SIGNIFICANCE OF CLASSICAL RULES IN PENNY FLIP GAME

S. Balakrishnan[a#] and R. Sankaranarayanan[b]

Department of Physics

National Institute of Technology

Tiruchirappalli 620015, India

a. physicsbalki@gmail.com b. sankar@nitt.edu

Present Address: Centre for Nonlinear Dynamics

School of Physics

Bharathidasan University

Tiruchirappalli 620024, India

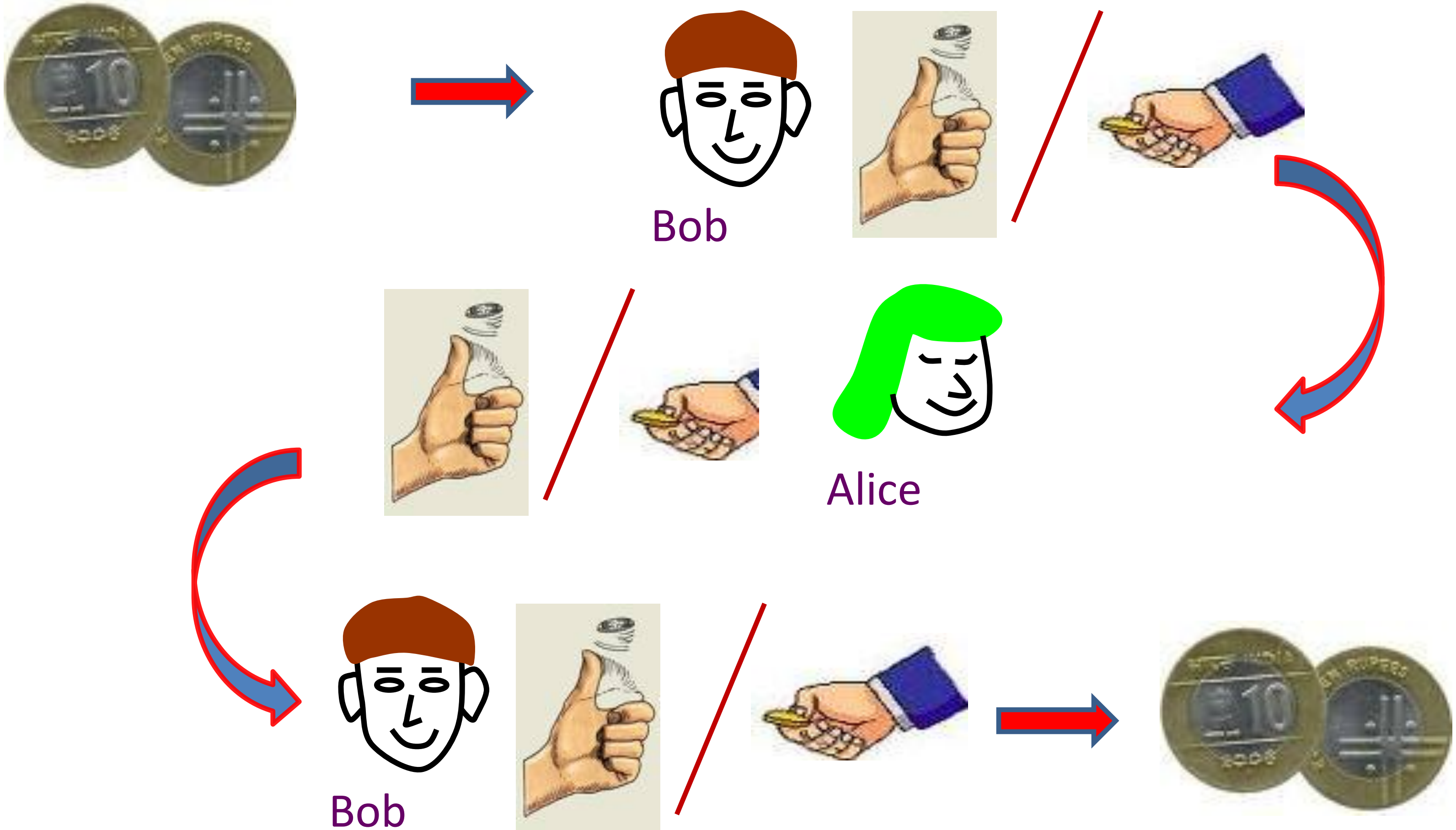
We study the quantum single penny flip game under various classical rules of the game. For every rule of the game, there exist unitary transformations which ensure the winning for quantum player. With the aim to understand the role of entanglement, we propose a quantization method for two penny flip problem in which quantum player is allowed to employ two-qubit entangling gate. While entangling gates are found to be not useful, local gates are necessary and sufficient to win the game. Further, importance of one qubit operations is indicated.

