Quantum States: A Pictorial Representation

Young Quantum - 2017

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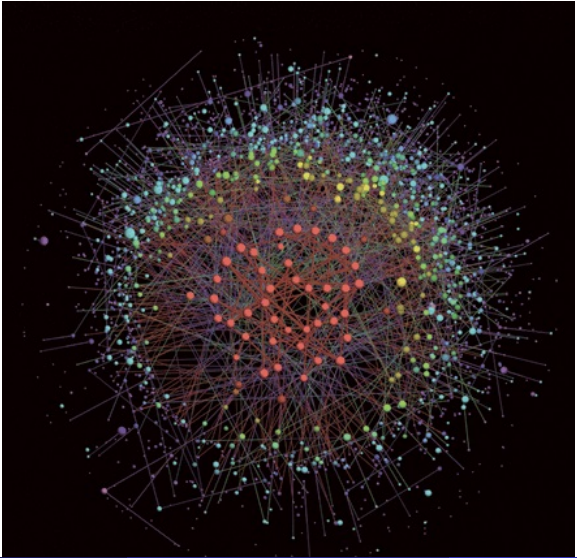
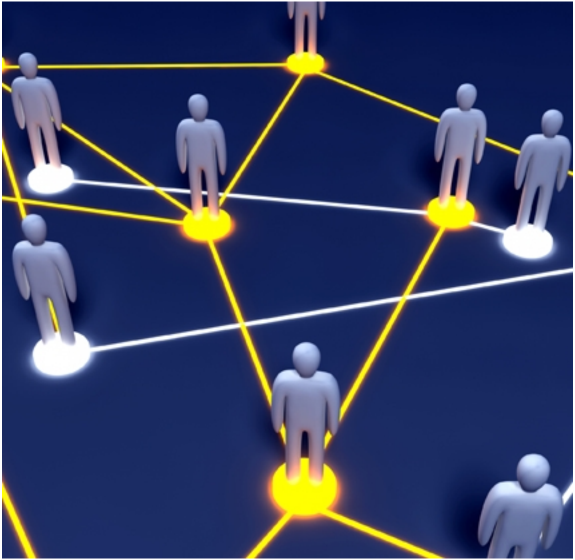
Combinatorial graph is the foundational stone of complex network



Friendship network is practically useful in everyday life.



Combinatorial graphs in quantum

Quantum graph [BK13]

Introduction to quantum graphs by G. Berkolaiko and P. Kuchment.

Continuous time quantum walk [Kon08]

"Quantum walks" by "N. Konno".

Quantum state transfer on graphs [God12]

"State transfer on graphs" by "Chris Godsil".

Quantum probability on graph [HO07]

"Quantum probability and spectral analysis of graphs" by "A. Hora, and "N. Obata".

Graph states: [HEB04]

"Multiparty entanglement in graph states" by Marc et.al.

Problems to attempt

1. How does a graph correspond a quantum state?
2. Can we represent unitary evolution graph-theoretically?
3. How the structure of a graph may infer properties of the corresponding quantum states?
    - Entanglement and separability of a state
    - Quantum discord

How does a graph correspond a quantum state?

Matrices related to graphs [Wes96, Bap14]

Adjacency matrix $A(G) = (a_{i,j})_{n \times n}$

Usual definition of simple graph G [Bap14]

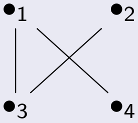
$$a_{ij} = \begin{cases} 1 & \text{If } (i,j) \in E(G) \\ 0 & \text{if } (i,j) \notin E(G) \end{cases}$$

Degree matrix

Degree d_i of a vertex $i \in V$ is given by $d_i = \sum_{j=1}^{n} |a_{ij}|$
Degree matrix of a graph G is $D(G) = diag\{d_1, d_2 \ldots d_n\}$.

