How to get entangled using dynamics

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Overview

1. Introduction: Why dynamics?

2. Nearly adiabatic dynamics: defect production

3. Non-integrable systems: a specific case study

4. Integrable spin models: Entanglement generation

5. Conclusion

Introduction: Why dynamics

1. *Progress with experiments: ultracold atoms can be used to study dynamics of closed interacting quantum systems.*

2. *Finding systematic ways of understanding dynamics of model systems and understanding its relation with dynamics of more complex models: concepts of universality out of equilibrium?*

3. *Understanding similarities and differences of different ways of taking systems out of equilibrium: reservoir versus closed dynamics and protocol dependence.*

4. *Key questions to be answered:*

   What is universal in the dynamics of a system following a quantum quench ?

   What are the characteristics of the asymptotic, steady state reached after a quench ? When is it thermal ?

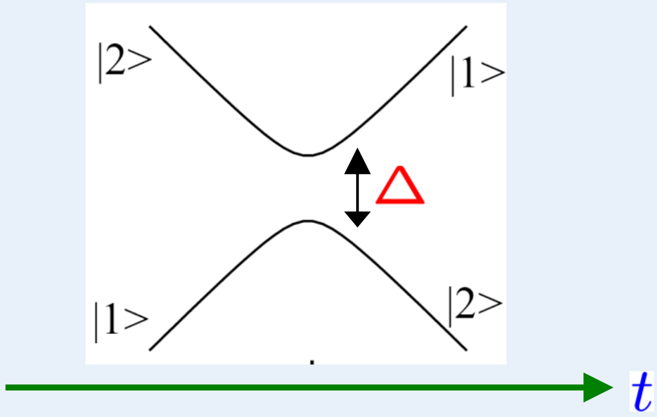
Nearly adiabatic dynamics: Scaling laws for defect production

Landau-Zenner dynamics in two-level systems

Consider a generic time-dependent Hamiltonian for a two level system

$$H = \tau_3 \lambda t/\tau + \Delta \tau_x$$

The instantaneous energy levels have an avoided level crossing at t=0, where the diagonal terms vanish.



The dynamics of the system can be exactly solved.

$$i\hbar\dot{c}_1 = \lambda t/\tau c_1 + \Delta c_2$$
$$i\hbar\dot{c}_2 = -\lambda t/\tau c_2 + \Delta c_1$$

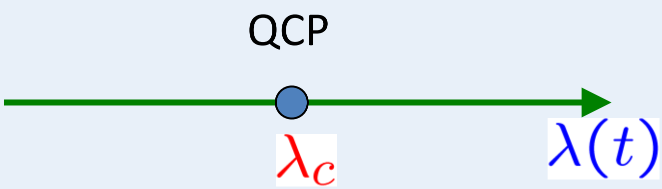
The probability of the system to make a transition to the excited state of the final Hamiltonian starting from The ground state of the initial Hamiltonian can be exactly computed

$$p = |c_1(t \to \infty)|^2 = \exp(-\pi\tau\Delta^2/\lambda)$$

Defect production and quench dynamics

Kibble and Zurek: Quenching a system across a thermal phase transition: *Defect production in early universe.*

Ideas can be carried over to T=0 quantum phase transition. The variation of a system parameter $\lambda(t)$ which takes the system across a quantum critical point at $\lambda = \lambda_c$



For adiabatic evolution, the system would stay in the ground states of the phases on both sides of the critical point.

The simplest model to demonstrate such defect production is

$$H = \sum_k \psi^\dagger(\mathbf{k}) \left[\tau_3 \lambda t/\tau + \Delta(\mathbf{k})\tau_+ + \Delta^*(\mathbf{k})\tau_-\right] \psi(\mathbf{k})$$

$\psi(\mathbf{k}) = (c_1(\mathbf{k}), c_2(\mathbf{k}))$

$\Delta(\mathbf{k}) \sim |\mathbf{k}|$ as $\lambda \to \lambda_c$

QCP with $z = \nu = 1$

Describes many well-known 1D and 2D models.

Specific Example: Ising model in transverse field

Spin Hamiltonian

$$H = J(-\sum_{\langle ij\rangle} S_i^z S_j^z + g \sum_i S_i^x)$$

Jordan-Wigner transformation:

$$s_i^x = (c_i + c_i^+)\prod_{j<i}(1 - 2c_j^+ c_j)$$

$$s_i^y = (c_i - c_i^+)\prod_{j<i}(1 - 2c_j^+ c_j)$$

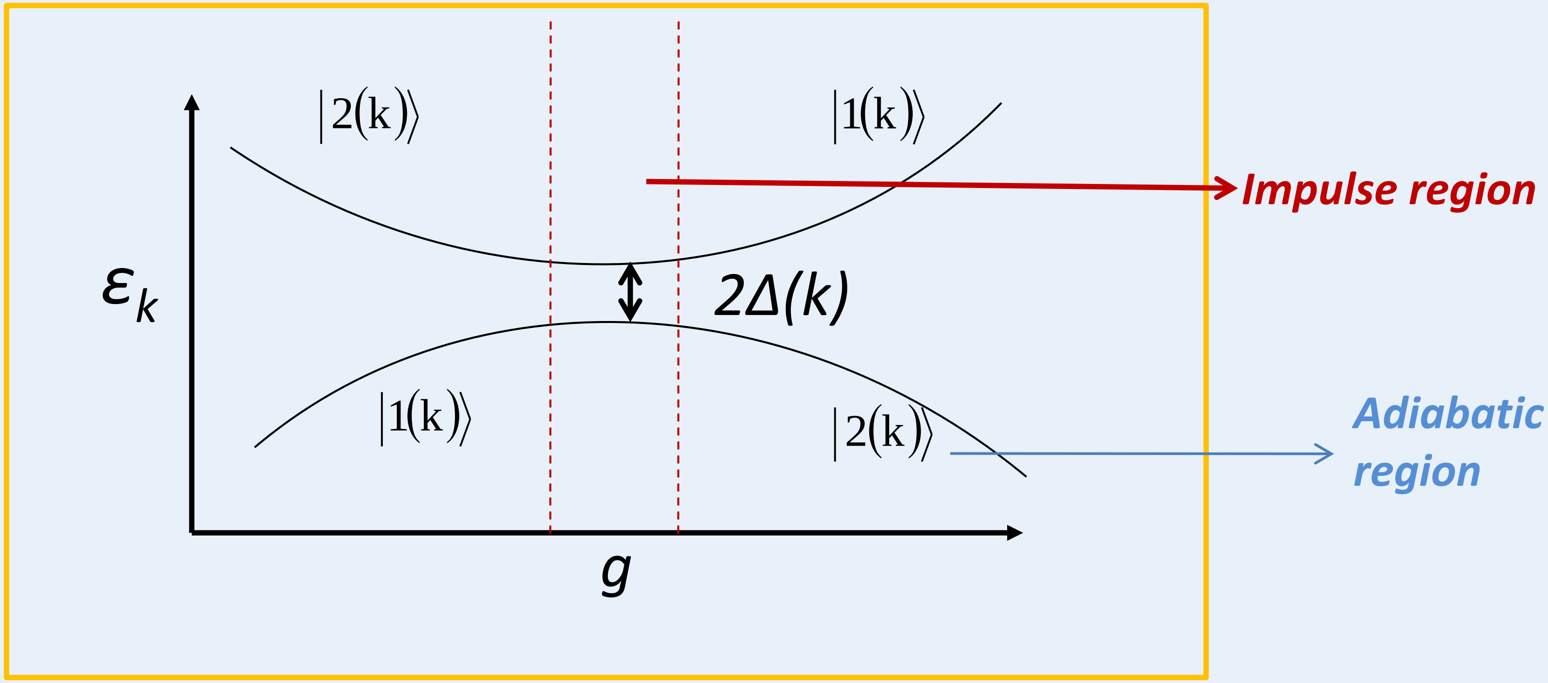
$$s_i^z = 1 - 2c_j^+ c_j$$

Hamiltonian in term of the fermions: [J=1]

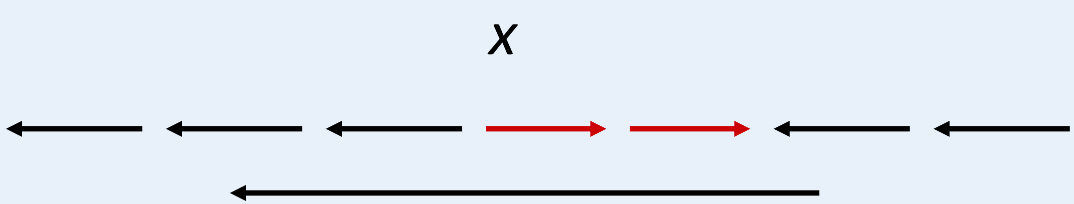
$$H = \sum_k \{2[g - \cos(ka)]c_k^+ c_k + \sin(ka)[c_k^+ c_{-k}^+ + c_{-k} c_k]\}$$

$$\varepsilon_k^{\pm} = \pm 2\sqrt{\left((g-\cos(k))^2 + (\sin(k))^2\right)}$$

$$g = g_0 \frac{t}{\tau}$$



Defect formation occurs mostly between a finite interval near the quantum critical point.



Exactly solvable problem:
A Landau-Zenner problem
for each k

$$i\partial_t \psi_k(t) = H(k;t)\psi_k(t)$$

The probability to end up in the excited state after the time evolution:

$$p_k = \exp(-2\pi\tau\sin^2(k))$$

Density of defects: sum probabilities over all k modes

$$\mathrm{n} = \sum_{\mathrm{k}} \mathrm{p}(\mathrm{k}) = \int \mathrm{dk}\ \mathrm{p_k}$$

For slow enough dynamics or large enough quench time, the maximum contribution to the defect density comes around k=0.

$$k \longrightarrow \sqrt{\tau}k$$

Scaling of defect density

$$n \sim \frac{1}{\sqrt{\tau}}$$

Generic critical points: A phase space argument

The system enters the impulse region when rate of change of the gap is the same order as the square of the gap.

$$d\ln(\Delta_{\vec{k}})/dt \;\geq\; \Delta_{\vec{k}}$$

For slow dynamics, the impulse region is a small region near the critical point where scaling works

$$\Delta_{\vec{k}} \;\sim\; \lambda^{z\nu}|t/\tau|^{z\nu}$$

The system thus spends a time T in the impulse region which depends on the quench time

$$T \sim \tau^{z\nu/(z\nu+1)}$$

In this region, the energy gap scales as

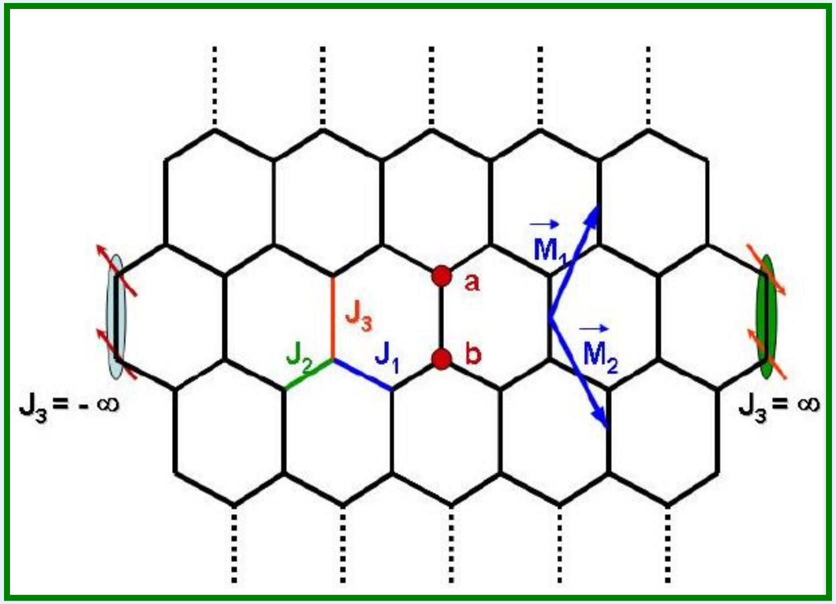
$$\Delta_{\mathbf{k}} \sim \tau^{-z\nu/(z\nu+1)}$$

Thus the scaling law for the defect density turns out to be

$$\Omega_n \sim |\mathbf{k}|^d \sim \Delta_{\mathbf{k}}^{d/z} \sim \tau^{-\nu d/(z\nu+1)}$$

Moving through a gapless line: Kitaev model

Kitaev Model in d=2



$$H = \sum_{j+l=\text{even}} (J_1 \sigma^x_{j,l}\sigma^x_{j+1,l} + J_2 \sigma^y_{j-1,l}\sigma^y_{j,l} + J_3 \sigma^z_{j,l}\sigma^z_{j,l+1})$$

Jordan-Wigner transformation

$$H_F = i \sum_{\vec{n}} [J_1 b_{\vec{n}} a_{\vec{n}-\vec{M}_1} + J_2 b_{\vec{n}} a_{\vec{n}+\vec{M}_2} + J_3 D_{\vec{n}} b_{\vec{n}} a_{\vec{n}}],$$

a and b represents Majorana Fermions living at the end sites of the vertical bonds of the lattice.

