Weak measurements in quantum state estimation

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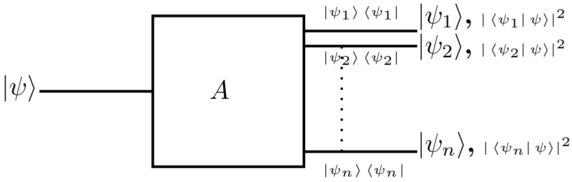
Measurement of an observable A



$$|\psi\rangle = \sum_{i=1}^{n} c_i |\psi_i\rangle \quad \text{where} \quad A|\psi_i\rangle = a_i|\psi_i\rangle \quad \text{and} \quad c_i = \langle\psi_i|\psi\rangle$$

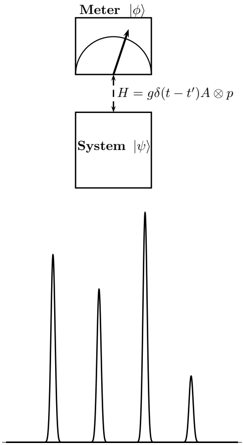
Probability of an outcome a_i (a final state $|\psi_i\rangle$)

$$P(i) = |c_i|^2 = |\langle\psi_i|\psi\rangle|^2$$

Expectation value of A

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_{i=1}^{n} a_i |c_i|^2$$

Von Neumann's model of measurement



Principal system state

$$|\psi\rangle = \sum_{i=1}^{n} c_i \,|\psi_i\rangle$$

Meter state

$$|\phi\rangle = \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{4}} \int dq\, e^{-\frac{\epsilon q^2}{4}}\, |q\rangle$$

$$[q,p] = \iota$$

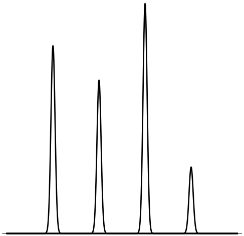
Interaction Hamiltonian between system and meter

$$H = g\delta(t-t')A\otimes p$$

Pre-measurement

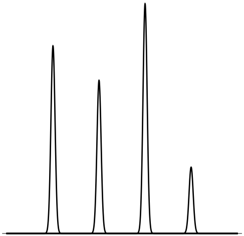
$$|\psi\rangle\otimes|\phi\rangle \to U\left(|\psi\rangle\otimes|\phi\rangle\right)U^{\dagger}$$
$$= \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{4}} \sum_{i=1}^{n} c_i \int dq\, e^{-\frac{\epsilon(q-ga_i)^2}{4}}\, |\psi_i\rangle\,|q\rangle$$

Von Neumann's model of measurement

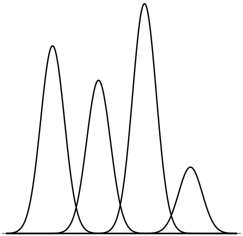


The model does not clarify the phenomenon of collapse. All it says is that when the meter shows a reading covered by one of the Gaussians, it should be associated with an outcome that defines the center of the Gaussian.

Strong vs weak



Strong measurement
$g \equiv$ large, $\epsilon \equiv$ large



Weak measurement
$g \equiv$ small, $\epsilon \equiv$ small

Strong vs Weak

- Strong/ projective measurements destroy the original state.
- Weak/ unsharp measurements come with a lot of inaccuracies, but are less invasive.

Effect of a weak measurement on a qubit with $g = 1$

Intial state

$$\rho_i = \frac{1}{2}\begin{pmatrix} 1+z & x-\iota y \\ x+\iota y & 1-z \end{pmatrix}$$

Final state

$$\rho_f = \frac{1}{2}\begin{pmatrix} 1+z & \left(1-\frac{\epsilon}{8}\right)(x-\iota y) \\ \left(1-\frac{\epsilon}{8}\right)(x+\iota y) & 1-z \end{pmatrix}$$

For $\epsilon \to 0$, $\rho_i \to \rho_f$.

