UNCERTAINTIES IN THE CLASSICAL REALM

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"The whole problem with the world is that fools and fanatics are always so certain of themselves, and wiser people so full of doubts". - Bertrand Russell (1872 - 1970)

Quantum Theory has been the crowning jewel of twenty first century modern physics. Ever since it's conception, it has been both the conservatives' nightmare and Turncoats' delight! It's enigmatic features have captured the attention and imagination of researchers. Topics ranging from the meaning and interpretation of the quantum theory to the correspondence to "our" classical world have ever since occupied the discussions at conferences and dinner tables alike.

Moreover, there have been many attempts to retrieve classical physics (CP) as a limiting case of quantum physics (QP). To this end, pedagogic discussions in several text books on QP, are essentially confined to the limit $\hbar \to 0$ and the Ehrenfest theorem in discussing the emergence of classical regime. While both these quantum-classical correspondences operate in their own domains of applicability, it has been identified that they are not universally satisfactory [1].

Quantum World → $\hbar \to 0$ & Ehrenfest theorem → Classical World

{For a system characterized by a conservative Hamiltonian $\hat{H}$,

$$\frac{d\langle\hat{x}\rangle}{dt}=\frac{\langle\hat{p}\rangle}{m}; \qquad \frac{d\langle\hat{p}\rangle}{dt}=-\left\langle\frac{dV(\hat{x})}{dx}\right\rangle=\langle F(\hat{x})\rangle\}$$

It may be pointed out that in stationary states of a symmetric Hamiltonian,

$$\boxed{\langle\hat{x}\rangle=0=\langle\hat{p}\rangle}$$

Which would correspond to a redundant result zero equals zero and hence the quantum classical correspondence via Ehrenfest theorem does not yield any information in the case of stationary states.

In the absence of commonly accepted notion of classical limit, it is important to recognize the quantum features that are expected to leave their imprints in the classical regime. ***It has been pointed out that the classical realm -- resulting from a quantum mechanical state -- is ought to correspond to an ensemble -- not a single particle.[1] The averages, variances and other higher order moments of the quantum and classical probability distributions are therefore expected to agree in the limiting case.***

In order to compare the statistical form of classical dynamics with the corresponding one in quantum dynamics, phase space probability distribution of the classical ensemble (a counterpart of the corresponding quantum state) needs to be identified.

The classical phase space probability distribution satisfies the Liouville equation and the phase space averages of the classical observables are shown to exhibit analogous dynamical behavior as that of the corresponding quantum case -- even when Ehrenfest's theorem breaks down[1]

CONSTANT ENERGY CLASSICAL PROBABILITY DISTRIBUTIONS CORRESPONDING TO QUANTUM MECHANICAL STATIONARY STATES

$$dP_{CL}(x,p)=\{H,P_{CL}(x,p)\}=0$$

$$P_{CL}\propto\delta\left[\frac{p^2}{2m}+V(x)-E\right]$$

$$P_{CL}(x)=\text{Constant}\cdot\int dp\,2m\,\delta(p^2+2m[V(x)-E])$$

$$\Rightarrow P_{CL}(x)=\frac{\text{N}}{\sqrt{E-V(x)}}\rightarrow\text{Position Probability Density Function}$$

Having recognized the classical probability density function, one would naturally be lead to ask[2]
1)what about the ***fluctuations*** in position and momentum variables?
2)How do they match with their Quantum counterparts?