D_n is independent of a and b and hence commutes with H_F: Special property of the Kitaev model

Ground state corresponds to D_n=1 on all links.

Solution in momentum space

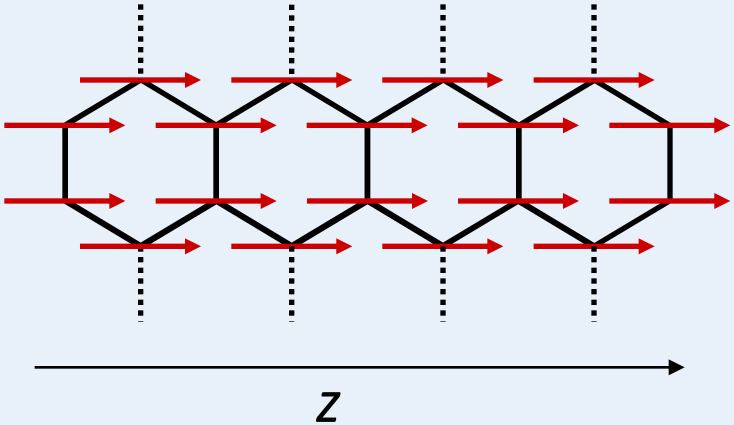
$$H_F = \sum_{\vec{k}} \psi'^{\dagger}_{\vec{k}} H'_{\vec{k}} \psi'_{\vec{k}},$$

$$H'_{\vec{k}} = 2[J_1 \sin(\vec{k}\cdot\vec{M}_1) - J_2 \sin(\vec{k}\cdot\vec{M}_2)]\sigma^1 + 2[J_3 + J_1 \cos(\vec{k}\cdot\vec{M}_1) + J_2 \cos(\vec{k}\cdot\vec{M}_2)]\sigma^3$$

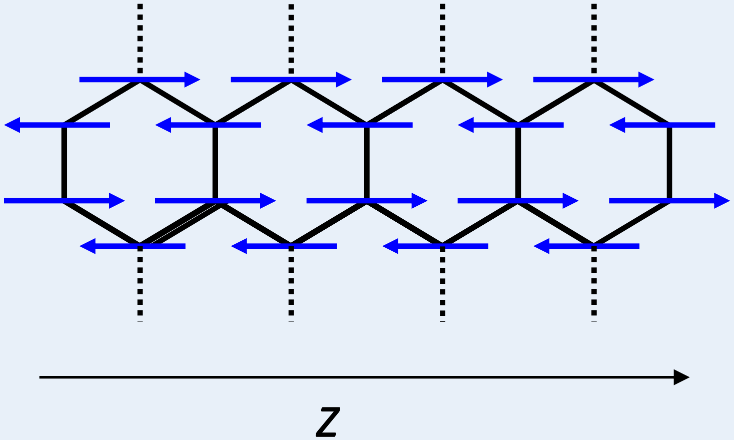
Off-diagonal element

Diagonal element

$$E_{\vec{k}} = 2[\{J_1 \sin(\vec{k}\cdot\vec{M}_1) - J_2 \sin(\vec{k}\cdot\vec{M}_2)\}^2 + \{J_3 + J_1 \cos(\vec{k}\cdot\vec{M}_1) + J_2 \cos(\vec{k}\cdot\vec{M}_2)\}^2]^{1/2}$$

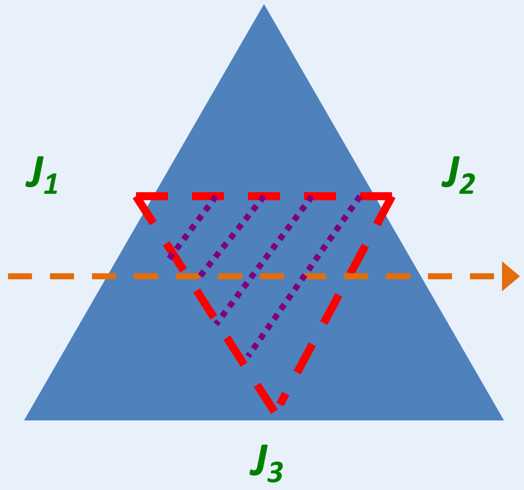


$J_3 \to -\infty$



$J_3 \to \infty$

Gapless phase when J_3 lies between (J_1+J_2) and $|J_1-J_2|$. The bands touch each other at special points in the Brillouin zone whose location depend on values of J_i s.



Quenching J_3 linearly takes the system through a critical line in parameter space and hence through the line

$$\sin(\mathbf{k}\cdot\mathbf{M}_1) = \frac{J_2}{J_1}\sin(\mathbf{k}\cdot\mathbf{M}_2)$$

in momentum space.

In general a quench of d dimensional system can take the system through a d-m dimensional gapless surface in momentum space.

For Kitaev model: d=2, m=1

For quench through critical point: m=d

Question: How would the defect density scale with quench rate?

Defect density for the Kitaev model

Solve the Landau-Zenner problem corresponding to H_F by taking $J_3(t) = Jt/\tau$

$$p(\mathbf{k}) = \exp\left[-\pi\tau \left(J_1 \sin\left(\mathbf{k}\cdot\mathbf{M_1}\right) - J_2 \sin\left(\mathbf{k}\cdot\mathbf{M_2}\right)\right)^2 / J\right]$$

$$n_d = \int_{BZ} d^2k\, p(\mathbf{k})/(4\pi^2 A)$$

For slow quench, contribution to n_d comes from momenta near the line $\sin(\mathbf{k}\cdot\mathbf{M_1}) = \frac{J_2}{J_1}\sin(\mathbf{k}\cdot\mathbf{M_2})$

$$n_d \simeq \int dk_\perp\, e^{-\pi\tau k_\perp^2 c} \sim (1/\sqrt{\tau})$$

For the general case where quench of d dimensional system can take the system through a d-m dimensional gapless surface with z=ν =1

