Approximate Quantum Error Correction: Theory and Applications

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Outline

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- Applications:
  - Numerical search for good quantum codes
  - Pretty-good state transfer over spin chains

Why QEC?

*"Noisy Intermediate-Scale Quantum (**NISQ**) technology will be available in the near future. Quantum computers with 50-100 qubits may be able to perform tasks which surpass the capabilities of today's classical digital computers, but noise in quantum gates will limit the size of quantum circuits that can be executed reliably.........*

Quantum Error Correction (is) our basis for thinking that quantum computers are scalable to large devices solving hard problems."

– John Preskill, Quantum Computing in the NISQ era and beyond.
(arxiv: 1801.00862)

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- Develop protocols to preserve quantum states with high *fidelity*, under reasonable assumptions about the noise.
- Example: Qubit subject to amplitude damping noise (spontaneous emission).

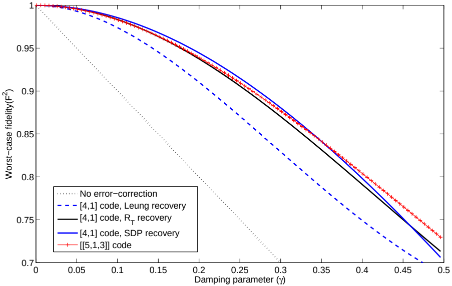
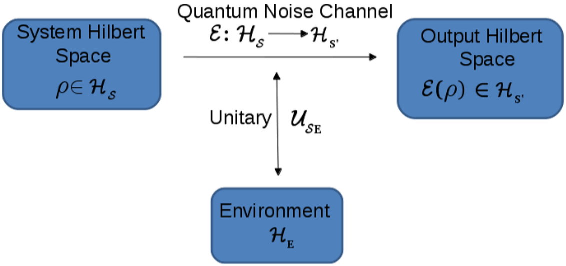


Figure: Fidelity vs Noise Strength for different QEC schemes under amplitude damping noise

Decoherence



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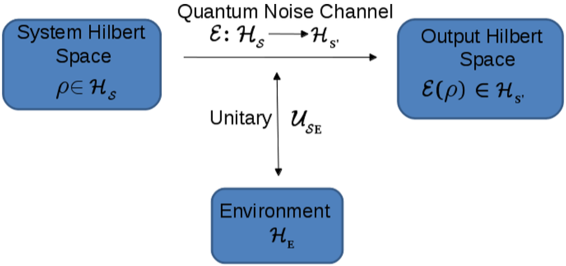

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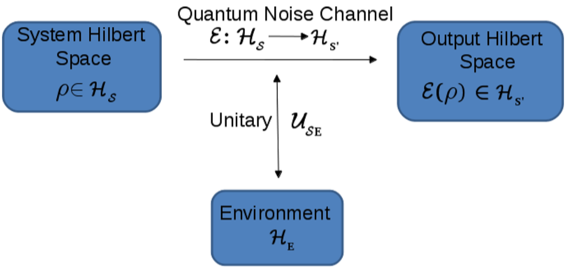

- Loss of coherence due to 'unwanted' interactions with a bath/environment $\Rightarrow$ Noise!
- Mathematically, this gives rise to completely positive, trace-preserving (CPTP) maps.

$$\mathcal{E}(\rho_S) = \mathrm{tr}_E[\,U_{SE}(\rho_S \otimes \Phi_E)U_{SE}^{\dagger})\,].$$

Quantum Channel

- Any physical process $\mathcal{E}$ on $\mathcal{H}_S$ must be,
  (i) **Completely positive (CP):** $\mathcal{E}(\rho) > 0$, for all $\rho > 0 \in \mathcal{B}(\mathcal{H}_s)$;
  And, $(\mathcal{E} \otimes \mathbb{I})$ is a positive map for any possible extension $\mathcal{H}_S \otimes \mathcal{H}_R$.
  $\Leftrightarrow$ Choi-Kraus-Sudarshan **operator-sum** representation:

$$\mathcal{E} \sim \{E_i\}_{i=1}^{N} : \;\; \mathcal{E}(\rho) = \sum_{i=1}^{N} E_i \rho E_i^{\dagger}.$$

