Universal detection of entanglement in two-qubit states using only two copies

Sibasish Ghosh

Optics & Quantum Information Group
The Institute of Mathematical Sciences
C. I. T. Campus, Taramani
Chennai - 600 113, India.

Joint work with Suchetana Goswami, Sagnik Chakraborty, and Archan S. Majumdar [arXiv:1808.08246 (quant-ph)]

Motivation

Motivation

- The 'separability problem' (to find out – without making any error – whether an arbitrary state of a given bi-partite (or, multi-partite) quantum system is *separable* or *entangled*) is known to be a computationally hard problem if the system dimension is greater than six.
- For two-qubit (or qubit-qutrit) states, the Peres-Horodecki criterion of positive partial transposition (PPT) provides a mathematical but universal charcterization for identifying entanglement in the states.
- But for universal *witnessing* of entanglement in an arbitrary two-qubit state physically, **four** copies of the state need to be supplied: $Tr[W_{A^{\otimes 4}B^{\otimes 4}}\rho_{AB}^{\otimes 4}] = \det(\rho_{AB}^{T_B})$.

Motivation (continued)

- Although one needs supply of large no. of copies of the state for universal entanglement detection, it may still be resource-efficient compared to state-tomography if the required no. of measurement settings is less than that for state-tomography.
- Unfortunately, for single copy usage of the states, no entanglement witnessing scheme is resource-efficient than state-tomography [Lu et al., *Phys. Rev. Lett.* (2016); Carmeli et al., *Phys. Rev. Lett.* (2016)].
- Using *weak values* and **two** copies of any two-qubit state, we provide here a universal entanglement witnessing scheme where the post-selection measurement (required for weak values) is made in the *computational* basis. It also provides *complete state information*.

Outline

Outline

- Separability vs. entanglement in two-qubit states
- Brief description about weak measurement and weak values
- Universal entanglement witnessing scheme in two-qubit states via weak values, using two copies of the state
- Implementing the scheme via local operations for pure states
- Robustness of the scheme
- Comparison with state tomography
- Conclusion

Separability vs. entanglement in two-qubit states

Separability vs. entanglement in two-qubit states

- With respect to the computational basis $\{|00\rangle_{AB}, |01\rangle_{AB}, |10\rangle_{AB}, |11\rangle_{AB}\}$, any two-qubit state ρ_{AB} is of the form: $\rho_{AB} = \sum_{i,j,\alpha,\beta=0}^{1} \rho_{i\alpha,j\beta} |i\rangle_A \langle j| \otimes |\alpha\rangle_B \langle\beta|$ with the coefficients $\rho_{i\alpha,j\beta}$ being complex numbers, satisfying the conditions for ρ_{AB} to be a density matrix.

- The partial transposition of ρ_{AB} with respect to the computational basis: $\rho_{AB}^{T_B} = \sum_{i,j,\alpha,\beta=0}^{1} \rho_{i\beta,j\alpha} |i\rangle_A \langle j| \otimes |\alpha\rangle_B \langle\beta|$.

- ρ_{AB} is separable iff $\det(\rho_{AB}^{T_B}) \geq 0$.

- The value of $\det(\rho_{AB}^{T_B})$ (if it is negative) can be used for quantification of entanglement in ρ.

Brief description about weak measurement and weak values

Brief description about weak measurement and weak values

- In the theory of weak measurement [Aharonov et al., *Phys. Rev. Lett.* (1988)], the *pointer* is prepared in an initial state $\mathcal{P}_{in}$ while the quantum system is *pre-selected* in any state ρ_S.
- The joint system-pointer state $\rho_S \otimes \mathcal{P}_{in}$ is then evolved through a weak interaction Hamiltonian $\epsilon H \otimes P_x$ for unit time $(\hbar = 1)$, P_x: momentum operator of pointer and ϵ: small positive.
- A projective measurement (*post-selection*) is then performed on the state of the system in an ONB $\{|u_k\rangle_S : k = 1, 2, \ldots, d_S\}$, resulting in the pointer state: $\mathcal{P}_f^{(k)} \approx \langle u_k|\rho|u_k\rangle e^{-i\epsilon\langle H\rangle_\rho^{(k)} P_x} \mathcal{P}_{in} e^{i\epsilon\langle H\rangle_\rho^{(k)} P_x}$ (for small ϵ).
- The *weak value* $\langle H\rangle_\rho^{(k)} = \mathrm{Tr}[H\rho|u_k\rangle\langle u_k|]/\langle u_k|\rho|u_k\rangle$.