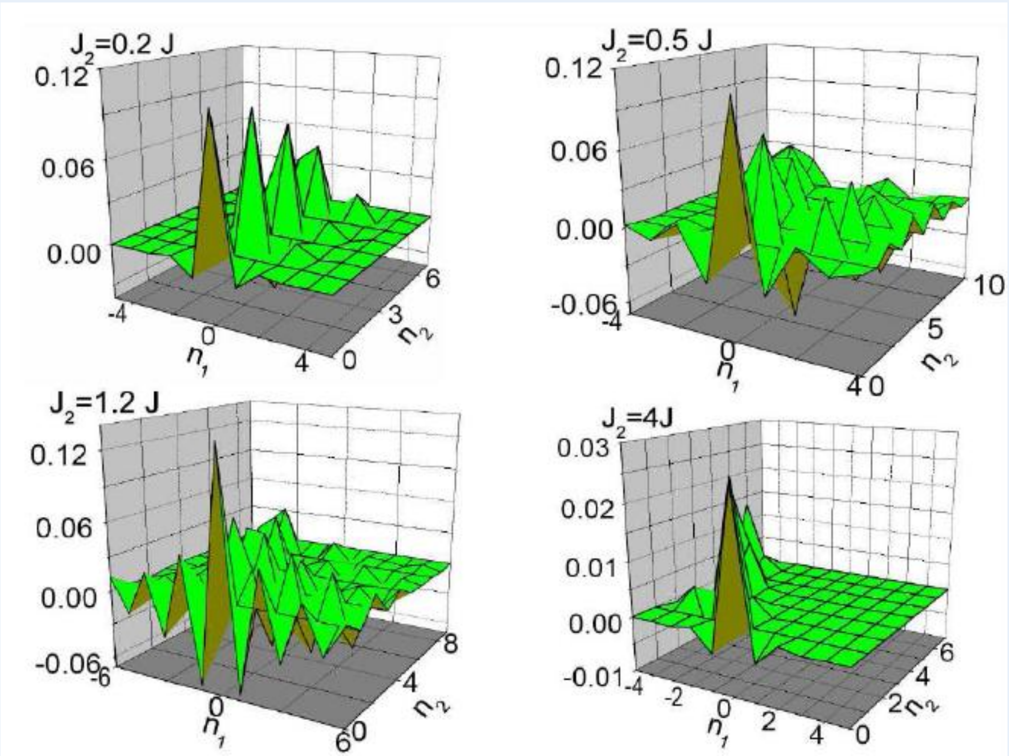
$$n_d \simeq \int d^m k\, e^{-\pi\tau \sum_{\alpha,\beta=1,m} g_{\alpha\beta} k_\alpha k_\beta} \sim (1/\tau)^{m/2}$$

It can be shown that if the surface has arbitrary dynamical and correlation length exponents , then the defect density scales as

$$n_d \sim (1/\tau)^{m\nu/(z\nu+1)}$$

Generalization of Polkovnikov's result for critical surfaces   *Phys. Rev. Lett. 100, 077204 (2008)*

Correlation functions in the Kitaev model



The non-trivial correlation as a function of spatial distance **r** is given in terms of Majorana fermion operators

$$\langle O_{\mathbf{r}} \rangle = i \left\langle b_{\mathbf{n}} a_{\mathbf{n}+\mathbf{r}} \right\rangle$$

Only non-trivial correlation function of the model

For the Kitaev model

$$O_{\mathbf{r}}^{2n} = 1, \quad \left\langle O_{\mathbf{r}}^{2n+1} \right\rangle = \langle O_{\mathbf{r}} \rangle^{2n+1}$$

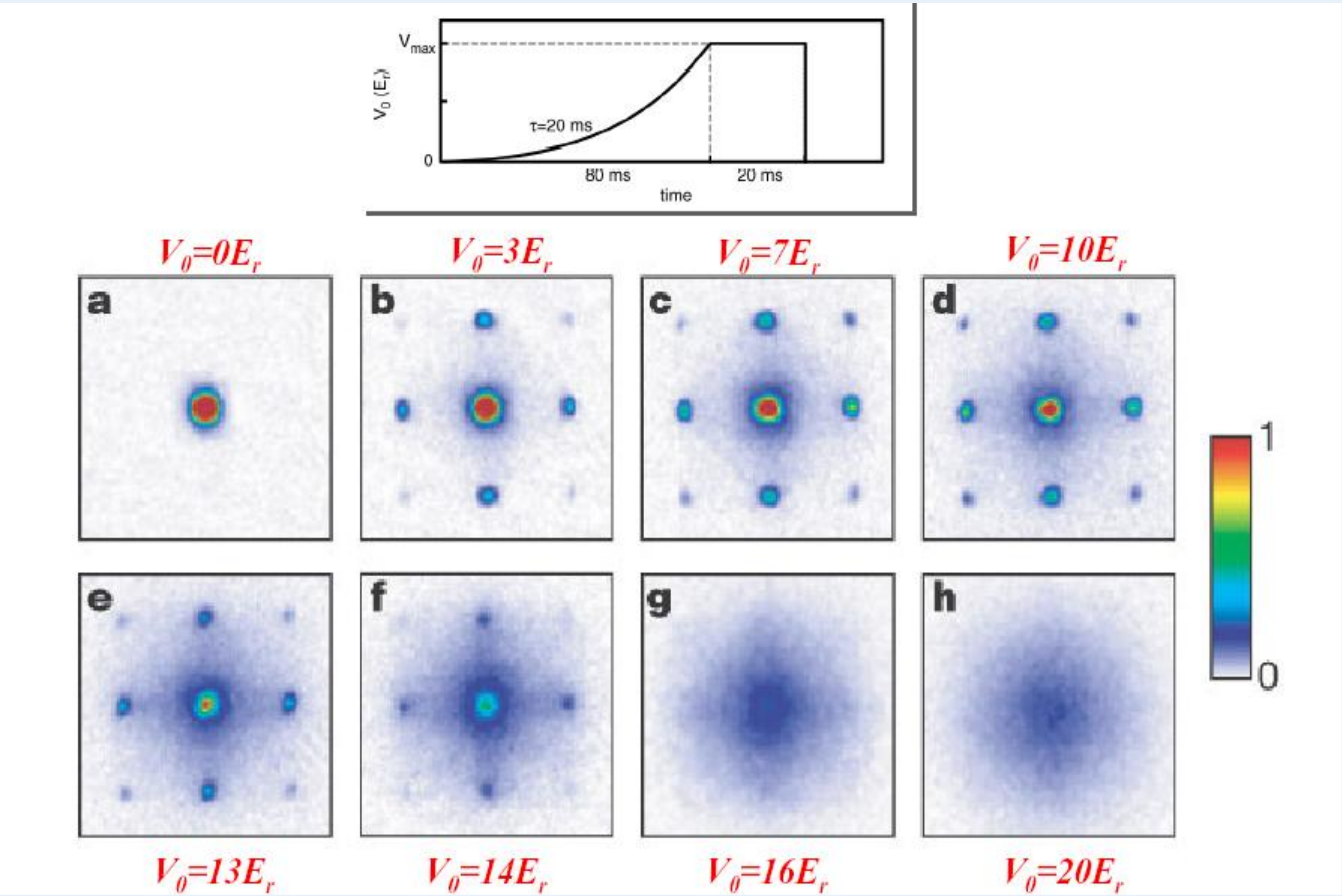
Plot of the defect correlation function sans the delta function peak for J_1=J and $J\tau$=5 as a function of J_2=J. Note the change in the anisotropy direction as a function of J_2.

