick.ac.uk/qinfo

Joining us

Quantum sensors for fundamental physics

We are looking for a post-graduate student to join the quantum information science group of Animesh Datta at the University of Warwick. The goals of this theoretical project are to produce the design principles for quantum sensors that can tackle some of the most fundamental open problems in physics. Instances include the direct detection of dark matter, testing the validity of quantum mechanics in macroscopic systems, searching for time variation of fundamental constants, and the direct detection of gravitational waves from exotic sources. The principle underlying all of these quests is the precise sensing of physical observables such as exquisitely small forces, phases, displacements and temperature.

The student must be interested in a close interplay of quantum metrology, quantum information science, quantum optics, and quantum mechanics.

www.warwick.ac.uk/qinfo