Brief description about weak measurement and weak values (continued)

- The weak value $\langle H \rangle_{\rho}^{(k)}$ is, in general, a complex number. Even if it is real, its value may lie beyond the spectrum of the system Hamiltonian H.

- By measuring the momentum and position shifts of the pointer state (through comparing $\mathcal{P}_{in}$ with $\mathcal{P}_f$), the real and imaginary parts of the weak value $\langle H \rangle_{\rho}^{(k)}$ can be determined [Jozsa, *Phys. Rev. A* **76**, 044103 (2007)]:

$$\langle \hat{q} \rangle_{\mathcal{P}_f} = \langle \hat{q} \rangle_{\mathcal{P}_{in}} + \epsilon \, \mathrm{Re} \left(\langle H \rangle_{\rho}^{(k)} \right) + \epsilon \, \mathrm{Im} \left(\langle H \rangle_{\rho}^{(k)} \right) \left(m \frac{d}{dt} \, \mathrm{Var}_q \right),$$

$$\langle \hat{p} \rangle_{\mathcal{P}_f} = \langle \hat{p} \rangle_{\mathcal{P}_{in}} + 2\epsilon \, \mathrm{Im} \left(\langle H \rangle_{\rho}^{(k)} \right) (\mathrm{Var}_p)$$

Brief description about weak measurement and weak values (continued)

- m: mass of the pointer, the time derivative is taken at $t = t_0$ – the instant of end measurement interaction, $\mathrm{Var}_q = \langle \hat{q}^2 \rangle_{\mathcal{P}_{in}} - \left(\langle \hat{q} \rangle_{\mathcal{P}_{in}} \right)^2$, etc.
- Real and imaginary parts of the weak value has been detected using Laguerre-Gaussian modes in the pointer state [Kobayashi et al., *Phys. Rev. A* (2014)].
- For detailed discussion on weak values, see [Dressel et al., *Rev. Mod. Phys.* **86**, 307 (2014)].

Universal entanglement witnessing scheme in two-qubit states via weak values, using two copies of the state

Universal entanglement witnessing scheme in two-qubit states via weak values, using two copies of the state

- Alice (A) and Bob (B) share **two** copies of a two-qubit state ρ_{AB}.
- With respect to the computational basis:

$$\rho = \begin{pmatrix} p & u & v & w \\ u^* & q & x & y \\ v^* & x^* & r & z \\ w^* & y^* & z^* & s \end{pmatrix}.$$

- $p, q, r, s \geq 0$ with $p + q + r + s = 1$. u, v, w, x, y, z are complex numbers, in general.
- $\rho \geq 0$ imposes further restrictions on $p, q, r, s, u, v, w, x, y, z$.

Universal entanglement $\cdots$ (general case: $pqrs > 0$)

- Then $\det(\rho_{AB}^{T_B}) =$

$$pqrs \Bigg(\frac{|uz|^2}{pqrs} - \frac{uvy^*z^*}{pqrs} - \frac{uw^*xz}{pqrs} - \frac{u^*v^*yz}{pqrs} - \frac{u^*wx^*z^*}{pqrs} + \frac{|vy|^2}{pqrs}$$
$$- \frac{vw^*x^*y}{pqrs} - \frac{v^*wxy^*}{pqrs} + \frac{|wx|^2}{pqrs} + \frac{uvw^*}{pqr} + \frac{u^*v^*w}{pqr} + \frac{uxy^*}{pqs}$$
$$+ \frac{u^*x^*y}{pqs} - \frac{|u|^2}{pq} + \frac{vx^*z^*}{prs} + \frac{v^*xz}{prs} - \frac{|v|^2}{pr} - \frac{|x|^2}{ps}$$
$$+ \frac{wy^*z^*}{qrs} + \frac{w^*yz}{qrs} - \frac{|w|^2}{qr} - \frac{|y|^2}{qs} - \frac{|z|^2}{rs} + 1 \Bigg).$$