$$\langle O_{\vec{r}} \rangle = -\delta_{\vec{r},\vec{0}} + \frac{2}{A} \int d^2\vec{k}\; p_{\vec{k}} \cos(\vec{k} \cdot \vec{r}),$$

Non-integrable systems: a specific case study

Dynamics of the Bose-Hubbard model

Bloch 2001

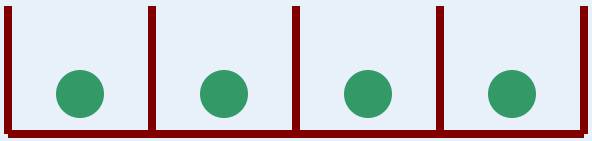


Transition described by the Bose-Hubbard model:

$$\mathcal{H} = \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} -J b^{\dagger}_{\mathbf{r}} b_{\mathbf{r}'} + \sum_{\mathbf{r}} [-\mu \hat{n}_{\mathbf{r}} + \frac{U}{2} \hat{n}_{\mathbf{r}} (\hat{n}_{\mathbf{r}} - 1)],$$

Mott-Superfluid transition: preliminary analysis

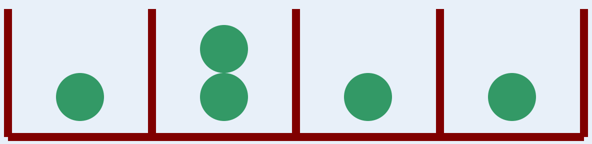
Mott state with 1 boson per site



$$\mathcal{H}_{\text{on}-\text{site}} = \frac{U}{2}\sum_i n_i(n_i-1) - \mu \sum_i n_i$$

Stable ground state for 0 < μ < U

Adding a particle to the Mott state

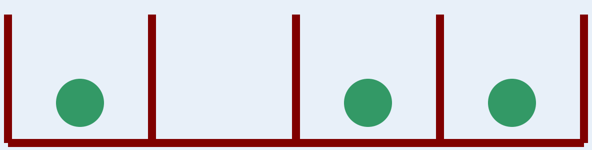


Mott state is destabilized when
the excitation energy touches 0.

$$\delta E_p = (-\mu + U) - 2zt$$

$$t_p^c = (-\mu + U)/2z$$

Removing a particle from the Mott state



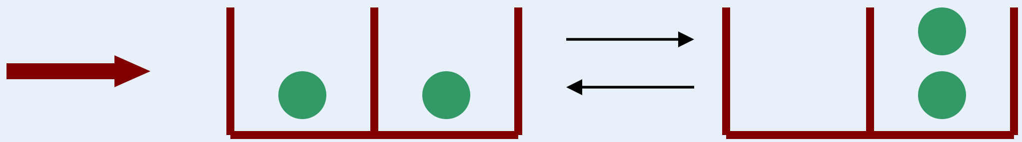
$$\delta E_p = \mu - \quad zt$$

$$t_c^h = \mu/\ z$$

Destabilization of the Mott state via addition of particles/hole: onset of superfluidity

Beyond this simple picture

Higher order energy calculation by Freericks and Monien: Inclusion of up to O(t^3/U^3) virtual processes.

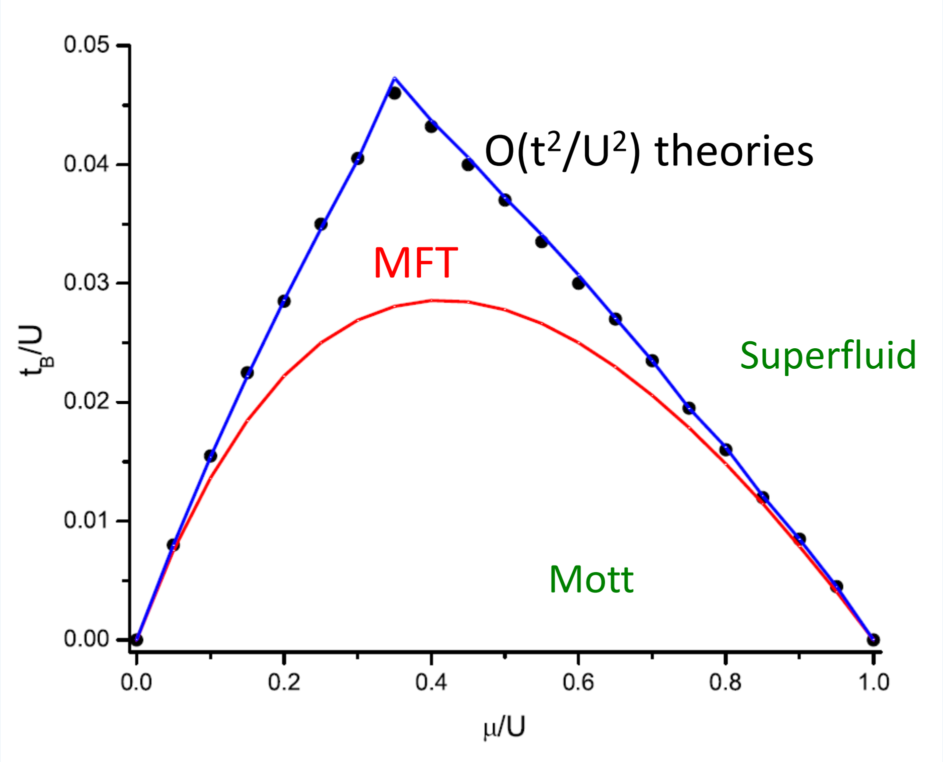


Mean-field theory (Fisher 89, Seshadri 93)

Quantum Monte Carlo studies for 2D & 3D systems: Trivedi and Krauth, B. Sansone-Capponegro

