Mutual uncertainty, conditional uncertainty and strong subadditivity

Sk Sazim

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http://www.hri.res.in/~qic

Uncertainty

- Which implies information. Statistical fluctuation
- Intrinsic and inebility
- It carries through time
- Measures: variance, entropy

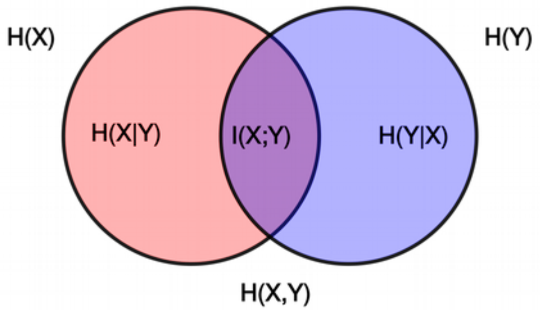
Motivations

- Sum uncertainty Relation: $\triangle(A+B) \leq \triangle A + \triangle B.$

$\longrightarrow$ $M(A:B) := \triangle A + \triangle B - \triangle(A+B).$

PLA, 367, 177

$$\triangle A_i^2 = \langle A_i^2 \rangle - \langle A_i \rangle^2, \text{ where } \langle A_i \rangle = \mathrm{Tr}[\rho A_i]$$



$$H = \frac{1}{2}P^2 + \frac{1}{2}X^2.$$

$$\triangle(A+B) = 0$$

1+1=0

Like entropy, the variance is

- Convex, $\Delta(\sum_i p_i A_i) \le \sum_i p_i \Delta(A_i)$

  $0 \le p_i \le 1$

  $\sum_i p_i = 1$

- Concave, for $\rho = \sum_\ell \lambda_\ell \rho_\ell,$ $\Delta(A)_\rho \ge \sum_\ell \lambda_\ell \Delta(A)_{\rho_\ell}$

- +ve
- Variance is the 2nd moment whereas entropy includes all.

→ Preparation uncertainty

Mutual and conditional uncertainty

- For $\{A_i; i = 1, 2, ..., n\}$,

$$M(A_1 : A_2 : \cdots : A_n) := \sum_{i=1}^{n} \triangle A_i - \triangle(\sum_{i=1}^{n} A_i).$$

- CU: $\triangle(A|B) : = \triangle(A + B) - \triangle B.$

- C-Variance: $\triangle(A|B)^2 := \triangle(A + B)^2 - \triangle B^2.$

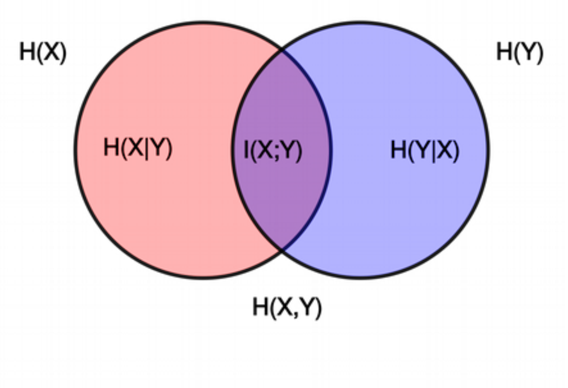
CU does't imply C-Var

Chain rule

- For sum uncertainty:

$$\triangle\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \triangle(A_i | A_{i-1} + \cdots + A_1).$$

$$\triangle(A + B) = \triangle(A) + \triangle(B|A)$$



Strong sub-additivity

- "Conditioning will not increase the entropy."

$$S(\rho_{1|23}) \leq S(\rho_{1|2}).$$

- **Theorem.3** *If* $M(B : C) = 0$, *then* $\triangle(A|B+C) \leq \triangle(A|B)$, *i.e., conditioning on more observables reduces the uncertainty.*

Implies: This guarantees the +vity of MU.

- Equivalent one:

Inequality.1 Discarding the observable, one cannot increase the mutual uncertainty, i.e., $M(A : B) \leq M(A : B + C)$.

