More Communication with Less Entanglement

Pankaj Agrawal
Institute of Physics
Bhubaneswar

February 25, 2012

Outline

Introduction

Multipartite States

Quantum Teleportation

Condition for Teleportation

Quantum Superdense Coding

Quantum Key Distribution

Cooperative Teleportation and Superdense Coding

Conclusions

Introduction

- One of the important problem in the field of Quantum Information is to understand the nature of entanglement in multipartite systems.
- One can do lot more when there are many particles and the nature of entanglement is more complex.
- Such systems exhibit some intriguing phenomena which don't occur for a bipartite (specially qubit) systems.
- In the case of multipartite system, in some situations one can do more communication with less entangled state.
- We consider teleportation, superdense coding, Quantum Key distribution to illustrate this feature.
- In the case of teleportation we also obtain a condition for the resource state. This shows the need for special amount of entanglement that is required for the teleportation of the state of n-qubit system with m terms.

Introduction

- Since the state requires specific values of entanglement, it is not surprising that some time more entangled state is not useful.
- We also show some states that can be used for superdense coding but are not useful for teleportation.
- This work has been done in collaboration with B. Pradhan, Satyabrata Adhikari and Arun Pati.

Outline

Introduction

Multipartite States

Quantum Teleportation

Condition for Teleportation

Quantum Superdense Coding

Quantum Key Distribution

Cooperative Teleportation and Superdense Coding

Conclusions

Multipartite States

- Our discussion will be valid for a n-qubit state.
- For teleportation and superdense coding, there are only two parties, Alice and Bob.
- with a two-qubit resource state, one can transmit at most one-qubit state, or two classical bits
- With a resource state with more qubits, one can teleport multi-qubit unknown state and send more than two classical bits
- For example, if we have a suitable four-qubit resource state, then one can teleport not only one-qubit state, but also a two-qubit state with two, three, or four superposed terms.
- One can also densecode three or four classical bits.
- A resource state with more than two qubits opens up many more possibilities.
- One can carry out cooperative teleportation and cooperative superdense coding and many other protocols.

States

We consider the following four-qubit entangled states

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \quad (1)$$

$$|W\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \quad (2)$$

$$|\Omega\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle) \quad (3)$$

$$|Q_4\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle) \quad (4)$$

$$|Q_5\rangle = \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle). \quad (5)$$

von Neumann Entropy

To see how entangled these states are, we compute entropy of all subsystems. In a bipartite partition, both subsystems will have identical entropies. Therefore, if we split the system in particle 1 on one hand and particles 2, 3 and 4 on the other, then the entropy of the particle 1, $S(\rho_1)$ will be the same as that of particle 234 subsystem, $S(\rho_{234})$.

	$S(\rho_1)$	$S(\rho_2)$	$S(\rho_3)$	$S(\rho_4)$	$S(\rho_{12})$	$S(\rho_{13})$	$S(\rho_{14})$
GHZ	1	1	1	1	1	1	1
Ω	1	1	1	1	2	2	1
W	0.81	0.81	0.81	0.81	1	1	1
Q_4	0.81	1	0.81	0.81	1.5	1.22	1.22
Q_5	0.81	1	1	1	1.5	1.5	1.5

Table 1: Entropies of the subsystems

Outline

Introduction

Multipartite States

Quantum Teleportation

Condition for Teleportation

Quantum Superdense Coding

Quantum Key Distribution

Cooperative Teleportation and Superdense Coding

Conclusions

Teleportation

- In the standard teleportation protocol, Alice wishes to transmit an unknown quantum state to Bob using an entangled state as a resource and classical communication.
- In the original protocol, Alice transmits unknown one-qubit state using a Bell state as a resource.e and using two classical bits of information.
- We now know that any entangled bipartite pure state is suitable for teleportation, which may be probabilistic.
- If we have a multipartite state as a resource, there exits many possibilities for exact teleportation.
- If there are only two parties, Alice can teleport a state of one or more qubits with varying number of terms.

