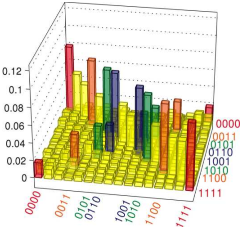
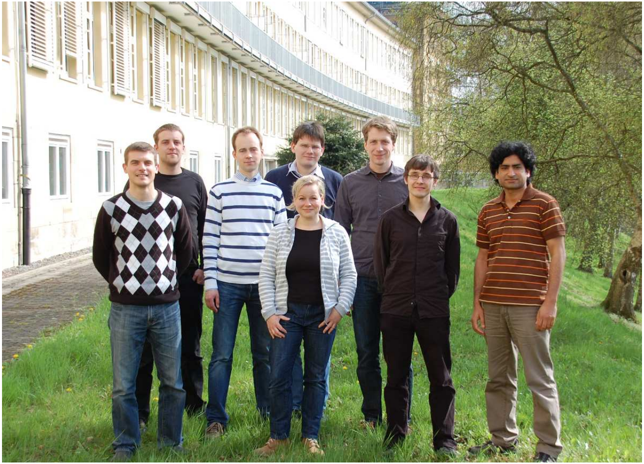
Characterizing multiparticle entanglement

Otfried Gühne
M. Hofmann, B. Jungnitsch, S. Niekamp, M. Kleinmann, T. Moroder
Experiments:
J. Barreiro & the group of Rainer Blatt,
H. Lu, W.B. Gao & the group of Jian-Wei Pan



Fachbereich Physik, Universität Siegen



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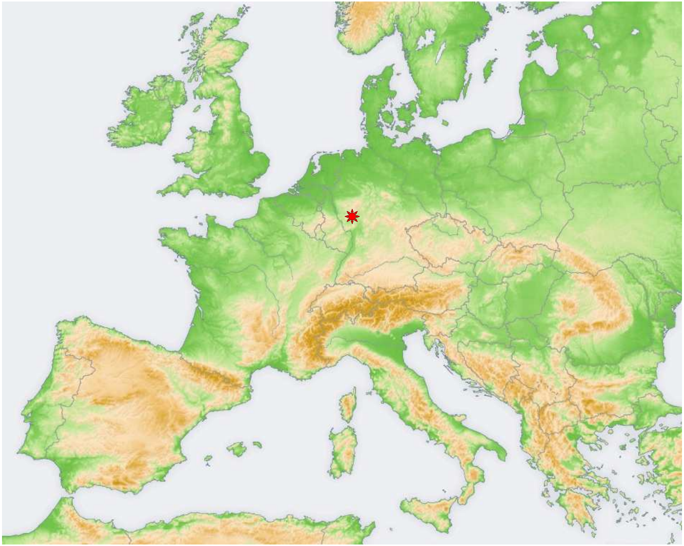
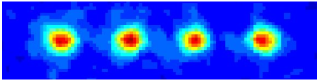
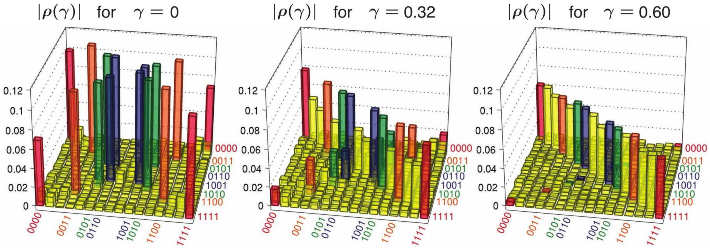


Overview

1. Motivation
2. Multiparticle entanglement: basic definitions
3. The partial tranposition and multiparticle entanglement
4. Proving separability of quantum states
5. Outlook

First motivation: a recent ion experiment



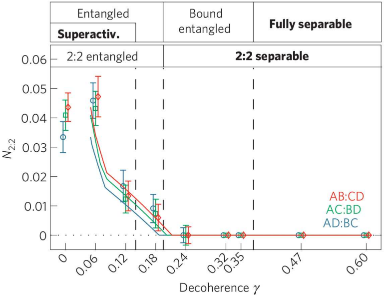


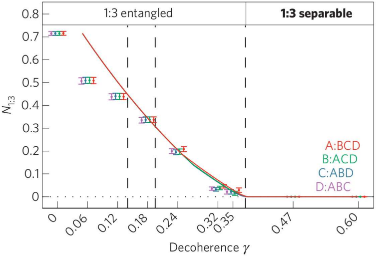
- Entangle four ions and switch on artificial decoherence.
- Entanglement dissappears somehow ...
- At some point, A vs. BCD is entangled, but AB vs. CD not.