- $\det(\rho_{AB}^{T_B})$ is homogeneous polynomial of degree four in the variables $p, q, \ldots, s$.
- Augusiak et al. [*Phys. Rev. A* (2008)] utilized this property to construct a universal witness operator – acting on **four** copies of ρ – to determine $\det(\rho_{AB}^{T_B})$.

Universal entanglement ⋯ (general case: $pqrs > 0$)

- Signature of $(1/pqrs)\,\det(\rho_{AB}^{T_B})$ determines that of $\det(\rho_{AB}^{T_B})$, and thereby, separability/entanglement of ρ_{AB}.

- Determining the values of

$$\frac{u^*}{p}, \frac{u}{q}, \frac{z^*}{r}, \frac{z}{s}, \frac{v^*}{p}, \frac{y^*}{q}, \frac{v}{r}, \frac{y}{s}, \frac{w^*}{p}, \frac{x^*}{q}, \frac{x}{r}, \text{ and } \frac{w}{s} \qquad (1)$$

will determine the value of $(1/pqrs)\,\det(\rho_{AB}^{T_B})$.

- Out of the aforesaid 12 quantities, 9 are independent (*e.g.*, u/q, z/s, and w^*/p can be expressed in terms of the remaining 9 quantities).

- However, this property does not help us in reducing the required number of copies (in our case, it is two) of the state.

Universal entanglement $\cdots$ (general case: $pqrs > 0$)

- Each term in eqn. (1) can be found as a weak value if we (i) consider two copies of ρ: $\rho_{AB} \otimes \rho_{A'B'}$, (ii) choose the system Hamiltonian H (acting on the four-qubit Hilbert space) suitably, and (iii) perform the post-selective measurement in the computational basis $\{|u_k\rangle : k = 1, 2, \ldots, 16\} = \{|0000\rangle, |0001\rangle, \ldots, |1111\rangle\}$ of the four qubits.
- We choose $H = |00\rangle_{AB}\langle 00| \otimes (H_1)_{A'B'} + |01\rangle_{AB}\langle 01| \otimes (H_1)_{A'B'} + |10\rangle_{AB}\langle 10| \otimes (H_2)_{A'B'} + |11\rangle_{AB}\langle 11| \otimes (H_3)_{A'B'}$
- $(H_1)_{A'B'} = I_{A'} \otimes (\sigma_x)_{B'}$, $(H_2)_{A'B'} = (\sigma_x)_{A'} \otimes I_{B'}$, and $(H_3)_{A'B'} = (\sigma_x)_{A'} \otimes (\sigma_x)_{B'}$.

Universal entanglement $\cdots$ (general case: $pqrs > 0$)

- One can then find out:

$$\frac{u^*}{p} = \langle H \rangle^{(1)}_{\rho\otimes\rho}, \quad \frac{u}{q} = \langle H \rangle^{(2)}_{\rho\otimes\rho}, \quad \frac{z^*}{r} = \langle H \rangle^{(3)}_{\rho\otimes\rho},$$

$$\frac{z}{s} = \langle H \rangle^{(4)}_{\rho\otimes\rho}, \quad \frac{v^*}{p} = \langle H \rangle^{(9)}_{\rho\otimes\rho}, \quad \frac{y^*}{q} = \langle H \rangle^{(10)}_{\rho\otimes\rho},$$

$$\frac{v}{r} = \langle H \rangle^{(11)}_{\rho\otimes\rho}, \quad \frac{y}{s} = \langle H \rangle^{(12)}_{\rho\otimes\rho}, \quad \frac{w^*}{p} = \langle H \rangle^{(13)}_{\rho\otimes\rho},$$

$$\frac{x^*}{p} = \langle H \rangle^{(14)}_{\rho\otimes\rho}, \quad \frac{x}{r} = \langle H \rangle^{(15)}_{\rho\otimes\rho}, \quad \frac{w}{s} = \langle H \rangle^{(16)}_{\rho\otimes\rho}.$$

