Distinguishing different classes of entanglement for three qubit pure states

Chandan Datta

Institute of Physics, Bhubaneswar

chandan@iopb.res.in

YouQu-2017, HRI

Overview

History

- In 1935, Einstein, Podolsky and Rosen (EPR)[1] encountered a spooky feature of quantum mechanics. Apparently, this feature is the most nonclassical manifestation of quantum mechanics.
- Later, Schrödinger coined the term entanglement to describe the feature[2].
- The outcome of the EPR paper is that quantum mechanical description of physical reality is not complete.
- Later, in 1964 Bell formalized the idea of EPR in terms of local hidden variable model[3]. He showed that any local realistic hidden variable theory is inconsistent with quantum mechanics.

[1] **Phys. Rev. 47**, **777 (1935)**.
[2] **Naturwiss. 23**, **807 (1935)**.
[3] **Physics (Long Island City**, **N.Y.) 1**, **195 (1964)**.

Motivation

- It is an essential resource for many information processing tasks, such as quantum cryptography[4], teleportation[5], super-dense coding[6] etc.
- With the recent advancement in this field, it is clear that entanglement can perform those task which is impossible using a classical resource.
- Also from the foundational perspective of quantum mechanics, entanglement is unparalleled for its supreme importance.
- Hence, its characterization and quantification is very important from both theoretical as well as experimental point of view.

---

[4] **Rev. Mod. Phys. 74**, **145 (2002)**.
[5] **Phys. Rev. Lett. 70**, **1895 (1993)**.
[6] **Phys. Rev. Lett. 69**, **2881 (1992)**.

Entanglement

- Quantum entanglement is the property of a quantum mechanical system of two or more particles.
- As a definition we can say that the particles are correlated in such a way that the state of one particle cannot be adequately described without full description of the other particles, even if the particles are spatially separated.
- If a joint system can not be written as $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$, then it is entangled.
- As an example, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$. We can't write the state as $|\psi\rangle = |A\rangle \otimes |B\rangle$.

LOCC

- In an entanglement theory, we deal with
  - Characterization : Decide which states are entangled.
  - Manipulation : Which operations are allowed.
  - Quantification : Order the states according to their entanglement.
- The most useful way to characterize bipartite and multipartite pure state entanglement is related to the study of equivalent relations under certain classes of operations, e.g., Local operation and classical communication (LOCC)[7].
- LOCC : In this method one party performs some local measurement on his/her subsystem and communicate the outcome classically to other parties. In the next step other parties perform some local operation depending on the measurements outcome of other parties.
- If $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$, then $|\psi\rangle$ is as useful as $|\phi\rangle$. $E(|\psi\rangle) \geqslant E(|\phi\rangle)$. This provides an operational ordering in the set of entangled states.

[7] **Phys. Rev. Lett. 78, 2031 (1996)**.

SLOCC

- For a single copy, two states are LOCC equivalent iff they are related by LU[8].
- But in the single copy restriction, even two bipartite pure states are not typically related by LU. To evade this difficulty we can take Stochastic LOCC (SLOCC)[8] instead of LOCC.
- SLOCC : Two entangled states are converted to each other by means of LOCC but with a non-vanishing probability of success.
- Two states are now called SLOCC equivalent if they can be obtained from each other under SLOCC, otherwise they are SLOCC inequivalent.

[8] **Phys. Rev. A 63, 012307 (2000)**.

SLOCC and 3 qubit classes

- For a three qubit pure states there are six SLOCC inequivalent classes : separable, 3 bi-separable and 2 genuinely entangled (GHZ and W)[9].
  - Separable : $|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3$.
  - Bi-separable : a) $|\psi\rangle_1 \otimes |\psi\rangle_{23}$, $|\psi\rangle_{23}$ is entangled.
    b) $|\psi\rangle_{12} \otimes |\psi\rangle_3$, $|\psi\rangle_{12}$ is entangled.
    c) $|\psi\rangle_{13} \otimes |\psi\rangle_2$, $|\psi\rangle_{13}$ is entangled.
  - Genuinely entangled : a) W class. The W state is defined as $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$.
    b) GHZ class. GHZ state can be written as $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

[9] **Phys. Rev. A 62, 062314 (2000)**.