INTERNATIONAL WORKSHOP ON QUANTUM INFORMATION

20 – 26 February 2012

Harish-Chandra Research Institute, Allahabad, India

CLASSICAL PENNY FLIP PROBLEM

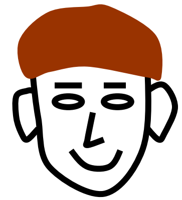


Same State of the coin → Bob wins

Otherwise → Alice wins

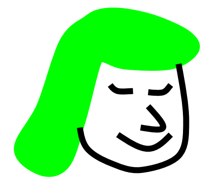
QUANTUM PENNY FLIP PROBLEM

$|\uparrow\rangle$ OR $|\downarrow\rangle$

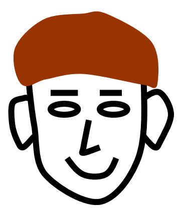


Hadamard gate

Bob



Alice



Hadamard gate

$|\uparrow\rangle$ OR $|\downarrow\rangle$

Bob

Bob always wins irrespective of Alice moves

GENERALIZATION OF QUANTUM PENNY FLIP

Game set up:

Head : $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Tail: $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In the density matrix formalism, $\rho_0 = |0\rangle\langle 0|$ and $\chi_0 = |1\rangle\langle 1|$

Allowed strategies for Alice:

Convex sum of flipping (*F*) the coin using the transformation σ_x with probability *p* and leaves the coin as it is (in other words, no flip (*N*)) using identity transformation with probability (*1-p*).

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \qquad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Allowed strategies for Bob:

Pure quantum strategy $k \in U(2)$.

$$k_j(\gamma_j, \theta_j, \alpha_j, \beta_j) = e^{i\gamma_j} \begin{pmatrix} e^{i\alpha_j} cos\,(\theta_j/2) & ie^{i\beta_j} sin\,(\theta_j/2) \\ ie^{-i\beta_j} sin\,(\theta_j/2) & e^{-i\alpha_j} cos\,(\theta_j/2) \end{pmatrix}$$

where $j = 1{,}2$, $e^{i\gamma_j}$ is the phase factor, $\theta_j \in [0, \pi]$ and $\alpha_j, \beta_j, \gamma_j \in [-\pi, \pi]$

Case 1: Initial state of the coin is head and final state is head.

Initial state of the coin: $\rho_0 = |0\rangle\langle 0|$

Action of Bob: $\rho_1 = k_1 \rho_0 k_1^\dagger.$

Action of Alice:

$$\rho_2 = pF\rho_1 F^\dagger + (1-p)N\rho_1 N^\dagger = pFk_1\rho_0 k_1^\dagger F^\dagger + (1-p)Nk_1\rho_0 k_1^\dagger N^\dagger.$$

Action of Bob: $\rho_3 = k_2 \rho_2 k_2^\dagger.$

If the final state of the coin is head, that is, $\rho_3 = \rho_0$, then Bob is the winner. Therefore, we have

$$pk_2\sigma_x k_1\rho_0 k_1^\dagger \sigma_x{}^\dagger k_2^\dagger + (1-p)k_2 k_1 \rho_0 k_1^\dagger k_2^\dagger = \rho_0.$$

This is possible, iff

$$k_2\sigma_x k_1\rho_0 k_1^\dagger \sigma_x{}^\dagger k_2^\dagger = \rho_0, \longrightarrow \quad (1)$$

and

$$k_2 k_1 \rho_0 k_1^\dagger k_2^\dagger = \rho_0. \longrightarrow \quad (2)$$

Eqn. (2) can be rewritten as $[k_2 k_1, \rho_0] = 0$, where $[.,.]$ represents usual commutation relation. Since $\rho_0 = \frac{1}{2}(I + \sigma_z)$, we have

$$\left[k_2 k_1, \frac{1}{2}(I+\sigma_z)\right] = \left[k_2 k_1, \frac{1}{2}I\right] + \left[k_2 k_1, \frac{1}{2}\sigma_z\right] = 0 \longrightarrow \quad (3)$$