- The measurement can be made weaker and weaker and hence can be made more and more non-invasive, but this will introduce more and more inaccuracy.

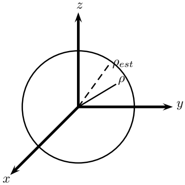
Outline

- State estimation of qubit using weak measurements
- State estimation of Gaussian state using weak measurements

State estimation of qubit using weak measurements

Problem

Can a trade-off of error and disturbance be reached so that weak measurements can be employed for state-estimation of a qubit if the ensemble size is low?



$$\rho = \frac{1}{2}(I + x\sigma_x + y\sigma_y + z\sigma_z)$$

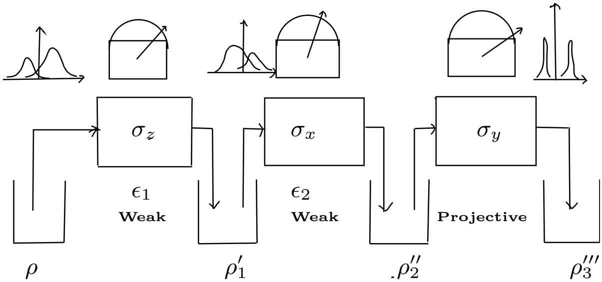
$$\langle\sigma_x\rangle = x, \quad \langle\sigma_y\rangle = y \quad \text{and} \langle\sigma_z\rangle = z$$

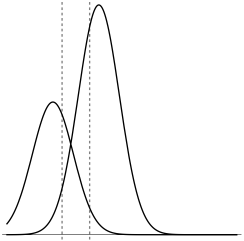
$$\rho \equiv (x, y, z) \quad \text{and} \quad \rho_{est} \equiv (x_{est}, y_{est}, z_{est})$$

A measure for efficacy of state estimation

$$f = 1 - [(x_{est} - x)^2 + (y_{est} - y)^2 + (z_{est} - z)^2]^{\frac{1}{2}}$$

State estimation of qubit using weak measurements





- Tuning ϵ $(g = 1)$
- State recycling
- Discarding ambiguous results

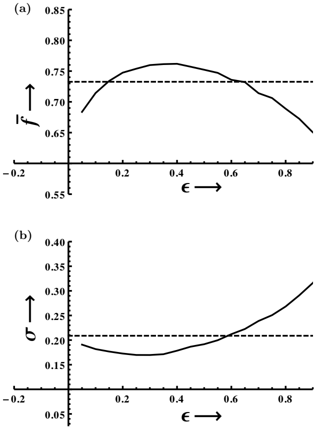
- Two individual randomly generated states

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1.399 & -0.385 + 0.042\iota \\ -0.385 - 0.042\iota & 0.601 \end{pmatrix} \quad \text{and}$$
$$\rho_2 = \frac{1}{2}\begin{pmatrix} 1.055 & -0.601 - 0.398\iota \\ -0.601 + 0.398\iota & 0.945 \end{pmatrix}$$

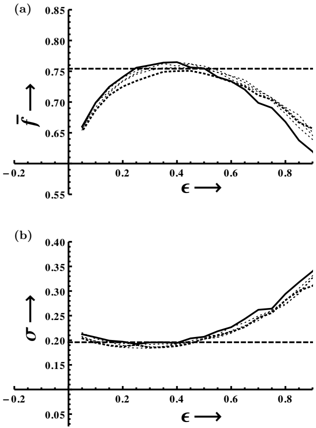
- 2000 random states in the Bloch sphere
- 500 random states on the subset of states $\langle \sigma_y \rangle = 0$

All the above experiments are repeated over 10000 runs to kill statistical fluctuations.