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  (ii) **Trace non-increasing:** $0 \leq \mathrm{tr}[\mathcal{E}(\rho)] \leq 1$ implies $\sum_i E_i^{\dagger} E_i \leq I_S$.
  Trace-Preserving (TP) map : $\mathrm{tr}[\mathcal{E}(\rho)] = 1 \Leftrightarrow \sum_i E_i^{\dagger} E_i = I_S$.

Example: Amplitude Damping Channel

- Characterizes the effects due to loss of energy from a quantum system. Single qubit Amplitude Damping Channel: $\mathcal{E}^{\rm AD} = \{E_0^{\rm AD}, E_1^{\rm AD}\}$

$$E_0^{\rm AD} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \; E_1^{\rm AD} = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

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- No linear combination of $E_0^{\rm AD}$ and $E_1^{\rm AD}$ gives an operator element proportional to I; Operator elements cannot be realized as scaled Pauli operators.

- **Codespace**: d-dim subspace $\mathcal{C}$ of $\mathcal{H}^{\otimes n}$, encoding a qu$d$it.
  Recovery map: $\mathcal{R}$ is any trace preserving CP map from $\mathcal{H}^{\otimes n}$ to $\mathcal{C}$.

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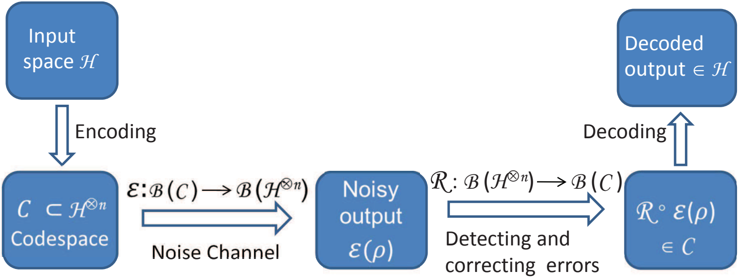

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$$PE_i^{\dagger} E_j P = \alpha_{ij} P,$$

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  - Easily verifiable, once P is given.
  - Linear - any channel whose operator elements are linear combinations of $\{E_i\}$ is also correctible. For correcting single qubit errors, sufficient to check for the "Pauli errors" !

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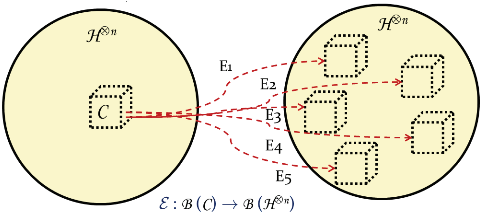

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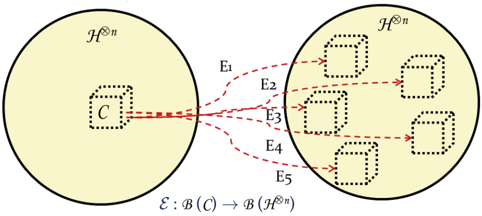

- Recovery map $\mathcal{R}_{\text{Perf}}$: $\{R_k = PU_k^\dagger\}$.
- Linearity of QEC condition, and, assumption of independent errors
  $\Rightarrow$ The shortest perfect QEC code to correct arbitrary single qubit errors requires 5 qubits (**Five-qubit code**[2,3])

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[2]Bennet *et al.*, Phys.Rev.A **54** 3824 (1996)

[3]Laflamme *et al.*, Phys. Rev. Lett. **77**, 198 (1996)

Beyond *Perfect* QEC

"Approximate quantum error correction can lead to better codes"[4]

- A 4-qubit code that corrects for single qubit amplitude damping errors:

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$
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Encodes 1 logical qubit in 4 physical qubits .

