Superposition of two single-qubit states with partial prior knowledge using NMR

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Superposition of unknown quantum states

- There is a no-go theorem stating that there exists no protocol to obtain a desired superposition of two unknown pure quantum states, $|\psi_1\rangle$ and $|\psi_2\rangle$ to get[1]:

$$|\Psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle. \tag{1}$$

- However, with partial prior knowledge, $|\langle\chi|\psi_1\rangle|$ and $|\langle\chi|\psi_2\rangle|$, one can obtain superposed state of the form,

$$|\Psi'\rangle = \frac{N_\psi}{N}\left(a\frac{\langle\chi|\psi_2\rangle}{|\langle\chi|\psi_2\rangle|}|\psi_1\rangle + b\frac{\langle\chi|\psi_1\rangle}{|\langle\chi|\psi_1\rangle|}|\psi_2\rangle\right),$$

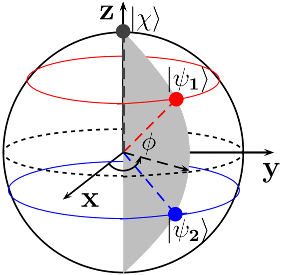
  where $|\chi\rangle$ is a known referential state.

- Such a protocol is proposed recently by Oszmaniec et al (*Phys. Rev. Lett.* **116**, 110403 (2016)).

[1] Phys. Rev. Lett. **116**, 110403 (2016).

The problem

- Input states: $|\psi_1\rangle$ and $|\psi_2\rangle$. Let $|\chi\rangle$ be a reference state, whose respective overlaps $\langle\chi|\psi_1\rangle$ and $\langle\chi|\psi_2\rangle$ are known.



- We presented an experimentally feasible protocol to superpose n pure states of a qu*d*it and experimentally performed the superposition of two pure single-qubit states on a two-qubit NMR quantum information processor[2].
- Calculated success probabilities for different prior information.

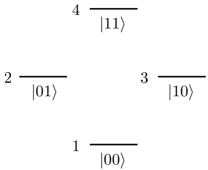
[2] S. Dogra, G. Thomas, S. Ghosh, and D. Suter, arXiv:1702.02418 [quant-ph] (2017).

Theoretical scheme

(A) Encoding

- Consider a system of two coupled spins 1/2 with the Hamiltonian

$$H = -\Omega_A A_Z - \Omega_X X_Z + J A_Z X_Z$$



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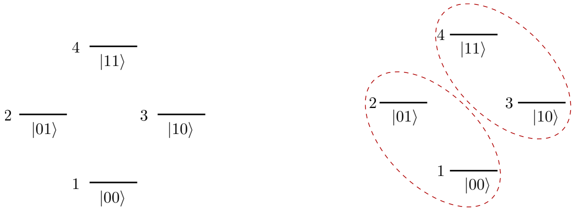

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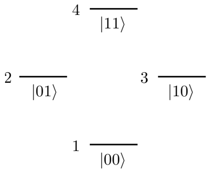

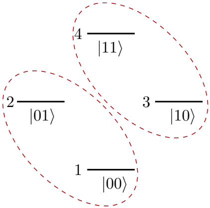

- Input states: $\psi_1 = c_{00}|0\rangle + c_{01}|1\rangle$ and $\psi_2 = c_{10}|0\rangle + c_{11}|1\rangle$.

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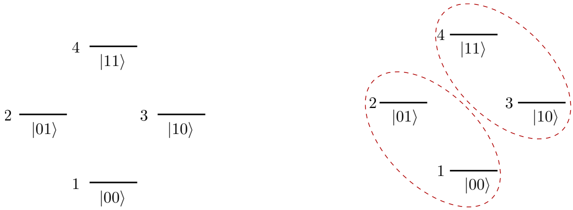

- Input states: $\psi_1 = c_{00}|0\rangle + c_{01}|1\rangle$ and $\psi_2 = c_{10}|0\rangle + c_{11}|1\rangle$. State of two-qubit system,

$$|\eta\rangle = a|0\rangle|\psi_1\rangle + b|1\rangle|\psi_2\rangle, \qquad \sqrt{a^2 + b^2} = 1.$$

- Here first qubit is considered as ancilla, and second qubit is considered as system-qubit.

