Symplectic Methods in the Theory of Quantum Correlations

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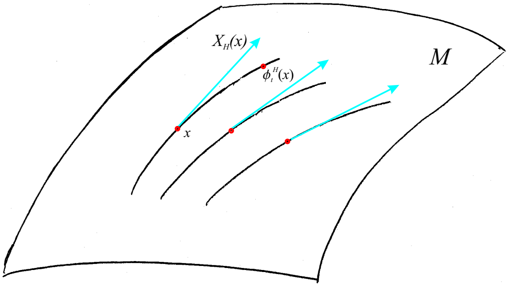
Description of Quantum Entanglement

- Description of quantum correlations in finite-dimensional composite quantum systems in terms of (differential, symplectic) geometry rather than linear algebra
- Problem 1: Construction of entanglement measures
- Problem 2: Local transformations (transforming a state of the total to another by means of local operations)
- Problem 3: States with symmetries: symmetric ("bosons"), antisymmetric ("fermions") where the Hilbert space is not the full tensor product but rather a subspace of states with the given symmetry

Symplectic geometry and classical mechanics

- Dynamics



- Flow (classical dynamics in the phase space M)

$$M \ni x \mapsto \phi_t^H(x) =: x(t) \in M$$

- Hamilton function

$$H : M \to \mathbb{R}$$

- Vector field (tangent to trajectories in the phase space = "velocity" in M)

$$X_H(x) = \frac{d}{dt}\Big|_{t=0} \phi_t^H(x)$$

Symplectic geometry and classical mechanics

- (M, ω) - symplectic manifold,
- $d\omega = 0$, ω - nondegenerate
- To find X_H for a given H we need ω:

$$dH = \omega(X_H, \cdot)$$

- Dynamics (flow)

$$\frac{d}{dt}x(t) = X_H(x(t))$$

- The Poisson bracket of two functions F, G on M

$$\{F, G\} = \omega(X_F, X_G) = X_F(G)$$

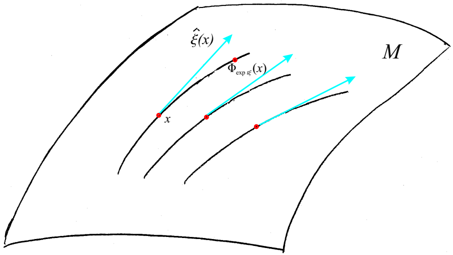
Symplectic group actions

- K – (compact, semisimple) Lie group + symplectic action on M (i.e. preserves the symplectic form ω = "canonical transformation")

$$K \times M \ni (g, x) \mapsto \Phi_g(x) \in M, \quad \Phi_{g_1 g_2} = \Phi_{g_1}(\Phi_{g_2}(x)), \quad \Phi_g^* \omega = \omega$$

- $\mathfrak{k}$ – Lie algebra of K
- Let $\xi \in \mathfrak{k}$, then
  - $\exp t\xi$ – a one parameter subgroup of K
  - $\Phi_{\exp t\xi}$ – a one parameter group of symplectomorphisms ("canonical transformations") of M
- Define a fundamental vector field $\hat{\xi}$ (= tangent to $\Phi_{\exp t\xi}(x) atx$)

$$\hat{\xi}(x) = \frac{d}{dt}\Big|_{t=0} \Phi_{\exp t\xi}(x).$$



- Is there a Hamilton function corresponding to the obtained vector field?

Symplectic group actions

- The answer is "yes"
- Locally there exists a Hamilton function $\mu_\xi : M \to \mathbb{R}$ for $\hat{\xi}$, i.e.

$$d\mu_\xi(\cdot) = \omega(\hat{\xi}, \cdot)$$

- This function can be chosen to be linear in ξ, i.e.

$$\mu_\xi(x) = \langle \mu(x), \xi \rangle, \quad \mu(x) \in \mathfrak{k}^*,$$

  where $\mathfrak{k}^*$ is the space dual to the Lie algebra $\mathfrak{k}$, i.e. the (linear) space of all linear functions on $\mathfrak{k}$ (remember $\mathfrak{k}$ is a vector space), and $\langle \, , \rangle$ is the pairing between $\mathfrak{k}$ and $\mathfrak{k}^*$.