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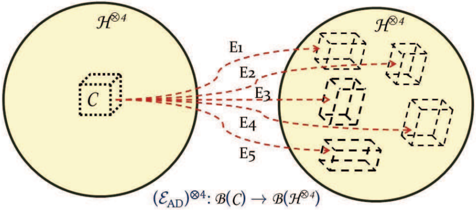

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  $\Rightarrow$ The [4,1] code is a shorter code of comparable fidelity!

Approximate Quantum Error Correction

- Channel $\mathcal{E}$ is *approximately correctible* on codespace $\mathcal{C}$ if $\exists$ a TPCP map $\mathcal{R}$ such that $F^2_{\min}[\mathcal{C}, \mathcal{R} \circ \mathcal{E}] \approx 1$

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- Actually, a triple optimization problem:

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- Finding the optimal recovery map:
  - Given a codespace $\mathcal{C}$ and a noise channel $\mathcal{E}$, the optimal recovery map ($\mathcal{R}_{\mathrm{op}}$) is defined as the recovery that gives the maximum worst-case fidelity -

$$\mathcal{R}_{\mathrm{op}}(\mathcal{C}, \mathcal{E}) = \max_{\mathcal{R}} \min_{|\psi\rangle} F^2[|\psi\rangle, \mathcal{R} \circ \mathcal{E}(|\psi\rangle)]$$

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- Analytically: channel-adapted recovery maps?
- Pretty-good recovery map: first proposed for an average measure of fidelity[7].

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[7] H. Barnum and E. Knill, JMP, **43**, 2097 (2002)

Analytical solution for AQEC [8]

- For any noise channel $\mathcal{E} \sim \{E_i\}_{i=1}^{N}$, and codespace $\mathcal{C}$, we define a channel-adapted recovery map – $\mathcal{R}_{\mathrm{P}}$ (Petz map) :

$$\mathcal{R}_{\mathrm{P}} \sim \{R_i\}_{i=1}^{N} \; , \; R_i \equiv P E_i^{\dagger} \mathcal{E}(P)^{-1/2}$$

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  (2) For any pair $(\mathcal{E}, \mathcal{C})$, $\mathcal{R}_{\mathrm{P}}$ achieves a worst-case fidelity close to that of the optimal recovery channel.
  (3) The perfect QEC conditions can be rewritten in terms of $\mathcal{R}_{\mathrm{P}}$. Perturbing these, leads to easily verifiable conditions for approximate QEC!

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    - Composed of three CP maps: $\mathcal{R}_{\rm T} = \mathcal{P} \circ \mathcal{E}^{\dagger} \circ \mathcal{N}$ – $\mathcal{P}$ is the projection onto $\mathcal{C}$, and $\mathcal{N}$ is the normalization map $\mathcal{N}(\cdot) = \mathcal{E}(P)^{-1/2}(\cdot)\mathcal{E}(P)^{-1/2}$

[8] H.K. Ng and P. Mandayam, Phys Rev A, **81**, 62342 (2010).

$\mathcal{R}_P$ achieves close-to-optimal fidelity

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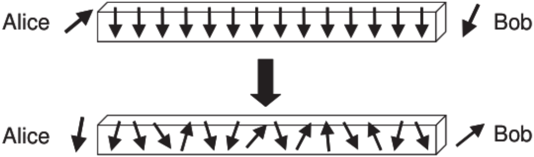
- For any noise-channel $\mathcal{E}$, $\mathcal{R}_{\rm P}$ does not perform much worse than $\mathcal{R}_{\rm op}$ - at most adds a factor of $(d+1)$ to the fidelity-loss.
- When $\eta_{\rm op} = 0$, $\eta_P = \eta_{\rm op}$ implying that $\mathcal{R}_{\rm P}$ is indeed the optimal recovery map for perfect QEC!

