Does CGLMP generalise Bell-CHSH in higher dimensions?

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The short answer? No.

The Question

- CGLMP inequality as measure of non-locality [1]
  - Dimension dependent inequality for arbitrary dimensional systems
  - Exceeds Tsirelson bound $2\sqrt{2}$ for Bell inequality
  - Works along Werner line: maximally entangled state + noise
  - Implicitly assumed by experimentalists in the field to be a generalization of the Bell inequality for higher level systems [eg. Lo (2016)]
- Question: are there states that are Bell-CHSH non-local but are CGLMP local?
- Specifically constructed pure and mixed states that are maximally Bell non-local to see how CGLMP fares
- Result: Contrary to expectations, CGLMP fails to correctly identify a large family of maximally non-local Bell states

[1] Kaszlikowski et. al., Phys. Rev. Lett. 85, 4418 (2000), Collins et. al., Phys. Rev. Lett. 88 040404 (2002)

A quick recap: The Bell-CHSH Inequality

- Consider: $M \times N$ level system with two subsystems $A(B)$ of $M(N)$ levels
- The Bell-CHSH operator: $\mathcal{B} = A_1 B_1 - A_1 B_2 + A_2 B_1 + A_2 B_2$
  - $A_{1,2}$ and $B_{1,2}$ are local observables for the two subsystems
  - Conditions $-1 \leq \langle A_i \rangle \leq 1$ and $-1 \leq \langle B_i \rangle \leq 1$.
- Local hidden variable models constrain the Bell-CHSH function as inequality: $\left| \langle \mathcal{B} \rangle \right| \leq 2$
  - A violation implies non-locality.
  - QM as a non-local theory yields the upper bound to a value $2\sqrt{2}$ [Tsirelson (1980)].
  - (Bound is absolute and independent of M and N)

The CGLMP prescription

- Analog of the Bell-CHSH operator: function I_N, defined for $N \times N$ level system by

$$I_N = \sum_{k=0}^{[N/2]-1} \left(1 - \frac{2k}{N-1}\right)\{[P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k)] - [P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) + P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1)]\}$$

- $P(A_i, B_i)$ – joint measurement probabilities for local observables A_i, B_i
- Measurement prescription [2] involves two local observers, Alice and Bob
  - The measurement bases for the observables A_i and B_i; $i = 1, 2$ are of the form
    $|K\rangle_{A,i} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} exp\left(i \frac{2\pi}{N} j(K + \alpha_i)\right) |j\rangle_A$
    $|L\rangle_{B,i} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} exp\left(i \frac{2\pi}{N} j(-L + \beta_i)\right) |j\rangle_B$.
  - All the observables have integer eigenvalues $0, 1, \cdots, N-1$.
  - Tune variable phases α_i, β_i of the states depending on the measurements they choose

[2] Durt et. al., Phys. Rev. A 64, 024101 (2001)

The CGLMP prescription (cont'd)

- Rules of classical probability $\implies |I_N| \leq 2$.
  - Interpreted as a locality condition; arises in measurements involving joint probabilities.
- For maximally entangled state, $I_{N\geq 3}$ exceeds Tsirelson bound
  - $I_{4\,\mathrm{max}} = 2.8962$
  - $I_{N\to\infty} = 2.9696$
- Interpretation: since $I_4 > 2\sqrt{2}$ for the Bell state, it follows that some noisy states will obey Bell-CHSH but violate CGLMP
  - Led those authors to claim that CGLMP prescription is more general and stronger than the Bell-CHSH prescription
    - True along Werner line
    - Note: experiments are also performed on maximally entangled states

Numerical Probe

- Known conditions on local observables to hit Tsirelson bound [Braunstein (1992), Popescu (1992)]:

$$\begin{aligned} \left\langle A_{1,2}^2 \right\rangle &= \left\langle B_{1,2}^2 \right\rangle = 1 \\ \left\langle \{A_1, A_2\} \right\rangle &\text{ or } \left\langle \{B_1, B_2\} \right\rangle = 0. \end{aligned} \tag{1}$$

- Conditions jointly constitute the definition of Clifford Algebra; the representations are essentially given by the standard Pauli matrices or their direct sums for each pair of observables.
- $\Longrightarrow$ maximally non-local Bell states are either coherent, or incoherent superpositions of Bell states in mutually orthogonal 2×2 sectors. (Explains why there are no fully entangled Bell states when N is odd.)
- Conveniently choose the observables [SU(4) - Sbaih (2013)]:

$$\begin{aligned} A_1 &= \frac{2}{\sqrt{3}}\lambda_8 + \frac{\sqrt{6}}{3}\lambda_{15}; \quad A_2 = (\lambda_4 + \lambda_{11}) \\ B_1 &= \frac{1}{\sqrt{2}}(A_1 + A_2); \quad B_2 = \frac{1}{\sqrt{2}}(A_2 - A_1) \end{aligned} \tag{2}$$

Numerical Probe (cont'd)

- Bell operator eigen-resolution:

$$\mathcal{B} = 2\sqrt{2}(\Pi_{\mathcal{H}_+} - \Pi_{\mathcal{H}_-}) \tag{3}$$

where $dim(\mathcal{H}_\pm) = 4$. The bases for $\mathcal{H}_+$ may be chosen to be

$$\begin{aligned} |\eta_1\rangle &= \frac{1}{\sqrt{2}}(|11\rangle + |33\rangle); \quad |\eta_2\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |32\rangle) \\ |\eta_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |23\rangle); \quad |\eta_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |22\rangle) \end{aligned} \tag{4}$$

