Quantum fidelity for Kitaev model in the thermodynamic limit

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V. Mukherjee and A. Dutta, Phys. Rev. B **83**, 214302 (2011)
V. Mukherjee, A. Dutta and D. Sen, Phys. Rev. B **85**, 024301 (2012)

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Plan of the Talk

- 1: Quantum Phase transition and Quantum Information theoretic Measures
- 2. Fidelity near a Quantum Critical Point; fidelity susceptibility.
- 3. Fidelity in the thermodynamic limit
- 4. Thermodynamic fidelity for Kitaev Model
- 5. Concluding Remarks

References:

A. Dutta, U. Divakaran, D. Sen, B. K. Chakrabarti, T. F. Rosenbaum and G. Aeppli, arxiv: arXiv:1012.0653.

Anatoli Polkovnikov, Krishnendu Sengupta, Alessandro Silva, Mukund Vengalattore, Rev. Mod. Phys.,Rev.Mod.Phys.**83**,863,2011

J. Dziarmaga, Advances in Physics 59, 1063 (2010).

Quantum Phase Transitions

- Zero-temperature transition: driven by quantum fluctuations

Paradigmatic model: Transverse Ising Model

$$H = -J_x \sum_{<ij>} \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z$$

Exactly solved in one dimension
For $h/J_x > 1$, spins are all aligned in the z

$$\langle \sigma_i^x \rangle = 0; \text{ Paramagnetic}$$

For $h/J_x < 1$, cooperative term dominates

$$\langle \sigma_i^x \rangle \neq 0; \text{ Ferromagnetic}$$

- Quantum phase transitions at $\lambda = h/J_x - 1 = 0$.

Quantum Phase Transitions: Critical Exponents

- Notion of Universality:

  Exponents depend on i) Symmetry ii) Dimensionality
- Diverging length Scale: $\xi \sim \lambda^{-\nu}$
- Diverging time Scale: $\xi_\tau \sim \xi^z$
- Energy gap scales as $\lambda^{\nu z}$
- At the quantum critical point gap scales k^z.

QPT and Quantum Information theoretic Measures:

Concurrence, negativity, entanglement entropy ...

The Quantum fidelity: The modulus of overlap of the wave function: Connected to....

- Scaling of the geometric phase near a QCP
- Sudden Quench of Small amplitude starting from a QCP
- Quantum Critical Environment and Loschmidt Echo

Quantum Discord

References: • A. Dutta, U. Divakaran, D. Sen, B. K. Chakrabarti, T. F. Rosenbaum and G. Aeppli, arxiv: arXiv:1012.0653.

The Quantum Fidelity

We consider the Hamiltonian

$$H(\lambda) = H_0 + \lambda H_I; \quad H(\lambda)|\psi_0(\lambda)\rangle = E_0|\psi_0(\lambda)\rangle$$

where $|\psi_0(\lambda)$ is the ground state wave function.

- λ is the driving term. The QCP is at $\lambda = 0$.
- The quantum fidelity: modulus of the overlap between two grouns state corresponding to parameters λ and $\lambda + \delta$

$$F(\lambda, \delta) = |\langle\psi_0(\lambda)|\psi_0(\lambda + \delta)\rangle|$$

- In the limit $N \to \infty$, the fidelity vanishes in the thermodynamic limit. Anderson's Orthogonality Catastrophe
- What happens for finite N? Indicator of Quantum Criticality

Finite N and $\delta \to 0$ limit: Fidelity Susceptibility

In this limit, one can express fidelity in the form

$$F(\lambda, \delta) = 1 - \frac{1}{2}\delta^2 L^d \chi_F(\lambda)$$

The quantity $\chi_F = -\frac{2}{L^d} \lim_{\delta \to 0} \frac{\ln F}{\delta^2} = -\frac{1}{L^d} \frac{\partial^2 F}{\partial \delta^2}$ is the fidelity susceptibility.

- $\chi_F \sim \lambda^{\nu d - 2}$ away from the QCP $|\lambda|^{-\nu} << L$.
- While at the QCP ($\xi >> L$): $\chi_F \sim L^{2/\nu - d}$.

Quantum Critical Scaling: $\nu d > 2$

Points to Note:

- The parameter δ is factored out. χ_F depends on λ only.
- The quantum fidelity is close to unity; Can not describe Anderson's Orthogonality Catastrophe

References:

P. Zanardi and N. Paunkovic, Phys. Rev. E **74**, 031123. (2007)
Gu, S. J., Int. J. Mod. Phys. B **24**, 4371 (2010).
Gritsev, V., and A. Polkovnikov, arXiv:0910.3692.