Results of tomography of qubit: With individual states

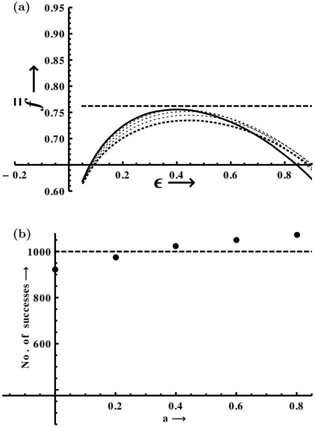


- Plots of (a) mean fidelity $\bar{f}$ and (b) standard deviation σ in fidelity with ϵ for ρ_1. The ensemble size is 30.

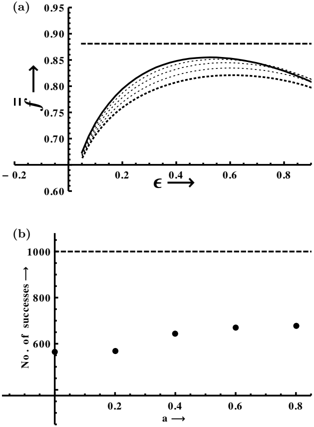


- Plots of (a) mean fidelity $\bar{f}$ and (b) standard deviation σ in fidelity with ϵ for ρ_2 with different amounts of discards. The ensemble size is 30.

Results of tomography of qubit: Averaged over 2000 states

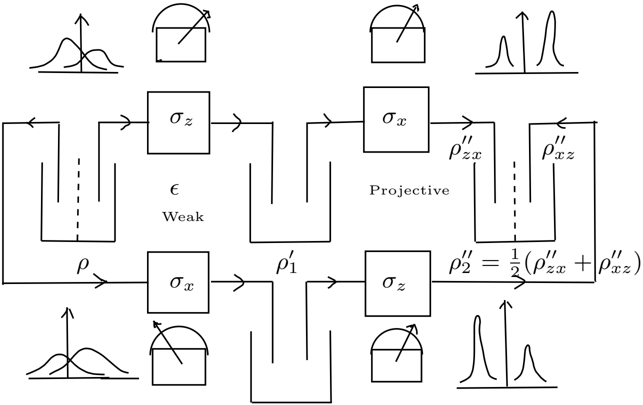


- Plots of (a) mean fidelity, averaged over 2000 random states, $\bar{\bar{f}}$ with ϵ for different discards and (b) number of successes for different discards. The ensemble size is 30

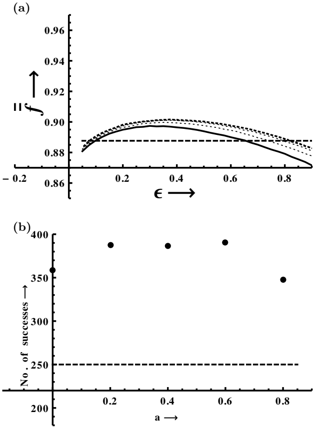


- Plots of (a) mean fidelity, averaged over 2000 random states, $\bar{\bar{f}}$ with ϵ for different discards and (b) number of successes for different discards. The ensemble size is 60

Estimation of qubit with $\langle \sigma_y \rangle = 0$ using weak measurements

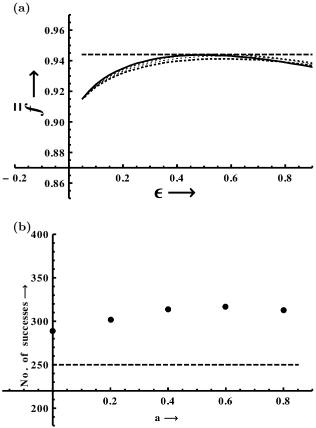


Results of tomography of qubit: States on a disc $\langle\sigma_y\rangle = 0$



- Plots of (a) mean fidelity, averaged over 2000 random states, satisfying $\langle\sigma_y\rangle = 0$, $\bar{\bar{f}}$ with ϵ for different discards and (b) number of successes for different discards. The ensemble size is 30

