Conclusive verification of bipartite bound entanglement

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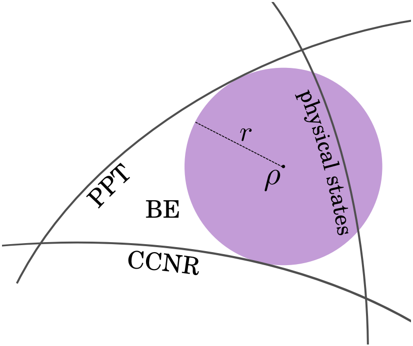
joint work with G. Sentís, J.N. Greiner, J. Shang, and J. Siewert



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Bound entanglement

An entangled state that is not distillable is bound entangled.

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Theorem (Horodecki *et al.*)

Any state with positive partial transpose (PPT) is undistillable, i.e.,

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$\hookrightarrow$ Two qutrits are the smallest system with bound entanglement.

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$$\rho_{ABCD} = \frac{1}{4}(\Phi^{+} + \Phi^{-} + \Psi^{+} + \Psi^{-}),$$

with $\Psi^{-} = |\psi^{-}\psi^{-}\rangle\langle\psi^{-}\psi^{-}|$, etc.

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- globally entangled
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Feels like cheating…

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Rigor of results.

These experiments employ

- a limited statistical analysis, or
- symmetry assumptions.

Certification of bound entanglement

Protocol in use.

1. Perform state tomography,
2. reconstruct state,
3. bootstrap, determine whether bound entangled,
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Problems

- Theorem: There can be no unbiased state reconstruction.
  [Schwemmer *et al.*, PRL (2015)]
- Bound entangled states are high-dimensional & nonconvex set.

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Disadvantages:

- slightly conservative
- requires to work in "Gaussian regime"

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(We only consider the bipartite case.)

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Lemma. If $\|\rho_0 - \tau\|_2 \leq r_0$ then, (d: dimension of joint system)

$$\lambda_{\min}[\Gamma(\tau)] \geq \lambda_{\min}[\Gamma(\rho_0)] - r_0\sqrt{1 - 1/d}.$$

Proof. Let $\rho_0 - \tau = r_0 X$ with $\|X\|_2 \leq 1$. Then

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Corollary.

All states around ρ_0 are undistillable, if

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Lemma. If $\|\rho_0 - \tau\|_2 \leq r_0$, then

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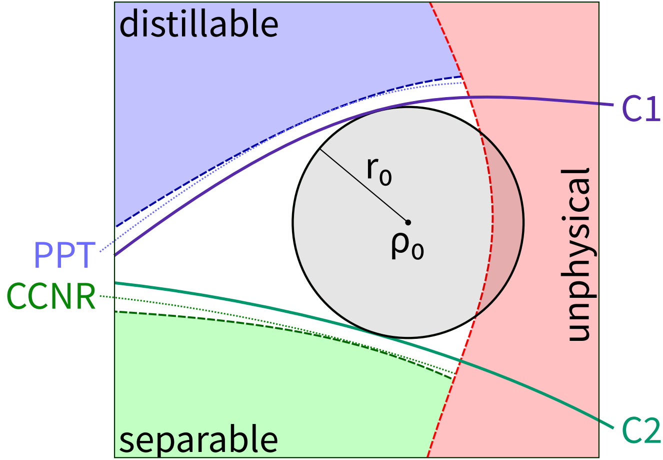
Corollary.

All states around ρ_0 are entangled, if

$$\|R(\rho_0)\|_1 > 1 + r_0\sqrt{d}.$$

Conditions



- C1: $\lambda_{\min}[\Gamma(\rho_0)] \geq r_0\sqrt{1-1/d}$. (⇒ PPT)
- C2: $\|R(\rho_0)\|_1 > 1 + r_0\sqrt{d}$. (⇒ CCNR entangled)

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- In principle, yields optimal state for given dimension.
- In practice, need to choose family of states with few parameters.

Example: Qutrits

Family of states: (contains Horodecki states)

$$\rho = a|\phi_3\rangle\langle\phi_3| + b\sum_{k=0}^{2}|k, k\oplus 1\rangle\langle k, k\oplus 1| + c\sum_{k=0}^{2}|k, k\oplus 2\rangle\langle k, k\oplus 2|,$$

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Optimal parameters

$a \approx$ 0.21289, $b \approx$ 0.04834, and $c \approx$ 0.21403.