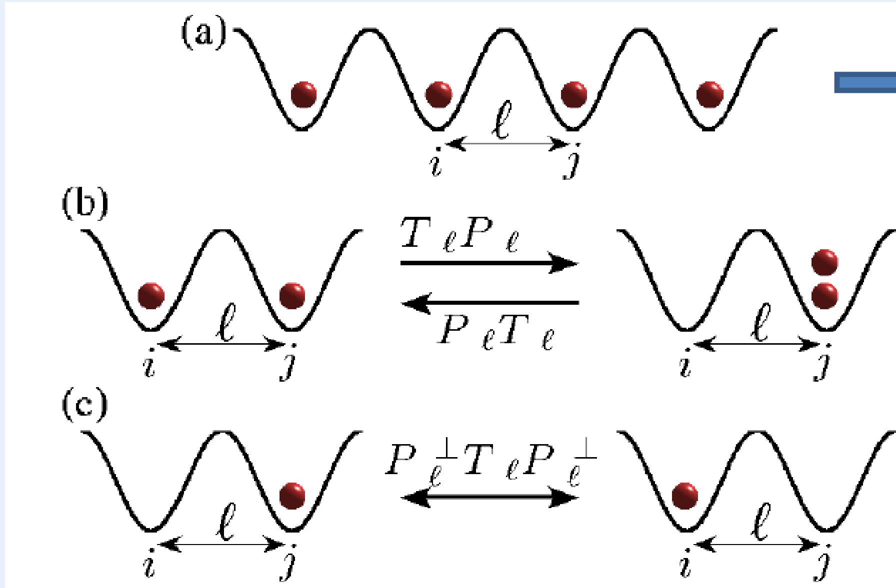
Predicts a quantum phase transition with z=2 (except at the tip of the Mott lobe where z=1).

Phase diagram for n=1 and d=3



No method for studying dynamics beyond mean-field theory

Distinguishing between hopping processes



Mott state

Distinguish between two types of hopping processes using a projection operator technique

Define a projection operator

$$P_\ell = |\bar{n}\rangle\langle\bar{n}|_{\mathbf{r}} \times |\bar{n}\rangle\langle\bar{n}|_{\mathbf{r}'}$$

Divide the hopping to classes (b) and (c)

$$T = \sum_{\langle \mathbf{r}\mathbf{r}'\rangle} -J b^\dagger_{\mathbf{r}} b_{\mathbf{r}'} = \sum_\ell T_\ell = \sum_\ell [(P_\ell T_\ell + T_\ell P_\ell) + P_\ell^\perp T_\ell P_\ell^\perp]$$

Building fluctuations over MFT

Design a transformation which eliminate hopping processes of class (b) perturbatively in J/U.

$$S \equiv S[J] = \sum_\ell i[P_\ell, T_\ell]/U$$

Obtain the effective Hamiltonian

$$H^* = \exp(iS)\mathcal{H}\exp(-iS)$$

$$\begin{aligned} H^* &= H_0 + \sum_\ell P_\ell^\perp T_\ell P_\ell^\perp - \frac{1}{U}\sum_\ell \Big[P_\ell T_\ell^2 + T_\ell^2 P_\ell \\ &- P_\ell T_\ell^2 P_\ell - T_\ell P_\ell T_\ell\Big] - \frac{1}{U}\sum_{\langle \ell\ell'\rangle}\Big[P_\ell T_\ell T_{\ell'} - T_\ell P_\ell T_{\ell'} \\ &+ \frac{1}{2}\Big(T_\ell P_\ell P_{\ell'} T_{\ell'} - P_\ell T_\ell T_{\ell'} P_{\ell'}\Big) + \text{h.c.}\Big] \qquad (2) \end{aligned}$$

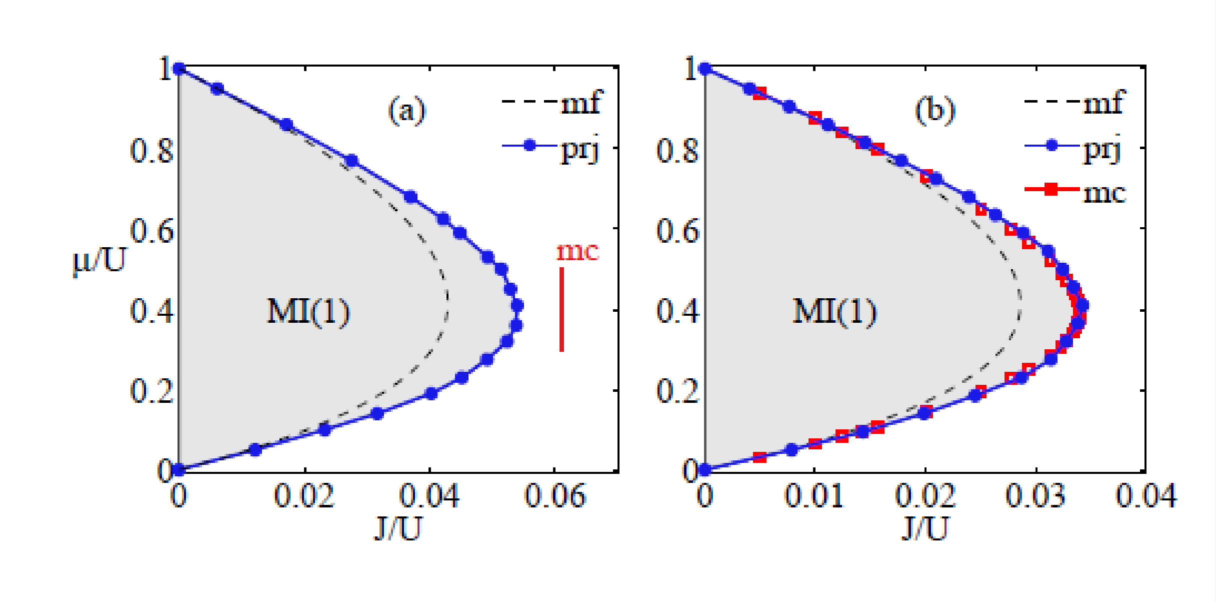
Use the effective Hamiltonian to compute the ground state energy and hence the phase diagram

$$E = \langle\psi|\mathcal{H}|\psi\rangle = \langle\psi'|H^*|\psi'\rangle + \mathrm{O}(z^3J^3/U^2)$$

$$|\psi'\rangle = \exp(iS)|\psi\rangle$$

$$|\psi'\rangle = \prod_{\mathbf{r}}\sum f_n^{(\mathbf{r})}|n\rangle$$

Equilibrium phase diagram



Reproduction of the phase diagram with remarkable accuracy in d=3: much better than standard mean-field or strong coupling expansion in d=2 and 3.