Scaling of Fidelity Susceptibility close to a QCP

Using the perturbation expansion:

$$\chi_F(\lambda) = \frac{1}{L^d} \sum_{m \neq 0} \frac{|\langle \psi_m(\lambda) | H_I | \psi_0(\lambda) \rangle|^2}{[E_m(\lambda) - E_0(\lambda)]^2}.$$

χ_F shows a scaling behavior with exponent given in terms of some of the critical exponents.
Compare χ_F with the ground state specific heat density

$$\chi_E = -\frac{1}{L^d} \frac{\partial^2 E_0}{\partial \lambda^2} = -\frac{2}{L^d} \sum_{m \neq 0} \frac{|\langle \psi_m(\lambda) | H_I | \psi_0(\lambda) \rangle|^2}{E_m(\lambda) - E_0(\lambda)}$$

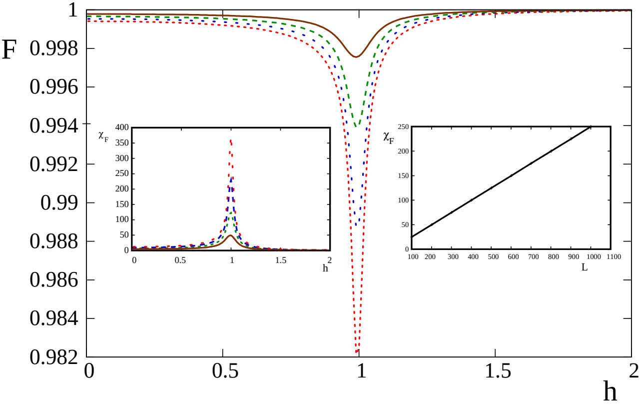
We note the difference in the denominator. Stronger divergence is expected.
Close to a QCP: $\chi_E \sim |\lambda|^{-\alpha}$ and the hyperscaling relation $2 - \alpha = \nu(d + z)$.

Verification for a transverse Ising Chain

We consider the transverse Ising Chain Hamiltonian

$$H = -\sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z$$

QCP at $h = h_c = 1$, exponents $\nu = z = 1$.



When does Fidelity suceptibility approach fail?

Recall the expansion

$$F(\lambda, \delta) = 1 - \frac{1}{2}\delta^2 L^d \chi_F(\lambda)$$

Close to the QCP:

$\chi_F(\lambda \simeq 0) \sim L^{2/\nu - d}$ so that $\delta^2 L^d \chi_F(\lambda \simeq 0) \sim \delta^2 L^{2/\nu}$

$\delta L^{1/\nu} \ll 1$, the fidelity susceptibility approach works!

$\delta L^{1/\nu} \gg 1$, one can not truncate the series at the order δ^2

Away from QCP, $L^d \delta^2 \lambda^{\nu d - 2} \gg 1$, one can not truncate

What about the other limit?

Fidelity in the thermodynamic limit

large N and δ small but finite: χ_F approach is not useful.

Fidelity per site $\mathcal{F}(\lambda, \delta) = \lim_{N\to\infty} F^{1/N}(\lambda, \delta)$ is finite.

Proposed Scaling Relation:

$$\ln F(\lambda, \delta) \simeq -L^d |\delta|^{d\nu} \mathcal{A}\left(\frac{\lambda}{|\delta|}\right)$$

Limiting situations

- At the QCP $\lambda = 0$, $\ln F(\lambda = 0, \delta) \simeq -L^d |\delta|^{d\nu}$.
Captures the ground state singularity
- $|\delta| << \lambda << 1$, $\ln F(\lambda, \delta) \simeq -L^d \delta^2 \lambda^{d\nu-2}$

Crossover to χ_F limit

- $|\delta|^\nu L << 1$; $\chi_F \sim \delta^2 L^{2/\nu-d}$
- If $L^d \delta^2 \lambda^{d\nu-2} << 1$, $F \simeq 1 - \delta^2 L^d \chi_F \simeq 1 - L^d \delta^2 \lambda^{d\nu-2}$

References

Rams and Damski,Phys. Rev. Lett. **106**, 055701 (2010)

How to arrive at the scaling realtion in the thermodynamic limit ?