Results of tomography of qubit: States on a disc $\langle\sigma_y\rangle = 0$



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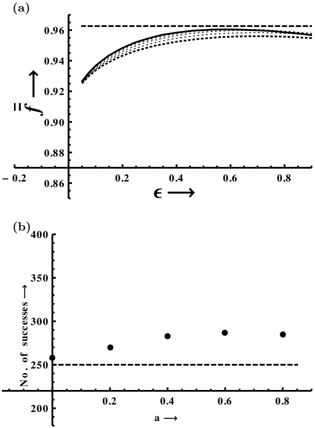


- Plots of (a) mean fidelity, averaged over 2000 random states, satisfying $\langle\sigma_y\rangle = 0$, $\bar{\bar{f}}$ with ϵ for different discards and (b) number of successes for different discards. The ensemble size is 90

One dimensional continuous variable quantum system

One-dimensional quantum systems with position and momentum operators $\hat{q}$ and $\hat{p}$ satisfy

$$[\hat{q}, \hat{p}] = \iota \ (\hbar = 1)$$

$$\hat{q}\,|q\rangle = q\,|q\rangle\,,\ \hat{p}\,|p\rangle = p\,|p\rangle$$

$$\langle q|\,q'\rangle = \delta(q - q'),\ \langle p|\,p'\rangle = \delta(p - p')\ and\ \langle q|\,p\rangle = e^{\iota pq}$$

Density operator in position and momentum bases:

$$\hat{\rho} = \int\ dqdq' f(q, q')\,|q\rangle\,\langle q'|\,,\ \hat{\rho} = \int\ dpdp'\tilde{f}(p, p')\,|p\rangle\,\langle p'|$$

Representation of the state in terms of Wigner distribution:

$$W(q, p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ dy\left\langle q - \frac{y}{2}\right|\hat{\rho}\left|q + \frac{y}{2}\right\rangle e^{\iota py}$$

Probability distributions:

$$P(q) = \int_{\infty}^{-\infty}\ W(q, p)dp,\ P(p) = \int_{-\infty}^{\infty}\ W(q, p)dq$$

Second order moments and uncertainty relations

Second order moments of position and momentum:

$$(\Delta q)^2 = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2, \ (\Delta p)^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$$

$$\Delta(q,p) = \frac{1}{2} \langle \{ \hat{q} - \langle \hat{q} \rangle, \hat{p} - \langle \hat{p} \rangle \} \rangle$$

Variance matrix:

$$V = \begin{pmatrix} (\Delta q)^2 & \Delta(q,p) \\ \Delta(q,p) & (\Delta p)^2 \end{pmatrix}$$

Uncertainty condition:

$$V + \frac{\iota}{2} \beta \geq 0$$

where

$$\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \beta^{-1} = \beta^T = -\beta, \ \text{Det}(\beta) = 1$$

Gaussian states

The class of Gaussian states includes all pure states with Gaussian wave-functions. It also includes the wider class of mixed states with Gaussian Wigner distributions.

$$W(\xi) = \frac{1}{\pi}\sqrt{|G|}e^{-\xi^T G \xi}$$

where

$$\xi = \begin{pmatrix} q \\ p \end{pmatrix}$$

$$V = \frac{1}{2}G^{-1},\ G = G^* = G^T$$

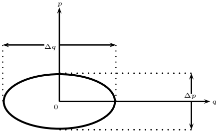
$$G = \hat{\mathcal{U}} S^T G_0 S \hat{\mathcal{U}}^{-1}$$

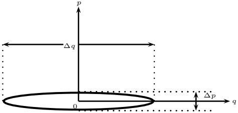
Gaussian states

We restrict ourselves to Gaussians states whose G is described by two parameters: squeezing and temperature.