J. Barreiro et al., Nature Physics 6, 943 (2010)

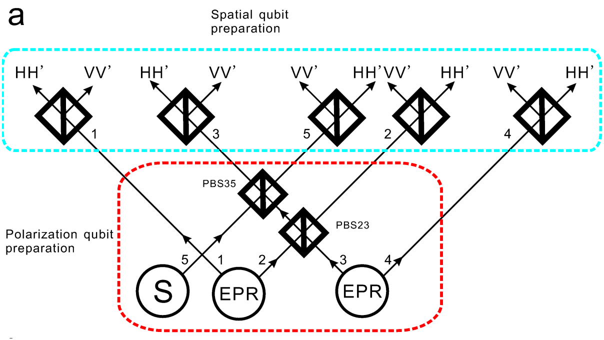
Questions for a theorist

- How can we prove that a state is entangled?
- How can we prove that a state is separable?
- What about statistical errors? (see the talk by Matthias Kleinmann)
- Is the reconstruction of a density matrix correct? (see the talk by Matthias Kleinmann)





Second motivation: a recent photon experiment



- Step 1: Generate GHZ states $|\psi\rangle = |00...0\rangle + |11...1\rangle$ with up to five photons using polarization.
- Step 2: Use hyperentanglement $\Rightarrow$ Up to ten qubits.

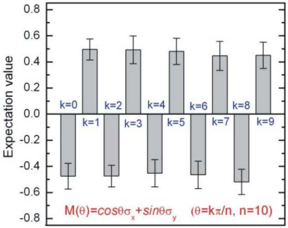
W.B. Gao et al., Nature Physics 6, 331 (2010)

Analysis of the data

Determining the fidelity

The fidelity needs N + 1 measurements, especially for $\theta = k\pi/N$

$$M_k = \left[\cos(\theta)\sigma_x + \sin(\theta)\sigma_y\right]^{\otimes N}.$$



Experimental fidelity:

$$F_{GHZ_{10}} = 0.561 \pm 0.019$$

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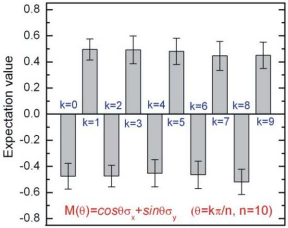

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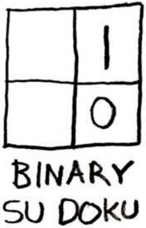
Theoretical issues

- The fidelity exceeds the critical value of 1/2 by 3 standard deviations only.
- The observable $(\sigma_x)^{\otimes 10}$ has $2^{10} = 1024$ possible results, but $\langle(\sigma_x)^{\otimes 10}\rangle$ is determined from ca. 350 copies only.
- Similarly: Throw a die four times, and estimate the probability distribution....

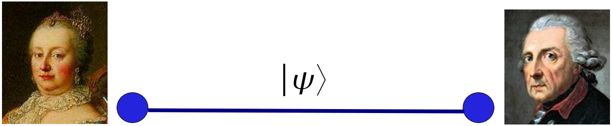
Basic facts about entanglement



What is entanglement?

The situation

Alice and Bob share a quantum state $|\psi\rangle$.



