Ensemble discrimination via selective random rotations and projective measurements [1]

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February 27, 2017

[1] viXra:1702.0086

Outline

What is the problem?

Ensemble $\mathcal{E}_1$

$N_{|0\rangle}$ number of $|0\rangle s$

and

$N_{|1\rangle}$ number of $|1\rangle s$

$$N_{|0\rangle} + N_{|1\rangle} = N,$$

$$N_{|0\rangle} / N \cong 1/2, \quad N_{|1\rangle} / N \cong 1/2,$$

$|0\rangle, |1\rangle$ are eigenkets of σ_z .

Ensemble $\mathcal{E}_2$

$N_{|+\rangle}$ number of $|+\rangle s$

and

$N_{|-\rangle}$ number of $|-\rangle s$

$$N_{|+\rangle} + N_{|-\rangle} = N,$$

$$N_{|+\rangle} / N \cong 1/2, \quad N_{|-\rangle} / N \cong 1/2,$$

$$|\pm\rangle = (|0\rangle \pm |1\rangle) / \sqrt{2}.$$

We need to discriminate between $\mathcal{E}_1$ and $\mathcal{E}_2$.

Individual v/s Collective control

Individual control

Collective control

Collective control - Nonselective operations((non)U)

Cannot address and control each of the N qubits in the ensemble separately:

$\mathcal{E}_1 \cong 1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| = I/2$

$|1\rangle \cdot |0\rangle |1\rangle \cdot |1\rangle$
$|1\rangle |1\rangle |0\rangle |1\rangle |0\rangle |0\rangle$
$\cdot \cdot |0\rangle |0\rangle \cdot$
$\cdot |0\rangle \cdot |0\rangle |1\rangle$
$|0\rangle \cdot$

For Alice, who prepares the state: Missing info=max, disorder=max

$\mathcal{E}_2 \cong 1/2|+\rangle\langle +| + 1/2|-\rangle\langle -| = I/2$

$|-\rangle |+\rangle |-\rangle \cdot |-\rangle$
$\cdot \cdot |+\rangle |-\rangle |-\rangle \cdot$
$|+\rangle |-\rangle \cdot |+\rangle |+\rangle$
$\cdot |-\rangle \cdot \cdot$
$|+\rangle |+\rangle |-\rangle |-\rangle$

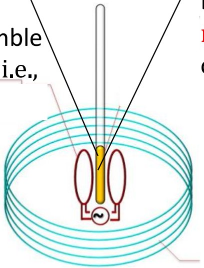
E.g., by measuring σ_z nonselectively on an ensemble of N identical copies of $|+\rangle$ i.e.,

$$|0\rangle\langle 0|+\rangle\langle +|0\rangle\langle 0| + |1\rangle\langle 1|+\rangle\langle +|1\rangle\langle 1| = I/2.$$

E.g., by measuring σ_x nonselectively on an ensemble of N identical copies of $|0\rangle$ i.e.,

$$|+\rangle\langle +|0\rangle\langle 0|+\rangle\langle +| + |-\rangle\langle -|0\rangle\langle 0|-\rangle\langle -| = I/2.$$

NMR spin ensemble $\longrightarrow$



Individual control - Selective operations((non) U)

Can address and control each of the N qubits in the ensemble separately ($\equiv$ going to 2^N-D Hilbert space):

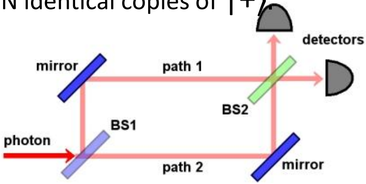
$$\mathcal{E}_1 = (|0\rangle|0\rangle|1\rangle\ldots|0\rangle)(|1\rangle|1\rangle|1\rangle\ldots|0\rangle)\ldots \qquad \mathcal{E}_2 = (|+\rangle|-\rangle|+\rangle\ldots|+\rangle)(|+\rangle|-\rangle|-\rangle\ldots|-\rangle)\ldots$$

$$\begin{array}{|c|c|c|c|c|}
\hline
|0\rangle & |1\rangle & |1\rangle & \cdots & |0\rangle \\
|0\rangle & |1\rangle & |0\rangle & & |0\rangle \\
|1\rangle & |1\rangle & |1\rangle & & |0\rangle \\
\vdots & \vdots & \vdots & & \vdots \\
|0\rangle & |0\rangle & |0\rangle & & |1\rangle \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|}
\hline
|+\rangle & |+\rangle & |-\rangle & \cdots & |+\rangle \\
|-\rangle & |-\rangle & |-\rangle & & |-\rangle \\
|+\rangle & |-\rangle & |+\rangle & & |-\rangle \\
\vdots & \vdots & \vdots & & \vdots \\
|+\rangle & |-\rangle & |+\rangle & & |-\rangle \\
\hline
\end{array}$$

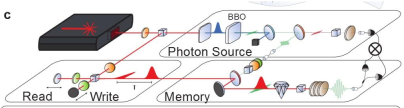
For Alice, who prepares the state: Missing info=0, disorder=0.