(B) Removing the impalpable global phases

- Assuming the global phases ($e^{\iota\gamma_1}$ and $e^{\iota\gamma_2}$) with states $|\psi_1\rangle$ and $|\psi_2\rangle$ respectively,

$$|\eta'\rangle = e^{\iota\gamma_1} a|0\rangle|\psi_1\rangle + e^{\iota\gamma_2} b|1\rangle|\psi_2\rangle.$$

- $e^{\iota\gamma_1} = \frac{\langle\chi|\psi_1\rangle}{|\langle\chi|\psi_1\rangle|}$ and $e^{\iota\gamma_2} = \frac{\langle\chi|\psi_2\rangle}{|\langle\chi|\psi_2\rangle|}$.
- A phase gate, $(\gamma_1 - \gamma_2)_z^1$ leads to,

$$|\eta''\rangle = e^{\iota(\frac{\gamma_1+\gamma_2)}{2}}(a|0\rangle|\psi_1\rangle + b|1\rangle|\psi_2\rangle).$$

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(C) Superposition

- A Hadamard gate on the first qubit leads to,

$$|\eta'''\rangle = |0\rangle \otimes (a|\psi_1\rangle + b|\psi_2\rangle) + |1\rangle \otimes (a|\psi_1\rangle - b|\psi_2\rangle).$$

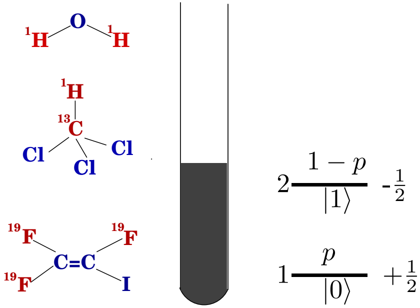
- First qubit measurement in basis $\{|0\rangle, |1\rangle\}$ gives the desired result,

$$N_\psi(a|\psi_1\rangle + b|\psi_2\rangle) \propto |\Psi\rangle.$$

Experimental realization

- We use a system of two nuclear spins and NMR as a tool to demonstrate the superposition of two single-qubit states.
- NMR quantum computer harnesses the intrinsic magnetic properties of the nuclear spins, manipulated by specifically tailored radio frequency pulses (quantum gates) to perform the desired computational tasks.
- The criteria for a quantum computer to work was put forward by David P. DiVincenzo in 2000. This involves: a working medium, ample coherence times, a set of implementable operations, ability to initialize the system in a desired state, and the ability to read the output.

- **System**
- Quantum Gates
- Initial state preparation
- Final readout



NMR Quantum Computation

- System
- **Quantum Gates**
- Initial state preparation
- Final readout

- Hadamard $\quad (\frac{\pi}{2})_y$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Controlled NOT $\quad (\frac{\pi}{2})^2_x \, \frac{1}{2J} \, (\frac{\pi}{2})^2_{-y}$

$$CNOT_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- System
- Quantum Gates
- **Initial state preparation**
- Final readout

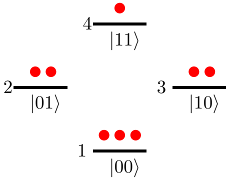
Pseudopure state $|\psi\rangle$

$$\rho = \frac{1-\epsilon}{n} I_{n\times n} + \epsilon|\psi\rangle\langle\psi|$$

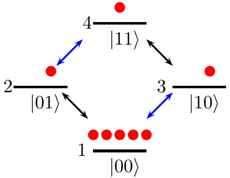
Eigenvalues in a two-qubit system:
$|00\rangle$: $\frac{1}{2}(\omega_1 + \omega_2 + \frac{1}{2}J_{12})$
$|01\rangle$: $\frac{1}{2}(\omega_1 - \omega_2 - \frac{1}{2}J_{12})$
$|10\rangle$: $\frac{1}{2}(-\omega_1 + \omega_2 - \frac{1}{2}J_{12})$
$|11\rangle$: $\frac{1}{2}(-\omega_1 - \omega_2 + \frac{1}{2}J_{12})$