Example

Consider the following graph



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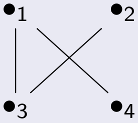

Adjacency matrix and degree matrix

$$A(G) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, D(G) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Laplacian and signless Laplacian matrix

Laplacian and Signless Laplacian Matrix

Laplacian matrix of a graph is $L(G) = D(G) - A(G)$.
Signless Laplacian matrix of a graph is $Q(G) = D(G) + A(G)$. [CS09]

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Examples

$$L(G) = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, Q(G) = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Quantum states represented by Laplacian matrices [BGS06, ABAK17]

Density matrix generated by Laplacian matrix

$$\rho_l(G) = \frac{1}{\text{trace}(L(G))} L(G).$$

Density matrix generated by signless Laplacian matrix

$$\rho_q(G) = \frac{1}{\text{trace}(Q(G))} Q(G).$$

Quantum states represented by Laplacian matrices [BGS06, ABAK17]

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Examples

$$\rho_l(G) = \frac{1}{6}\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \rho_q(G) = \frac{1}{6}\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Example: Bell states

Representation with state vectors

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

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Density matrix representation

$$\rho = |\phi\rangle\langle\phi| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{bmatrix}$$

Example: Bell states

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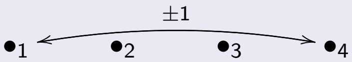
Representation with graphs

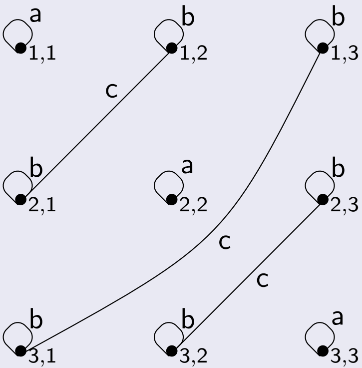


Some remarks

- This graph theoretic representation is beneficial to represent mixed quantum states.
- Corresponding to every graphs there is a quantum state.
- But, a general quantum state may not be represented as a density matrix corresponding to a graph.
- Density matrix corresponding to a graph is not unique.
- Different vertex labellings on a graph generates different quantum states.

An example of a mixed state

Consider the following graph



c = 3x - 1

b = 3 - x

a = 2 + 2x

It represents the Werner state

$$\rho_{x,3} = \frac{3-x}{3^3-3} I + \frac{3x-1}{3^3-3} F, \text{ where } F = \sum_{i,j=1}^{3} |i\rangle\langle j| \otimes |j\rangle\langle i|.$$

Can we represent unitary evolutions graph theoretically?

Why may it be important?

Three fundamentals of quantum computation [Fey82, D^{+}00]

- **Initialization** of the system.
- **Unitary evolution** of the system
- **Measurement** with respect to a number of observables.

Graphs theoretic counterparts of quantum gates [DAB16]

A number of questions regarding unitary operations of graph

Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ we have density matrices $\rho(G)$ and $\rho(H)$.

- When there exists a unitary operator U such that $\rho(H) = U^\dagger \rho(G) U$?
- When U is a local unitary operator?
- Given an unitary operator U and a graph G how to construct another graph H such that $\rho(H) = U^\dagger \rho(G) U$?

Graph Switching

$G = (V(G), E(G))$ and $H = (V(H), E(H))$ are switching equivalent graphs if $V(H) = V(G)$ and $E(H)$ is constructed from $E(G)$ after removing/adding some weighted edges and/or altering weights of the edges in G.

Graph theoretic procedure for generating Bell state

$$|10\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$$

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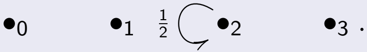
Graph, corresponding to state $|10\rangle\langle 10|$



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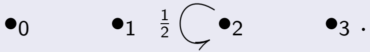

Graph after completing Hadamard operation

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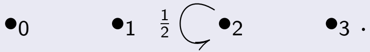
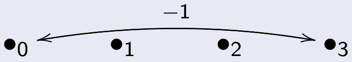


Graph after completing Hadamard operation

Applying CNOT operation we get a graph representing $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$



How structure of a graph may infer properties of the corresponding quantum states?