E.g., by measuring σ_z selectively on an ensemble of N identical copies of $|+\rangle$.

E.g., by measuring σ_x selectively on an ensemble of N identical copies of $|0\rangle$.



No interaction b/w qubits. Hence we call ensemble.
Information loss: Individual to collective control.



Discrimination via variance of sample mean

- $\langle \sigma_z \rangle_{|\psi\rangle} = p^+ \times (+1) + p^- \times (-1)$,
  $(\Delta\sigma_z)^2_{|\psi\rangle} = \langle \sigma_z^2 \rangle_{|\psi\rangle} - \langle \sigma_z \rangle^2_{|\psi\rangle}$.
- $S = (T^+ - T^-)/M$, $\langle S \rangle = \langle \sigma_z \rangle_{|\psi\rangle}$, $\Delta S^2 = (\Delta\sigma_z)^2_{|\psi\rangle}/M$. As M increases, ΔS^2 decreases, and hence S approaches $\langle \sigma_z \rangle_{|\psi\rangle}$.
- Hence we implicitly neglect variance ΔS^2.
- We will show that, even though variances got via $\mathcal{E}_1$ and $\mathcal{E}_2$ tends to zero, their ratio does not tend to one, due to reduction of variance got via $\mathcal{E}_1$.
- $N_{|0\rangle}$ does not converge to $N/2$ (but may diverge, we exploit this), where as $N_{|0\rangle}/N$ converges to $1/2$ as N increases.

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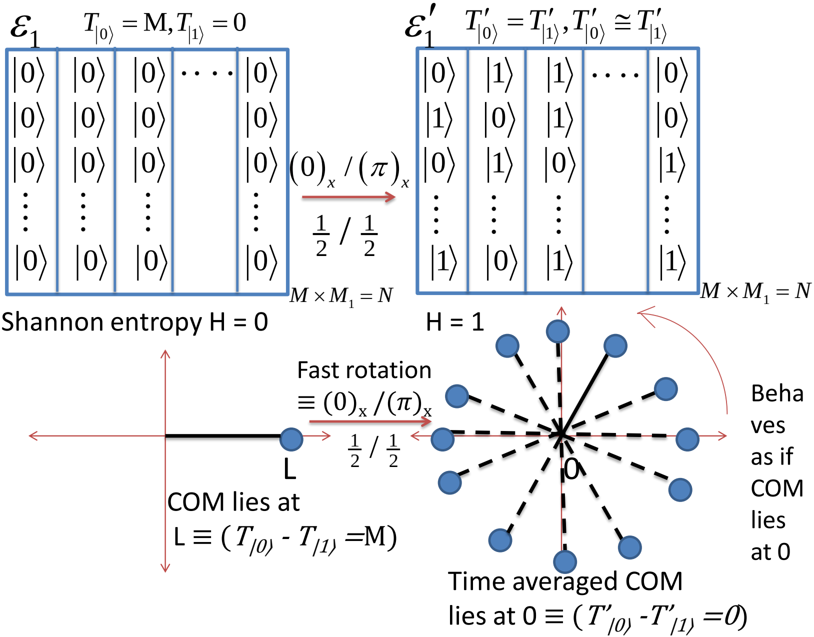
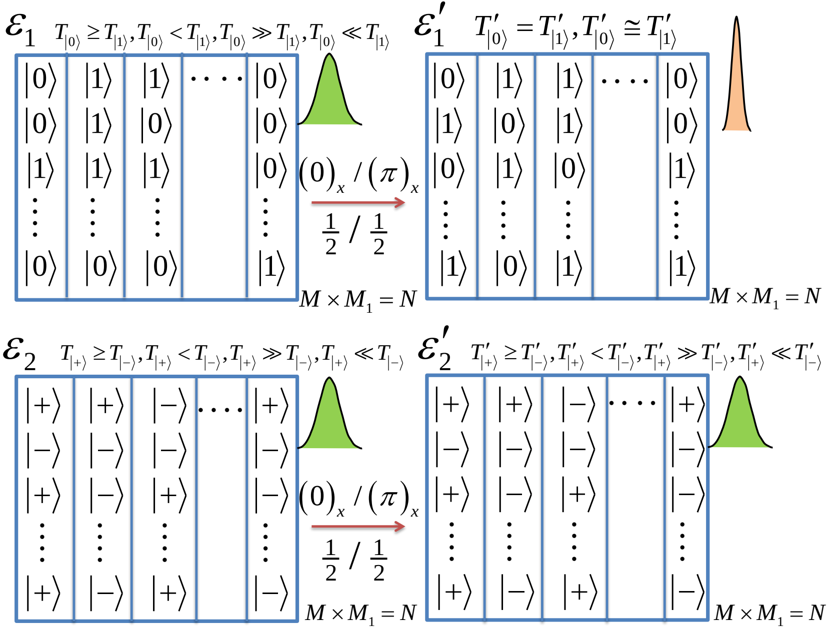
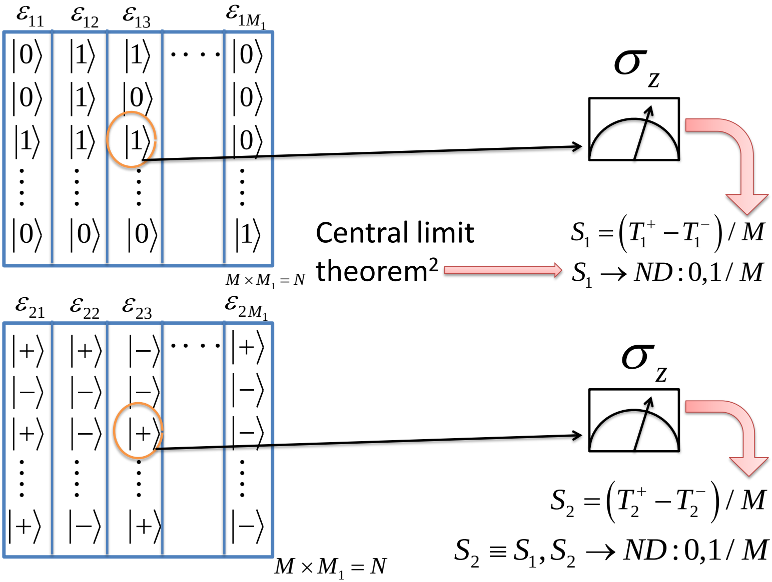
Hypothetical extreme case