- The remaining four weak values $\langle H \rangle^{(j)}_{\rho\otimes\rho}$ (for $j = 5, 6, 7, 8$) are redundant.
- Our scheme thus leads to determination of the signature of $\det(\rho_{AB}^{T_B})$: universal entanglement witness

Universal entanglement $\cdots$ (general case: $pqrs > 0$)

- Using the aforesaid expressions, one can now find out:
$\frac{q}{p} = \frac{\langle H \rangle^{(1)}_{\rho\otimes\rho}}{(\langle H \rangle^{(2)}_{\rho\otimes\rho})^*}$, $\frac{r}{p} = \frac{\langle H \rangle^{(9)}_{\rho\otimes\rho}}{(\langle H \rangle^{(11)}_{\rho\otimes\rho})^*}$, and $\frac{s}{p} = \frac{\langle H \rangle^{(13)}_{\rho\otimes\rho}}{(\langle H \rangle^{(16)}_{\rho\otimes\rho})^*}$.
- Together with the condition: $p + q + r + s = 1$, one can now find the laues of p, q, r, s.
- Hence, we can find out values of all the entries $p, u, v, \ldots, s$ of the two-qubit state ρ: complete state tomography

Universal entanglement $\cdots$ (special case: $pqrs = 0$)

- Physically this scenario refers to receiving no signal at the pointer, for the corresponding measurement outcome, before the weak interaction is switched on.
- For example, if $p = 0$, no signal is received at the pointer for the outcome $|u_1\rangle = |0000\rangle$.
- For $p = 0$, $\rho \geq 0$ demands that u, v, and w must be zero.
- Hence, in this case, $\det(\rho_{AB}^{T_B}) = -|x|^2 qr$.
- If now $qr \neq 0$, ρ is entangled (separable) for $\frac{x^*}{q} = \langle H \rangle_{\rho \otimes \rho}^{(14)}$ to be non-zero (zero). If $qr = 0$, ρ is separble.

Implementing the scheme via local operations for pure states

Implementing the scheme via local operations for pure states (one copy is enough!)

- Given that ρ is pure: $\rho = |\Psi\rangle\langle\Psi|$ with $|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, where $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$.
- ρ is separable iff $ad - bc = 0$.
- We have here: $p = |a|^2$, $q = |b|^2$, $r = |c|^2$, $s = |d|^2$, $u = ab^*$, and $z = cd^*$.
- The system Hamiltonian for weak mesurement may now be chosen as: $H' = I_A \otimes I_B \otimes I_{A'} \otimes (\sigma_x)_{B'}$: the corresponding unitary operator acts locally on all the four qubits: $\langle H' \rangle_{\rho\otimes\rho}^{(k=k_1 k_2 k_3 k_4)} = \frac{{}_{A'B'}\langle k_3 k_4|(I_{A'} \otimes (\sigma_x)_{B'})|\Psi\rangle_{A'B'} \times \langle\Psi|k_3 k_4\rangle_{A'B'}}{|\ {}_{A'B'}\langle k_3 k_4|\Psi\rangle_{A'B'}|^2}$ for $k_1, k_2, k_3, k_4 = 0, 1$.

Implementing the scheme via local operations for pure states (one copy is enough!)