Transitions:
Qubit-1: $\omega_1 \pm \frac{1}{2}J_{12}$
Qubit-2: $\omega_2 \pm \frac{1}{2}J_{12}$

Thermal equilibrium



Pseudo-pure state



- System
- Quantum Gates
- Initial state preparation
- **Final readout**

Tomography

$$\rho = \begin{pmatrix} a_{11} & a_{12} + \iota b_{12} & a_{13} + \iota b_{13} & a_{14} + \iota b_{14} \\ a_{12} - \iota b_{12} & a_{22} & a_{23} + \iota b_{23} & a_{24} + \iota b_{24} \\ a_{13} - \iota b_{13} & a_{23} - \iota b_{23} & a_{33} & a_{34} + \iota b_{34} \\ a_{14} - \iota b_{13} & a_{24} - \iota b_{24} & a_{34} - \iota b_{34} & a_{44} \end{pmatrix}$$

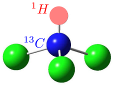
Fidelity:

$$F = \frac{Tr(\rho^{\dagger}_{\text{theory}}\rho_{\text{expt}})}{\sqrt{(Tr(\rho^{\dagger}_{\text{theory}}\rho_{\text{theory}}))}\sqrt{(Tr(\rho^{\dagger}_{\text{expt}}\rho_{\text{expt}}))}}$$

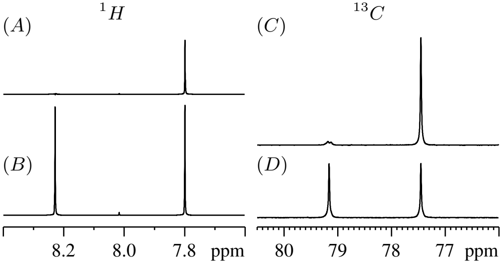
where ρ_{theory} and ρ_{expt} denote the theoretical and experimental density matrices respectively.

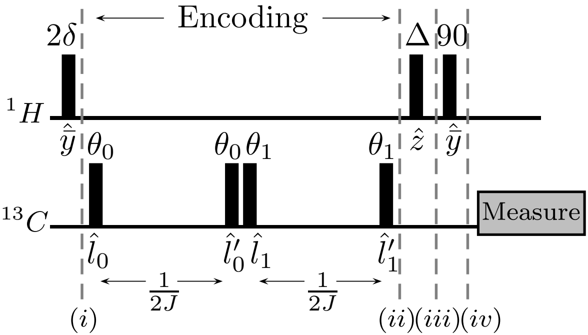
Experimental system

- A two-qubit system:



- (A), (C): Pseudopure state NMR spectra with a small detection angle
  (B), (D): Thermal equilibrium NMR spectra

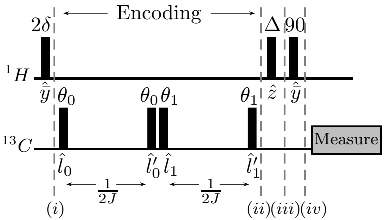




$\hat{l}_0 = \cos(\frac{3\pi}{2} + \phi_0)\hat{x} + \sin(\frac{3\pi}{2} + \phi_0)\hat{y}$,
$\hat{l}'_0 = \cos(\phi_0)\hat{x} + \sin(\phi_0)\hat{y}$,
$\hat{l}_1 = \cos(\pi + \phi_1)\hat{x} + \sin(\pi + \phi_1)\hat{y}$, and
$\hat{l}'_1 = \cos(\frac{\pi}{2} + \phi_1)\hat{x} + \sin(\frac{\pi}{2} + \phi_1)\hat{y}$.