Reduction in variance of sample mean (σ_z)



Technical details: Notation and definition



Central limit theorem[2]

$$S_1 = \left(T_1^+ - T_1^-\right)/M$$

$$S_1 \rightarrow ND : 0,1/M$$

$$S_2 = \left(T_2^+ - T_2^-\right)/M$$

$$S_2 \equiv S_1, S_2 \rightarrow ND : 0,1/M$$

2

[2] S. Ross, A first course in probability (Pearson, 2012).

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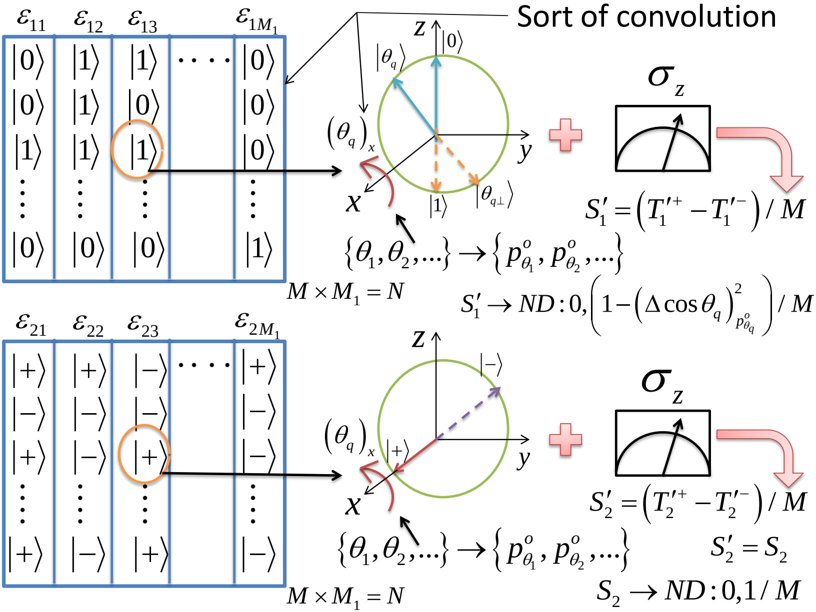

- Note that even though no change in mean upon applying $(\theta_q)_x$s, variance has reduced.
- Measuring σ_z selectively on $|\theta_q\rangle$s and $|\theta_{q\perp}\rangle$s is equivalent to tossing differently biased coins (i.e., independently distributed (id) random variables).
- Applying central limit theorem to independently distributed (id) random variables [3], we obtain effective mean

$$\begin{aligned}\mu_{eff} &= \sum_q (p_q\langle\sigma_z\rangle_{|\theta_q\rangle} + p_{q\perp}\langle\sigma_z\rangle_{|\theta_{q\perp}\rangle}) \\ &= \sum_q (p_q - p_{q\perp})\cos\theta_q. \end{aligned} \quad (1)$$

where $p_q = M'_q(T_1^+, p_{\theta_q})/M$ ($p_{q\perp} = M'_{q\perp}(T_1^-, p_{\theta_q})/M$), M'_q ($M'_{q\perp}$) is the total number of $|\theta_q\rangle$s ($|\theta_{q\perp}\rangle$s). $\sum_q(M'_q + M'_{q\perp}) = M$.

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- Using Bayes rule

$$p_q = p_1^+ p_{\theta_q}, p_{q\perp} = p_1^- p_{\theta_q}, \tag{2}$$

where $p_1^\pm = T_1^\pm/M$, and $p_{\theta_q} = m_q/M$, m_q is the total number of times $(\theta_q)_x$ is applied, $\sum_q m_q = M$.