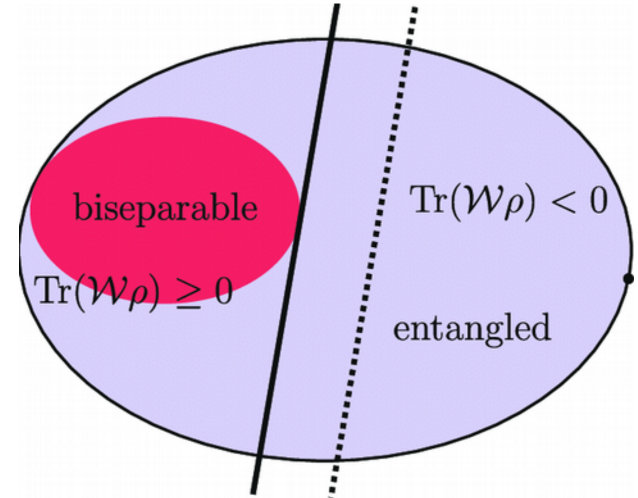
$$I(\rho_{12}) \leq I(\rho_{1(23)})$$

Physical implications

- Detection of entanglement
- and steering phenomenon

PPT criteria

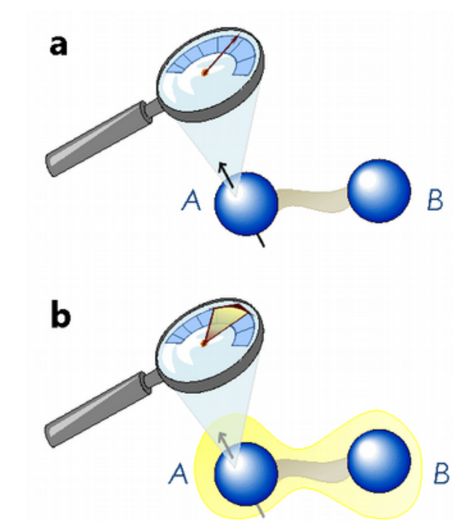
Rev. Mod. Phys. **81**, 865



Entanglement detection

Tomography

Phys. Rev. A **68**, 032103



$$||T||_{KF} \leq \frac{d(d-1)}{2}$$

Quantum Inf. Comput. **7**, 624

$$||X||_{KF} := \sum_i \lambda_i(X) = \mathrm{Tr}[\sqrt{X^\dagger X}],$$

N parties & d minension

$$\rho = \frac{1}{d^N}\mathbb{I}_{d^N} + \frac{1}{2d^{N-1}}[\vec{\sigma}.\vec{r}\otimes\mathbb{I}_d^{\otimes N-1} + \cdots + \mathbb{I}_d^{\otimes N-1}\otimes\vec{\sigma}.\vec{r_N}] + \frac{1}{4d^{N-2}}\sum_{ij}[t_{ij0\cdots0}\sigma_i\otimes\sigma_j\otimes\mathbb{I}_d^{\otimes N-2} + \cdots$$

$$+t_{0\cdots0ij}\mathbb{I}_d^{\otimes N-2}\otimes\sigma_i\otimes\sigma_j] + \cdots + \frac{1}{2^N}\sum_{i_1\cdots i_N} t_{i_1\cdots i_N}\sigma_{i_1}\otimes\cdots\otimes\sigma_{i_N},$$

$$||T^{(k)}||_{KF} \leq \sqrt{(1/2^k)d^k(d-1)^k}$$

Quant. Inf. and Comp. **8**, 0773

$$||T||_{KF} \leq \frac{d(d-1)}{2}$$

Entanglement in Higher dimension

Theorem-4 *For two qudit separable states and the set of observables* $\{A_i\}$ *and* $\{B_i\}$ *described above,* $\sum_i \triangle(A_i|B_i)^2 \geq 2(d-1)$. *This criteria is equivalent to* $||T||_{KF} \leq \frac{2(d-1)}{d} - \frac{1}{2}(|\vec{r_1}| - |\vec{r_2}|)^2$.