Extensions of the AQEC formalism

- Approximate subsystem codes
  (P. MAndayam and H.K.Ng, Phys Rev A 86(1), 012335 (2012).)
- Continuous-variable extensions of the pretty-good recovery map
  (L Lami, S. Das and M. Wilde, J Phys A 51 (12) , 125301 (2018).)
- Connections to ETH and translational-invariant manybody systems
  (F. BRandao, E. Crosson et al. arxiv quant-ph: 1710.04631)

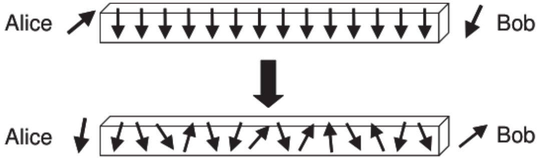
Applications

Quantum state transfer over 1-d spin chain



- Transfer of information from one spin-site 's' ("sender") to another spin site 'r' ("receiver"), via the natural, Hamiltonian dynamics of the chain. Example: state transfer via a 1-d Heisenberg chain[9].

[9] Sougato Bose, Phys. Rev. Lett. 91, 207901 (2003).



- Transfer of information from one spin-site 's' ("sender") to another spin site 'r' ("receiver"), via the natural, Hamiltonian dynamics of the chain. Example: state transfer via a 1-d Heisenberg chain[9].
- Consider a general spin-preserving Hamiltonian on a 1-d spin chain:

$$\mathcal{H} = -\sum_k J_k \left(\sigma_x^k \sigma_x^{k+1} + \sigma_y^k \sigma_y^{k+1}\right) - \sum_k \tilde{J}_k \sigma_z^k \sigma_z^{k+1} + \sum_k B_k \sigma_k^z,$$

where, $\{J_k\} > 0$ and $\{\tilde{J}_k\} > 0$.

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State transfer protocol as a quantum channel

- Spin chain is initialised to the ground state $|00\ldots0\rangle$. Sender encodes $|\psi_{\rm in}\rangle = a|0\rangle + b|1\rangle$ at the $s^{\rm th}$ site.

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- The state of the spin chain after time t is,

$$|\Psi(t)\rangle = e^{-i\mathcal{H}t}|\Psi(0)\rangle$$

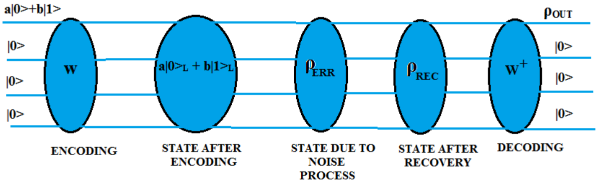
Reduced state of the $r^{\rm th}$ spin at the receiver's site is thus obtained as

$$\rho_{\rm out} = {\rm tr}_{1,2,\ldots,r-1,r+1,\ldots,N}(\rho(t)) = \mathcal{E}(\rho_{\rm in}) = \sum_{k=0,1} E_k \rho_{\rm in} E_k^\dagger,$$

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & f^N_{r,s}(t) \end{pmatrix}, E_1 = \begin{pmatrix} 0 & \sqrt{1-|f^N_{r,s}(t)|^2} \\ 0 & 0 \end{pmatrix}.$$

$f^N_{r,s}(t) = \langle \mathbf{r}|e^{-i\mathcal{H}t}|\mathbf{s}\rangle$ is the transition amplitude between the $r^{\rm th}$ site and the $s^{\rm th}$ site.

Pretty-good state transfer via adaptive QEC



[10] ¡1¿A.Jayashankar and P.Mandayam, Physical Review A 98,052309 (2018).

