A Common Framework for Non-classicality

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Why ?

- The widely divergent use of the term *non-classicality*.
- A hierarchical structure shown by certain *manifestations* of *non-classicality*.
- Is there an underlying framework from which all the manifestations of nonclassicality may emerge ?
- Our approach: Distinction between classical and quantum probabilities.

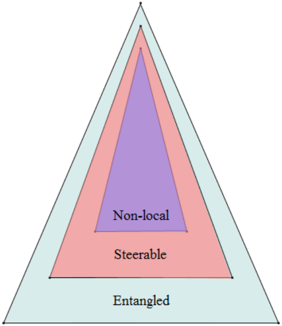
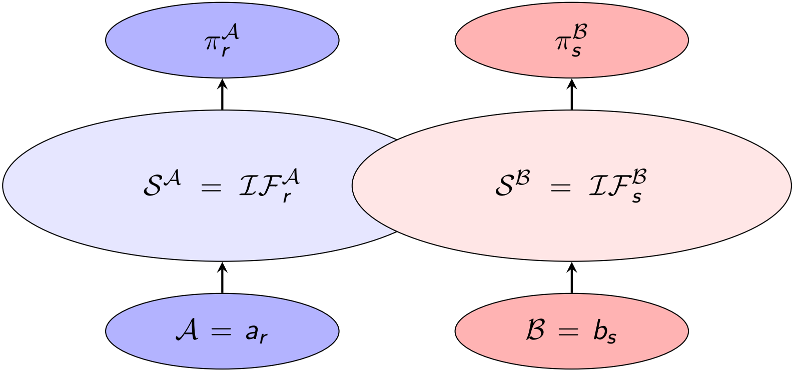


Figure: The manifestations of nonclassicality

Classical Probability vs. Quantum Probability

- Observables: $\mathcal{A}$; $\mathcal{B}$
- Outcomes: $\{a_1, a_2, ..., a_N\}$; $\{b_1, b_2, ..., b_N\}$.
- Event 1: $\mathcal{A} = a_r$
  Indicator function: $\mathcal{IF}_r^A \rightarrow$ Region in the classical phase space $\mathcal{S}_r^A$
  In QM: $\mathcal{IF}_r^A \rightarrow \pi_r^A$.
- Event 2: $\mathcal{B} = b_s$
  Indicator function: $\mathcal{IF}_s^B \rightarrow$ Region in the classical phase space $\mathcal{S}_s^B$
  In QM: $\mathcal{IF}_s^B \rightarrow \pi_s^B$.
- Event 3: $\mathcal{A} = a_r$ and $\mathcal{B} = b_s$
  Indicator function: $\mathcal{IF}_{rs}^{AB} \rightarrow \mathcal{S}_r^A \cap \mathcal{S}_s^B$
  In QM: $\mathcal{IF}_{rs}^{AB} \rightarrow$???.

Classical Probability Vs. Quantum Probability



If, $\mathcal{S}^{\mathcal{A}} = \mathcal{IF}_r^{\mathcal{A}} \rightarrow \pi_r^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}} = \mathcal{IF}_s^{\mathcal{B}} \rightarrow \pi_s^{\mathcal{B}}$, $\mathcal{S}^{\mathcal{A}} \cap \mathcal{S}^{\mathcal{B}} = \mathcal{IF}_{rs}^{\mathcal{AB}} \rightarrow$???

Pseudo-Projection Operators

$$\mathcal{IF}_{rs}^{AB} \rightarrow \Pi_{rs}^{AB} \equiv \frac{1}{2}\left[\pi_r^A \pi_s^B + \pi_s^B \pi_r^A\right] \qquad (1)$$

- The hermitian representative of $\mathcal{IF}_{rs}^{AB}$ is not a projection operator in general. $(\Pi_{rs}^{AB})^2 \neq \Pi_{rs}^{AB}$.
- Π_{rs}^{AB} is not a positive operator.
- Nevertheless Π_{rs}^{AB} preserves total probability and reduces to a valid projection operator for commuting observables.
- We call Π_{rs}^{AB}, pseudo-projection operators.
- Given a state ρ, $\mathcal{P}_{rs}^{\rho} = Tr[\Pi_{rs}^{AB}\rho]$ are the corresponding pseudprobabilities. The collection of all $\{\mathcal{P}_{rs}^{\rho}\}$ for all outcomes of $\mathcal{A}$ and $\mathcal{B}$ constitutes the pseudo-joint-probability scheme.