Tangle and GHZ class

- Tangle[10] was introduced in the context of distributed entanglement, to quantify the amount of three-way entanglement in a three qubit state.
- The tangle is defined as

$$\tau = C^2_{A(BC)} - C^2_{AB} - C^2_{AC}, \tag{1}$$

  where C_{AB} and C_{AC} denote the concurrence of the entangled state between the qubits A and B and between the qubits A and C respectively. The concurrence $C_{A(BC)}$ refers to the entanglement of qubit A with the joint state of qubits B and C.
- Any three qubit pure state can be written as Acin's canonical form

$$\begin{aligned} |\psi\rangle &= \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle \\ &+\lambda_4|111\rangle, \end{aligned} \tag{2}$$

  where $\lambda_i \geqslant 0$, $\sum_i \lambda_i^2 = 1$, $\theta \in [0, \pi]$.

[10] **Phys. Rev. A 61**, **052306 (2000)**.

Tangle and GHZ class

- The tangle for the state $|\psi\rangle$ given in (2) is found out to be

$$\tau_\psi = 4\lambda_0^2\lambda_4^2. \tag{3}$$

- We can measured it experimentally if we take the expectation value of the operator

$$O = 2(\sigma_x \otimes \sigma_x \otimes \sigma_x). \tag{4}$$

$$\langle O\rangle_\psi = \langle\psi|O|\psi\rangle = 4\lambda_0\lambda_4 = 2\sqrt{\tau_\psi}. \tag{5}$$

- Hence, from the above equation it is clear that by measuring the expectation value of O, one can easily calculate the value of the tangle.
- As tangle is nonzero only for GHZ class, by measuring $\langle O\rangle_\psi$ one can separate GHZ class from other classes.

W class

- Let us consider two quantity P and Q which can be defined as $P = \langle\psi|O_1|\psi\rangle\langle\psi|O_2|\psi\rangle = \langle O_1\rangle_\psi \langle O_2\rangle_\psi$ and $Q = \langle O_1\rangle_\psi + \langle O_2\rangle_\psi + \langle O_3\rangle_\psi$, where $O_1 = 2(\sigma_x \otimes \sigma_x \otimes \sigma_z)$, $O_2 = 2(\sigma_x \otimes \sigma_z \otimes \sigma_x)$ and $O_3 = 2(\sigma_z \otimes \sigma_x \otimes \sigma_x)$.

Theorem

Any three qubit state belong to W class if (i) $\tau_\psi = 0$ and (ii) $P \neq 0$.

- Any three qubit pure state, which is in W class can be written as,

$$|\psi\rangle_W = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle. \tag{6}$$

- For this state $\tau_{\psi_W} = 0$ and $P = 16\lambda_0^2\lambda_2\lambda_3 \neq 0$.

Biseparable class

Lemma

Any three qubit state is biseparable in 1 *and* 23 *bipartition if (i)* $\tau_\psi = 0$*, (ii)* $\langle O_1 \rangle_\psi = 0$*, (iii)* $\langle O_2 \rangle_\psi = 0$ *and* $\langle O_3 \rangle_\psi \neq 0$*.*

- Any pure three qubit state which, is biseparable in 1 and 23 bipartition, can be written as $|0\rangle(\alpha|00\rangle + \beta|11\rangle)$, upto some local unitary transformation.

$$|\psi\rangle_{1|23} = |1\rangle(\lambda_1|00\rangle + \lambda_4|11\rangle). \tag{7}$$

- As $\lambda_0 = 0$, the tangle is zero for this class of state. $\langle O_1 \rangle_\psi = 0$, $\langle O_2 \rangle_\psi = 0$ and $\langle O_3 \rangle_\psi = 4\lambda_1\lambda_4$. Hence, $P = 0$ and $Q \neq 0$.