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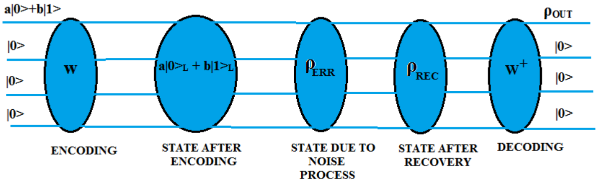

We propose a QEC protocol based on[10]

- The approximate 4-qubit code

$$|0_L\rangle = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle \right),$$
$$|1_L\rangle = \frac{1}{\sqrt{2}}\left(|1100\rangle + |0011\rangle \right).$$

- Adaptive recovery: $\mathcal{R}(.) = \sum_i PE_i^\dagger \mathcal{E}(P)^{-1/2}(.)\mathcal{E}(P)^{-1/2}E_iP$, where $P = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$ is the projection on the code space $\mathcal{C}$.

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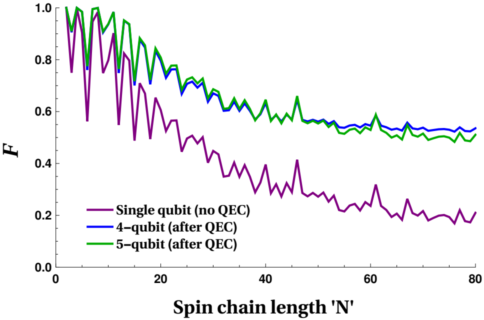
Pretty good state transfer via adaptive QEC[11]



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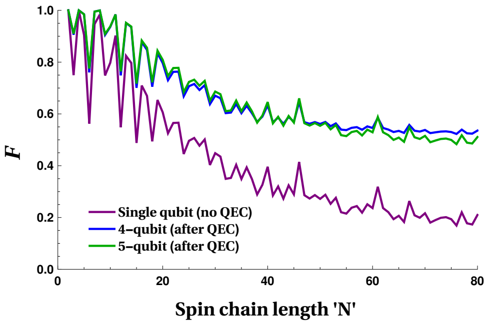

- The fidelity of s to site r state transfer with adaptive QEC, under a spin-conserving Hamiltonian, after time t:

$$F^2_{\min} \approx 1 - \frac{7p^2}{4} + O(p^3),\ (p = 1 - |f^N_{r,s}(t)|^2).$$

Without QEC: $F^2_{\min} = 1 - p$.

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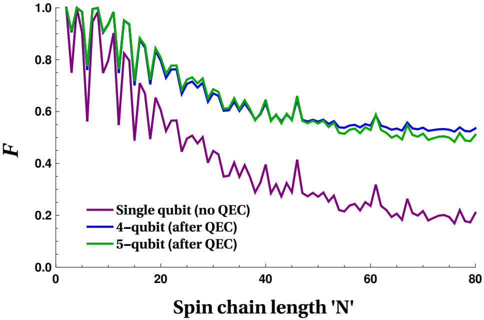

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- Can be extended to disordered 1-d spin-chains (Akshaya's poster!).

[11] A.Jayashankar and P.Mandayam, Physical Review A 98,052309 (2018).

Finding *good* AQEC codes

- Pick a d-dimensional subspace $\mathcal{C} \subseteq (\mathcal{H})^{\otimes n}$.
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- Computing η_{P} is hard in general - requires a maximization over all states in the codespace.

- A simple solution for qubit codes: $\mathcal{R}_P \circ \mathcal{E}$ is a qubit map.

Optimizing the fidelity for qubit codes

- Given a pair of codewords $|v_1\rangle, |v_2\rangle$,

$$\begin{aligned}\sigma_0 &= |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| \equiv I_2 \\ \sigma_x &= |v_1\rangle\langle v_2| + |v_2\rangle\langle v_1|, \\ \sigma_y &= -i(|v_1\rangle\langle v_2| - |v_2\rangle\langle v_1|), \\ \sigma_z &= |v_1\rangle\langle v_1| - |v_2\rangle\langle v_2|\end{aligned}$$

- Expressing the initial state $|\psi\rangle\langle\psi|$ as

$$\rho = \frac{1}{2}(I + \mathbf{s}.\sigma) = \frac{1}{2}\vec{s}.\vec{\sigma} \quad (1)$$

where $\mathbf{s}$ is a real 3-element unit vector (s_1, s_2, s_3), $\vec{s} \equiv (1, \mathbf{s})$ and $\vec{\sigma} \equiv (I, \sigma_1, \sigma_2, \sigma_3)$.