Experimental design

- Various different pairs of states are encoded.
- A two-qubit tomography at the end of step (iii).
- A two-qubit tomography at the end of step (iv).
- A single-qubit partial quantum state tomography mimicing the single-qubit measurement.
- Single-qubit measurement: subspace spanned by $|00\rangle$, $|01\rangle$.
- Normalization: $G_z(\frac{\pi}{2})_y^1$

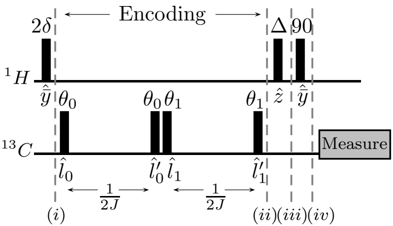


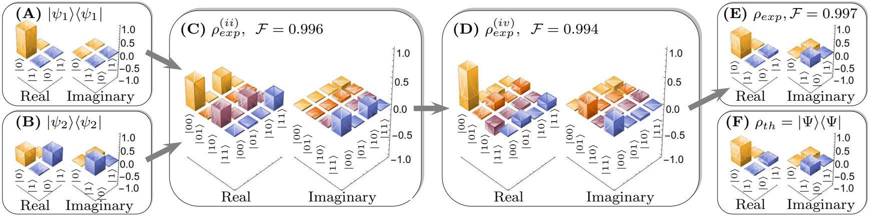
$$\rho = \begin{pmatrix} a_{11} & a_{12}+\iota b_{12} & a_{13}+\iota b_{13} & a_{14}+\iota b_{14} \\ a_{12}-\iota b_{12} & a_{22} & a_{23}+\iota b_{23} & a_{24}+\iota b_{24} \\ a_{13}-\iota b_{13} & a_{23}-\iota b_{23} & a_{33} & a_{34}+\iota b_{34} \\ a_{14}-\iota b_{13} & a_{24}-\iota b_{24} & a_{34}-\iota b_{34} & a_{44} \end{pmatrix}$$

(i) Direct readout, and (ii) $G_z(\frac{\pi}{2})_y^2$

Example

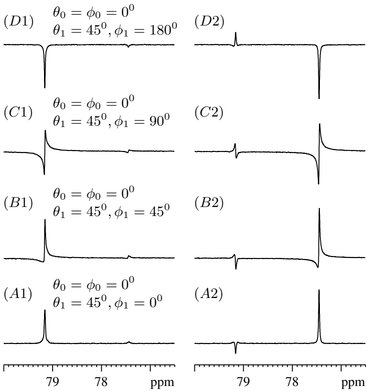
- $|\psi_1\rangle = |0\rangle$
- $\frac{1}{\sqrt{2}}(|0\rangle + e^{\frac{\iota\pi}{2}}|1\rangle)$
- $a = b$





Experimental outcomes

- Input states:
  $|\psi_1\rangle = |0\rangle$
  $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{\iota\phi_2}|1\rangle)$
- $|\Psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle$
- $a = b$



	ϕ_2	$\mathcal{F}_c$	$\rho_{th} = \|\Psi\rangle\langle\Psi\|$	$\rho_{exp} + \frac{1}{4}\mathbb{I}_{2\times 2}$	$\mathcal{F}_{final}$
1	0	0.993	$\begin{pmatrix} 0.73 & 0.3 \\ 0.3 & 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.77 & 0.28 \\ 0.28 & 0.10 \end{pmatrix}$	0.996
2	$\frac{\pi}{4}$	0.990	$\begin{pmatrix} 0.73 & 0.21 - 0.21i \\ 0.21 + 0.21i & 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.76 & 0.19 - 0.23i \\ 0.19 + 0.23i & 0.08 \end{pmatrix}$	0.995
3	$\frac{\pi}{2}$	0.996	$\begin{pmatrix} 0.73 & 0. - 0.3i \\ 0. + 0.3i & 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.76 & 0. - 0.3i \\ 0. + 0.3i & 0.07 \end{pmatrix}$	0.997
4	π	0.994	$\begin{pmatrix} 0.73 & -0.3 \\ -0.3 & 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.76 & -0.29 \\ -0.29 & 0.08 \end{pmatrix}$	0.997

Experimental outcomes....different weights

$$|\psi_1\rangle = \frac{1}{2}(|0\rangle + \sqrt{3}|1\rangle)$$
$$|\psi_2\rangle = \frac{1}{2}(\sqrt{3}|0\rangle + |1\rangle)$$

Table : Here are the various experimental outcomes. $|\Psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle$. As per column 4, values of $2\delta = 90^0$, 53.13^0, and 36.87^0 correspond to the weights $a = b$, $a = 2b$ and $a = 3b$ respectively. $\mathcal{F}_c$ is the fidelity with which input states are encoded and and $\mathcal{F}_{final}$ is the fidelity of final superposed state.