Fidelity per site: $\mathcal{F}(\lambda+\delta, \lambda-\delta) = -\lim_{L\to\infty} \frac{lnF}{L^d}$

The Scaling Ansatz

- $\mathcal{F}(\lambda+\delta, \lambda-\delta) = b^{-d} f\left((\lambda+\delta)b^{1/\nu}, (\lambda-\delta)b^{1/\nu}\right)$.
- $\lambda = c\delta$ and set the scale of renormalization $|\delta| b^{1/\nu} = 1$:
- $\mathcal{F}(\lambda+\delta, \lambda-\delta) = |\delta|^{d\nu} f\,(c+1, c-1)$.
- $\lambda = 0$ implies $c = 0$, $\ln F(\lambda = 0, \delta) \simeq -L^d |\delta|^{d\nu}$
- For $\lambda \neq 0$, set $(\lambda+\delta)b^{1/\nu} = 1$, and expand the scaling function.

Verification

Isolated Critical Points:

- For transverse Ising Spin Chain: One observes the crossover to small size limit $\delta L << 1$.

- Away from the QCP $F(h, \delta) \simeq \exp(-L\delta^2/16|\lambda|)$; this reduces to the fidelity susceptibility result in the when L finite and $\delta \to 0$.

- verified for massless Dirac fermions

Two dimensional Kitaev Model allows us to study

- Anistropic Quantum Critical Point

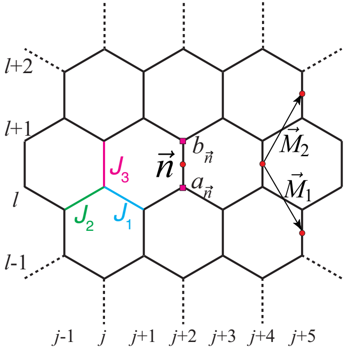
- Extended Gapless Region

References

Rams and Damski,Phys. Rev. Lett. **106**, 055701 (2010)

Mukherjee, Dutta and Sen, Phys. Rev. B **85**, 024301 (2012)

Kitaev Model on a honeycomb lattice



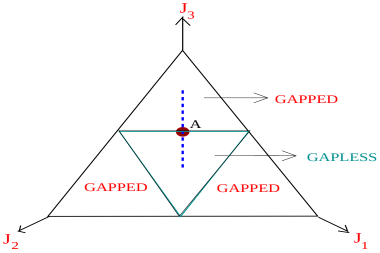
$$H_{2d} = \sum_{j+l=\mathrm{even}} \left(J_1 \sigma^x_{j,l}\sigma^x_{j+1,l} + J_2 \sigma^y_{j-1,l}\sigma^y_{j,l} + J_3 \sigma^z_{j,l}\sigma^z_{j,l+1}\right),$$

$$H' = \sum_{\vec{k}} \begin{pmatrix} a^{\dagger}_{\vec{k}} & b^{\dagger}_{\vec{k}} \end{pmatrix} H_{\vec{k}} \begin{pmatrix} a_{\vec{k}} \\ b_{\vec{k}} \end{pmatrix},$$

$$\begin{aligned} H_{\vec{k}} &= \alpha_{\vec{k}}\, \sigma^1 + \beta_{\vec{k}}\, \sigma^2, \\ \alpha_{\vec{k}} &= 2[J_1 \sin(\vec{k} \cdot \vec{M}_1) - J_2 \sin(\vec{k} \cdot \vec{M}_2)], \\ \beta_{\vec{k}} &= 2[J_3 + J_1 \cos(\vec{k} \cdot \vec{M}_1) + J_2 \cos(\vec{k} \cdot \vec{M}_2)]. \end{aligned}$$

$$E^{\pm}_{\vec{k}} = \pm \sqrt{\alpha^2_{\vec{k}} + \beta^2_{\vec{k}}}.$$

Phase Diagram: Gapless Phase for $|J_1 - J_2| \leq J_3 \leq (J_1 + J_3)$



Anisotropic Quantum Critical Point

$$\alpha_{\vec{k}} = \sqrt{3}(J_2 - J_1)dk_x + 3(J_1 + J_2)dk_y,$$
$$\beta_{\vec{k}} = J_1\left(\frac{\sqrt{3}}{2}dk_x - \frac{3}{2}dk_y\right)^2 + J_2\left(\frac{\sqrt{3}}{2}dk_x + \frac{3}{2}dk_y\right)^2,$$

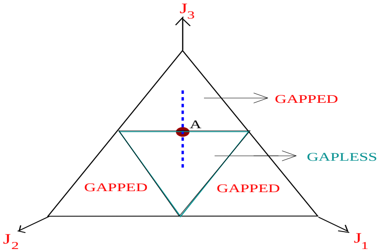
$\alpha_{\vec{k}}$ varies linearly in one particular direction in the plane of (dk_x, dk_y), while $\beta_{\vec{k}}$ varies quadratically in any direction.