Nonclassicality: Definitions

Definition 1.

For a given ρ, if the set of all pseudoprobabilities, $\{\mathcal{P}^{\rho}_{rs}\}$ has no non-negative entries, there exists a classical joint probability scheme that mimics all it's properties. The corresponding state ρ is deemed classical.

Definition 2.

Naturally ρ is nonclassical even if one $\mathcal{P}^{\rho}_{rs}$ is negative.

Example: A Qubit

- The state: $\rho = \frac{1}{2}\left(1 + \vec{P}\cdot\vec{\sigma}\right)$
- Observables: $A_i = \vec{\sigma}\cdot\hat{m}_i \quad ; \quad (i = 1, 2, 3)$.
- Projectors: $\pi_i^{\pm} = \frac{1}{2}\left(1 \pm \vec{\sigma}\cdot\hat{m}_i\right) \implies Tr[\rho\pi_i^{\pm}] \geq 0, \forall |\vec{P}|$.
- Bilinear Pseudoprojectors:
  $\Pi_{ij}^{\pm\pm} = \frac{1}{2}\left(1 \pm \vec{\sigma}\cdot(\pm\hat{m}_i \pm \hat{m}_j)\right) \implies Tr[\rho\Pi_{ij}^{\pm\pm}] \geq 0, \forall |\vec{P}| \leq \frac{1}{\sqrt{2}}$.
- Trilinear Pseudoprojectors: $\Pi_{123}^{\pm\pm\pm} =$
  $\frac{1}{2}\left(1 \pm \vec{\sigma}\cdot(\pm\hat{m}_1 \pm \hat{m}_2 \pm \hat{m}_3)\right) \implies Tr[\rho\Pi_{123}^{\pm\pm\pm}] \geq 0, \forall |\vec{P}| \leq \frac{1}{\sqrt{3}}$.
- For the given state ρ, if a pseudo JPS corresponding to trilinear pseudoprojectors has all nonzero entries, the entries in a pseudo JPS corresponding to all possible pairs of bilinear pseudoprojectors has to be positive. The converse however is not true.

Bipartite Systems

$$\Pi^{(\alpha\beta)}(\hat{m}_1 \ldots \hat{m}_\alpha; \hat{n}_1 \ldots \hat{n}_\beta) = \Pi^{(\alpha)}(\hat{m}_i \ldots \hat{m}_\alpha) \otimes \Pi^{(\beta)}(\hat{n}_1 \ldots \hat{n}_\beta)$$

- The pseudoprojection operator for a bipartite system is simply the direct product of the pseudoprojections in the individual subsystems.
- The positivity of the corresponding pseudo JPS determines the classicality for the bipartite states, following Definition-1.
- The conditions of nonclassicality that we achieve from above are an independent set of conditions. Blind to the conventionally known criterion.
- Can the pseudoprobabilities behave as a repository for known measures of nonclassicality like nonlocality etc. ?

Bipartite Systems (Contd.)

- Additional Condition:
  $\int d\mu(\hat{m}_1 \ldots \hat{m}_\alpha; \hat{n}_1 \ldots \hat{n}_\beta)\mathcal{P}_\rho^{\alpha\beta}(\hat{m}_1 \ldots \hat{m}_\alpha; \hat{n}_1 \ldots \hat{n}_\beta) \geq 0 \ \ ; \ \ \int d\mu = 1.$
- This tantamounts to the construction of operators:
  $W^{\alpha\beta} = \int d\mu(\hat{m}_1 \ldots \hat{m}_\alpha; \hat{n}_1 \ldots \hat{n}_\beta)\Pi^{\alpha\beta}(\hat{m}_1 \ldots \hat{m}_\alpha; \hat{n}_1 \ldots \hat{n}_\beta)$,
  such that $Tr[W^{\alpha\beta}\rho] \geq 0$
- For a suitable choice of measures, $W^{\alpha\beta}$ behaves as a witness for known nonclassicality criterion.
- Also by varying α, β values in each of the subsystems, one can explore the hierarchy among the various manifestations of nonclassicality.