Allows for straightforward generalization for treatment of dynamics

Non-equilibrium dynamics

Consider a linear ramp of $J(t)=J_i+(J_f-J_i)\,t/\tau$.
For dynamics, one needs to solve the Sch. Eq.

$$i\hbar\partial_t|\psi\rangle = \mathcal{H}[J(t)]|\psi\rangle$$

Make a time dependent transformation to address the dynamics by projecting on the instantaneous low-energy sector.

$$|\psi'\rangle = \exp(iS[J(t)])|\psi\rangle$$

$$(i\hbar\partial_t + \partial S/\partial t)|\psi'\rangle = H^*[J(t)]|\psi'\rangle$$

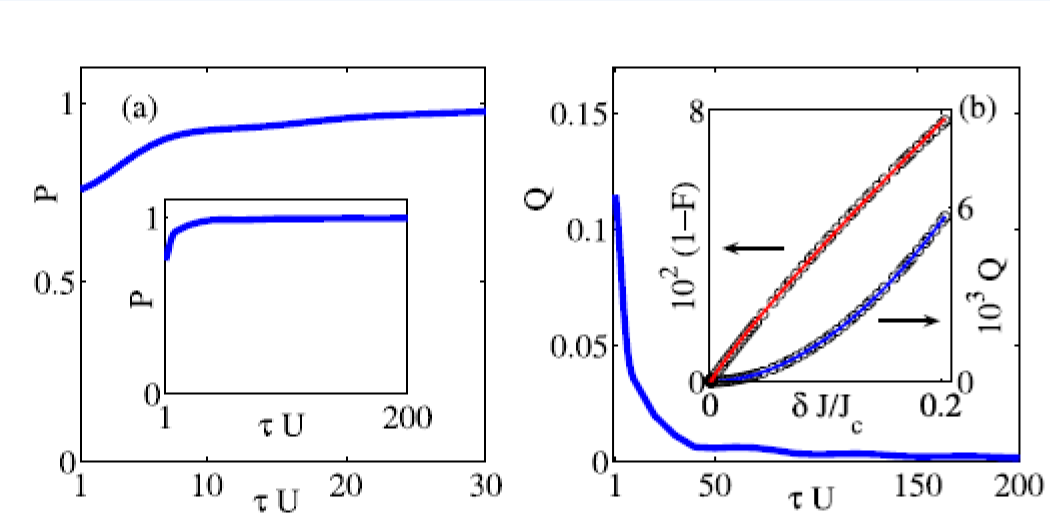
Takes care of particle/hole production due to finite ramp rate

The method provides an accurate description of the ramp if $J(t)/U <<1$ and hence can treat slow and fast ramps at equal footing.

$$\begin{aligned} i\hbar\partial_t f_n^{(\mathbf{r})} &= \delta E[\{f_n(t)\};J(t)]/\delta f_n^{*(\mathbf{r})} + i\hbar\frac{(J_f-J_i)}{U\tau} \\ &\times \sum_{\langle\mathbf{r}'\rangle_\mathbf{r}} \sqrt{n} f_{n-1}^{(\mathbf{r})}\left[\delta_{n\bar{n}}\varphi_{\mathbf{r}'\bar{n}} - \delta_{n,\bar{n}+1}\varphi_{\mathbf{r}',\bar{n}-1}\right] \\ &+\sqrt{n+1} f_{n+1}^{(\mathbf{r})}\left[\delta_{n\bar{n}}\varphi^*_{\mathbf{r}',\bar{n}-1} - \delta_{n,\bar{n}-1}\varphi^*_{\mathbf{r}'\bar{n}}\right] \end{aligned}$$

$$\varphi_\mathbf{r}[\Phi_\mathbf{r}] = \langle\psi'|b_\mathbf{r}|\psi'\rangle[\langle\psi'|b_\mathbf{r}^2|\psi'\rangle]$$

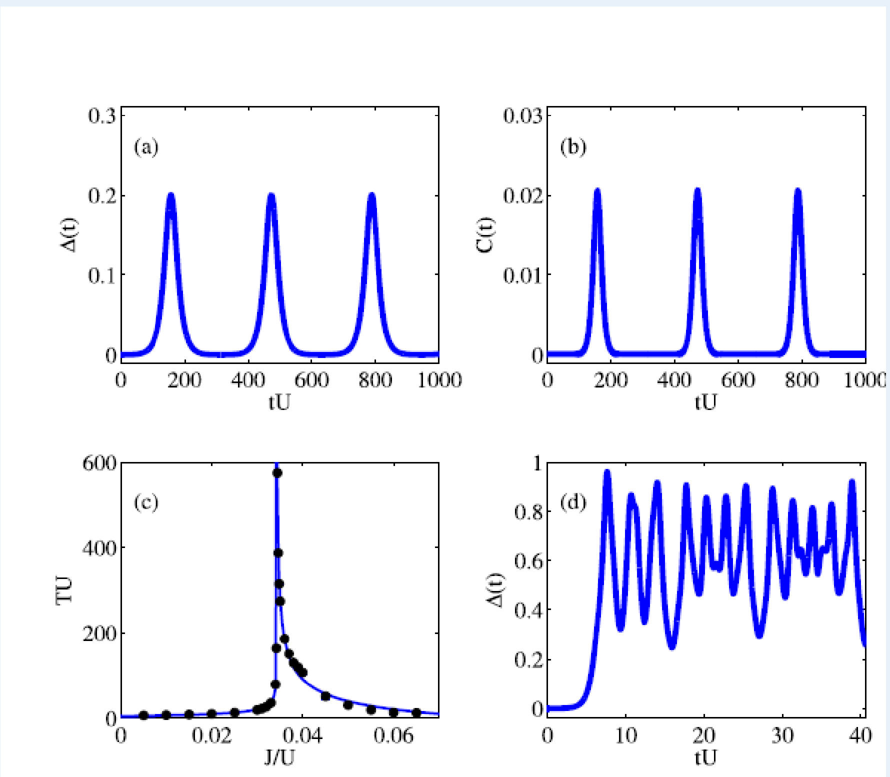
$$E = \langle\psi|\mathcal{H}|\psi\rangle$$



$$P = 1 - |\langle \psi_G | \psi(t_f) \rangle|^2$$

$$Q = \langle \psi_c | \mathcal{H}[J_f] | \psi_c \rangle - E_G[J_f]$$

Absence of critical scaling: may be understood as the inability of the system to access the critical (k=0) modes.



$$\Delta_{\mathbf{r}}(t) = \langle \psi(t) | b_{\mathbf{r}} | \psi(t) \rangle = \langle \psi'(t) | b'_{\mathbf{r}} | \psi'(t) \rangle$$

$$C_{\mathbf{r}}(t) = \langle \psi'(t) | b'_{\mathbf{r}} b'_{\mathbf{r}} | \psi'(t) \rangle - \Delta^2_{\mathbf{r}}(t)$$

Fast quench from the Mott to the SF phase; study of superfluid dynamics.