$a:b$	2δ	$\mathcal{F}_c$	$\rho_{th} = \|\Psi\rangle\langle\Psi\|$	$\rho_{exp} + \frac{1}{4}\mathbb{I}_{2\times 2}$	$\mathcal{F}_{final}$
1 : 1	90^0	0.993	$\begin{pmatrix} 0.47 & 0.47 \\ 0.47 & 0.47 \end{pmatrix}$	$\begin{pmatrix} 0.47 & 0.46 \\ 0.46 & 0.51 \end{pmatrix}$	0.998
2 : 1	53.13^0	0.988	$\begin{pmatrix} 0.35 & 0.42 \\ 0.42 & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.37 & 0.42 \\ 0.42 & 0.52 \end{pmatrix}$	0.999
3 : 1	36.87^0	0.984	$\begin{pmatrix} 0.28 & 0.37 \\ 0.37 & 0.48 \end{pmatrix}$	$\begin{pmatrix} 0.31 & 0.37 \\ 0.37 & 0.5 \end{pmatrix}$	0.999

Experimental outcomes...with assumed global phases

- $|\psi_1\rangle = e^{\iota\gamma_1}(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle)$ and $|\psi_2\rangle = e^{\iota\gamma_2}(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle)$

γ_1	γ_2	$\mathcal{F}_c$	$\rho_{th} = \|\Psi\rangle\langle\Psi\|$	$\rho_{exp} + \frac{1}{4}\mathbb{I}_{2\times 2}$	$\mathcal{F}_{final}$
0	0	0.993	$\begin{pmatrix} 0.47 & 0.47 \\ 0.47 & 0.47 \end{pmatrix}$	$\begin{pmatrix} 0.47 & 0.46 \\ 0.46 & 0.51 \end{pmatrix}$	0.998
0	$\frac{2\pi}{3}$	0.991	$\begin{pmatrix} 0.47 & 0.47 \\ 0.47 & 0.47 \end{pmatrix}$	$\begin{pmatrix} 0.46 & 0.46 \\ 0.46 & 0.46 \end{pmatrix}$	0.999

- $|\psi_1\rangle = \frac{e^{\iota\gamma_1}}{2}(|0\rangle + e^{\frac{\iota\pi}{4}}\sqrt{3}|1\rangle)$, $|\psi_2\rangle = \frac{e^{\iota\gamma_2}}{2}(\sqrt{3}|0\rangle + e^{\frac{\iota 2\pi}{3}}|1\rangle)$

$$\rho_{th} = |\Psi\rangle\langle\Psi| = \begin{pmatrix} 0.47 & 0.12 - 0.36i \\ 0.12 + 0.36i & 0.31 \end{pmatrix}$$

γ_1	γ_2	$\mathcal{F}_c$	$\rho_{exp} + \frac{1}{4}\mathbb{I}_{2\times 2}$	$\mathcal{F}_{final}$
0	0	0.988	$\begin{pmatrix} 0.44 & 0.21 - 0.33i \\ 0.21 + 0.33i & 0.35 \end{pmatrix}$	0.974
0	$\frac{2\pi}{3}$	0.989	$\begin{pmatrix} 0.46 & 0.18 - 0.29i \\ 0.18 + 0.29i & 0.29 \end{pmatrix}$	0.981

Superposition of n-qudits

- d-dimensional states: $|\Psi_1\rangle_d$, $|\Psi_2\rangle_d$, $\ldots$ $|\Psi_n\rangle_d$
- a_1, $a_2, \ldots a_n$ be the desired weights
- Consider d-dimensional referential state $|\chi\rangle_d$.
- Known overlaps, $|\langle\chi|\Psi_j\rangle_d|^2 = c_j$, where $j \in \{1, 2, \ldots, n\}$
- We begin with the initial state,

$$\frac{1}{N}(a_1'|0\rangle_n + a_2'|1\rangle_n + \cdots + a_n'|n-1\rangle_n) \otimes |\Psi_1\rangle_d \otimes \cdots \otimes |\Psi_n\rangle_d$$