- Corresponding to any quantum channel Φ, we have,

$$\mathcal{M}_{\alpha\beta} \equiv \frac{1}{2}\text{tr}\left\{\sigma_\alpha \Phi(\sigma_\beta)\right\}$$

Fidelity for a state $|\psi\rangle \in \mathcal{C}$ under the map Φ can be written as,

$$F^2(|\psi\rangle, \Phi) = \frac{1}{2}s^T \mathcal{M} s,$$

Numerical search for good codes[12]

- For $\Phi = \mathcal{R}_P \circ \mathcal{E}$ which is not only trace preserving but also unital ($\mathcal{R}_P \circ \mathcal{E}(P) = P$), $\mathcal{M}$ takes the form,

$$\mathcal{M} = \left(\begin{array}{c|c} 1 & 0 \ldots 0 \\ \hline 0 & \\ \vdots & \mathcal{T} \\ 0 & \end{array}\right)$$

- Defining $\mathcal{T}_{sym} \equiv \frac{1}{2}(\mathcal{T} + \mathcal{T}^T)$, fidelity becomes

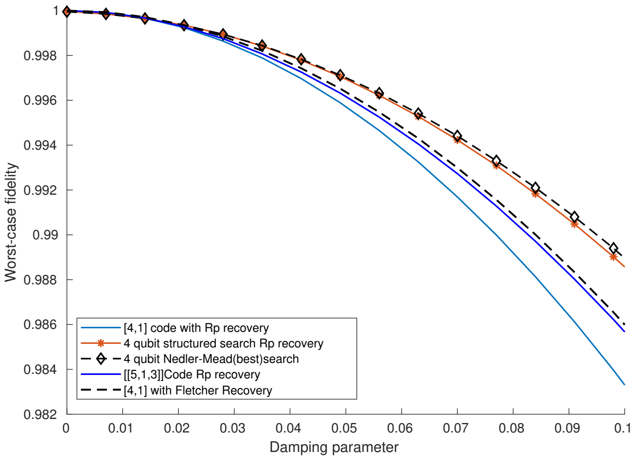
$$\begin{aligned} F^2(|\psi\rangle, \Phi) &= \frac{1}{2}(1 + \mathbf{s}^T \mathcal{T}_{sym} \mathbf{s}), \\ \min_{|\psi\rangle \in \mathcal{C}} F^2(|\psi\rangle, \Phi) &= \frac{1}{2}(1 + t_{min}). \end{aligned}$$

Fidelity loss: $\eta_\Phi = 1 - F_\Phi = \frac{1}{2}(1 - t_{min})$, where t_{min} is the smallest eigenvalue of $\mathcal{T}_{\rm sym}$ corresponding to the map Φ.

[12] Anjala MB. Akshaya J, P Mandayam and H.K. Ng, in preparation.

Nelder-Mead search

- A pair of N-qubit code-words $\{|v1\rangle, |v2\rangle\}$ are chosen by searching through the parameter space of $SU(2^N)$ using Neldear-Mead search.
- E.g. Codes for amplitude damping channel.



Structured search

- Example of a 3-qubit structured code via numerical gradient search:

$$|0_L\rangle = \begin{pmatrix} -0.0127 + 0.0756i \\ -0.5870 + 0.3695i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0259 + 0.0516i \\ 0.3847 + 0.6014i \end{pmatrix}, \; |1_L\rangle = \begin{pmatrix} 0 \\ 0 \\ -0.1516 + 0.0564i \\ -0.3291 - 0.1774i \\ 0.4911 + 0.7628i \\ -0.0440 - 0.0954i \\ 0 \\ 0 \end{pmatrix}$$

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- Open Question: Integrating AQEC with fault-tolerance : first level of concatenation of a FT protocol?

Thank You!