- In this way we obtain a map $\mu : M \to \mathfrak{k}^*$.

$$M \ni x \mapsto \mu(x) \in \mathfrak{k}^*$$

  called the moment (or momentum) map

Coadjoint action. Symplectic structure on coadjoint orbits

- The group K acts in a natural way on its algebra $\mathfrak{k}$

$$\mathrm{Ad}_g X = gXg^{-1}.$$

- The coadjoint action Ad^*_g on $\mathfrak{k}^*$ is the dual one

$$\langle \mathrm{Ad}^*_g \alpha, X\rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}} X\rangle = \langle \alpha, g^{-1}Xg\rangle,$$

- Coadjoint orbits $\Omega_\alpha = \{\mathrm{Ad}^*_g \alpha, g \in K\}$, are symplectic manifolds
- The symplectic form ω at $\beta \in \Omega_\alpha$ is

$$\omega(\tilde{X}, \tilde{Y})(\beta) = \langle \beta, [X, Y]\rangle$$

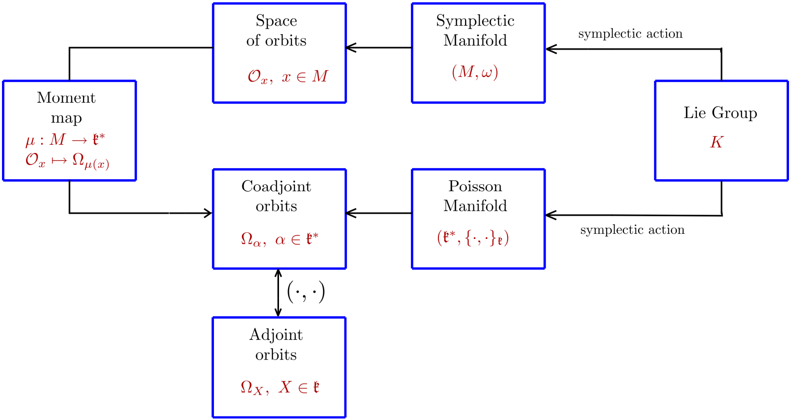
  where $\tilde{X}$ is the fundamental vector field of the coadjoint action $\mathrm{Ad}^*_{\exp tX}\beta$ by the one-parameter group $\exp tX$

$$\tilde{X}(\beta) := \frac{d}{dt}\Big|_{t=0} e^{-tX}\beta e^{tX}$$

- μ can be chosen equivariant with respect to the coadjoint action of K, i.e.

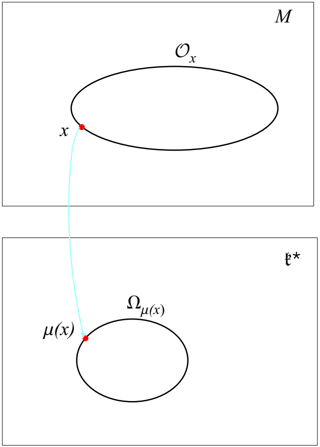
$$\mu\left(\Phi_g(x)\right) = \mathrm{Ad}^*_g \mu(x),$$

Geometric Structure



Orbits of group actions and the moment map

- Two symplectic structures connected by the moment map:
  - on M
  - on coadjoint orbits
- An orbit $\mathcal{O}_x$ of K action on M is mapped onto a coadjoint orbit $\Omega_{\mu(x)}$ w $\mathfrak{k}^*$ via momentum map μ.