- Note: within each sector, *all* states, both pure and mixed, violate the Bell-CHSH inequality maximally.
- The Nelder-Mead optimization technique [Nelder 1965] employed to search for max I_4 per state,
  - Search over 4-D parameter space spanned by phases $\{\alpha_{1,2}, \beta_{1,2}\}$
  - Note: Same technique used in CGLMP papers; numerical results verified

Pure States

- Pure Bell state can take form

$$|\Psi\rangle_{\mathcal{H}_+} = \sum_i c_i |\eta_i\rangle \quad ; \quad \sum_i |c_i|^2 = 1 \tag{5}$$

- 1000 pure states were sampled randomly through uniform distributions
  - Only 8.9% violated CGLMP

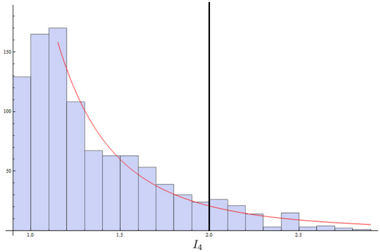


Figure : I_4 values for 1000 randomly sampled pure states, with a bin-width of 0.1. Red: Polynomial fit showing population decay. Black: $I_4 = 2$, demarcation between local and non-local states.

Mixed States

- Sample 100 random mixed states of the form:

$$\rho_{\mathcal{H}_+} = \sum_i p_i |\eta_i\rangle\langle\eta_i|. \tag{6}$$

- The CGLMP prescription fails to identify non-locality, this time more dramatically

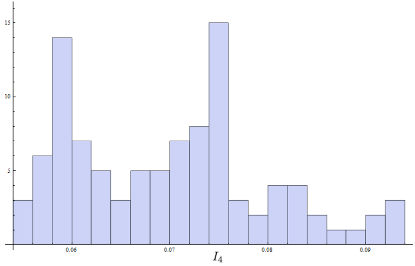
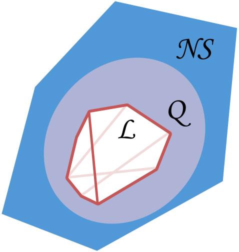


Figure : I_4 values for 100 randomly sampled mixed states, with a bin width 0.002.

This isn't surprising. Let's see why.

Significance of Bell Inequalities



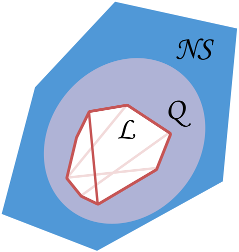
- Consider complete space of *behaviours* system, i.e., all possible sets of joint probabilities $\mathbf{p}=\{p(ab|xy)\}$: a, b –outputs, x, y – inputs of the two subsystems. (Essentially the correlation space.)
- Can be broken up into three non-mutually exclusive categories, depending on conditions that are imposed on the overall system: no-signalling, local, and quantum

- The locality condition constitutes a polytope in the behaviour space, of whose vertices are local deterministic behaviours
- Hyperplanes characterising this polytope constitute a set of inequalities: reveal the non-local nature of a system depending on violation or satisfaction
- In the literature, this set is casually termed "the Bell inequalities" though in actuality they are facet inequalities, combinations of which create a Bell inequality
- One is the Bell-CHSH form and another is the CGLMP form
- In this way it is unsurprising that CGLMP is not a generalization of Bell-CHSH, and works along the Werner line in ways the Bell-CHSH falls short. They address different directions in the behaviour space. [Froissard (1981), Garg and Mermin (1984), Pitowsky (1989), Peres (1999), Werner and Wolf (2001)]

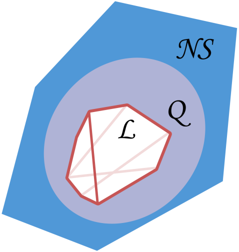
Conclusion

- Contrary to expectations, the CGLMP prescription fails to correctly identify a large family of maximally non-local Bell states
- The CGLMP and Bell-CHSH inequalities are merely two different facets of the Local polytope in the correlation space
- The family of facet inequalities (dubbed 'Bell inequalities' in literature) to give complete distinction of non-locality is as yet to be explored

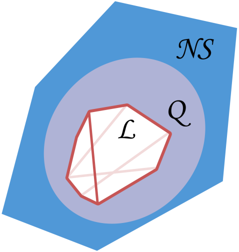
Thank You



- *No-signalling*– Natural limitation that essentially implies: any local marginal probability of one subsystem is not influenced by the measurement setting of the second. Mathematically: $\sum p(ab|xy) = \sum p(ab|xy')\forall$ outputs a, b and inputs x, y, y'. [Tsirelson (1980), Popescu and Rohrlich (1994)]



- *Quantum*– Consists of behaviours governed by QM, i.e., either 1) a tensor product structure can be constructed between the two systems' positive operator valued measurements i.e. $p(ab|xy) = \mathrm{tr}(\rho_{AB} M_{a|x} \otimes M_{b|y})$ or 2) that their local observables as orthogonal projectors commute $[M_{a|x}, M_{b|y}] = 0$. [Tsirelson (1993)]



- *Local*– Locality implies existence of a set of past factors λ shared by the two subsystems that taken together jointly and causally influence the outcomes of measurements performed on each part of the system. Formally, a Local Hidden Variable Theorem: $p(ab|xy) = \int_\Lambda d\lambda q(\lambda) p(a|x,\lambda) p(b|y,\lambda)$, $p(*|*,\lambda)$ – the local marginal probabilities, $q(\lambda)$ – probability density of hidden variables λ in space Λ