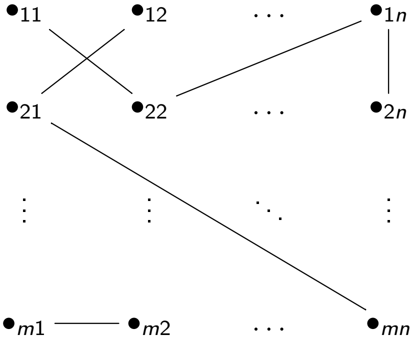
Quantum separability and entanglement

Partition on the vertex set

Organize the $m \times n$ vertices into vertex clusters as follows.

$$
\begin{array}{lcccc}
C_1 = & \{\bullet_{11}, & \bullet_{12} & \cdots & \bullet_{1n}\} \\
C_2 = & \{\bullet_{21}, & \bullet_{22} & \cdots & \bullet_{2n}\} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_m = & \{\bullet_{m1}, & \bullet_{m2} & \cdots & \bullet_{mn}\}
\end{array}
$$

Horizontal, vertical and tilled edges



Partial transpose on matrices

Partition of the adjacency matrix into blocks

$$A(G) = \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,m} \\ A_{2,1} & A_{2,2} & \dots & A_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m,1} & A_{m,2} & \dots & A_{m,m} \end{bmatrix},$$

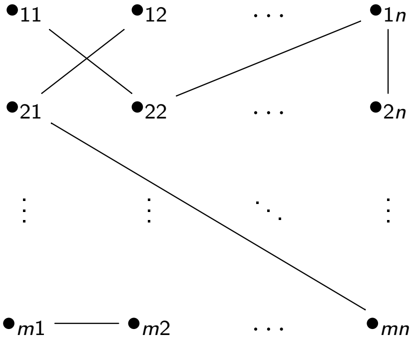
$A_{i,j}$ are matrices of order $n \times n$, corresponds to C_i.

Partial transpose on $A(G)$ wrt second subsystem

$$A(G) = \begin{bmatrix} A^t_{1,1} & A^t_{1,2} & \dots & A^t_{1,m} \\ A^t_{2,1} & A^t_{2,2} & \dots & A^t_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ A^t_{m,1} & A^t_{m,2} & \dots & A^t_{m,m} \end{bmatrix}.$$

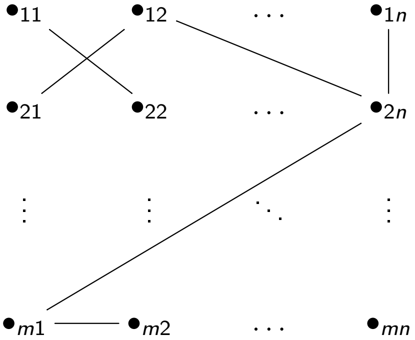
Partial transpose on graph [Wu06]

Modify the tilled edges only! Here is the original graph:



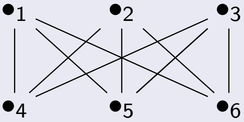
Partial transpose on graph [Wu06]

Modify the tilled edges only! Here is the graph after partial transpose:

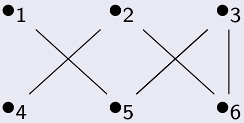


Some examples of locally symmetric graph [DABS16]

Complete bipartite graph



A path of even order



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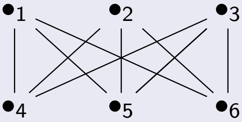

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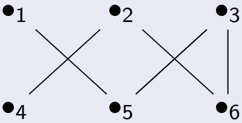


How to construct?

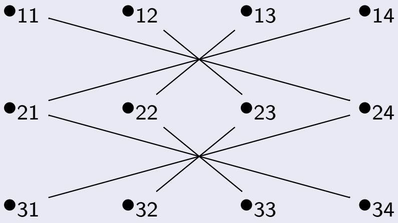
Tensor product of two graphs are locally symmetric.

Separability condition for general bipartite system [DABS16]

$\rho(G)$ is separable if the following holds

- There is no Horizontal edge.
- Either there is no edge between layers of C_i and C_j or pattern of edge distribution is same.
- Degree of all the vertices of a layer are equal.