- A: Anisotropic QCP $d = 2$, $m = 1$. $\nu_{||} = 1/2$ and $\nu_{\perp} = 1$.

Hickichi, Suzuki and Sengupta, Phys. Rev. B **82**, 174305 (2010)

Anisotropic Quantum Critical Point

<u>Phase Diagram</u>: Gapless Phase for $|J_1 - J_2| \leq J_3 \leq (J_1 + J_3)$



- Case I: One state in the gapped phase and other at the gapless phase
- Case II: Both in the gapless phase
- Case III: Both in the gapped phase.

How to calculate the Fidelity:

Ground State: $|\Psi\rangle \; = \; \prod_{\vec{k}} \; \left[\frac{1}{2} \; (a^{\dagger}_{\vec{k}} - e^{i\theta_{\vec{k}}} \; b^{\dagger}_{\vec{k}}) \; (\; a'^{\dagger}_{\vec{k}} + i \; b'^{\dagger}_{\vec{k}})\right] \; |\Phi\rangle.$

Ground State Fidelity: $F^2 = \prod_k |\langle \Psi^+ | \Psi^- \rangle|^2 = \prod_k \cos^2 \left(\frac{\theta^+_{\vec{k}} - \theta^-_{\vec{k}}}{2} \right),$

$|\Psi^{\pm}\rangle = |\Psi(\lambda \pm \delta\rangle$ with $\lambda = J_{3,c} - J_3$.

$$\cos\theta^{\pm}_{\vec{k}} \; = \; \frac{\alpha^{\pm}_{\vec{k}}}{E^{\pm}_{\vec{k}}} \quad \text{and} \quad \sin\theta^{\pm}_{\vec{k}} \; = \; \frac{\beta^{\pm}_{\vec{k}}}{E^{\pm}_{\vec{k}}},$$

$$\ln F \simeq \delta^2 L^2 \int_{\pi/L}^{\pi-\pi/L} \int_{\pi/L}^{\pi-\pi/L} dk_x dk_y \; \frac{\alpha^2_{\vec{k}}}{\alpha^2_{\vec{k}} + \beta^2_{\vec{k}}}.$$

Generalized Scaling: AQCP

The QCP is characterized by two set of critical exponents: $\nu_{||}$ and $\nu_{\perp}$.

$$
\begin{aligned}
S(\lambda+\delta,\lambda-\delta) \; &= L_{||}^{-m} L_{\perp}^{-(d-m)} \\
& f((\lambda+\delta)L_{||}^{1/\nu_{||}}, (\lambda+\delta)L_{\perp}^{1/\nu_{\perp}}, (\lambda-\delta)L_{||}^{1/\nu_{||}}, (\lambda-\delta)L_{\perp}^{1/\nu_{\perp}}),
\end{aligned}
$$

Rescaling $L_{||}(L_{\perp})$ to $b_{||}$ $(b_{\perp})$ with $(\lambda+\delta)b_{||}^{1/\nu_{||}} = (\lambda+\delta)b_{\perp}^{1/\nu_{\perp}} = 1$,

$$
S(\lambda+\delta,\lambda-\delta) = (\lambda+\delta)^{\nu_{||}m+\nu_{\perp}(d-m)} f\left(1, \frac{\lambda-\delta}{\lambda+\delta}\right).
$$

Expand in the limit, $\delta/\lambda \to 0$

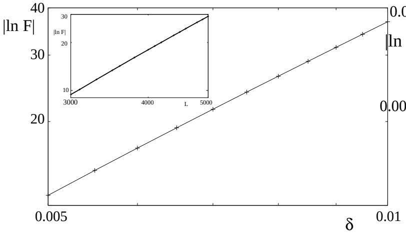
Modified Scaling Relation

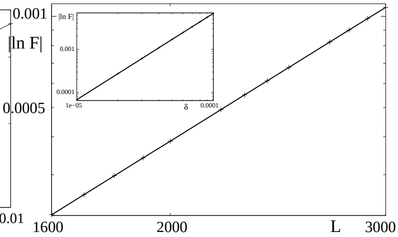
- At the critical point $\ln F(\delta, -\delta) \sim -L_{||}^{m} L_{\perp}^{d-m} \delta^{\nu_{||} m+(d-m)\nu_{\perp}}$
- For $1 \gg \lambda \gg \delta$, $\ln F(\delta, -\delta) \sim -\delta^2 L_{||}^{m} L_{\perp}^{d-m} \lambda^{\nu_{||} m+(d-m)\nu_{\perp}-2}$