Definition: A pure state $|\psi\rangle$ is separable iff it is a product state:

$$|\psi\rangle = |a\rangle_A |b\rangle_B = |a, b\rangle.$$

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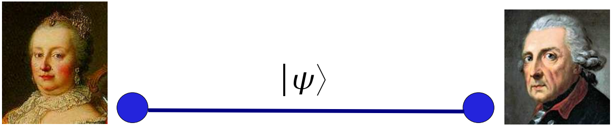

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Mixed states: Ask for convex combinations. ϱ is separable iff

$$\varrho = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \quad \text{with} \quad p_i \geq 0, \ \sum_i p_i = 1.$$

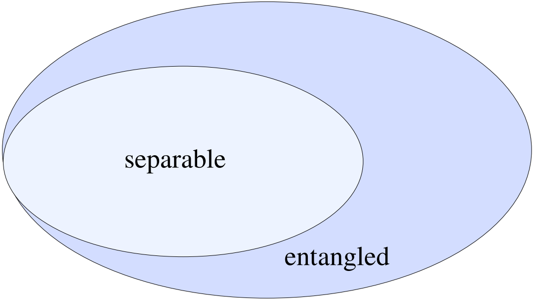
Interpretation: Entanglement cannot be produced by local operations and classical communication (LOCC).

R. Werner, PRA 40, 4277 (1989).

The separability problem

Open question: Given a state ϱ is it entangled or not?

Geometrical picture: The set of separable states is a convex set.



The PPT criterion

Are there simple criteria to prove that a state is entangled?

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Transposition and partial transposition

- Transposition: The usual transposition $X \mapsto X^T$ does not change the eigenvalues of the matrix X
- For a product space one can also consider the partial transposition. If $X = A \otimes B$:

$$X^{T_B} = A \otimes B^T$$

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Partial transposition and separability

Theorem. If a state is separable, then its partial transposition has no negative eigenvalues ("the state is PPT" or $\varrho^{T_B} \geq 0$).
Proof:

$$\varrho_{sep}^{T_B} = \sum_k p_k \varrho_A \otimes \varrho_B^T = \sum_k p_k \varrho_A \otimes \tilde{\varrho}_B \geq 0.$$

Remark: For two qubits: ϱ is PPT $\Leftrightarrow$ ϱ is separable.

A. Peres, PRL 77, 1413 (1996)

Entanglement witnesses

An observable $\mathcal{W}$ is an entanglement witness, if

$$Tr(\mathcal{W}\varrho) \begin{cases} \geq 0 & \text{for all separable } \varrho_s, \\ < 0 & \text{for one entangled } \varrho_e. \end{cases}$$

If $Tr(\mathcal{W}\varrho)$ is measured:

$$Tr(\mathcal{W}\varrho) \begin{cases} < 0 \Rightarrow & \varrho \text{ is entangled}, \\ \geq 0 \Rightarrow & \text{no detection}. \end{cases}$$

Horodecki$^{\otimes 3}$, PLA 223 (1996); B.M. Terhal, PLA 271 (2000); O. Gühne, G. Tóth, Phys. Rep. 474 (2009).

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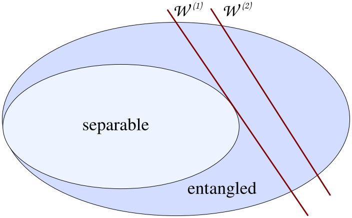

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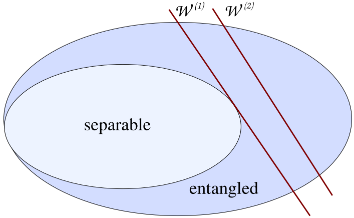
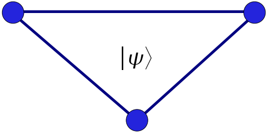


- For any entangled ϱ there is a witness.
- Witnesses can be optimized ($\mathcal{W}^{(1)}$ optimal, $\mathcal{W}^{(2)}$ not!).
- One can construct nonlinear entanglement witnesses
  O. Gühne, N. Lütkenhaus, PRL 96, 170502 (2006)

Three parties



There are different possibilities of states

$$|\psi^{\mathrm{fs}}\rangle = |000\rangle \qquad |\psi^{\mathrm{bs}}\rangle = |000\rangle + |110\rangle = (|00\rangle + |11\rangle) \otimes |0\rangle$$

$$|\psi^{\mathrm{me}}\rangle = |000\rangle + |111\rangle$$

Multipartite entanglement

Definition

A pure N-qubit state $|\psi\rangle$ is *k-separable,* if we can write

$$|\psi^{(n)}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes ... \otimes |\phi_k\rangle,$$

that is, the system can be divided into k uncorrelated parts.