Example

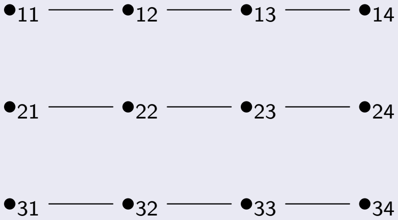


Separability condition for general bipartite system [DABS16]

m copies of any graph taken together is separable

$mG = G + G + \dots G(m-$ times). $\rho(mG)$ is a separable state in $\mathcal{H}^n \otimes \mathcal{H}^m$.

Example

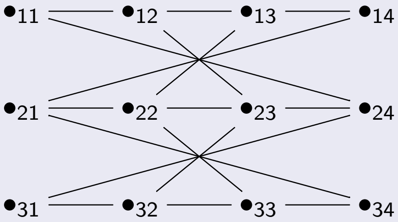


Sufficient conditions for separability for general bipartite system [DABS16]

The following composition of two graphs is separable

H is a locally symmetric graph with m clusters representing a separable state. Graph $G \bowtie H$ is constructed by placing m different copies of G on m layers of H. Then the state related to $G \bowtie H$ is separable.

Example



How structure of a graph may infer properties of the corresponding quantum states?

Quantum discord

Quantum discord and state with zero quantum discord

Quantum discord of ρ acting on $\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$

$$\mathcal{D}_{\{k_B\}}(\rho) = S(\sum_k p_{k_B} S(\rho_{k_B})) - \left[S(\rho) - S(\rho_B)\right].$$

where, $p_{k_B} = \text{trace}_A(\langle k_B | \rho | k_B \rangle)$.

Pointer state [HWZ11]

$$\rho = \sum_i p_i \rho_i^{(a)} \otimes |k_b\rangle \langle k_b| .$$

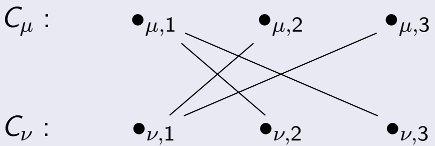
Relationship with the block matrices and bipartite subgraphs

Adjacency matrix of a bipartite graph

$$A(\langle C_\mu, C_\nu \rangle) = \begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix}$$

An example

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



Graphical representation of commutative matrices [DAB17]

Commutative matrix

$$AB = BA$$

Graphical alternative

Let $G(A) = \langle C_\mu, C_\nu \rangle$ and $G(B) = \langle C_\alpha, C_\beta \rangle$, respectively. Then $AB = BA$ if and only if,

$$\#(\text{nbd}(v_{\mu i}) \cap \text{nbd}(v_{\beta j})) = \#(\text{nbd}(v_{\nu j}) \cap \text{nbd}(v_{\alpha i})).$$

Graphical representation of Normal matrices [DAB17]

Normal matrix

$$AA^\dagger = A^\dagger A$$

Graphical alternative

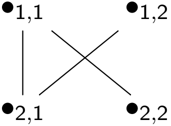
Let $A = [a_{ij}] \in \{0,1\}^{n \times n}$ and $G_A = \langle C_\mu, C_\nu \rangle$ be the bipartite graph corresponding to A. Then A is normal if and only if for every i and j with $1 \leq i, j \leq n$,

$$\#(\text{nbd}(v_{\mu i}) \cap \text{nbd}(v_{\mu j})) = \#(\text{nbd}(v_{\nu i}) \cap \text{nbd}(v_{\nu j})).$$

Graph theoretical perspective to quantum discord [DAB17]

- Graph theoretic zero discord states can be identified by comparing the neighborhoods of the vertices.
- A measure of graph theoretic discord is constructed for all quantum states related to graphs.

Separable state with non-zero discord



$$\rho_l(G) = \frac{1}{6}\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$ do not commute, hence $\mathcal{QD}(G) \neq 0$ although $\rho(G) = \frac{1}{5}L(G)$ represents a 2-qubit separable state.

Significant concluding remarks

1. Combinatorial properties are useful in the investigation of the properties of mixed quantum states.
2. A model of quantum computation may be proposed with graphs.

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Thank you