Orbits of group actions and the moment map

- In general, however, it is not a diffeomorphism between $\mathcal{O}_x$ and $\Omega_{\mu(x)}$.
- There exist two interesting subgroups of K to consider
  - stabilizer of the points on the orbit of the action on M (i.e. a subgroup which does not move them), $\mathrm{Stab}(x)$
  - stabilizer of the elements of the coadjoint action on the corresponding coadjoint orbit, $\mathrm{Stab}(\mu(x))$
- An orbit $\mathcal{O}_x$ on M is diffeomorphic to the corresponding coadjoint orbit $\Omega_{\mu(x)}$ iff both stabilizers are equal.
- If this is the case the orbit on M is symplectic (since the corresponding coadjoint orbit is), i.e. the symplectic form on M restricted to this orbit is nondegenerate (Kostant-Sternberg).
- Otherwise the orbit $\mathcal{O}_x$ is not symplectic, the symplectic form is degenerate and the dimension of the degeneracy can be a useful characterization of orbits.

$$\mathrm{D}(x) = \dim(\mathcal{O}_x) - \dim(\Omega_{\mu(x)}) = \dim(\mathrm{Stab}(\mu(x))) - \dim(\mathrm{Stab}(x))$$

- In general coadjoint orbits encode only partial information about orbits in M.

Quantum mechanics

- Pure states - points in the projective space $\mathbb{P}(\mathcal{H})$, where $\mathcal{H}$ - underlying Hilbert space
- The projective space $M = \mathbb{P}(\mathcal{H})$ is a symplectic manifold
  - Any vector from $T_{[v]}\mathbb{P}(\mathcal{H})$ (tangent space at the point $[v]$) can be written as $[Av]$, where $A \in \mathfrak{su}(\mathcal{H})$ and
$$\omega([Av], [Bv]) = -\frac{i\langle [A\,,\,B]v|v\rangle}{2\langle v|v\rangle}$$
- The unitary group $SU(\mathcal{H})$ acts on $M = \mathbb{P}(\mathcal{H})$ preserving ω.
- The moment map for this action
$$\mu\big([v]\big)(X) = -\frac{i}{2}\frac{\langle v|X|v\rangle}{\langle v|v\rangle},$$

Quantum mechanics. Separable and entangled states

- Composite (L-partite) systems

$$\mathcal{H}_c = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L$$

(for simplicity, let's assume all $\mathcal{H}_i$ are the same, say $\mathcal{H} \simeq \mathbb{C}^N$)

- Quantum state is separable (not entangled) iff it is simple tensor

$$v = v_1 \otimes v_2 \cdots \otimes v_L, \quad v_j \in \mathcal{H}_j$$

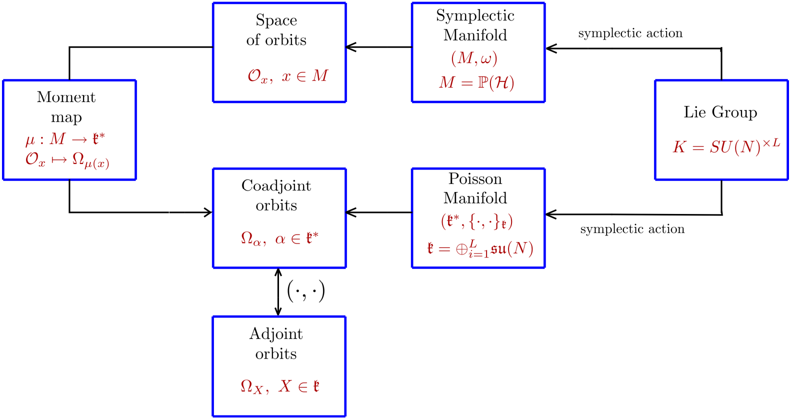
otherwise it is entangled.