Mixed states: Ask for convex combinations $\varrho^{(k)} = \sum_i p_i |\psi_i^{(k)}\rangle\langle\psi_i^{(k)}|$.

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Examples for four qubits:

$\lvert\psi_{fs}\rangle = \lvert 0000\rangle$	is fully separable,
$\lvert\psi_{ts}\rangle = \lvert 00\rangle \otimes (\lvert 00\rangle + \lvert 11\rangle)$	is 3-separable,
$\lvert\psi_{bs}\rangle = \lvert 0\rangle \otimes (\lvert 000\rangle + \lvert 111\rangle)$	is biseparable,
$\lvert GHZ_4\rangle = \lvert 0000\rangle + \lvert 1111\rangle$	is truly multipartite entangled.

A. Acin, D. Bruß, M. Lewenstein, A. Sanpera, PRL 87, 040401 (2001).

What are the interesting multiqubit states?

- The GHZ states violate Bell inequalities maximally:

$$|GHZ\rangle = |0000\rangle + |1111\rangle$$

- The W-states are robust against qubit loss:

$$|W\rangle = |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle$$

- The cluster states are useful for the one-way quantum computer:

$$|CL\rangle = |0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle$$

- The Dicke states are often easy to prepare:

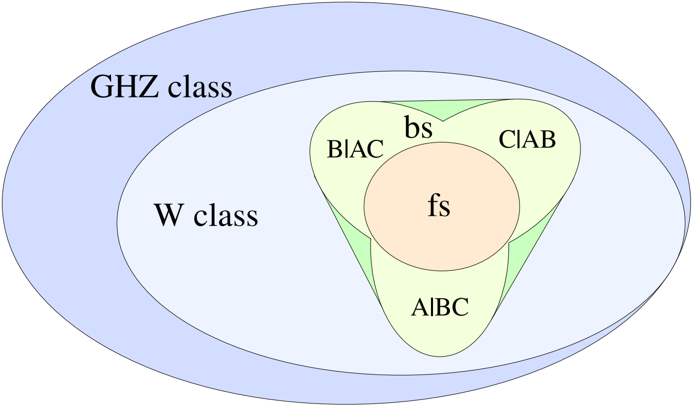
$$|D\rangle = |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle$$

- The singlet states are $U \otimes ... \otimes U$ invariant:

$$|\psi^{(4)}\rangle = |0011\rangle + |1100\rangle - \frac{1}{2}(|10\rangle + |10\rangle) \otimes (|10\rangle + |10\rangle)$$

Classification of mixed three-qubit states

Consider convex combinations of all possible states:



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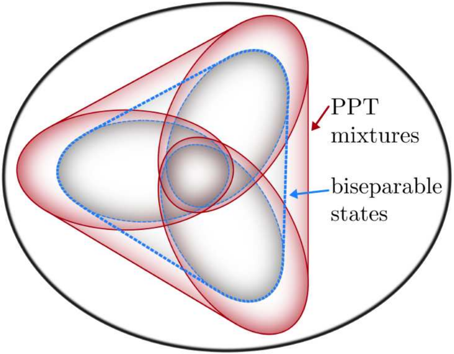
Generalizing the PPT criterion to multiparticle entanglement

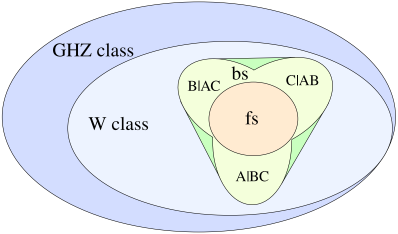


The problem

Separability criteria

- There are simple criteria, which can be used to show entanglement for two particles.
- Can we derive some simple separability criteria for genuine multipartite entanglement?
- The problem are mixtures of different bipartitions:

$$\varrho^{\mathrm{bs}} = p_1 \varrho^{\mathrm{sep}}_{A|BC} + p_2 \varrho^{\mathrm{sep}}_{B|AC} + p_3 \varrho^{\mathrm{sep}}_{C|AB}.$$