➢While the first commutation is obviously zero, the second commutation can be zero for $k_2 k_1 = exp\ (i\varphi\sigma_z/2)$ where $0 \leq \varphi \leq \pi$. Therefore, $k_2 = exp\ (i\varphi\sigma_z/2)k_1^\dagger$ By substituting this in Eqn. (2), we have

$$[k_1^\dagger \sigma_x k_1, \rho_0] = \left[k_1^\dagger \sigma_x k_1, \frac{1}{2} I\right] + \left[k_1^\dagger \sigma_x k_1, \frac{1}{2}\sigma_z\right] = 0 \longrightarrow \quad (4)$$

➢ Second term is zero for $k_1^\dagger \sigma_x k_1 = exp(i\eta\sigma_z)$ where $0 \leq \eta \leq 2\pi$. Using the general form of k_1, we have

$$\left.\begin{array}{l}
\text{(i) } sin\theta_1\, sin(\alpha_1 + \beta_1) = e^{i\eta}, \\[2ex]
\text{(ii) } -sin\theta_1\, sin(\alpha_1 + \beta_1) = e^{-i\eta}, \\[2ex]
\text{(iii) } e^{2i\beta_1} sin^2\left(\frac{\theta_1}{2}\right) + e^{-2i\alpha_1} cos^2\left(\frac{\theta_1}{2}\right) = 0, \\[2ex]
\text{(iv) } e^{-2i\beta_1} sin^2\left(\frac{\theta_1}{2}\right) + e^{2i\alpha_1} cos^2\left(\frac{\theta_1}{2}\right) = 0,
\end{array}\right\} \longrightarrow \quad (5)$$

From (i) and (ii), we have $\eta = \pi/2$ or $3\pi/2$, implying that $k_1^{\dagger}\sigma_x k_1 = \sigma_z$. With this, the above simultaneous equations can be satisfied for $\theta_1 = \pi/2$ and $\alpha_1 + \beta_1 = \pi/2$ or $-3\pi/2$. Using this, we have found that

$$\boxed{u_1 = k_1 = \frac{e^{i\gamma_1}}{\sqrt{2}}\begin{pmatrix} e^{i\alpha_1} & -e^{-i\alpha_1} \\ e^{i\alpha_1} & e^{-i\alpha_1} \end{pmatrix};\ u_2 = k_2 = exp\left(\frac{i\varphi\sigma_z}{2}\right)u_1^{\dagger}.}$$

- The well known Meyer's strategy of Hadamard transformation can be obtained as a special case for the values of $\alpha_1 = \pi/2$, $\gamma_1 = -\pi/2$ and $\varphi = 0$.

Case 2:* *Initial state of the coin is tail and final state is tail.

Replacing ρ_0 with χ_0 and proceeding as earlier, and noting $\chi_0 = \frac{1}{2}(I - \sigma_z)$, the solution is the same as that of the previous one.

Case 3: Initial state of the coin is head and final state is tail.

In this case, we have

$$p k_2 \sigma_x k_1 \rho_0 k_1^\dagger {\sigma_x}^\dagger k_2^\dagger + (1-p) k_2 k_1 \rho_0 k_1^\dagger k_2^\dagger = \chi_0.$$

This is possible, iff

$$k_2 \sigma_x k_1 \rho_0 k_1^\dagger {\sigma_x}^\dagger k_2^\dagger = \chi_0 \longrightarrow \quad (6)$$

and

$$k_2 k_1 \rho_0 k_1^\dagger k_2^\dagger = \chi_0 \longrightarrow \quad (7)$$

The above equation implies that $k_2 k_1 \rho_0 = \chi_0 k_2 k_1$.

Since $\rho_0 = \frac{1}{2}(I + \sigma_z)$ and $\chi_0 = \frac{1}{2}(I - \sigma_z)$, we have

$$\left[k_2 k_1, \frac{1}{2} I\right] + \left\{k_2 k_1, \frac{1}{2}\sigma_z\right\} = 0 \longrightarrow \quad (8)$$

where $\{\,.,.\}$ represents anti-commutation relation. Similarly, Eqn. (6) becomes

$$\left[k_2 \sigma_x k_1, \frac{1}{2} I\right] + \left\{k_2 \sigma_x k_1, \frac{1}{2}\sigma_z\right\} = 0. \longrightarrow \quad (9)$$

While the commutation relations are zero, anti-commutations can be zero for suitable operators and can be identified from the following argument.

- We may note that the action of Alice have no effect on the superposition state of the coin. In order to identify the quantum strategy in favour of Bob, he should transform the initial state of head into a superposition state by employing the operator u_1 as

$$\boxed{u_1|0\rangle = \frac{e^{i\gamma_1}e^{i\alpha_1}}{\sqrt{2}}(|0\rangle + |1\rangle)}$$

This state is unaffected by Alice's move and, in his next turn Bob should change the state to tail. Equivalently, it is necessary to find an operator such that

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \to |1\rangle$$

Using the general form of unitary operator, the required operator is found to be

$$\boxed{u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(I - i\sigma_y)}$$

Recognizing $k_1 = u_1$ and $k_2 = u_3$, it is simple to check that Eqns. (8) and (9) are satisfied. Hence, Bob can beat the classical Alice by employing the operators u_1 followed by u_3.