- $$\Rightarrow \mu_{eff} = S_1 \langle \cos\theta_q \rangle_{p_{\theta_q}} \tag{3}$$

  where $\langle \cos\theta_q \rangle_{p_{\theta_q}} = \sum_q p_{\theta_q} \cos\theta_q$.

- $S_1 \to$ ND : $0, 1/M$,
  $p_1^\pm \to$ ND : $1/2, 1/(4M)(\because T_1^\pm \to$ ND : $M/2, M/4)$, and $p_{\theta_q} \to$ ND : $p^o_{\theta_q}, \sigma^2_{m_q}/M^2(\because m_q \to$ ND : $p^o_{\theta_q} M, \sigma^2_{m_q})$ where $\sigma^2_{m_q} \sim M$. Hence we need to take care of the variance (however small) present in them.

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- Applying central limit theorem to id random variables, we obtain effective variance [4]

$$\begin{aligned}(\Delta\sigma_z)^2_{eff} &= \sum_q (p_q(\Delta\sigma_z)^2_{|\theta_q\rangle} + p_{q\perp}(\Delta\sigma_z)^2_{|\theta_{q\perp}\rangle}) \\ &= \sum_q (p_q + p_{q\perp})\sin^2\theta_q = 1 - \langle\cos^2\theta_q\rangle_{p_{\theta_q}}, (4)\end{aligned}$$

where $(\Delta\sigma_z)^2_{|\theta_q\rangle} = \langle\sigma_z^2\rangle_{|\theta_q\rangle} - \langle\sigma_z\rangle^2_{|\theta_q\rangle}$, $\langle\cos^2\theta_q\rangle_{p_{\theta_q}} = \sum_q p_{\theta_q}\cos^2\theta_q$.

- $(\Delta\sigma_z)^2_{eff} \neq \langle\sigma_z^2\rangle_{\rho'_{1j}} - \langle\sigma_z\rangle^2_{\rho'_{1j}}$ where $\rho'_{1j} = \sum_q (p_q|\theta_q\rangle\langle\theta_q| + p_{q\perp}|\theta_{q\perp}\rangle\langle\theta_{q\perp}|)$, because in going from $\mathcal{E}'_{1j}$ to ρ'_{1j}, there is information loss.
- According to central limit theorem, in the large M limit, probability distribution of effective sample mean S'_1, for given values of p_{θ_q}s and S_1 (i.e., for given values of m_qs and T_1^+ i.e., for a given $\mathcal{E}'_{1j}$), tends to normal distribution[4] i.e, $S'_1 \to \text{ND} : \mu_{eff}, (\Delta\sigma_z)^2_{eff}/M$.

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- Resultant probability density of S'_1 is given by

$$f(S'_1) = \int \prod_{i,i\neq I} \{dp_{\theta_i}(\mathrm{Nd}(p_{\theta_i}) : p^o_{\theta_i}, \sigma^2_{m_i}/M^2)\} \times dS_1(\mathrm{Nd}(S_1) : 0, 1/M) \times \left(\mathrm{Nd}(S'_1) : S_1\langle\cos\theta_q\rangle_{p_{\theta_q}}, (1 - \langle\cos^2\theta_q\rangle_{p_{\theta_q}})/M\right), \tag{5}$$

where $(\mathrm{Nd}(x) : \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-(x-\mu)^2/(2\sigma^2))$ (i.e., Normal probability density function with mean μ and variance σ^2), $dx(\mathrm{Nd}(x) : \mu, \sigma^2)$ is the probability of obtaining value x of normally distributed random variable x.

- Integrating out S_1 we get

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where $(\Delta\cos\theta_q)^2_{p_{\theta_q}} = \langle\cos^2\theta_q\rangle_{p_{\theta_q}} - \langle\cos\theta_q\rangle^2_{p_{\theta_q}}$.

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$$f(S_1') = \int \prod_{i,i\neq I} \{dp_{\theta_i}(\mathrm{Nd}(p_{\theta_i}) : p^o_{\theta_i}, \sigma^2_{m_i}/M^2)\}$$
$$\times dS_1(\mathrm{Nd}(S_1) : 0, 1/M)$$
$$\times \left(\mathrm{Nd}(S_1') : S_1 \langle \cos\theta_q \rangle_{p_{\theta_q}}, (1 - \langle \cos^2\theta_q \rangle_{p_{\theta_q}})/M\right), \qquad (5)$$

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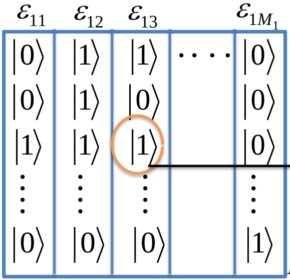
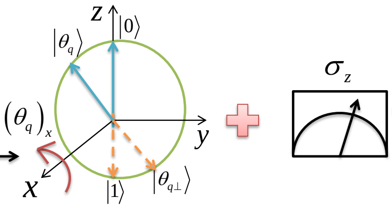
$$\times (\mathrm{Nd}(S_1') : 0, (1 - (\Delta \cos\theta_q)^2_{p_{\theta_q}})/M), \qquad (6)$$

where $(\Delta \cos\theta_q)^2_{p_{\theta_q}} = \langle \cos^2\theta_q \rangle_{p_{\theta_q}} - \langle \cos\theta_q \rangle^2_{p_{\theta_q}}$.