Biseparable class

Lemma

Any three qubit state is biseparable in 12 and 3 bipartition if (i) $\tau_\psi = 0$, (ii) $\langle O_1 \rangle_\psi \neq 0$, (iii) $\langle O_2 \rangle_\psi = 0$ and $\langle O_3 \rangle_\psi = 0$.

- The state, which belongs to 12 and 3 bipartition can be written as

$$|\psi\rangle_{12|3} = (\lambda_0|00\rangle + \lambda_3|11\rangle)|0\rangle. \tag{8}$$

- As $\lambda_4 = 0$, the tangle is zero for this class of state. $\langle O_1 \rangle_\psi = 4\lambda_0\lambda_3$, $\langle O_2 \rangle_\psi = 0$ and $\langle O_3 \rangle_\psi = 0$. Hence, $P = 0$ and $Q \neq 0$.

Biseparable class

Lemma

Any three qubit state is biseparable in 13 and 2 bipartition if (i) $\tau_\psi = 0$, (ii) $\langle O_1 \rangle_\psi = 0$, (iii) $\langle O_2 \rangle_\psi \neq 0$ and $\langle O_3 \rangle_\psi = 0$.

- The state belongs to 13 and 2 bipartition can be written as

$$|\psi\rangle_{13|2} = \lambda_0|000\rangle + \lambda_2|101\rangle. \tag{9}$$

- The tangle is zero as $\lambda_4 = 0$. $\langle O_1 \rangle_\psi = 0$, $\langle O_2 \rangle_\psi = 4\lambda_0\lambda_2$ and $\langle O_3 \rangle_\psi = 0$. Hence, $P = 0$ and $Q \neq 0$.
- Combining these three lemmas we can state a theorem as

Theorem

Any three qubit pure state is biseparable if, (i) $\tau_\psi = 0$, (ii) $P = 0$ and (iii) $Q \neq 0$.

Separable class

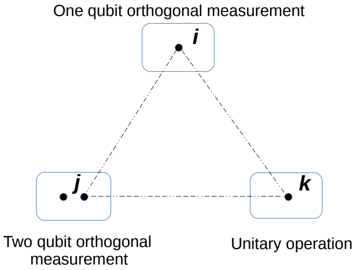
Theorem

Any three qubit state is separable if (i) $\tau_\psi = 0$,(ii) $P = 0$ and (iii) $Q = 0$.

- Any seperable three qubit pure state can be written as $|0\rangle|0\rangle|0\rangle$, after applying some appropriate local unitary operation.
- For these kind of state it can be easily shown that τ_ψ, P and Q all are zero.
- Hence, using the above theorems and lemmas we can classify all the classes present in a three qubit pure state.

Experimental measure of fidelity for a teleportation scheme

- Here we will discuss a teleportation scheme using a three qubit pure state as studied earlier by Lee. *et. al.*[a]
- The faithfulness of this teleportation scheme depends on the single qubit measurement on i'th qubit and the compound state of the system j and k.



[a] **Phys. Rev. A 72**, **024302 (2005)**.

Experimental measure of fidelity for a teleportation scheme

- Partial tangle : $\tau_{ij} = \sqrt{C^2_{i(jk)} - C^2_{ik}} = \tau_{ji}$.

$$\begin{aligned}
\tau_{12} &= 2\lambda_0\sqrt{\lambda_3^2 + \lambda_4^2}, \\
\tau_{23} &= 2\sqrt{\lambda_0^2\lambda_4^2 + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3\lambda_4 \cos\theta}, \\
\tau_{31} &= 2\lambda_0\sqrt{\lambda_2^2 + \lambda_4^2}. \qquad (10)
\end{aligned}$$

- These partial tangles are related to teleportation fidelity F_i as $\tau_{jk} = 3F_i - 2$, where F_i is the maximal teleportation fidelity over the resulting two qubit systems j and k after the orthogonal measurement on i'th party.