Case 4: Initial state of the coin is tail and final state is head.

In this case, we have

$$pk_2\sigma_x k_1\chi_0 k_1^\dagger \sigma_x{}^\dagger k_2^\dagger + (1-p)k_2k_1\chi_0k_1^\dagger k_2^\dagger = \rho_0.$$

This is satisfied, iff

$$\left[k_2k_1, \frac{1}{2}I\right] - \left\{k_2k_1, \frac{1}{2}\sigma_z\right\} = 0 \longrightarrow (10)$$

and

$$\left[k_2\sigma_x k_1, \frac{1}{2}I\right] - \left\{k_2\sigma_x k_1, \frac{1}{2}\sigma_z\right\} = 0. \longrightarrow (11)$$

➢ Since the above conditions are the same as that of Eqns. (8) and (9), the winning strategies of Bob are the same as that of the previous case.

Table 1 Penny flip game with different rules

Rule of the game		Bob's quantum strategy	
Initial state	Final state	k_1	k_2
Head	Head	$u_1 = \frac{e^{i\gamma_1}}{\sqrt{2}}\begin{pmatrix} e^{i\alpha_1} & -e^{-i\alpha_1} \\ e^{i\alpha_1} & e^{-i\alpha_1} \end{pmatrix}$	$u_2 = exp(i\varphi\sigma_z/2)u_1^{\dagger}$
Tail	Tail		
Head	Tail	u_1	$u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
Tail	Head		

❖ For every rule of the game, there exist unitary operators which ensure the winning of quantum player.

❖ This observation in the penny flip problem demonstrates that classical rule of the game is insignificant.

QUANTUM TWO PENNY PROBLEM

Game set up

1) Set the state of the coins
2) Bob will employ nonlocal two-qubit operation U_1 (thereby he can produce entanglement between the coins)
3) Alice employs her classical probabilistic operations on the coins
4) And finally Bob employs two-qubit operation U_2
5) If the final state of the coins is the same as that of initial state, then Bob is declared as a winner.

$$U_j(c_{1j}, c_{2j}, c_{3j}) = \begin{pmatrix} e^{-\frac{ic_{3j}}{2}} c_j^- & 0 & 0 & -ie^{-\frac{ic_{3j}}{2}} s_j^- \\ 0 & e^{\frac{ic_{3j}}{2}} c_j^+ & -ie^{\frac{ic_{3j}}{2}} s_j^+ & 0 \\ 0 & -ie^{\frac{ic_{3j}}{2}} s_j^+ & e^{\frac{ic_{3j}}{2}} c_j^+ & 0 \\ -ie^{-\frac{ic_{3j}}{2}} s_j^- & 0 & 0 & e^{-\frac{ic_{3j}}{2}} c_j^- \end{pmatrix}$$

Where $c_j^{\pm} = cos[(c_{1j} \pm c_{2j})/2]$ $s_j^{\pm} = sin[(c_{1j} \pm c_{2j})/2]$ and $j = 1, 2$

Here, c_{1j}, c_{2j}, c_{3j} are the geometrical points of a two qubit gate

$c_{1j} = c_{2j} = c_{3j} = 0 \Rightarrow I \otimes I \Rightarrow$ No Flip

$c_{1j} = \pi, c_{2j} = c_{3j} = 0 \Rightarrow (\sigma_x \otimes \sigma_x) \Rightarrow$ Flip

Initial state of the coins: $\rho_0 = |mn\rangle\langle mn|$

Action of Bob: $\rho_1 = U_1 \rho_0 U_1^\dagger$.