- Here $a = 0$ iff no signal is received at the pointer for the outcome $|0000\rangle$; $b = 0$ iff no signal is received at the pointer for the outcome $|0101\rangle$; $c = 0$ iff no signal is received at the pointer for the outcome $|1010\rangle$; $d = 0$ iff no signal is received at the pointer for the outcome $|1111\rangle$.
- If $pqrs = 0$ (*i.e.*, $abcd = 0$), one can check easily whether $ad - bc = 0$.
- If $pqrs \neq 0$ (*i.e.*, $abcd \neq 0$), we check whether the weak values $u/q = \langle H' \rangle^{(2)}_{\rho \otimes \rho} = \langle (I \otimes \sigma_x) \rangle^{(k_3 k_4 = 01)}_{\rho}$ and $z/s = \langle H' \rangle^{(4)}_{\rho \otimes \rho} = \langle (I \otimes \sigma_x) \rangle^{(k_3 k_4 = 11)}_{\rho}$ are equal. Equality $\Leftrightarrow$ ρ is separable.

Robustness of the scheme

Robustness of the scheme: Interaction

- Assume that an erroneous system Hamiltonian H_e is being implemented during the weak interaction instead of the actual system Hamiltonian H where $||H - H_e||_1 \leq \delta$ $(||A||_1 = \mathrm{Tr}(\sqrt{A^\dagger A}))$.
- Error in the k-th weak value: $\Delta_k \equiv |\langle H\rangle^{(k)}_{\rho\otimes\rho} - \langle H_e\rangle^{(k)}_{\rho\otimes\rho}| = \frac{|\langle u_k|(\rho\otimes\rho)(H-H_e)|u_k\rangle|}{\langle u_k|(\rho\otimes\rho)|u_k\rangle} \leq \frac{|\langle u_k|(\rho\otimes\rho)(H-H_e)|u_k\rangle|}{m}$ with $m \equiv \min\{p^2, pq, pr, \ldots, s^2\}$ which is always positive.
- Note that the weak value for the k-th post-selection measurement outcome is measured only when $\langle u_k|(\rho \otimes \rho)|u_k\rangle \neq 0$.
- Spectral decomposition: $H - H_e = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ with $\{|\psi_i\rangle : i = 1, 2, \ldots, 16\}$ being an ONB.

Robustness of the scheme: Interaction (continued)

- Then $||H - H_e||_1 = \sum_i |\lambda_i|$.
- So $\Delta_k \leq \frac{|\sum_i \lambda_i \langle u_k|(\rho\otimes\rho)|\psi_i\rangle \times \langle\psi_i|u_k\rangle|}{m} \leq$ $\frac{1}{m} \times \sum_i |\lambda_i| \times |\langle u_k|(\rho \otimes \rho)|\psi_i\rangle \times \langle\psi_i|u_k\rangle|$.
- As ρ is a state, we have: $|\langle u_k|(\rho \otimes \rho)|\psi_i\rangle \times \langle\psi_i|u_k\rangle| \leq 1$.
- Hence we have: $\Delta_k \leq \frac{1}{m} \times \sum_i |\lambda_i| \leq \frac{||H-H_e||_1}{m} \leq \frac{\delta}{m}$.
- Thus our scheme is robust to errors arising out of inappropriate choice of weak interaction.

Robustness of the scheme: Post-selection

- Assume now that the measurement $\mathcal{M}_z = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on each of the four qubits is noisy (unsharp): $\mathcal{M}_z$ being replaced by $\mathcal{M}_z(\lambda) = \{E_0(\lambda) \equiv (1-\lambda)|0\rangle\langle 0| + \lambda I_2, E_1(\lambda) \equiv (1-\lambda)|1\rangle\langle 1| + \lambda I_2\}$, $I_2 = |0\rangle\langle 0| + |1\rangle\langle 1|$, $0 < \lambda \leq 1$.
- Then the erroneous $k = (k_1, k_2, k_3, k_4)$-th weak-value $\langle H\rangle_{\rho\otimes\rho}^{(k_1,k_2,k_3,k_4)}(\lambda) \approx \langle H\rangle_{\rho\otimes\rho}^{(k)} + \frac{\lambda}{1-\lambda} \times [\{(\langle k_1 k_2 k_3 0|H\rho^{\otimes 2}|k_1 k_2 k_3 0\rangle + \langle k_1 k_2 k_3 1|H\rho^{\otimes 2}|k_1 k_2 k_3 1\rangle) + \cdots\}/\langle k_1 k_2 k_3 k_4|H\rho^{\otimes 2}|k_1 k_2 k_3 k_4\rangle - \{(\langle k_1 k_2 k_3 0|\rho^{\otimes 2}|k_1 k_2 k_3 0\rangle + \langle k_1 k_2 k_3 1|\rho^{\otimes 2}|k_1 k_2 k_3 1\rangle) + \cdots\}/\langle k_1 k_2 k_3 k_4|\rho^{\otimes 2}|k_1 k_2 k_3 k_4\rangle]$ (upto 1st order in $\lambda/(1-\lambda)$)
- Thus for small λ, the post-selection is robust, in general.