Resultant variance





$$M \times M_1 = N$$

$$(\Delta\sigma_z)^2_{eff} = 1 - \langle \cos^2\theta_q \rangle_{p_{\theta_q}}$$

$$\mu_{eff} = \langle \cos\theta_q \rangle_{p_{\theta_q}} S_1$$ ←Swaying of center of Gaussian

$$S_1 = (T_1^+ - T_1^-)/M$$

$$\langle \cos\theta_q \rangle^2_{p_{\theta_q}} / M \quad + \quad (\Delta\sigma_z)^2_{eff} / M$$

$$= \left(1 - (\Delta\cos\theta_q)^2_{p_{\theta_q}}\right)/M$$

- Consider $\theta_q = \theta_0, \forall q$ (i.e., no randomness). Then Eq. (6) reduces to $f(S_1') = (\mathrm{Nd}(S_1') : 0, (1-0)/M) = g(S_2')$, hence no discrimination.
- Consider the simplest case: $\{\theta_1, \theta_2\} \to \{p^o_{\theta_1}, p^o_{\theta_2}\}$.
-

$$\Rightarrow f(S_1') \approxeq \int_{p^o_{\theta_1}-\epsilon}^{p^o_{\theta_1}+\epsilon} dp_{\theta_1} \delta(p_{\theta_1} - p^o_{\theta_1})$$
$$\times(\mathrm{Nd}(S_1') : 0, (1 - p_{\theta_1}(1-p_{\theta_1})(\cos\theta_1 - \cos\theta_2)^2)/M)$$
$$= (\mathrm{Nd}(S_1') : 0, (1 - (\Delta\cos\theta_q)^2_{p^o_{\theta_q}})/M), \quad (7)$$

where $\epsilon > 0$ ($\because$ no swaying of center of Gaussian in Eq. (6)).

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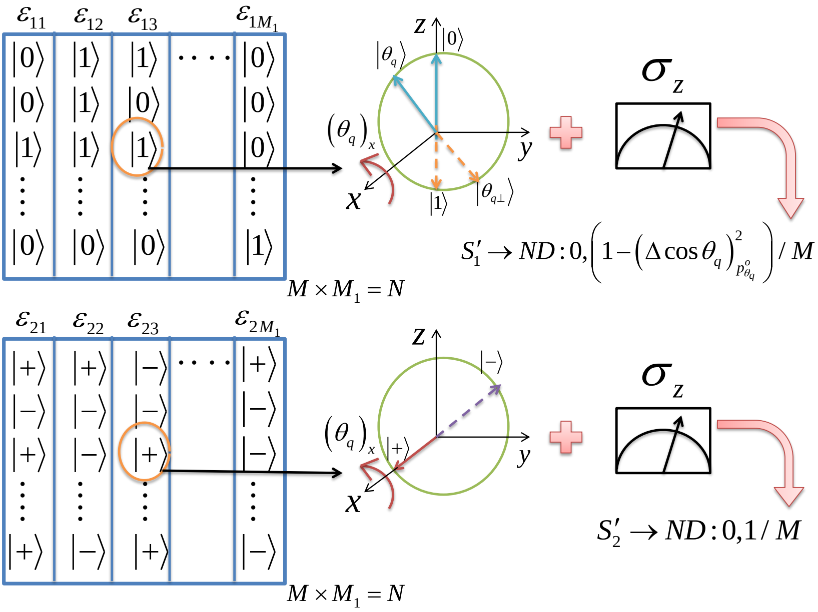
Summary so for....



Nonlinearity in action

- We will show how nonlinearity is reducing the variance. We have $\Delta S_1'^2 \approx (1 - (\Delta \cos\theta_q)^2_{p^o_{\theta_q}})/M =$ $(\langle\cos\theta_q\rangle^2_{p^o_{\theta_q}} + \langle\sin^2\theta_q\rangle_{p^o_{\theta_q}})/M$ (Eq. (7)).
- Let $\{\theta_1(=0), \theta_2(=\pi/2)\} \to \{p^o_0, p^o_{\pi/2}\}$.
- $\Rightarrow \Delta S_1'^2 \approx [(p^o_0 \cos 0 + p^o_{\pi/2}\cos(\pi/2))^2 + p^o_0 \sin^2 0 + p^o_{\pi/2}\sin^2(\pi/2)]/M = (p^{o2}_0 + p^o_{\pi/2})/M < 1/M$.