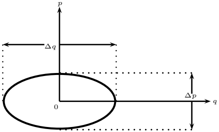
$$\hat{\mathcal{U}} = I,\ S = \begin{pmatrix} e^{-u} & 0 \\ 0 & e^{u} \end{pmatrix},\ G_0 = \kappa I,\ \kappa = \tanh\left(\frac{\omega}{2kT}\right)$$

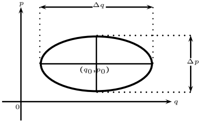
$$V = \begin{pmatrix} (\Delta q)^2 & 0 \\ 0 & (\Delta p)^2 \end{pmatrix}$$





Gaussian states





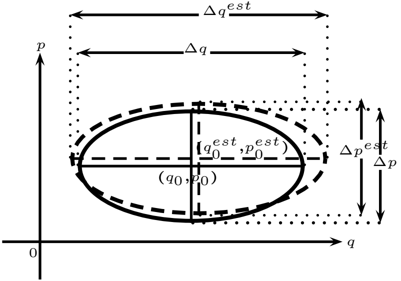
Action of a displacement operator:

$$\hat{D}(q_0, p_0) = e^{\iota(p_0 \hat{q} - q_0 \hat{p})}$$

$$\hat{\xi} = \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \rightarrow \hat{\hat{\xi}} = \hat{D}(q_0, p_0)\hat{\xi}\hat{D}(q_0, p_0)^{-1} = \begin{pmatrix} \hat{q} - q_0 \\ \hat{p} - p_0 \end{pmatrix}$$

Problem

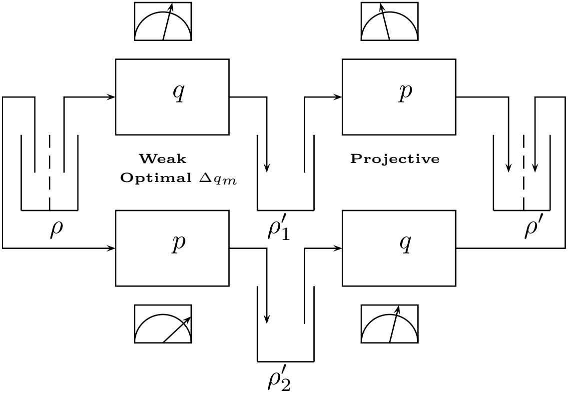
Can a trade-off of error and disturbance be reached so that weak measurements can be employed for state-estimation of a Gaussian state if the ensemble size is low?



Measure of efficacy

$$
\begin{aligned}
d_1 &= \left(q_0 - q_0^{est}\right)^2 + \left(p_0 - p_0^{est}\right)^2 \\
d_2 &= \left(\Delta q - \Delta q^{est}\right)^2 + \left(\Delta p - \Delta p^{est}\right)^2
\end{aligned}
$$

State estimation of Gaussian state using weak measurements



- Tuning Δq_m $(g = 1)$
- State recycling

Composite system and meter:

$$\Xi = \begin{pmatrix} q \\ q_{m_1} \\ p \\ p_{m_1} \end{pmatrix} \rightarrow \tilde{\Xi} = \begin{pmatrix} q - q_0 \\ q_{m_1} \\ p - p_0 \\ p_{m_1} \end{pmatrix}$$

$$W(\Xi) = \frac{1}{\pi^2}\sqrt{|G|}e^{-\tilde{\Xi}^T G \tilde{\Xi}}$$

with

$$G = Diag\left((\Delta q)^2, (\Delta q_{m_1})^2, (\Delta p)^2, (\Delta p_{m_1})^2\right)$$

Interaction Hamiltonian:

$$\hat{H} = \delta(t - t_1)\hat{q} \otimes \hat{p_{m_1}},\ \hat{U} = e^{-\iota \int H dt}$$

Symplectic transformation:

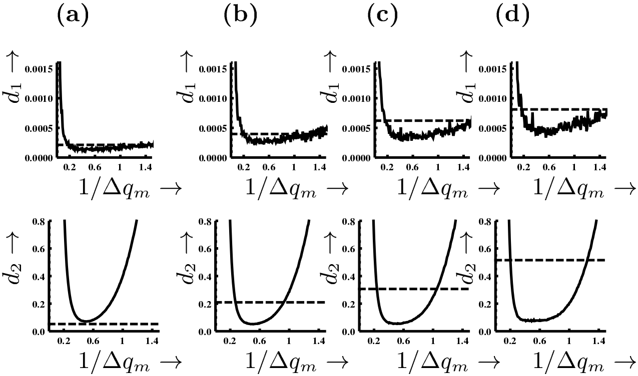
$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \Xi \rightarrow \mathcal{S}\Xi$$

where

$$\mathcal{S}^T \beta_2 \mathcal{S} = \mathcal{S}\beta_2\mathcal{S}^T = \beta_2 \text{ with} \beta_2 = \begin{pmatrix} 0_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 0_{2\times 2} \end{pmatrix}$$

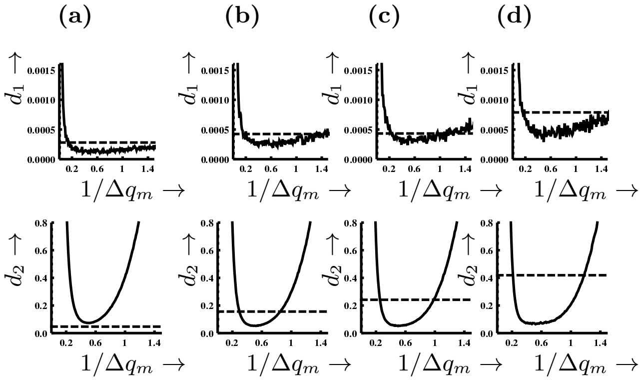
Three sets (corresponding to three different temperatures T) of 100 random Gaussian states of different squeezings are generated by varying the parameter u. The meter states are also chosen to be squeezed coherent states, satisfying $\Delta q_m \Delta p_m = \frac{1}{2}$ with $\Delta q_m \neq \Delta p_m$. The spreads of the meter states are controlled by varying the squeezing along Δq_m.

Results of tomography of Gaussian state



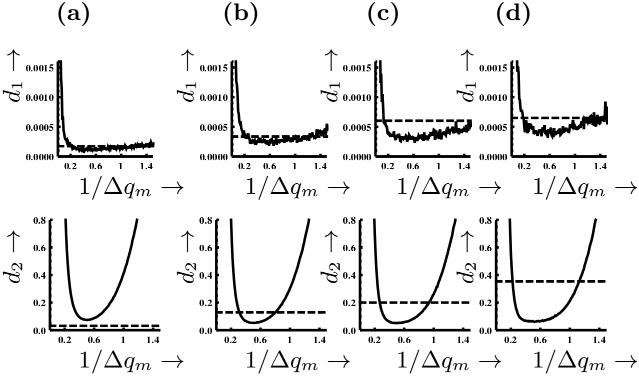
- Plots of d_1 and d_2 vs ϵ for $\kappa = 0.8$, ensemble sizes 20,10,8,6.

Results of tomography of Gaussian state [5]



- Plots of d_1 and d_2 vs ϵ for $\kappa = 0.9$, ensemble sizes 20,10,8,6.

Results of tomography of Gaussian state



- Plots of d_1 and d_2 vs ϵ for $\kappa = 1$, ensemble sizes 20,10,8,6.

Results presented can be found in...

- Das, Debmalya and Arvind, Estimation of quantum states by weak and projective measurements, Phys. Rev. A, 89, 6, 2014.
- Das, Debmalya and Arvind, Weak measurement-based state estimation of Gaussian states of one-variable quantum systems, arXiv:1601.01936, 2016 (accepted in J. Phys. A: Math. Theor.).

Summary

- We have presented a protocol for estimating the state of a qubit, using weak measurements. We show that this gives an advantage over the ususal projective measurement based protocol when the number of copies available is small.
- The advantage of the protocol further increases if we have some prior partial information about the state of the qubit.
- The scheme has been extended to the regime of Gaussian states in which we show that Gaussian states without rotation can be estimated more efficiently that projective measurements for small ensembles.

Thank You