Experimental measure of fidelity for a teleportation scheme

- Let us define two new operators as $O_4 = 2(\sigma_z \otimes \sigma_y \otimes \sigma_y)$ and $O_5 = 2(\sigma_z \otimes \sigma_y \otimes \sigma_x)$.
- $\langle O_4 \rangle_\psi = 4(\lambda_1 \lambda_4 \cos\theta - \lambda_2 \lambda_3)$.
  $\langle O_5 \rangle_\psi = 4\lambda_1 \lambda_4 \sin\theta$.
- $\tau_{12} = \frac{1}{2}\sqrt{\langle O \rangle_\psi^2 + \langle O_1 \rangle_\psi^2} = 3F_3 - 2$.
  $\tau_{23} = \frac{1}{2}\sqrt{\langle O \rangle_\psi^2 + \langle O_4 \rangle_\psi^2 + \langle O_5 \rangle_\psi^2} = 3F_1 - 2$.
  $\tau_{31} = \frac{1}{2}\sqrt{\langle O \rangle_\psi^2 + \langle O_2 \rangle_\psi^2} = 3F_2 - 2$.
- Since the teleportation fidelities are related with some functions of these expectation values, so we can say that the teleportation fidelities for this teleportation scheme can be computed experimentally.

Experimental measure of fidelity for a teleportation scheme

- **Result-1 :** If all the partial tangles are equal to zero then the state is a separable one. As, the expectation value of O_4 and O_5 are zero for separable state.
- **Result-2:** If at least one partial tangle is equal to zero then the three qubit state is a biseparable state.
- **Result-3:** If each partial tangle is not equal to zero then the state is a three qubit genuine entangled state.

Experimental measure of fidelity for a teleportation scheme

Corollary

Any three qubit pure genuinely entangled state is useful in this teleportation scheme.

- Three qubit genuinely entangled states consist of GHZ-class and W-class.
- $F_k = \frac{2}{3} + \frac{\tau_{ij}}{3}$.
- **GHZ-class :** For GHZ-class states $\langle O \rangle_\psi$ is nonzero and hence, $\tau_{ij} > 0$. If $\tau_{ij} > 0$, $F_k > \frac{2}{3}$ and hence the resource state consist of qubits i and j is suitable for teleportation.

Experimental measure of fidelity for a teleportation scheme

- **W-class :** For W-class states, $\tau = 0$ and hence $\langle O \rangle_\psi = 0$.
- $\tau_{12} = \frac{1}{2}\langle O_1 \rangle_\psi$.
  $\tau_{23} = \frac{1}{2}\sqrt{\langle O_4 \rangle_\psi^2 + \langle O_5 \rangle_\psi^2}$.
  $\tau_{31} = \frac{1}{2}\langle O_2 \rangle_\psi$.
- $P = \langle O_1 \rangle_\psi \langle O_2 \rangle_\psi = 4\tau_{12}\tau_{31}$. For W-class states, $\tau_{12} \neq 0$ and $\tau_{31} \neq 0$ as $P \neq 0$. Hence, F_3 and F_2 are greater than $\frac{2}{3}$.
- The sturucture of W-class states tells us that λ_0, λ_2 and λ_3 can not be zero simultaneously. Hence, $\langle O_4 \rangle_\psi$ is nonzero. Therefore, $\tau_{23} \neq 0$ and hence, F_1 is greater than $\frac{2}{3}$.
- Thus for this teleportation scheme, all states in W-class are useful for teleportation.

Conclusion

- We have constructed some operators, which can be used to distinguish six SLOCC inequivalent classes of entanglement present in pure three qubit states.
- These operators contain only Pauli matrices and hence are easily implementable in experiments.
- We can detect the type of entanglement present in a three qubit pure state experimentally.
- Also we have shown that, these operators can be used to measure the fidelity of a teleportation scheme.
- We believe that, there are other such applications, where we can use our operators effectively.

Thank You