Single frequency pattern near the critical Point; more complicated deeper in the SF phase.

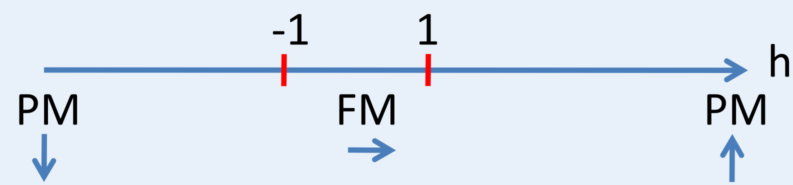
Strong quantum fluctuations near the QCP; justification of going beyond mft.

Entanglement generation

Entanglement generation in transverse field anisotropic XY model

$$H = \frac{J}{4}\sum_n [(1+\gamma)\sigma_n^x\sigma_{n+1}^x + (1-\gamma)\sigma_n^y\sigma_{n+1}^y + h\sigma_n^z],$$

Quench the magnetic field h from large negative to large positive values.



One can compute all correlation functions for this dynamics in this model. (Cherng and Levitov).

$$\alpha_n = \int_0^\pi \frac{dk}{\pi} p_k \cos(nk),$$

No non-trivial correlation between the odd neighbors.

$$\langle\sigma_i^z\rangle = 1 - 2\alpha_0, \quad \langle\sigma_i^z\sigma_{i+n}^z\rangle = \langle\sigma_i^z\rangle^2 - 4\alpha_n^2.$$

$$\langle\sigma_i^+\sigma_{i+2}^-\rangle = \alpha_2\langle\sigma_i^z\rangle,$$

$$\langle\sigma_i^+\sigma_{i+4}^-\rangle = (\alpha_4\langle\sigma_i^z\rangle - 2\alpha_2^2)\langle\sigma_i^z\sigma_{i+2}^z\rangle,$$

Single-site entanglement: the linear entropy or the Single site concurrence

$$\rho_i = (I + \sigma^z\langle\sigma_i^z\rangle)/2.$$

$$\mathcal{C}^{(1)} = \sqrt{4\det\rho_i} = 2\sqrt{\alpha_0(1-\alpha_0)}$$

Finite for all finite non-zero quench rate

What's the bipartite entanglement generated due to the quench between spins at i and i+n?

Measures of bipartite entanglement in spin ½ systems

Concurrence (Hill and Wootters)

Consider a wave function for two spins and its spin-flipped counterpart

$$|\tilde{\psi}\rangle = \sigma_y|\psi\rangle \quad C = |\langle\psi|\tilde{\psi}\rangle|$$

C is 1 for singlet and 0 for separable states
Could be a measure of entanglement

Use this idea to get a measure for mixed state of two spins : need to use density matrices

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\,\rho\,(\sigma_y \otimes \sigma_y)$$

$$\text{Eigenvaluesof}\,\rho\tilde{\rho} = \lambda_{1..4}$$

$$C = \text{Max}\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$$

Negativity (Peres)

Consider a mixed state of two spin ½ particles and write the density matrix for the state.

Take partial transpose with respect to one of the spins and check for negative eigenvalues.

$$N = \text{Max}\{0, \sum_i |\lambda_i^n|\}$$

Note: For separable density matrices, negativity is zero by construction

$$\rho = \sum_k w_k \rho_{1k} \otimes \rho_{2k}$$

$$\rho^T = \sum_k w_k (\rho_{1k})^T \otimes \rho_{2k}$$

Steps:

1. Compute the two-body density matrix

$$a^n_{\pm} = \langle \frac{1}{4}(1 \pm \sigma^z_i)(1 \pm \sigma^z_{i+n}) \rangle,$$

$$a^n_0 = \langle \frac{1}{4}(1 \pm \sigma^z_i)(1 \mp \sigma^z_{i+n}), \rangle \quad b^n_{1(2)} = \langle \sigma^-_i \sigma^{-(+)}_{i+n} \rangle$$

$$\rho^n = \begin{pmatrix} a^n_+ & 0 & 0 & b^n_1 \\ 0 & a^n_0 & b^n_2 & 0 \\ 0 & b^{n*}_2 & a^n_0 & 0 \\ b^{n*}_1 & 0 & 0 & a^n_- \end{pmatrix}$$

2. Compute concurrence and negativity as measures of two-site entanglement from this density matrix

$$C^n = \max\{0, 2(|b^n_2| - \sqrt{a^n_+ a^n_-})\}$$

Results

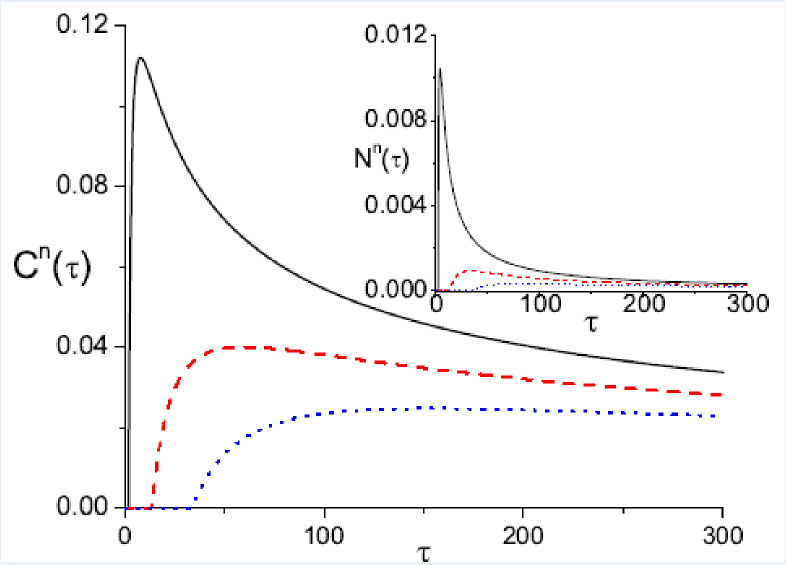
Properties of bipartite entanglement

a. *Finite only between even neighbors*

b. ***Requires a critical quench rate above which it is zero.***

$$\gamma^2\tau_c^n = 1.96,\ 13.6,\ \text{and}\ 33.8,\ \text{for}\ n = 2,\ 4,\ \text{and}\ 6$$

c. ***Ratio of entanglement between even neighbors can be tuned by tuning the quench rate.***



d. ***The entanglement necessarily involves N>2 spins for sufficiently fast quench rates.***

e. ***Analogous study for the 2D Kitaev model shows that the bipartite entanglement vanishes for any quench rate.***