$$||X||_{KF} := \sum_i \lambda_i(X) = \mathrm{Tr}[\sqrt{X^\dagger X}]$$

Compare with the extant criteria

$$||T||_{KF} \leq \frac{d(d-1)}{2}$$

$$A_i = \tilde{A}_i \otimes \mathbb{I}_d$$

$$\tilde{A}_i = \vec{a}_i.\vec{\sigma}$$

$$\tilde{A}_i = \sum_j \Theta_{ij}\sigma_j, \text{ where } \Theta \in SO(d^2-1).$$

$$\mathrm{Tr}[\tilde{A}_i \tilde{A}_j] = 2\delta_{ij}$$

- It detects bound entangled states also.
- For $\rho = \frac{1}{4}[\mathbb{I}_4 + \frac{2}{5}(1-\alpha)\sigma_3 \otimes \mathbb{I}_2 - \frac{3}{5}(1-\alpha)\mathbb{I}_2 \otimes \sigma_3 - \alpha \sum_{i=1}^{3} \sigma_i \otimes \sigma_i],$

entangled for $\alpha > \frac{1}{19(5\sqrt{6}-6)}$ → (0.3288) PPT criteria.

Theorem-4 → $\alpha > \frac{49}{74+5\sqrt{221}} \simeq 0.3303$

- Bound entangle state $\rho = \frac{1}{4}[\mathbb{I}_9 - \sum_i^4 |\psi_i\rangle\langle\psi_i|]$

$||T||_{KF} \simeq 3.1603,$ Theorem-4 able to detect its entanglement

$|\psi_0\rangle = |0\rangle(|0\rangle - |1\rangle)/\sqrt{2}, |\psi_1\rangle = (|0\rangle - |1\rangle)|2\rangle/\sqrt{2}, |\psi_2\rangle = |2\rangle(|1\rangle - |2\rangle)/\sqrt{2}$

$|\psi_3\rangle = (|1\rangle - |2\rangle)|0\rangle/\sqrt{2}$ $(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)/3$

N-qubit state

Proposition.1.– For pure N-qubit states with all pairwise correlation tensors of the form $T^{(2)} = \vec{r}_i \vec{r}_j^{\mathsf{T}}$ $(i \neq j)$ and the set of N observables $\{A_i\}$, the mutual uncertainty is $M(A_1 : \cdots : A_N) = N - \sqrt{N}$, where r_i is the Bloch vector of i^{th} subsystem.

- For pure two qubits: $C = \frac{1}{2t}[2 + M(M-4)].$

where $C = |\langle \Psi | \sigma_2 \otimes \sigma_2 | \Psi^* \rangle|$,

Experimentally measurable with two obsevables.

Concurrence

$T^{(2)} \longrightarrow [t_{0 \cdots 0ij}]$

$$A_1 = \vec{a}_1.\vec{\sigma} \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \cdots, A_2 = \mathbb{I}_2 \otimes \vec{a}_2.\vec{\sigma} \otimes \mathbb{I}_2 \otimes \cdots$$