- Separability (entanglement) of states is invariant under the direct product (local unitary operations)

$$K = SU(N) \times \cdots \times SU(N)$$

- Using our machinery we can attack some interesting problems
  - How the degeneracy of the symplectic form of an orbit is connected with entanglement properties of the states on it?
  - How to check that two states are locally unitary equivalent (i.e., they belong to the same orbit of K)
  - How to define quantum correlations if there is no tensor product structure (indistinguishable particles)

Geometric structure - Quantum Entanglement



Two-partite systems

- Hilbert space $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$ with the natural action of $K = SU(N) \times SU(N)$, $\mathfrak{k} = \mathrm{Span}_{\mathbb{R}}\{X_1 \otimes I, I \otimes X_2\}$, where $X_1, X_2 \in \mathfrak{su}(N)$
- Quantum state $|\Psi\rangle$
$$|\Psi\rangle = \sum_{i,j=1}^{N} C_{ij}|i\rangle \otimes |j\rangle, \quad C_{ij} \in \mathbb{C}.$$
can be transformed by the K-action to the canonical ('Schmidt') form
$$|u\rangle = \sum_i \lambda_i |i\rangle \otimes |i\rangle$$
- Let m_i = the multiplicity of distinct λ_i, and m_0 corresponds to $\lambda_0 = 0$
- Dimension of degeneracy
$$D(|\Psi\rangle) = \dim(\mathcal{O}_{|\Psi\rangle}) - \dim(\mu(\mathcal{O}_{|\Psi\rangle})) = \sum_{n=1}^{r} m_n^2 - 1.$$
- Separable states form the only symplectic orbit $D(|\Psi\rangle) = 0$
- The degree of degeneracy is a well defined entanglement measure

Maximally entangled state

- Maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle.$$

- The restriction of symplectic form ω to $\mathcal{O}_{|\Psi\rangle}$ is zero and

$$\dim \mathcal{O}_{|\Psi\rangle} = \frac{1}{2} \dim \mathbb{P}(\mathcal{H})$$

- The orbit through the maximally entangled state is lagrangian (maximally "non-symplectic")

Symplectic geometry of entanglement - many particles

- L-particles, $\mathcal{H} = \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N$, $K = SU(N)^{\times L}$
- The orbit of separable states is symplectic (the degeneracy of ω equals 0, $D(|\Psi\rangle) = 0$). Moreover it is the only symplectic orbit
- It is an orbit of the Perelomov coherent states for the irreducible representation of K on $\mathcal{H}$ (the orbit through the highest weight vector of the representation).
- The degree of degeneracy is again a well defined entanglement measure

$|GHZ_L\rangle$ state

- The $|GHZ_L\rangle$ state of L-qubits

$$|GHZ_L\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes L} + |1\rangle^{\otimes L}\right)$$

- For $|GHZ_L\rangle$ we have

$$\omega|_{\mathcal{O}_{|GHZ_L\rangle}} = 0$$

- Dimension $\dim\mathcal{O}_{|GHZ_L\rangle} = 2L+1$, when $L > 2$ and $\dim\mathcal{O}_{|GHZ_2\rangle} = 3$
- When $L = 2$ or $L = 3$ then an orbit $\mathcal{O}_{|GHZ_L\rangle}$ is lagrangian
- When $L > 3$ the dimension of the orbit $\mathcal{O}_{|GHZ_L\rangle}$ is too small to be lagrangian (although the degeneracy is maximal). It is a matter of taste whether we can call such states maximally entangled (there other with the same property of maximal degeneracy)

Local Unitary Equivalence

- Two states are Locally Unitary equivalent (LU-equivalent) iff

$$|v_1\rangle = U_1 \otimes \ldots \otimes U_L |v_2\rangle.$$

- Or equivalently

$$[v_1] = [U_1 \otimes \ldots \otimes U_L v_2].$$

- A pure state of a bipartite system

$$|\Psi\rangle = \sum_{ij} c_{ij} |e_i\rangle \otimes |f_j\rangle$$

  can be transformed by local unitary transformations to the canonical ('Schmidt') form

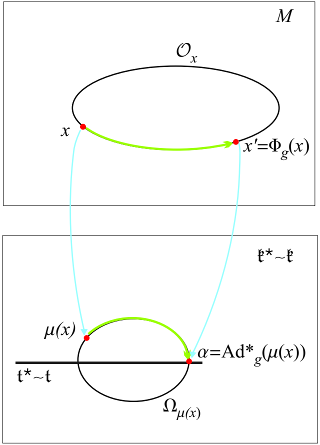
$$|u\rangle = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle$$

- Two states are locally unitary equivalent iff they have the same Schmidt decomposition

- There is no direct analog of the Schmidt decomposition for systems with more than 2 components.