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Smoothing out nonuniformities

$\mathcal{E}_1$ $\quad T_{|0\rangle} \geq T_{|1\rangle}, T_{|0\rangle} < T_{|1\rangle}, T_{|0\rangle} \gg T_{|1\rangle}, T_{|0\rangle} \ll T_{|1\rangle}$

$$\begin{array}{|c|c|c|c|c|}
\hline
|0\rangle & |1\rangle & |1\rangle & \cdots & |0\rangle \\
|0\rangle & |1\rangle & |0\rangle & & |0\rangle \\
|1\rangle & |1\rangle & |1\rangle & & |0\rangle \\
\vdots & \vdots & \vdots & & \vdots \\
|0\rangle & |0\rangle & |0\rangle & & |1\rangle \\
\hline
\end{array}$$

$M \times M_1 = N$

$$\xrightarrow[\frac{1}{2}\,/\,\frac{1}{2}]{(0)_x\,/\,(\pi)_x}$$

$\mathcal{E}_1'$ $\quad T'_{|0\rangle} = T'_{|1\rangle}, T'_{|0\rangle} \cong T'_{|1\rangle}$

$$\begin{array}{|c|c|c|c|c|}
\hline
|0\rangle & |1\rangle & |1\rangle & \cdots & |0\rangle \\
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\hline
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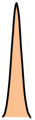


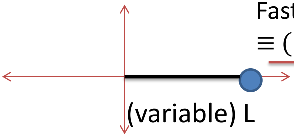
$\{0,\pi\} \rightarrow \{\frac{1}{2},\frac{1}{2}\}$

$\Delta S_1'^2 \cong$

$0 / M$

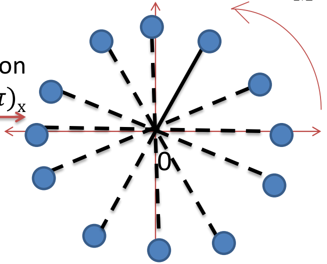
$M \times M_1 = N$

$T_{|0(1)\rangle} = T_1^{+(-)}, T'_{|0(1)\rangle} = T_1'^{+(-)}$



Fast rotation

$$\xrightarrow[\frac{1}{2}\,/\,\frac{1}{2}]{\equiv (0)_x\,/\,(\pi)_x}$$



Behaves as if COM lies at 0

COM lies at

L $\equiv$ ($T_{|0\rangle} \geq T_{|1\rangle}$, $T_{|0\rangle} < T_{|1\rangle}$, $T_{|0\rangle} \gg T_{|1\rangle}$, $T_{|0\rangle} \ll T_{|1\rangle}$)

Time averaged COM lies at 0, $\cong 0 \equiv$ ($T'_{|0\rangle} = T'_{|1\rangle}$, $T'_{|0\rangle} \cong T'_{|1\rangle}$)

MATLAB simulation results

- MATLAB generates standard uniformly distributed Pseudo Random Numbers (PRN) drawn from the open interval $(0, 1)$.
- We want to simulate σ_z measurement on $|\chi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$.
- If we get a PRN in the interval $(0, \cos^2(\theta/2))$, then it is equivalent to getting outcome $+1$. Else -1.
- We simulated the case $\{\theta_1(=0), \theta_2(=\pi)\} \to \{p^o_{\theta_1}(=1/2), p^o_{\theta_2}(=1/2)\}$.
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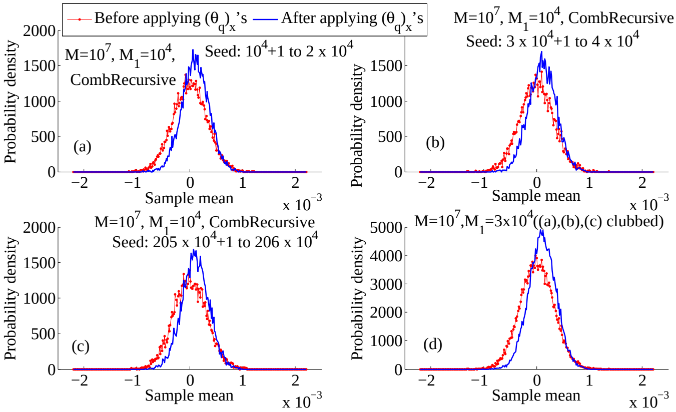
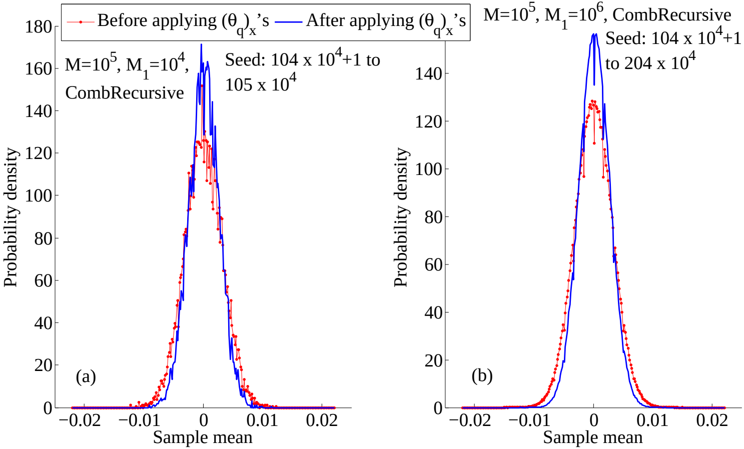