Teleportation

- Let us now consider the four-qubit states to illustrate the various possibilities. We will also see that a state with more entanglement is sometimes not suitable for the teleportation.
- Let us now consider the teleportation of the state of the most general two-qubit state.

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle. \qquad (6)$$

- One can check that, one can construct a suitable measurement basis only in the case of Ω-state as a resource with Alice holding the qubits '12' or '13' only. These partitions have entropy as 2. Alice makes a four-particle measurement in a suitable basis and sends four classical bits of information to Bob. Bob then can convert his two-qubit state to that of unknown state with suitable unitary transformations.

Teleportation

- Let us know consider a state with only two terms

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle. \tag{7}$$

- One can check that GHZ-state and Ω-state can be used for teleporting this state to Bob with two classical bits of communication.
- However, it is not possible to construct suitable measurement basis, to teleport this state using Q4-state or Q5-state.
- If we look at the entropy table for 2:2 partitions – '12', '13' and '14', we see that **the Q4 and Q5 states are more entangled, but they are not suitable for teleportation**.
- So, more entanglement does not necessarily mean more communication.

Teleportation

- Let us now discuss the teleportation of a one-qubit state using the considered four qubit states.
- All partitions of the GHZ-state and Ω-state are suitable for teleportation.
- None of the partitions of the W-state are suitable.
- Only those partitions of the Q4 ans Q5 states are suitable for teleportation where Bob's qubit has entropy as one.
- Basically, if the Bob's qubit has entropy one, the state with particular partition is suitable for teleportation, otherwise it is not.

Outline

Introduction

Multipartite States

Quantum Teleportation

Condition for Teleportation

Quantum Superdense Coding

Quantum Key Distribution

Cooperative Teleportation and Superdense Coding

Conclusions

Condition for Teleportation

- We will discuss teleportation of a pure state with the resource state that is also pure.
- We will consider exact teleportation, i.e., teleportation with unit probability and unit fidelity.
- Question we pose is: **if we are given a n-qubit state with m terms to teleport, what kind of resource state is needed** ?
- Or, in reverse, **given a resource state, what are the states that can be teleported using this resource** ?
- Once we know the answer to this question, we can immediately see why sometimes a more entangled state is less suitable.

Condition for Teleportation

Let us consider the following n-qubit state with m terms

$$|\Psi\rangle_n = \sum_{k=1}^{m} \alpha_k |\eta_k\rangle_n. \tag{8}$$

Following conditions are to be satisfied.

$$\langle \eta_k|\eta_l\rangle = \delta_{kl}, \qquad \sum_{k=1}^{m} |\alpha_k|^2 = 1. \tag{9}$$

Given a resource state of p qubits, if it is suitable to teleport $|\Psi\rangle_n$, we should be able to write it as

$$|R\rangle_p = \frac{1}{\sqrt{m}} \sum_{l=1}^{m} |\chi_l\rangle_{p-n} |\eta_l\rangle_n. \tag{10}$$

Here the states $|\chi_l\rangle$ are not necessarily normalized or orthogonal.

Condition for Teleportation

The combined state is

$$
\begin{aligned}
|\Psi\rangle_n|R\rangle_p &= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\sum_{i=1}^{m}\alpha_i|\eta_i\rangle_n|\chi_k\rangle_{p-n}|\eta_k\rangle_n, \\
&= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\sum_{i=1}^{m}|\eta_i\rangle_n|\chi_k\rangle_{p-n}\alpha_i|\eta_k\rangle_n. \qquad (11)
\end{aligned}
$$

Alice will now make a measurement in an orthonormal basis $|\theta_l\rangle_p$. Therefore, we should be able to write

$$
|\eta_i\rangle_n|\chi_k\rangle_{p-n} = \frac{1}{\sqrt{m}}\sum_{l=1}^{m^2} C_{ik,l}|\theta_l\rangle_p \qquad (12)
$$

$C_{ik,l}$ is an interesting object. For each l, it is a mxm matrix in ik space. It is also a m^2xm^2 matrix with row label as ik. We need to find a condition on $C_{ik,l}$ such that we indeed have a suitable measurement basis.