For the AQCP in the Kitaev model:

- A: Anisotropic QCP $d = 2$, $m = 1$. $\nu_{||} = 1/2$ and $\nu_{\perp} = 1$.
- For $\lambda = J_3 - J_{3c} = 0$, $\ln F \sim \delta^{3/2} L^2$; Thermodynamic limit:
- Small system size limit: $\ln F \sim -\delta^2 L^2 \chi_F(\lambda = 0) \sim -\delta^2 L^{5/2}$
- $\lambda \neq 0$, $\ln F(\delta, -\delta) \sim \delta^2 L_{||}^{m} L_{\perp}^{d-m} \lambda^{-1/2}$

One state in the gapped phase and other in the gapless phase along the vertical line





Thermodynamic limit $\delta L >> 1$

$\ln F \sim -\delta^{3/2} L^2$

Fidelity Susceptibility limit $\delta L << 1$

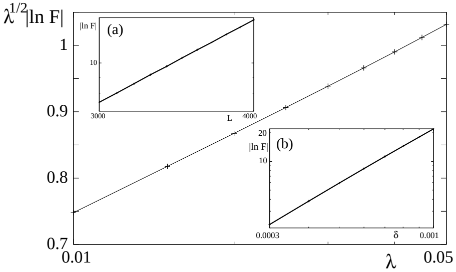
$\ln F \sim -L^{5/2} \delta^2$

The scaling behavior for both the limits is dictated by the AQCP.

Crossover: $\delta L^{1/\nu_\perp} = \delta L \sim 1$.

Both states in the gapless phase: Surprise Emerges

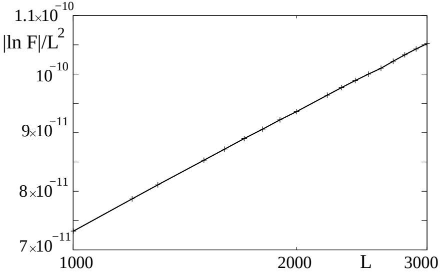
Consider the limit $\delta L^{1/\nu_\perp} = \delta L \gg 1$ and $\lambda \gg L^{-1/\nu_\perp}$



- $\ln F \sim -\delta^2 L^2 \lambda^{-1/2} \ln \lambda$
- $\lambda^{-1/2}$ is the signature of the AQCP
- $\ln \lambda$ correction due to the fact that we are in the gapless phase!

Both states in the gapless phase: Surprise Emerges

The other limit $\delta L^{1/\nu_\perp} = \delta L \ll 1$ and $\lambda \gg L^{-1/\nu_\perp}$



- $\ln F \sim -\delta^2 L^2 \lambda^{-1/2} \ln L \ln \lambda$,
- An additional $\ln L$ correction term
- When $\lambda \leq L^{-1/\nu_\perp} = L^{-1}$, crossover to $\ln F \sim L^{5/2}$.

Is it a thermodynamic to non-thermodynamic crossover?

Apparently yes

- for $\delta L >> 1$, $\ln F \sim L^2$ Thermodynamic limit?
- for $\delta L << 1$, $\ln F \sim L^2 \ln L$ non-Thermodynamic limit?
- The crossover is dictated by $\delta L^{1/\nu_\perp} \sim 1$
- AQCP is the dominant critical point dictating the scaling with λ in the gapless phase.

But...

- one can not extend the derivation to arrive at the scaling to the gapless phase!!
- Scaling with δ is identical.

At the moment the situation is murky.

Inside the gapped phase

$\lambda < 0$ and choose $\lambda \gtrsim L^{-1}$ and $\lambda \gg \delta$

$$\ln F \sim -\delta^2 L^d \lambda^{\nu_{||} m + \nu_\perp (d-m) - 2} \sim \delta^2 L^2 \lambda^{-1/2},$$

It is the same as the scaling of the fidelity susceptibility!

When $\lambda \leq L^{-1/\nu_\perp} = L^{-1}$, crossover to $\ln F \sim L^{5/2}$

Concluding Comments

- Fidelity: Indicator of Quantum Criticality
- Small size and $\delta \to 0$ limit: Fidelity Susceptibility χ_F
  - Displays quantum critical scaling with critical exponents
- Thermodynamic limit; Large N, finite δ
  - Scaling with δ near QCP and AQCP
  - Crosses over to small size limit
- There is possibly a crossover even in the gapless phase!