Results from some exactly solvable 1 D conservative Hamiltonian systems

Attributes	Simple Harmonic Oscillator	Particle in an Infinite potential well	Bouncing Ball
Classical Probability Density function- $P_{CL}(x)$	$P_{CL}(x)=\frac{1}{\pi\sqrt{A^2-x^2}},\ \lvert x\rvert\le A$; $P_{CL}(x)=0$ otherwise	$P_{CL}(x)=\frac{1}{L},\ \lvert x\rvert\le\frac{L}{2}$; $P_{CL}(x)=0$ otherwise	$P_{CL}(z)=\frac{1}{2A\left(\sqrt{1-z/A}\right)},\ 0\le z\le A$; $P_{CL}(z)=0$ otherwise
Stationary state Wave function	$\psi_n(x)=\sqrt{\frac{\sqrt{\frac{m\omega}{\pi\hbar}}}{2^n n!}}H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right)e^{-\frac{m\omega x^2}{2\hbar}}$	$\psi_n(x)=\sqrt{\frac{2}{L}}\text{Cos}\left(\frac{n\pi x}{L}\right),\ n=1,3,5\ldots$; $\psi_n(x)=\sqrt{\frac{2}{L}}\text{Sin}\left(\frac{n\pi x}{L}\right),\ n=2,4,6\ldots$	$\psi_n(z)=N_n Ai(z)\rightarrow$ Airy Functions
Dimensionless Variables	$X=\frac{x}{A};P=\frac{p}{\sqrt{2mE}}=\frac{p}{m\omega A}$	$X=\frac{x}{(L/2)};\ P=\frac{p}{\sqrt{2mE}}=\frac{p}{\lvert p\rvert}$	$Z=\frac{z}{A};\ P=\frac{p}{\sqrt{2mE}}=\frac{p}{\sqrt{2m^2gA}}$
	$\hat{X}=\frac{\hat{x}}{A_n}=\hat{x}\sqrt{\frac{m\omega}{(2n+1)\hbar}};\ \hat{P}=\frac{\hat{p}}{\sqrt{2mE_n}}=\frac{\hat{p}}{\sqrt{(2n+1)\hbar m\omega}}$	$\hat{X}=\frac{\hat{x}}{(L/2)};\ \hat{P}=\frac{\hat{p}}{\sqrt{2mE_n}}=\frac{\hat{p}}{(n\pi\hbar/L)}$	$\hat{Z}=\frac{\hat{z}}{\left(E_n/mg\right)};\ \hat{P}=\frac{\hat{p}}{\sqrt{2mE_n}}$
Classical Moments	$\langle X\rangle=0=\langle P\rangle;\ \langle X^2\rangle=\langle P^2\rangle=\frac{1}{2}$	$\langle X\rangle=0=\langle P\rangle;\ \langle X^2\rangle=\frac{1}{3},\langle P^2\rangle=1$	$\langle Z\rangle=\frac{2}{3},\langle P\rangle=0,\ \langle Z^2\rangle=\frac{8}{15},\langle P^2\rangle=\frac{1}{3}$
Quantum Moments	$\langle\hat{X}\rangle=0=\langle\hat{P}\rangle;\ \langle\hat{X}^2\rangle=\langle\hat{P}^2\rangle=\frac{1}{2}$	$\langle\hat{X}\rangle=0=\langle\hat{P}\rangle;\ \langle\hat{X}^2\rangle=\frac{1}{3}-\frac{2}{(n\pi)^2},\langle\hat{P}^2\rangle=1$	$\langle\hat{Z}\rangle=\frac{2}{3},\langle\hat{P}\rangle=0,\ \langle\hat{Z}^2\rangle=\frac{8}{15},\langle\hat{P}^2\rangle=\frac{1}{3}$
Uncertainties	$(\Delta X)^2(\Delta P)^2=\frac{1}{4}=(\Delta\hat{X})^2(\Delta\hat{P})^2$	$(\Delta X)^2(\Delta P)^2=\frac{1}{3}=\lim_{n\to\infty}(\Delta\hat{X})^2(\Delta\hat{P})^2$	$(\Delta Z)^2(\Delta P)^2=\frac{4}{135}=(\Delta\hat{Z})^2(\Delta\hat{P})^2$

That's Interesting!!

Need to (a) explore analogous results for non-quadratic potentials (b) investigate other imprints emerging from quantum theory in classical regime(signatures of entanglement in classical domain?)

1.L. E. Ballentine, Y. Yang, J. P. Zibin, "Inadequacy of Ehrenfests theorem to characterize the classical regime," Phys. Rev. A **50**, 2854(1994)
2. A. R. Usha Devi and H. S. Karthik, "Uncertainty product of position and momentum in the realm of classical dynamics," To appear in American Journal of Physics,arXiv:1108.2682.