Condition for Teleportation

$$
\begin{aligned}
|\Psi\rangle_n |R\rangle_p &= \frac{1}{m}\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m^2} C_{ik,l}|\theta_l\rangle_p \alpha_i |\eta_k\rangle_n \\
&= \frac{1}{m}\sum_{l=1}^{m^2}|\theta_l\rangle_p \sum_{k=1}^{m}\sum_{i=1}^{m} C_{ik,l}\alpha_i|\eta_k\rangle_n
\end{aligned}
$$

For teleportation to succeed, we should have,

$$
\sum_{k=1}^{m}\sum_{i=1}^{m} C_{ik,l}\alpha_i|\eta_k\rangle_n = V^l \sum_{n=1}^{m}\alpha_n|\eta_n\rangle_n \tag{13}
$$

Taking its adjoint and scalar product with it, we get,

$$
\sum_{k=1}^{m}\sum_{i=1}^{m}\sum_{i'=1}^{m} C_{ik,l}C^*_{i'k,l}\alpha_i\alpha^*_{i'} = 1 \tag{14}
$$

Condition for Teleportation

For each l,

$$\sum_{i=1}^{m}\sum_{i'=1}^{m}(CC^\dagger)_{ii',l}\alpha_i\alpha_{i'} = 1 \tag{15}$$

For this equation to be satisfied,

$$(CC^\dagger)_{ii',l} = \delta_{ii'} \tag{16}$$

This implies that C is unitary in ik space for each l for teleportation to succeed. Let us now see what it means for the resource state.

Condition for Teleportation

Taking the adjoint and taking the scalar product,

$$\langle \eta_{i'} | \eta_i \rangle \langle \chi_{k'} | \chi_k \rangle = \frac{1}{m} \sum_{l=1}^{m^2} \sum_{l'=1}^{m^2} C_{ik,l} C^*_{i'k',l'} \langle \theta_l | \theta_{l'} \rangle \tag{17}$$

$$\delta_{ii'} \langle \chi_{k'} | \chi_k \rangle = \frac{1}{m} \sum_{l=1}^{m^2} C_{ik,l} C^*_{i'k',l} \tag{18}$$

Multiplying by $\delta_{i'i}$ and summing over i and i', we get,

$$\langle \chi_{k'} | \chi_k \rangle = \frac{1}{m^2} \sum_{l=1}^{m^2} (C^\dagger C)_{k'k,l} \tag{19}$$

Since C is unitary in the subspace, we get

$$\langle \chi_{k'} | \chi_k \rangle = \delta_{k'k} \tag{20}$$

Condition for Teleportation

This implies that $|\chi_k\rangle$ should be orthonormal. Therefore, the resource state should have the form

$$|R\rangle_p = \frac{1}{\sqrt{m}} \sum_{l=1}^{m} |\chi_l\rangle_{p-n} |\eta_l\rangle_n. \tag{21}$$

with both $|\chi_k\rangle$ and $|\eta_l\rangle$ being orthonormal.
So, the reduced density matrix of the Bob's qubits would be

$$\rho_{\text{bob}} = \frac{1}{m} \sum_{l=1}^{m} |\eta_l\rangle\langle\eta_l|. \tag{22}$$

This is a completely mixed state with entropy $\log_2 m$. This is the requirement.

Condition for Teleportation

- What we have shown is that if we wish to teleport a n-qubit state with m terms, then we should be able to distribute resource states qubits in such a way such that Bob's n qubits have entropy $\log_2 m$.
- Given a resource state, we can compute entropy of all the partitions. If there is a partition where Bob's n qubits have entropy as $\log_2 m$, then the state with m terms can be teleported with that partition.