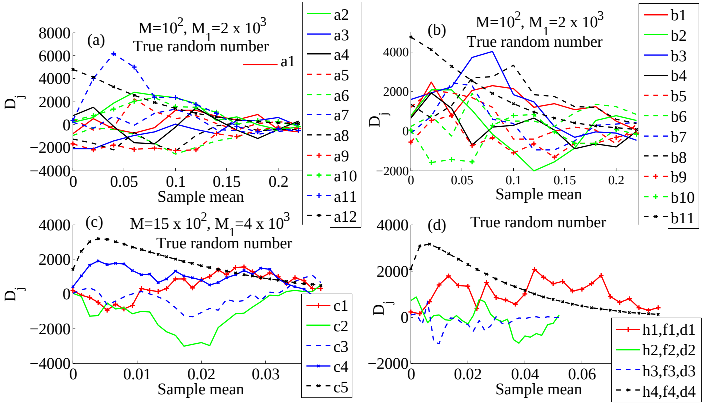
(a)$A_g = 0.6795$ (theory 0.6826895). $A_f = 0.7445$ (theory 1).
$A'_g = 0.6795$, and $A'_f = 0.785$.
(b)$A_g = 0.6787, A_f = 0.739, A'_g = 0.6793, A'_f = 0.777$.
(c)$A_g = 0.685, A_f = 0.7492, A'_g = 0.6828, A'_f = 0.7855$.
(d)$A_g = 0.6811, A_f = 0.7442, A'_g = 0.6811, A'_f = 0.7824$.

MATLAB simulation results



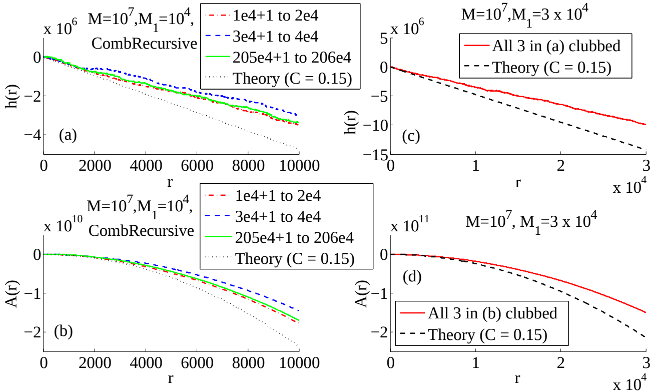
(a) $A_g = 0.6854, A_f = 0.7833$. (b) $A_g = 0.683967, A_f = 0.77997, A'_g = 0.683286, A'_f = 0.780642$.

MATLAB simulation results-True random numbers



D_j is the difference in area under the Gaussians $\times 10^5$ i.e., $D_j = (\sum_{S_1'=-a_j}^{a_j} f(S_1') - \sum_{S_1=-a_j}^{a_j} g(S_1))\delta S \times 10^5$ where δS is the smallest element (step size) on x-axis (sample mean) considered for plotting, and $a_j = j \times \delta S, j = 1, 2, ...$ Theoretical curve: $(D_j)_{thoery}/10, \Delta S_1'^2 \to 0.1^2/M$ instead of $0/M$.

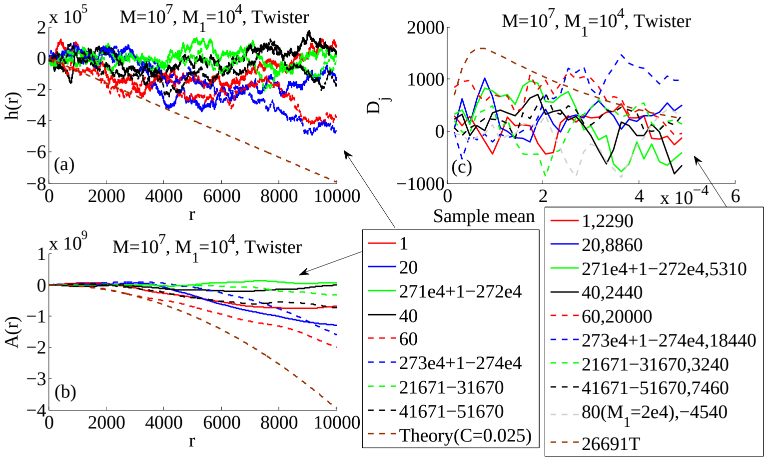
MATLAB simulation results



$$h(r) = \sum_{i=1}^{r}(|T'^{+}_{1i} - T'^{-}_{1i}| - |T^{+}_{1i} - T^{-}_{1i}|), \qquad (8)$$

where $r = 1, 2, ..., M_1$.

MATLAB simulation results



26691T : $(D_j)_{thoery}/100, \Delta S_1'^2 \to 0.1^2/M$ instead of $0/M$.