The task

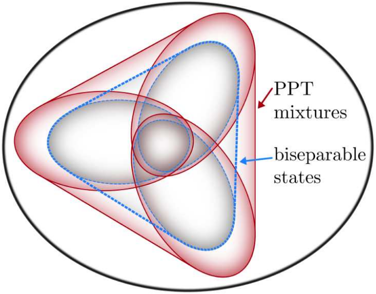
Idea

Replace separable states by PPT states. Instead of biseparable states,

$$\varrho^{\mathrm{bs}} = p_1 \varrho^{\mathrm{sep}}_{A|BC} + p_2 \varrho^{\mathrm{sep}}_{B|AC} + p_3 \varrho^{\mathrm{sep}}_{C|AB},$$

consider PPT mixtures

$$\varrho^{\mathrm{pmix}} = p_1 \varrho^{\mathrm{ppt}}_{A|BC} + p_2 \varrho^{\mathrm{ppt}}_{B|AC} + p_3 \varrho^{\mathrm{ppt}}_{C|AB}.$$



The resulting method

Classification via witnesses

A state ϱ is not a PPT mixture, if and only if $Tr(\varrho\mathcal{W}) < 0$ for

$$\mathcal{W} = P_A + Q_A^{T_A} = P_B + Q_B^{T_B} = P_C + Q_C^{T_C}$$

with $P_i, Q_i \geq 0$.

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Main advantages

- This problem can be solved efficiently via semidefinite programming.
- In practice, it requires only few lines of code in Matlab.
  ($\Rightarrow$ the program `PPTmixer` on the web)
- Numerically, it works for ≤ 7 qubits. Analytically, up to "∞" qubits.
- One can also solve it, if only some expectation values (and not the whole ϱ) are known.
- The amount of the violation is an entanglement monotone.

Results

Noise robustness

The noise robustness increases drastically: Consider

$$\varrho(p) = p\mathbb{1}/8 + (1-p)|\psi\rangle\langle\psi|$$

and compute maximal p_{tol} :

state	tolerances p_{tol}	
	new	before
$\vert GHZ_3\rangle^\star$	0.571	0.571
$\vert GHZ_4\rangle^\star$	0.533	0.533
$\vert W_3\rangle^\star$	0.521	0.421
$\vert W_4\rangle$	0.526	0.444
$\vert Cl_4\rangle^\star$	0.615	0.533
$\vert D_{2,4}\rangle$	0.539	0.381
$\vert \Psi_{S,4}\rangle$	0.553	0.317

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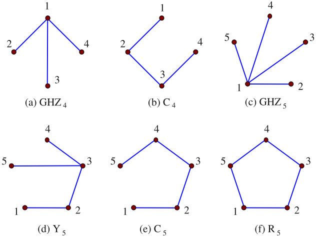
Cluster states



For cluster states of N qubits, the noise robustness is at least:

$$p_{\mathrm{tol}} = \frac{1}{1 - \frac{1}{2^{N-1}} + (N/3+1)\frac{1}{2^{N/3}}} \xrightarrow{N\to\infty} 1$$

$\Rightarrow$ An exponential improvement compared with existing results!

B. Jungnitsch et al., PRL 106, 190502 (2011).

Graph-diagonal states

Graph-diagonal states

- For any graph, there is a graph-state basis $|G_i\rangle$; e.g. the GHZ basis: $|000\rangle \pm |111\rangle$, $|001\rangle \pm |110\rangle$, etc.
- Consider states diagonal in this basis:

$$\varrho = \sum_i \gamma_i |G_i\rangle\langle G_i|$$

Results

- For GHZ diagonal states: ϱ is separable $\Leftrightarrow \gamma_i \leq 1/2$

  O. Gühne, M. Seevinck, NJP 12, 053002 (2010).

- For four-qubit cluster diagonal states, the approach of PPT mixtures solves the problem.

  O. Gühne et al., arXiv:1107.4863

- Its also a solution for all five-qubit graph states mixed with white noise and some other graphs.

Is the criterion necessary und sufficient?

Permutation invariant states

For permutation invariant states of three qubits

$$\varrho = \pi_{ij} \varrho \pi_{ij}$$

the PPT mixer is necessary and sufficient for multiparticle entanglement.

