Channel discrimination in the context of Deutsch-Jozsa problem

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Introduction

Deutsch's algorithm was one of the first to demonstrate that quantum circuits can outperform classical ones. It solves the problem of differentiating between two binary functions (even or odd) in a single evaluation, compared to two (or n/2+1 in general) in the classical case.

Viewing Deutsch's problem as channel discrimination

The output of Deutsch's algorithm corresponds to two states associated with two different functions(even or odd)

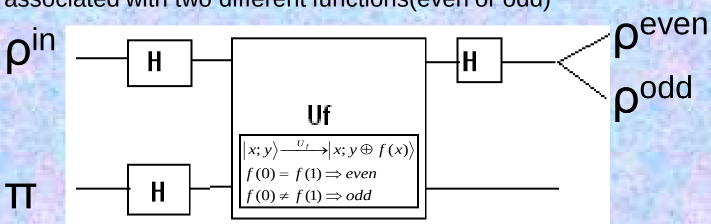

fig 1

The two outputs (for even or odd functions) are seen to be the output of two different channels. Now the performance of Deutsch's algorithm can be quantified in terms of minimizing the probability of error in channel discrimination/hypothesis testing [1]

Hypothesis testing

Hypothesis testing involves discriminating between the two states by minimizing the error in distinguishing these two states i.e. by minimizing the probability of error.

Errors could occur due to

- reading an odd function as even
- reading an even function as odd.

The probability of error is given by [1]

$$P_e = \frac{1}{2}[1 - \frac{1}{2}\left\|\rho^{odd} - \rho^{even}\right\|]$$

where, tracenorm $\|A\| = Tr\sqrt{A^{+}A}$

Comparing Probability of error and Holevo bound

The probability of error defined above is used here to check the performance of the algorithm for different input states of both the qubits. The results are then compared with the Holevo bound –the usually employed quantifier.

Consider the density matrices (see fig 1)

$$\Pi = \begin{bmatrix} p & \beta \\ \beta^* & 1-p \end{bmatrix} \quad \rho^{in} = \begin{bmatrix} q & \gamma \\ \gamma^* & 1-q \end{bmatrix} = \rho^{even} \quad \rho^{odd} = \begin{bmatrix} q & 2\gamma \operatorname{Re}\beta \\ 2\gamma^* \operatorname{Re}\beta & 1-q \end{bmatrix}$$

We obtain probability of error as,

$$P_e = \frac{1}{2}\left[1 - |\gamma|(1 - 2\operatorname{Re}\beta)\right]$$

while Holevo bound to differentiate between two states (ρ^{odd} and ρ^{even}) is given by

$$\chi = S(\rho) - \frac{1}{2}[\,S(\rho^{odd}\,) + S(\rho^{even}\,)]$$

where, $$\rho = \frac{(\rho^{odd} + \rho^{even})}{2}$$

$$S(\rho) = -Tr\ [\rho \log_2 \rho]$$

Note that $$0 \le \chi \le 1$$

whereas, $$0 \le P_e \le \frac{1}{2}$$

Π	ρ^{in}	Probability of error p_e	Holevo bound χ
$\lvert -\rangle\langle -\rvert$, $p=\frac{1}{2}, \beta=-\frac{1}{2}$	$\lvert +\rangle\langle +\rvert$ $q=\frac{1}{2}, \gamma=\frac{1}{2}$	0	1
	$\lvert -\rangle\langle -\rvert$ $q=\frac{1}{2}, \gamma=-\frac{1}{2}$	0	1
	$\begin{bmatrix} s & 0 \\ 0 & 1-s \end{bmatrix}$ $q=s, \gamma=0$	0.5	0
	$\frac{(\lvert 00\rangle+\lvert 11\rangle)(\langle 00\rvert+\langle 11\rvert)}{2}$	0	1
$\lvert +\rangle\langle +\rvert$, $p=\frac{1}{2}, \beta=\frac{1}{2}$	$\lvert +\rangle\langle +\rvert$ $q=\frac{1}{2}, \gamma=\frac{1}{2}$	0.5	0
	$\lvert -\rangle\langle -\rvert$ $q=\frac{1}{2}, \gamma=-\frac{1}{2}$	0.5	0
	$\begin{bmatrix} s & 0 \\ 0 & 1-s \end{bmatrix}$ $q=s, \gamma=0$	0.5	0
	$\frac{(\lvert 00\rangle+\lvert 11\rangle)(\langle 00\rvert+\langle 11\rvert)}{2}$	0.5	0
$\begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$, $p=p, \beta=0$	$\lvert +\rangle\langle +\rvert$ $q=\frac{1}{2}, \gamma=\frac{1}{2}$	0.25	0.311
	$\lvert -\rangle\langle -\rvert$ $q=\frac{1}{2}, \gamma=-\frac{1}{2}$	0.25	0.311
	$\frac{(\lvert 00\rangle+\lvert 11\rangle)(\langle 00\rvert+\langle 11\rvert)}{2}$	0.25	0.311

Future investigations:

- Extension to DJ algorithm with multiqubits
- role of entanglement in DJ channel discrimination
- discrimination with many copies of the input bit (evaluation of quantum Chernoff bound on probability of error.

References:

[1] C.W. Helstrom, Quantum Detection and Estimation Theory (Academic Press, New York, 1976)

[2] David Deutsch and Richard Jozsa, Proceedings of the Royal Society of London A 439: 553. (1992)

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