Nonlocality and W^{12}

- $\Pi^{12}(\hat{m}_1; \hat{n}_1 \hat{n}_2) = \frac{1}{8}\left\{1 + \vec{\sigma} \cdot \hat{m}_1\right\} \otimes \left\{1 + \vec{\Sigma} \cdot (\hat{n}_1 + \hat{n}_2)\right\}$
- $\mu(\hat{m}_1; \hat{n}_1, \hat{n}_2) = \mu(-\hat{m}_1; -\hat{n}_1, -\hat{n}_2) = \mu(\hat{m}_2; \hat{n}_1, -\hat{n}_2) = \mu(-\hat{m}_2; -\hat{n}_1, \hat{n}_2) = 1/4.$
- $W^{12} = \frac{1}{16}\left\{2 + (\vec{\sigma} \cdot \hat{m}_1)(\vec{\Sigma} \cdot (\hat{n}_1 + \hat{n}_2)) + (\vec{\sigma} \cdot \hat{m}_2)(\vec{\Sigma} \cdot (\hat{n}_1 - \hat{n}_2))\right\}$
- W^{12} is the standard Bell-CHSH witness.
- An observation: The extent of 'negativity' in this pseudo probability scheme - $\sum_i |\mathcal{P}_\rho|_i - 1$ is exactly the advantage that one gets while playing a CHSH game with a shared Bell state over the best classical strategy.

Construction of Other $W^{\alpha\beta}$

- $\vec{M}^{(\alpha)} = \pm\hat{m}_1 ... \pm \hat{m}_\alpha \quad ; \quad \vec{N}^{(\beta)} = \pm\hat{n}_1 ... \pm \hat{n}_\beta .$
- $W^{13} = \frac{1}{16}\Big\{2 + (\vec{\sigma}\cdot\hat{m})(\vec{\Sigma}\cdot\vec{N}^{(3)}) + (\vec{\sigma}\cdot\hat{m}')(\vec{\Sigma}\cdot\vec{N}'^{(3)})\Big\}$
- $W^{22} = \frac{1}{16}\Big\{2 + (\vec{\sigma}\cdot\vec{M}^{(2)})(\vec{\Sigma}\cdot\vec{N}^{(2)}) + (\vec{\sigma}\cdot\vec{M}'^{(2)})(\vec{\Sigma}\cdot\vec{N}'^{(2)})\Big\}$
- $W^{23} = \frac{1}{32}\Big\{3 + (\vec{\sigma}\cdot\vec{M}^{(2)})(\vec{\Sigma}\cdot\vec{N}^{(3)}) + (\vec{\sigma}\cdot\vec{M}'^{(2)})(\vec{\Sigma}\cdot\vec{N}'^{(3)}) + (\vec{\sigma}\cdot\vec{M}''^{(2)})(\vec{\Sigma}\cdot\vec{N}''^{(3)})\Big\}$
- $W^{33} = \frac{1}{64}\Big\{3 + (\vec{\sigma}\cdot\vec{M}^{(3)})(\vec{\Sigma}\cdot\vec{N}^{(3)}) + (\vec{\sigma}\cdot\vec{M}'^{(3)})(\vec{\Sigma}\cdot\vec{N}'^{(3)}) + (\vec{\sigma}\cdot\vec{M}''^{(3)})(\vec{\Sigma}\cdot\vec{N}''^{(3)})\Big\}$

Werner states

- $\varrho_w = \frac{1}{4}\left\{1 - \eta\vec{\sigma}\cdot\vec{\Sigma}\right\}$
- $Tr[W^{12}\varrho_w] \geq 0 \;\; \forall \;\; \eta \leq \frac{1}{\sqrt{2}}$
- $Tr[W^{13}\varrho_w] \geq 0 \;\; \forall \;\; \eta \leq \frac{1}{\sqrt{3}}$
- $Tr[W^{22}\varrho_w] \geq 0 \;\; \forall \;\; \eta \leq \frac{1}{2}$
- $Tr[W^{23}\varrho_w] \geq 0 \;\; \forall \;\; \eta \leq \frac{1}{\sqrt{6}}$
- $Tr[W^{33}\varrho_w] \geq 0 \;\; \forall \;\; \eta \leq \frac{1}{3}$

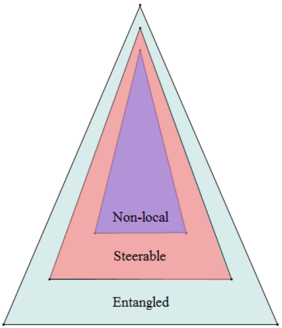


Figure: Manifestations of nonclassicality

Thank You !