Outline

Introduction

Multipartite States

Quantum Teleportation

Condition for Teleportation

Quantum Superdense Coding

Quantum Key Distribution

Cooperative Teleportation and Superdense Coding

Conclusions

Superdense Coding

- This has been one of the pioneering protocol in the field of quantum information. We shall consider the original protocol.
- We have two parties, Alice and Bob. Alice wishes to communicate classical information to Bob by sending qubits. If these qubits are not entangled, then she can communicate at most one classical bit by sending one qubit to Bob.
- As we know, by using entangled state, she can encode more classical information. If Alice and Bob have one qubit each which are in a Bell state, then Alice can communicate two classical bits to Bob.
- If Alice and Bob share a partially entangled state, Alice can still transmit more than one classical bit. Basically *all* entangled states can be dense coded, however probabilistically.

Superdense Coding

- In general, if Alice can transmit more than n classical bits by sending n qubits, the resource state is useful for dense coding.
- We are considering four-qubit states. So Alice can send at most four classical bits to Bob, She has option of sending one, two, or three qubits.

In the case of the states under consideration, our observations are following:

- With GHZ state, by sending n qubits, Alice can transmit $n+1$ classical bits. So deterministic dense coding is always possible, but the maximal dense coding, i.e., sending four classical bits with two quantum bits are not possible.

Superdense Coding

- With the W-state dense coding is possible only when Alice can send two qubits to Bob. She can then transmit three classical bits. In other scenarios, she cannot send $n+1$ or more classical bits by sending n qubits. This is because other partitions have entropy less than one. However, by suitably modifying W-state, one can construct W-class states which are suitable for superdense coding.
- The cluster state is the best. Two of its partitions '12' and '13' allow us to transmit four classical bits by sending two qubits. This is because these partitions give rise to entropy of two for the subsystems. Other partitions behave like GHZ state. So, by sending n qubits, Alice can transmit $n+1$ classical bits.
- These states are useful, since the subsystems have nice values of entropies.

Superdense Coding

- **The Q4-state exhibit the phenomena of less communication with more entanglement**
- If we consider the partitions ‘13’ and ‘14’, the subsystems have entropy 1.22, which is more than GHZ and W-states. But if we apply local unitaries on these two qubits, we get four orthogonal states. So if we use these partitions, one can transmit only two classical bits by sending two qubits. So there is no superdense coding in these situations for Q4-state.
- However, partition ‘12’, with entropy 1.5 is useful for dense coding. As we saw, this partition was not useful for teleportation.
- In the case of sending one or three qubits to Bob, the protocol will work when Alice either sends qubits ‘134’ or qubit ‘2’. In all other cases, the protocol would not work, since the subsystems have entropy less than one.

Superdense Coding

- In the case of Q5-state, we will have superdense coding for all the partitions, when Alice sends two qubits. Here entropy of the subsystems is 1.5. However, as we saw, these partitions are not suitable for teleportation.
- In the case of sending one or three qubits to Bob, the protocol will work for all the partitions except when Alice either sends qubits ‘234’ or qubit ‘1’.
- As we saw, superdense coding does not work in the case of Q4-state even when it is more entangled. One should be able to construct many more such states where such a phenomenon occurs.

Outline

Introduction

Multipartite States

Quantum Teleportation

Condition for Teleportation

Quantum Superdense Coding

Quantum Key Distribution

Cooperative Teleportation and Superdense Coding

Conclusions

Quantum Key Distribution

- Let us consider two four-qubit states such as $|GHZ\rangle$ and $|Q_4\rangle$. Let us suppose that the two distant partners Alice and Bob possesses two qubits each.
- From the Table 1, we see that Q4-state is more entangled in these partitions, then the GHZ state.
- Still we will see that in a QKD scheme, Q4-state cannot be used, while GHZ-state can be used.
- To see the protocol, let us rewrite the GHZ-state as

$$|GHZ\rangle_{1234} = \frac{1}{\sqrt{2}}(|00\rangle_{13} \otimes |00\rangle_{24} + |11\rangle_{13} \otimes |11\rangle_{24}) \quad (23)$$

The qubits 1 and 3 hold by Alice and the remaining two qubits 2 and 4 are with Bob.