Quantum steering

- In nutshell

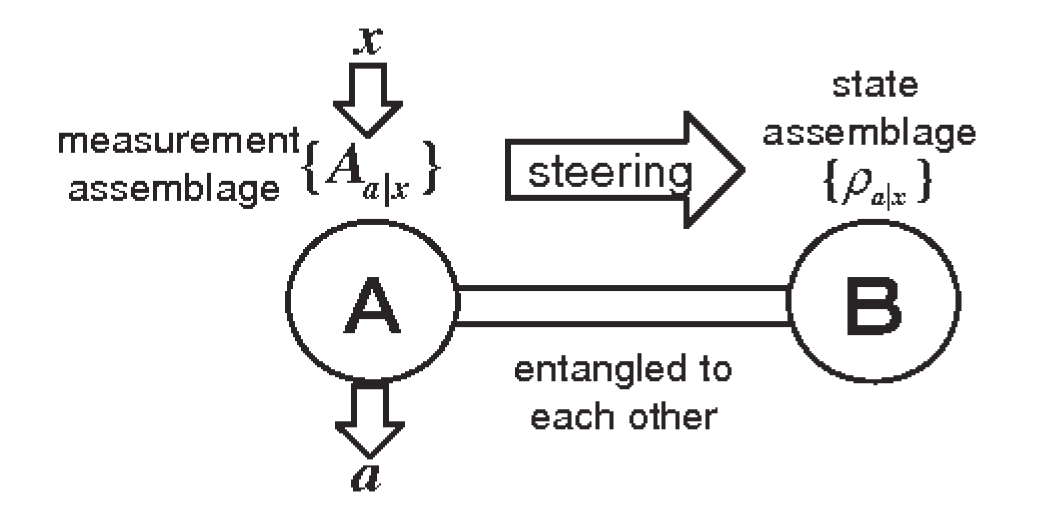
$\tilde{\rho}_1^e = \sum_\mu p(\mu)\mathcal{P}(e|E,\mu)\rho_2^Q(\mu)$, where $F = \{p(\mu), \rho_2^Q(\mu)\}$

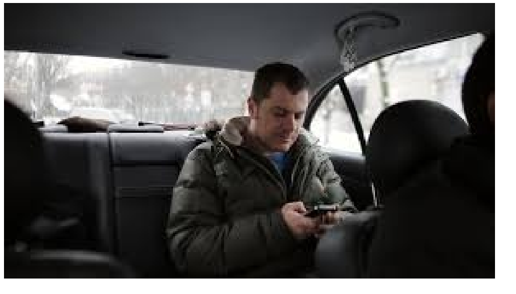
an ensemble prepared by Alice

and $\mathcal{P}(e|E,\mu)$

stochastic map.

if Bob cannot find such F and $\mathcal{P}(e|E,\mu)$, → steering





Detecting steering

- SDP

- Local uncertainty at Bob's (Reid's criteria)

Phys. Rev. A **40**, 913

if $\mathrm{Cov}(A, C) \neq 0,$ $\triangle_{\mathrm{inf}} A = \sqrt{\langle A - A_{est}(C) \rangle^2},$

Much easier than SDP

Steering contd..

Proposition.2.– For any bipartite quantum state and any two observables, A and B, if $M_{\inf}(A:B) < 0$, then the quantum state can demonstrate steering.

$$M_{\inf}(A:B) = \triangle_{\inf} A + \triangle_{\inf} B - \triangle(A+B).$$

- Werner state:

$$\rho_W = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{4}\mathbb{I}_4,$$

$p > \frac{1}{3}$ Entangled

$$M_{\inf}(A:B) = \sqrt{1-p^2} - 1/\sqrt{2}.$$

$$A = \sigma_x/2$$
$$B = \sigma_z/2.$$

$p > 1/\sqrt{2}$ Steerable

Non-Gaussian state

- 2-mode squeeze vacuum `-’ a single photon

$$W(X_1, P_{X_1}, X_2, P_{X_2}) = \frac{1}{\pi^2}\exp[2\sinh(2\alpha)(X_1X_2 - P_{X_1}P_{X_2}) - \cosh(2\alpha)\sum_{i=1}^{2}(X_i^2 + P_{X_i}^2)][-\sinh(2\alpha)\{(P_{X_1} - P_{X_2})^2 - (X_1 - X_2)^2\} + \cosh(2\alpha)\{(P_{X_1} - P_{X_1})^2 + (X_1 - X_2)^2\} - 1], \quad (18)$$

$$\triangle_{\inf}X_1^2\triangle_{\inf}P_{X_1}^2 = \frac{9}{2[3\cosh(4\alpha) + 5]}$$