Comparison with state tomography

Comparison with state tomography: Pure state case

- Two copies of $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ (with $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$ but a, b, c, d are otherwise arbitrary) are supplied with.
- Only <u>one</u> **global** measurement in the **entangled** basis $\{|0000\rangle, |0101\rangle, |1010\rangle, |1111\rangle, (1/\sqrt{2})(|0001\rangle + |0100\rangle), (1/\sqrt{2})(|0010\rangle + |1000\rangle), (1/\sqrt{2})(|0011\rangle + |1100\rangle), (1/\sqrt{2})(|0110\rangle + |1001\rangle), (1/\sqrt{2})(|0111\rangle + |1101\rangle), (1/\sqrt{2})(|1011\rangle + |1110\rangle)\}$ is sufficient to know a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, and cd $\Rightarrow a, b, c, d$ uniquely.
- In our case, it requires **local** unitary interaction on the four qubits followed by only <u>one</u> **local** measurement in the basis $\{|0000\rangle, |0001\rangle, \cdots, |1111\rangle\}$.
- Mixed state case is complicated to analyse.

Conclusion

Conclusion

- We provided here a scheme for detection of entanglement in any two-qubit state in a state-independent (*i.e.*, universal) way.
- Our scheme uses only two copies of the state, for the general case while it requires only one copy in the case of pure states.
- The post-selection measurement is done in the computational basis: just one measurement set-up is required.
- Although the system Hamiltonian required for the weak interaction can not be, in general, implemented locally, it can be done so in the case of pure states.
- Our scheme is robust against errors in sytem Hamiltonian.
- Our scheme leads to complete state tomography.

Open questions

- Resource comparison of our scheme with the existing other schemes is needed to be done.
- Possibility for extending to measurement-device-independent universal entanglement detection scheme (for general measurement errors)? [For standard measurement, with four copies of arbitrary two-qubit states, see: *Phys. Rev. A* **96**, 052323 (2017).]

Open questions (continued)

- Possibility for extending to higher dimensions?
- Such a universal entanglement witnessing scheme may not exist in higher dimension – as a single-letter (or, finitely many letters) formula – ρ_{AB} is entangled iff $\det(\rho_{AB}^{T_B}) < 0$ – is not there in higher dimension.
- Nevertheless, one may detect PPT-ness/NPT-ness of a two-qudit state in a universal manner using weak values [for standard measurement: *Phys. Rev. A* **96**, 052323 (2017)].

Open questions (continued)

- Measurement of concurrence of arbitrary two-qubit pure state was done by Zhou and Sheng for atomic entanglement [*Phys. Rev. A* **90**, 042301 (2014)].
- Optical realization of the universal MDIEW scheme with four copies of two-qubit state – using polarization and OAM ($l = 1$) degrees of photons – is currently underway.
- Possibility for photonic realization of the present work:

  • Two photons each being prepared in one and the same joint state of polarization and OAM ($l = 1$) degrees of freedom.

  • Conditioned on the states of the 1st photon, Hamiltonian interactions on the two degrees of freedom of the 2nd photon.

  • Separate measurement of these two degrees of freedom for the individual photons.

Thank you!