Canonical forms

- In general an orbit $\mathcal{O}_x$ does not contain any distinguished point, but corresponding coadjoint orbit $\Omega_{\mu(x)}$ has such a point
- Each orbit of the coadjoint action intersects the subspace $\mathfrak{t}^*$ in $\mathfrak{k}^*$ which is dual to the maximal commutative subalgebra of $\mathfrak{k}$
- Let $x \in M$, then $\mu(x) \in \mathfrak{k}^*$ and there is $g \in K$ such that $\mathrm{Ad}_g^*(\mu(x)) \in \mathfrak{t}^*$.
  Let us call $x' = \Phi_g(x)$ - canonical form of x (up to a factor – the collection of reduced density matrices in their diagonal forms)



Canonical form

Properties of canonical form

- It is given up to the action of $G = \mathrm{Stab}(\mu(x'))/\mathrm{Stab}(x')$
- Points on the same orbit have the same (modulo G) canonical forms
- For x, y on the same orbit $\mu(x') = \mu(y')$ (x', y' are in the same fiber of μ)
- In other words, if $|v\rangle$ and $|u\rangle$ are LU-equivalent then their canonical forms belong to the same fiber of the moment map, $\mu([v']) = \mu([u'])$, but this can happen also for LU-nonequivalent states
- The problem arises since there might be different orbits in M which are mapped by μ on the same coadjoint orbit

LU-equivalence and fibers of momentum map

- ... but if the fiber of the moment map lies entirely in the orbit (i.e., the tangent space to the fiber is a subspace of the tangent space of the orbit) then all states in the fiber are LU-equivalent
- Fact: the tangent space to the fiber of μ over $[v]$ is contained in the ω-orthogonal complement of the tangent space at $[v]$ to the orbit through $[v]$
- If the orbit is coisotropic (i.e., at each point the tangent space to it contains its ω - orthogonal complement) then the states in the same fiber are LU-equivalent
- Example: The orbit through $|GHZ_3\rangle$ is coisotropic - using this fact we find simple solution of LU-equivalence problem for any three qubit states

LU-equivalence and fibers of momentum map

- It turns out that even in two-partite case not all orbits are coisotropic although μ fibers are contained in them.

- We need some more subtle analysis of the structure of the fiber - this can give some information whether the canonical form gives sufficient information about LU-equivalence or what we should know in addition to decide whether two states are LU-equivalent

- Example (Brion's theorem)
  - Together with the group $K = SU(N)^{\times L}$ we consider its complexification $K^{\mathbb{C}} = SL(N, \mathbb{C})$
  - $K^{\mathbb{C}}$ contains a group B of upper-triangular matrices (Borel group – can be defined more abstractly)
  - If B has an open orbit in $M = \mathbb{P}(\mathcal{H}$ the fibers of μ are fully contained in single orbits
  - Conclusion: the canonical form determines LU-equivalence – this is the case, eg., of bipartite systems of arbitrary dimensions or GHZ_3 states

Summary and other applications

The presented construction works for

- arbitrary number of subsystems
- arbitrary (finite) dimensions of subsystems
- for systems without the full tensor product structure by adapting appropriately the group K
- *mutatis mutandis* can be adapted to mixed states (but this is another just begun story ...)

- A Sawicki, A Huckleberry, M Kuś Symplectic Geometry of Entanglement Commun. Math. Phys Volume 305, Number 2, 441-468 (2011)
- A Sawicki, M Kuś Geometry of the local equivalence of states J. Phys. A: Math. Theor. 44 495301 (2011),