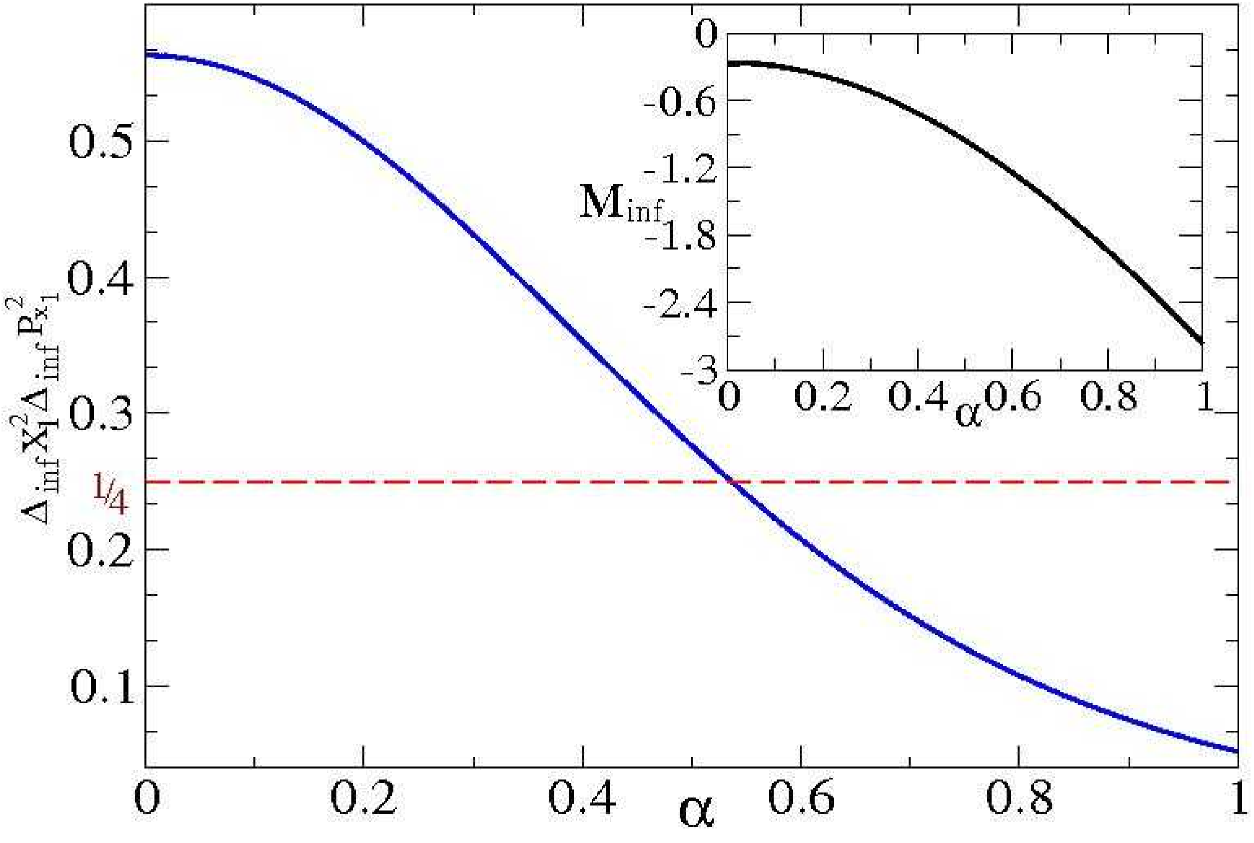
$$M_{\inf}(X_1 : P_{X_1}) = \frac{\sqrt{3}}{2}\left(\frac{1}{\eta_-} + \frac{1}{\eta_+}\right) - (\eta_+ + \eta_-),$$

$$\eta_\pm = \sqrt{\cosh(2\alpha) \pm \cosh(\alpha)\sinh(\alpha)}.$$

Contd.

$$\triangle_{\inf} X_1^2 \triangle_{\inf} P_{X_1}^2 = \frac{9}{2[3\cosh(4\alpha)+5]}.$$