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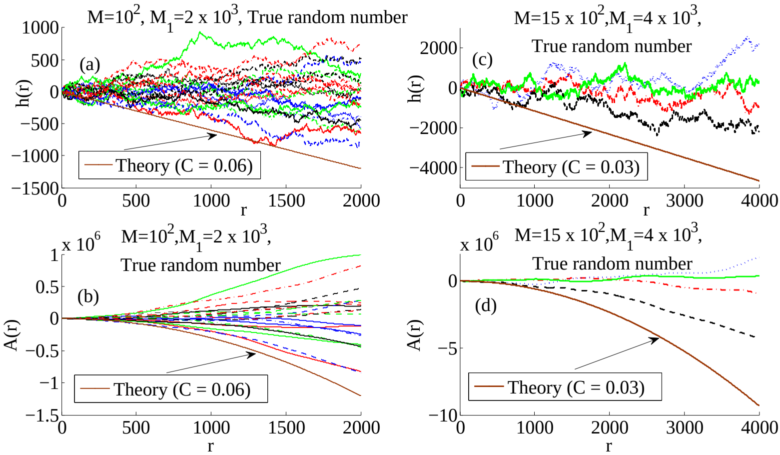

Conclusion

- We showed that, if we have individual control, then we can discriminate between two ensembles via selective random rotations, which otherwise (i.e., without individual control) cannot be discriminated, as both are maximally mixed.
- Numerical simulation results support theoretical predictions.
- However the origin of nonlinear effect (reduction in variance) which leads to discrimination is not clear.
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THANK YOU

Single copy picture in 2^N-D Hilbert space

There are two sets: $\mathcal{F}_1 = \{|0\rangle^{\otimes N}, |0\rangle^{\otimes N-1}|1\rangle, ..., |1\rangle^{\otimes N}\}$, and $\mathcal{F}_2 = \{|+\rangle^{\otimes N}, |+\rangle^{\otimes N-1}|-\rangle, ..., |-\rangle^{\otimes N}\}$.
$\mathcal{F}_i$ is a complete set of orthonormal basis states in 2^N dimensional (2^N-D) Hilbert space, $i = 1, 2$. E.g., for $N = 2$, $\mathcal{F}_1 = \{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\}$, and $\mathcal{F}_2 = \{|+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|-\rangle\}$.
Let $|\phi_{ij}\rangle \in \mathcal{F}_i,\ i = 1, 2, j = 1, 2, ..., 2^N$. Even though $|\langle\phi_{1j}|\phi_{2k}\rangle|$ tends to zero in the limit $N \to \infty$, $|\phi_{1j}\rangle$ can never become perfectly orthogonal to $|\phi_{2k}\rangle$, because the set $\mathcal{F}_i$ is already complete, $i = 1, 2$. Hence $\mathcal{F}_1$ and $\mathcal{F}_2$ together constitute a set of nontrivial nonorthogonal states.
Alice gives Bob, a *single copy* of $|\phi_{ij}\rangle$ chosen with probability $1/2^N$ (i.e., all the states are equally likely to be chosen) from $\mathcal{F}_i$, $i = 1$ or 2.

Hence $|\phi_{1j}\rangle$ ($|\phi_{2j}\rangle$) is nothing but the renormalized post measurement state of measuring σ_z (σ_x) selectively (i.e., locally) on each of the N qubits in the state $|+\rangle^{\otimes N}$ ($|0\rangle^{\otimes N}$), $|0\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$.
Alice tells Bob the way she chose the state from one of $\mathcal{F}_1, \mathcal{F}_2$, but she do not tell him exactly from which set she chose the state. Hence Bob is aware of $\mathcal{F}_1, \mathcal{F}_2$, and Alice's state choosing procedure.
Bob has a single copy of the unknown state $|\phi_{ij}\rangle, i = 1$ or 2. We are going to show that, in the limit $N \to \infty$, even though Bob cannot know the unknown state exactly, still he can know deterministically whether it was chosen from $\mathcal{F}_1$ or $\mathcal{F}_2$ (and hence it is deterministic but inexact nonorthogonal state discrimination).

In density matrix formulation, Bob's unknown state is given by:

$$\rho_i = \sum_{j=1}^{2^N} \frac{1}{2^N} |\phi_{ij}\rangle\langle\phi_{ij}| = \frac{\mathbb{1}_{2^N}}{2^N}, i = 1 \text{ or } 2 \tag{9}$$

where $\mathbb{1}_n$ is $n \times n$ identity matrix. Note that ρ_i represents the state of a single copy of one of $|\phi_{ij}\rangle$s, $j = 1, 2, ..., 2^N$, which Bob has got, taking into consideration the probability $(1/2^N)$ with which he obtains it. ρ_i represents the state of an ensemble with individual control, but not collective control. Mixedness of ρ_i represents Bob's ignorance about the single copy of the state he has got. Hence it can be purified by selective projective measurement unlike in nonselective ensemble measurement