Quantum Key Distribution

- In the Bell basis $\{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\}$, $|GHZ\rangle_{1234}$ state can be re-written as

$$|GHZ\rangle_{1234} = \frac{1}{\sqrt{2}}(|\phi^+\rangle_{13} \otimes |\phi^+\rangle_{24} + |\phi^-\rangle_{13} \otimes |\phi^-\rangle_{24}) \quad (24)$$

- In the next step, Alice randomly perform measurements on the particles in either the computational basis or in the Bell basis. The information is encoded by using the two binary digits 0 and 1. In the computational basis (Bell basis), if the measurement outcome is $|0\rangle_1 \otimes |0\rangle_3$ ($|\Phi^+\rangle_{13}$) then 0 is encoded and if the measurement outcome is $|1\rangle_1 \otimes |1\rangle_3$ ($|\Phi^-\rangle_{13}$) then 1 is encoded. After Alice's measurement, Bob also randomly choose either the computational basis or the Bell basis and then perform measurement on the particles in that basis.
- After this, Alice publicly announce the basis in which she measured the state of the particles but not declare the measurement outcome through authenticated channel.

Quantum Key Distribution

- If Bob find that his measurement basis matches with Alice's basis then Bob inform Alice to keep the data, otherwise the data will be thrown out. In this way, quantum key can be distributed between Alice and Bob. Therefore, $|GHZ\rangle$ state can be used in generating the quantum key.
- To see the usefulness of the Q4-state we rewrite the four qubit state $|Q_4\rangle$ in the computational basis as well as in the Bell basis.

$$|Q_4\rangle = \frac{1}{2}(|00\rangle_{13}|00\rangle_{24} + |00\rangle_{13}|11\rangle_{24} + |10\rangle_{13}|00\rangle_{24} + |11\rangle_{13}|10\rangle_{2}$$

$$\begin{aligned}|Q_4\rangle &= \frac{1}{4}[(2|\phi^+\rangle_{13} + 2|\phi^-\rangle_{13} + |\psi^+\rangle_{13} + |\psi^-\rangle_{13})|\phi^+\rangle_{24} + \\ &(|\psi^+\rangle_{13} + |\psi^-\rangle_{13})|\phi^-\rangle_{24} + (|\psi^+\rangle_{13} - |\phi^-\rangle_{13})|\psi^+\rangle_{24} \\ &-(|\psi^+\rangle_{13} - |\psi^-\rangle_{13})|\psi^-\rangle_{24}]\end{aligned} \quad (26)$$

Quantum Key Distribution

- From eq. (25) and eq. (26), it is clear that one cannot generate key even if their basis matches. Therefore, we have shown that although the four qubit state $|Q_4\rangle$ has more entanglement than $|GHZ\rangle$ state but the former cannot be used in our QKD protocol while the latter can.

Outline

Introduction

Multipartite States

Quantum Teleportation

Condition for Teleportation

Quantum Superdense Coding

Quantum Key Distribution

Cooperative Teleportation and Superdense Coding

Conclusions

Cooperative Teleportation and Superdense Coding

- In these protocols, there are more than two parties. Qubits are distributed among these parties.
- Alice teleports a state to Bob, or dense codes the classical information for Bob. However, Bob can receive the state or the classical information only with the cooperation of the other parties.
- In the case of these protocols also, GHZ-state and Ω-state are more successful than the Q4 and Q5 states.
- One can construct states with higher number of qubits, say six qubits. One can again see the phenomena of 'less is more', by distributing more than one qubit to each party.

Outline

Introduction

Multipartite States

Quantum Teleportation

Condition for Teleportation

Quantum Superdense Coding

Quantum Key Distribution

Cooperative Teleportation and Superdense Coding

Conclusions

Conclusions

- We have considered a set of four qubit states to illustrate some of the intriguing features of the multipartite entangled states.
- We have seen that there are states which have more entanglement but are less suitable for teleportation and superdense coding.
- We have also seen that there are states which are suitable for superdense coding, but are not suitable for teleportation.
- We have also obtained a condition to determine the suitability of a resource state to teleport a n-qubit state with m terms.