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X-states

For X-states with many qubits

$$\varrho = \begin{bmatrix} \ddots & & & \iddots \\ & & & \\ \iddots & & & \ddots \end{bmatrix}$$

the PPT mixer is necessary and sufficient for multiparticle entanglement.

But...

... some entangled five-qubit states are not detected by the PPT mixer!

Incomplete information

Consider a Dicke state mixed with white noise:

$$\varrho(p) = p\mathbb{1}/16 + (1-p)|D_4\rangle\langle D_4|$$

with $|D_4\rangle = |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle$.

- If the observables $O_1 = XXXX$ and $O_2 = YYYY$ are measured: $p = 0.29$ can be tolerated.
- If in addition $O_3 = ZZZZ$ is measured: $p = 0.38$ can be tolerated.
- If in addition $O_4 = XXZZ$ and $O_5 = XXYY$ (& permutations) are measured: $p = 0.45$ can be tolerated.
- If one has complete knowledge on ϱ: $p = 0.54$ can be tolerated.

Experimentalists can learn during their experiment, whether further measurements are necessary.

The multiparticle negativity

The entanglement monotone

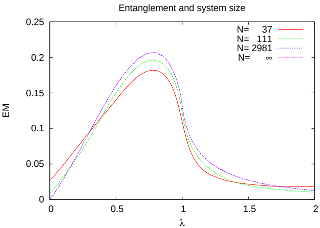
The quantity

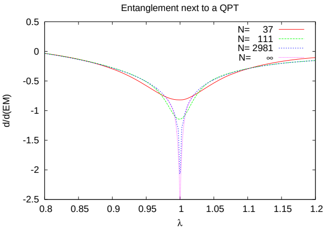
$$\mathcal{E}(\varrho) = |\min[Tr(\varrho\mathcal{W})]|$$

with $\mathcal{W} = P_A + Q_A^{T_A} = P_B + Q_B^{T_B} = P_C + Q_C^{T_C}$ and $P_i, Q_i \geq 0$ is a computable entanglement monotone for genuine multiparticle entanglement.

Application to the Ising model

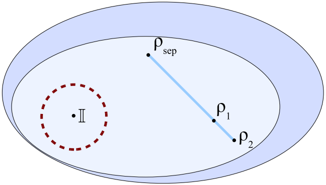
Investigate the scaling of the measure in the ground state:





$$H = -\lambda \sum_i X_i X_{i+1} - \sum_i Z_i$$

Proving separability of quantum states



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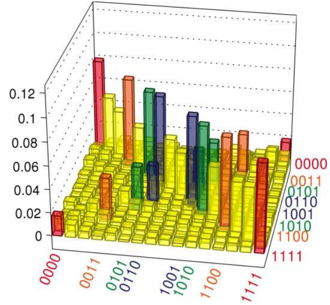
- But given a density matrix, how can we prove that it is *separable?*
- We have to write it as

$$\varrho_{exp} = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|.$$

- For experimental density matrices this is a hopeless task.



Two facts

Convexity

Let ϱ_{sep} be separable, and let

$$\varrho_1 = \varrho_2 + \varepsilon \varrho_{sep}$$
$$\Leftrightarrow \quad \varrho_2 = \varrho_1 - \varepsilon \varrho_{sep}$$

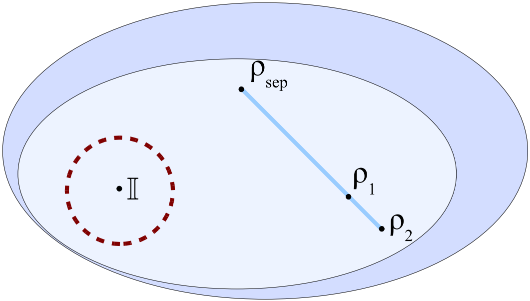
Then, if ϱ_2 is separable, ϱ_1 is separable, too.

Highly mixed states

If a state is close to the maximally mixed state, then it is separable. For instance, in an $N \times M$ system:

$$Tr(\varrho^2) \leq \frac{1}{NM-1} \Rightarrow \varrho \text{ is separable.}$$

L. Gurvits, L., H. Barnum, PRA 66, 062311 (2002)



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$$\max_{|\phi\rangle=|a\rangle|b\rangle} |\langle\phi|\varrho_i|\phi\rangle|$$

   and find $|\phi_i\rangle = |a_i\rangle|b_i\rangle$ with a high overlap with ϱ_i.