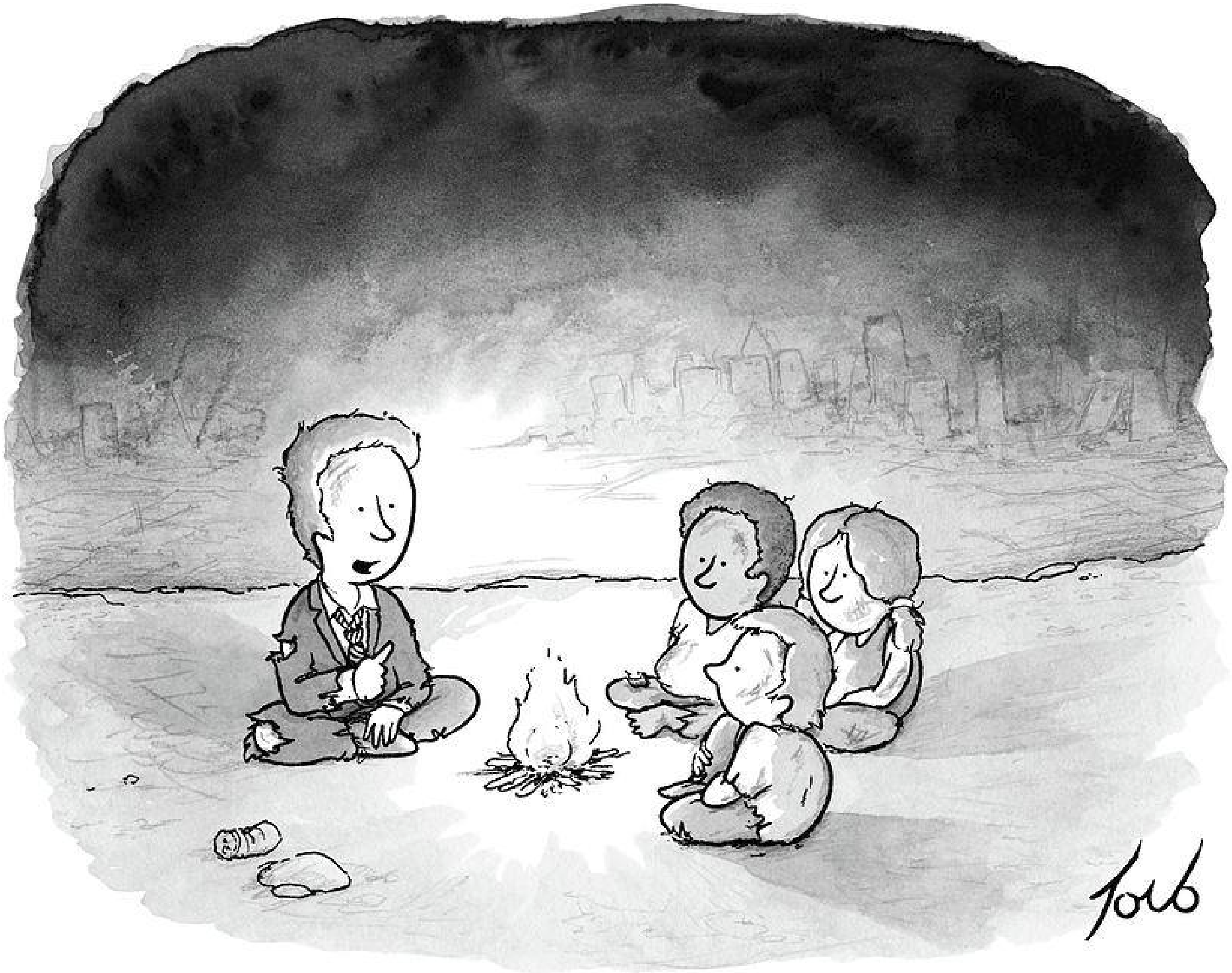
- Where Reid's criteria fails, it doesn't.



$$\triangle_{\inf} X_1^2 \triangle_{\inf} P_{X_1}^2 \geq 1/4$$

Phys. Rev. A **40**, 913

Thanks.



"Yes, the planet got destroyed, but for a beautiful moment in time we created a lot of value for shareholders."