$N = \sqrt{\sum_{j=1}^{n} a_j'^2}$ and $a_k' = \frac{a_k}{\sqrt{\prod_{(j\neq k, j=1)}^{n} c_j}}$

Superposition of n-qudits

1. **Controlled-swap operations**[3], $\mathcal{CS}^1_{2,3}\ \mathcal{CS}^1_{2,4}\dots\mathcal{CS}^1_{2,n}$

$$
\begin{aligned}
1/N \qquad & (a'_1|0\rangle_n \otimes |\Psi_1\rangle_d \otimes |\Psi_2\rangle_d \otimes |\Psi_3\rangle_d \otimes \cdots \otimes |\Psi_n\rangle_d \\
& + a'_2|1\rangle_n \otimes |\Psi_2\rangle_d \otimes |\Psi_1\rangle_d \otimes |\Psi_3\rangle_d \otimes \cdots \otimes |\Psi_n\rangle_d \\
& + \dots\dots\dots \\
& + a'_n|n-1\rangle_n \otimes |\Psi_n\rangle_d \otimes |\Psi_3\rangle_d \otimes \cdots \otimes |\Psi_1\rangle_d).
\end{aligned}
$$

2. **Projective measurements,** $I_{n\times n} \otimes I_{d\times d} \otimes \bigotimes_{k=3}^{n} (|\chi\rangle_d\langle\chi|_d)_k$

$$
\frac{1}{N}\sum_{k=1}^{n}\left(a_k\left(\prod_{(j\neq k, j=1)}^{n} \frac{\langle\chi|\Psi_j\rangle_d}{\sqrt{c_j}}\right)|k-1\rangle_n|\Psi_k\rangle_d\right)\bigotimes_{m=1}^{n-1}|\chi\rangle_d
$$

$$
\frac{1}{N}(a_1|0\rangle_n|\Psi_1\rangle_d + a_2|1\rangle_n|\psi_2\rangle_d + \cdots + a_n|n-1\rangle_n|\psi_n\rangle_d)
$$

3. **Fourier transformation**

$$
\frac{1}{N\sqrt{n}}\sum_{j=0}^{n-1}\left(|j\rangle_n \otimes \sum_{k=1}^{n}\left(f^{j(k-1)}a_k|\Psi_k\rangle_d\right)\right)
$$

4. **Projective measurement:** $|0\rangle_n\langle 0|_n \otimes \mathbb{I}_{d\times d}$

[3] Phys. Rev. Lett. **116**, 110403 (2016).

Superposition of n-qudits

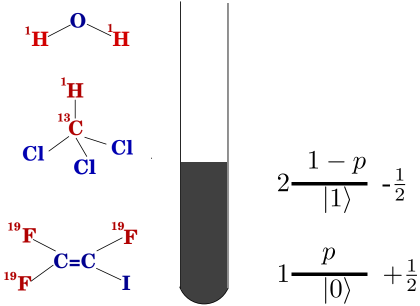
Final state: $|\Psi\rangle = \frac{N_\Psi}{N\sqrt{n}} \sum_{k=1}^{n} a_k \left(\prod_{(j\neq k, j=1)}^{n} \frac{\langle \chi | \Psi_j \rangle_d}{\sqrt{c_j}} \right) |\Psi_k\rangle_d$

Success probability[4],

$$P = \frac{N_\Psi^2}{N^2 n} = \frac{\prod_{j=1}^{n} c_j}{\sum_{j=1}^{n} a_j^2 c_j} \frac{N_\Psi^2}{n}.$$

[4] S. Dogra, G. Thomas, S. Ghosh, and D. Suter, arXiv:1702.02418 [quant-ph] (2017).

Thank you!



1H O 1H
1H 13C Cl Cl Cl
19F 19F C=C 19F I
2 $1-p$ $|1\rangle$ $-\frac{1}{2}$
1 p $|0\rangle$ $+\frac{1}{2}$