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   and find $|\phi_i\rangle = |a_i\rangle|b_i\rangle$ with a high overlap with ϱ_i.
3. Find an $\varepsilon_i \geq 0$ such that

$$\varrho_{i+1} := (1 + \varepsilon_i)\varrho_i - \varepsilon_i|\phi_i\rangle\langle\phi_i|$$

   has no negative eigenvalues and $Tr(\varrho_i^2) \geq Tr(\varrho_{i+1}^2)$ holds.

- $|\langle\phi_i|\varrho_i|\phi_i\rangle|$ large $\Rightarrow$ overlap with the biggest eigenvector large.
- So $\lambda_{\max}(\varrho_{i+1}) \leq \lambda_{\max}(\varrho_i)$, and from normalization $\lambda_{\min}(\varrho_{i+1}) \geq \lambda_{\min}(\varrho_i)$.
- Hence, ϱ_{i+1} is closer to the maximally mixed state than ϱ_i.

The algorithm

1. Take the given data ϱ_{exp} as ϱ_i with $i = 1$.
2. Consider the optimization problem

$$\max_{|\phi\rangle = |a\rangle|b\rangle} |\langle\phi|\varrho_i|\phi\rangle|$$

   and find $|\phi_i\rangle = |a_i\rangle|b_i\rangle$ with a high overlap with ϱ_i.
3. Find an $\varepsilon_i \geq 0$ such that

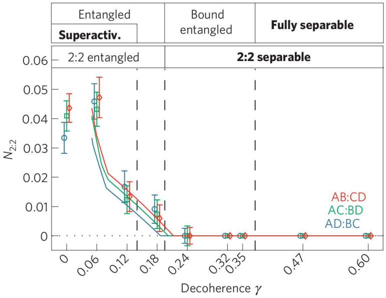
$$\varrho_{i+1} := (1+\varepsilon_i)\varrho_i - \varepsilon_i|\phi_i\rangle\langle\phi_i|$$

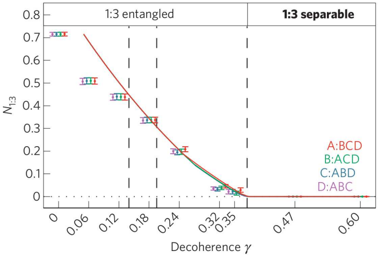
   has no negative eigenvalues and $Tr(\varrho_i^2) \geq Tr(\varrho_{i+1}^2)$ holds.

   - $|\langle\phi_i|\varrho_i|\phi_i\rangle|$ large $\Rightarrow$ overlap with the biggest eigenvector large.
   - So $\lambda_{\mathrm{max}}(\varrho_{i+1}) \leq \lambda_{\mathrm{max}}(\varrho_i)$, and from normalization $\lambda_{\mathrm{min}}(\varrho_{i+1}) \geq \lambda_{\mathrm{min}}(\varrho_i)$.
   - Hence, ϱ_{i+1} is closer to the maximally mixed state than ϱ_i.
4. Check, whether ϱ_{i+1} is sufficiently mixed. If yes, then ϱ_{i+1} is separable, and also ϱ_i and finally ϱ_{exp} separable. If no, iterate.

Practical issues

- The algorithm can be implemented with few lines of code
- Usually ca. 50 - 100 iterations.
- Calculations up to 6 qubits easy.
- Its also possible to extend it to prove full separability or W-class entanglement.





J. Barreiro et al., Nature Physics 6, 943 (2010), H. Kampermann et al., in preparation

Conclusion

Conclusion

- The PPT criterion can be extended to the multipartite case.
- Separability can be proven with a simple algorithm.

Open Questions

- Is the multipartite PPT criterion necessary and sufficient for three qubits?
- What states are robust under decoherence, if the multipartite entanglement monotone is considered?

Conclusion

Literature:

- J. Barreiro et al., Nature Physics 6, 943 (2010)
- B. Jungnitsch et al., PRL 106, 190502 (2011)
- H. Kampermann et al., quant-ph/thisweek

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MARIE CURIE ACTIONS