Action of Alice:

$$\rho_2 = p(\sigma_x \otimes \sigma_x)\rho_1(\sigma_x \otimes \sigma_x)^\dagger + (1-p)(I \otimes I)\rho_1(I \otimes I)^\dagger$$

Action of Bob: $\rho_3 = U_2 \rho_2 U_2^\dagger$

- For Bob to win, the final state of the coins should be ρ_0 . That is,

$$\rho_3 = pU_2(\sigma_x \otimes \sigma_x)U_1\rho_0 U_1^\dagger(\sigma_x \otimes \sigma_x)^\dagger U_2^\dagger + (1-p)U_2U_1\rho_0U_1^\dagger U_2^\dagger = \rho_0$$

The above equation is satisfied, iff

$$U_2U_1\rho_0U_1^\dagger U_2^\dagger = \rho_0 \Rightarrow [U_2U_1, \rho_0] = 0$$

$$U_2(\sigma_x \otimes \sigma_x)U_1\rho_0U_1^\dagger(\sigma_x \otimes \sigma_x)^\dagger U_2^\dagger = \rho_0 \Rightarrow [U_2(\sigma_x \otimes \sigma_x)U_1, \rho_0] = 0$$

For the initial state of $|00\rangle$ as well as $|11\rangle$, the above conditions read as

$$c_{11} + c_{12} - c_{21} - c_{22} = \pi \text{ and } c_{11} + c_{12} - c_{21} - c_{22} = 2\pi$$

It is clear that the above conditions cannot be satisfied simultaneously for any entangling operators.

For the initial state of $|01\rangle$ as well as $|10\rangle$, we have

$$c_{11} + c_{12} + c_{21} + c_{22} = \pi \text{ and } c_{11} + c_{12} + c_{21} + c_{22} = 2\pi$$

It is clear that the above conditions cannot be satisfied simultaneously.

❖ However, quantum Bob can still win by employing suitable local operations on the two coins. From the analysis of single penny problem, we have identified suitable unitary operations (refer Table 1) employing which Bob can win the two penny problem as well. *Table 2 indicates that Bob can employ appropriate local operations on the coins to win the classical player. Therefore, local operations are necessary and sufficient for the quantum player to beat the classical opponent in the penny flip problem.*

Table 2 Winning advice for Bob

Rule of the game		Bob's quantum strategy	
Initial state	Final state	U_1	U_2
Head-Head	Head-Head	$u_1 \otimes u_1$	$u_2 \otimes u_2$
Tail-Tail	Tail-Tail		
Head-Tail	Head- Tail		
Tail-Head	Tail-Head		

CONCLUSION

❖ General unitary operations, for the given classical rule, are derived and employing which quantum player can always win the single penny game. *Thus we have shown the winning strategies for quantum player irrespective of the classical rule of the game.*

❖ With the aim to understand the role of entanglement, we propose a quantization method for two penny flip problem. However, we demonstrate that entanglement does not help the quantum player to produce the desired final state.

❖ Nevertheless, by employing local operations quantum player can win against the classical player. Therefore, *local gates are necessary and sufficient to win the two penny game.*

❖ While one qubit operations are critically important in the quantum circuit constructions, penny flip problem is yet another instance where importance of one qubit operations is revealed. In this context, one qubit operations deserve detailed investigations in game theory.

References

[1] Von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Wiley, (1967)

[2] Nash, J.: Proc. National Academy of Sciences. **36**, 48 (1950)

[3] Hong Guo, Juheng Zhang, Koehler, G. J.: Dec. Supp. Sys. **46**, 318 (2008)

[4] Eisert, J., Wilkens, M., Lewenstein, M.: Phys. Rev. Lett. **83**, 3077 (1999)

[5] Benjamin, S. C., Hayden, P. M.: Phys. Rev. A. **64**, 030301(R) (2001)

[6] Marinatto, L., Weber, T.: Phys. Letts. A. **272**, 291 (2000)

[7] Du, J., Xu, X., Li, H., Zhou, X., Han, R.: Phys. Letts. A. **289**, 9 (2001)

[8] Flitney, A. P., Abbott. D.: Proc. R. Soc. London. **459**, 2463 (2003)

[9] Meyer, D. A., Phys. Rev. Lett. **82**, 1052 (1999)

[10] Chappell, J. M., Iqbal, A., Lohe, M. A., Smekal, L. V.: J. Phys. Soc. Jpn. **78**, 054801 (2009)

[11] Heng-Feng Ren, Qing-Liang Wang.: Int. J. Theor. Phys. **47**, 1828 (2008)

[12] Wang, X.-B., Kwek, L. C., Oh, C. H.: Phys. Lett. A. **278**, 44 (2009).

[13] van Enk, S. J. and R. Pike.: Phys. Rev. A **66**, 024306 (2002)

[14] Rezakhani, A. T.: Phys. Rev. A **70**, 052313 (2004)

[15] Zhang, J., Vala, J., Whaley, K.B., Sastry, S.: Phys. Rev. A **67**, 042313 (2003)

Acknowledgements

S. B acknowledges Council of Scientific and Industrial Research, New Delhi, India for the financial support .