$\hookrightarrow$ $r_0 \approx$ 0.02345

- $\mathrm{rank}(\rho) = 7$.
- Value of r_0 is (basically) tight w.r.t. CCNR and PPT.

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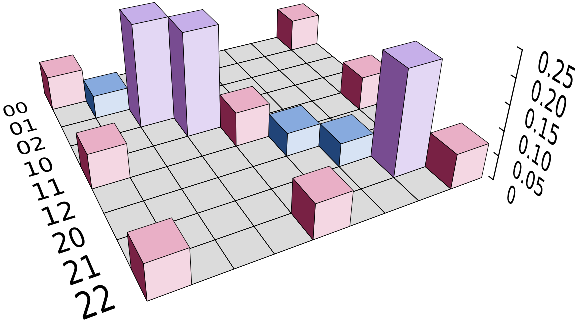

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$$\rho = \sum_k x_k g_k \otimes g_k,$$

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Optimal states

- $\mathrm{rank}(\rho) < 9$ yields $r_0 = 0$.
- $\mathrm{rank}(\rho) = 9$ yields $r_0 \approx 0.0161$.
- $\mathrm{rank}(\rho) \geq 10$ yields $r_0 \approx 0.0214$.

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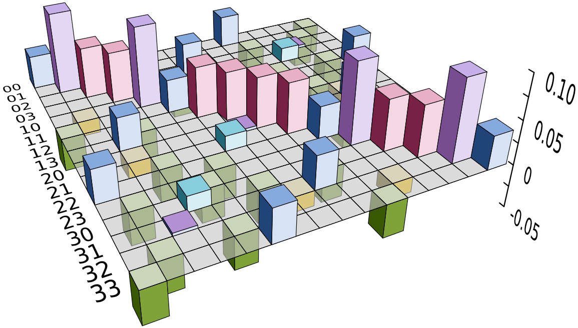

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Statistical parameters:

- distribution of raw data (Poissonian, multinomial, …)
- preprocessing method $\quad$ (raw data) $\mapsto \boldsymbol{x}$.
- (Covariance matrix Σ of x.)
- Quadratic test function $\quad \hat{t} \colon \boldsymbol{x} \mapsto t$.
- Threshold significance, yielding critical value t^*.

Evaluation of the data

Choice of test function

A good choice of the test function is

$$\hat{t}(\boldsymbol{x}) = \|\Sigma^{-1/2}[\boldsymbol{x}_0 - \boldsymbol{x}]\|_2,$$

with $\boldsymbol{x}_0$ the expected value of $\boldsymbol{x}$ for ρ_0.

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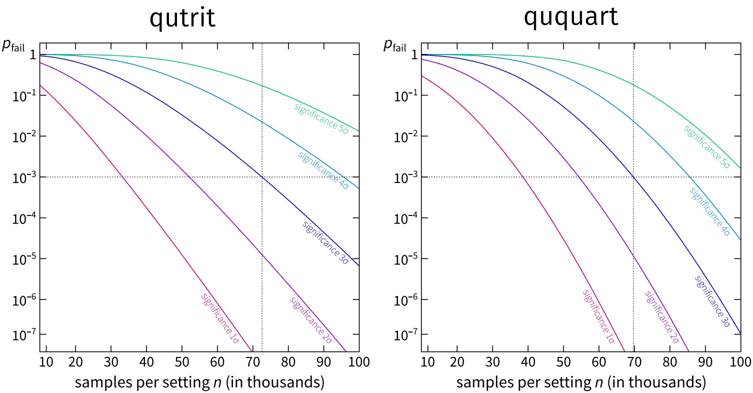
Even with $\|\rho_0 - \rho_{\text{exp}}\|_2 \leq r_0$, there is a chance that $\hat{t}(\boldsymbol{x}) > t^*$.
These unlucky cases become less likely with more samples.

Precision requirements



Probability p_{fail} to obtain data

- that does not confirm bound entanglement
- at a level of significance of $k\sigma$ standard deviations
- assuming 5% (2.5%) white noise for qutrit (ququart) case.

Summary

- For suitably parametrized states, it is possible to find ρ_0 and r_0, such that

$$\|\rho_0 - \tau\|_2 \leq r_0 \quad \Longrightarrow \quad \tau \text{ is bound entangled.}$$

- For qutrits and qubits, $r_0 \approx 0.02$.
- With tomographic data, we obtain a p-value for the null hypothesis “the state is not bound entangled.”
- In realistic scenarios, $\sim 10^5$ samples per setting are required to certify bound entanglement with 3σ significance.

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