Aspects of quantum nets on phase spaces based on $\mathbb{F}_{2^n}$ and $\mathbb{Z}_{2^n}$

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Wigner Distributions on $\mathbb{R}$

Consider a quantum system whose classical configuration space $Q = \mathbb{R}$ is the real line $\mathbb{R}$. Let $\{\ |q\rangle \mid q \in \mathbb{R}\}$ denone the coordinate basis in the corresponding Hilbert space $\mathcal{H}$:

$$\langle q|q'\rangle \;=\; \delta\left(q - q'\right), \int_{-\infty}^{\infty} dq|q\rangle\langle q| = \mathbb{I};$$

Given the coordinate basis, one defines a ‘momentum basis’ $\{\ |p\rangle \mid p \in \mathbb{R}\}$ related to it by Fourier transformation:

$$|p\rangle \;=\; \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq\ e^{iqp/\hbar}|q > \ ;$$

$$\langle p|p'\rangle \;=\; \delta\left(p - p'\right), \int_{-\infty}^{\infty} dp|p\rangle\langle p| = \mathbb{I};$$

We may arrange the values of q and p in the usual Cartesian fashion and call it the 'classical phase space' associated with the quantum system. (Note that this classical phase space is not always the same as T^*Q).
In 1932, Wigner introduced a quantum analogue of the classical phase space distribution which associates with any quantum state $\widehat{\rho}$ a function $W_{\widehat{\rho}}(q,p)$ as follows:

$$
\begin{aligned}
\widehat{\rho} \mapsto W_{\widehat{\rho}}(q,p) &= \mathrm{Tr}\left\{\widehat{\rho}\,\widehat{W}(q,p)\right\} \;; \\
\widehat{W}(q,p) &= \frac{1}{(2\pi\hbar)} \int_{-\infty}^{\infty} dq' \; |q + \frac{1}{2}q'\rangle\langle q - \frac{1}{2}q'| \; e^{i\,pq'/\hbar},
\end{aligned}
$$

The operators $\widehat{W}(q,p)$ will be referred to as phase point operators

The Wigner distribution defined above has the following properties

1. Reality : $W_{\widehat{\rho}}(q,p) = W_{\widehat{\rho}}(q,p)^*$.
2. Marginals property : Average of the Wigner distribution along a line in phase space yields a probability density
3. Traciality: $\mathrm{Tr}\,\{\widehat{\rho}'\widehat{\rho}\} =$
$$\frac{1}{(2\pi\hbar)} \int\limits_{-\infty}^{\infty} dq \int\limits_{-\infty}^{\infty} dp \; W_{\widehat{\rho}'}(q,p) \; W_{\widehat{\rho}}(q,p).$$
4. $W_{\widehat{\rho}}(q,p)$ not necessarily positive for all $\widehat{\rho}$. For pure states $|\psi >\in \mathcal{H}$ the Wigner distribution is positive if and only if the state is a Gaussian state. (The Wigner distribution for such states is itself a Gaussian).

Correspondingly, for the phase point operators,

1. Hermiticity : $\widehat{W}(q,p) = \widehat{W}^{\dagger}(q,p)$.
2. Marginals property : Average of the phase point operator along a line in phase space yields a Projector

Wigner distributions have played an important role in semi classical approximations, classical optics etc. Of late they came into prominence in Quantum Information Theory largely due to the work of R. Simon [Phys. Rev. Lett. **84**, 2726 (2000)] and also Duan et. al [Phys. Rev. Lett. **84**, 2722 (2000)] in the context of continuous variable entanglement where necessary and sufficient conditions for entanglement in two mode Gaussian pure states were derived.

We now focus on phase space descriptions of d-state quantum stystems. It turns out that it is sufficient to examine the case $d = \mathfrak{p}^n$.

Phase space description of d-state quantum systems for $d = \mathfrak{p}^n$

Two possibilities naturally arise:

- q, p take values in the **field** $\mathbb{F}_{\mathfrak{p}^n}$
- q, p take values in the **ring** $\mathbb{Z}_{\mathfrak{p}^n}$

[A field, like $\mathbb{R}$ or $\mathbb{C}$, is a set whose elements form an abelian group under addition (additive identity denoted by '0') and the non zero elements form an abelian group under multiplication (multiplicative identity denoted by '1'). A ring is the same as a field except that not all elements have multiplicative inverses]

The first case was examined in detail by Gibbons et al [Phys Rev A **70**, 062101 (2004)] and by Wootters [IBM. Res. Dev. **48**, 99 (2004); quant-ph/0406032] leading to the notion of Quantum Nets.

An alternative algebraic approach to Wigner distributions developed by us [S.C, N.M. and R.S., J. Phys A **43**, 075302, (2010) is capable of handling both the cases within the same framework. Here we will compare the two approaches for the first case when $d = 2^n$ and highlight the differences with the second case for the same dimensions. These arise from the differences in the geometry of the corresponding phase spaces. In the field case the usual geometric propositions hold

- Two points define a line
- Parallel lines have no points in common
- Any two non parallel lines intersect at exactly one point

As a result

- The phase space has exactly $d+1$ isotropic lines –'straight' lines through the origin.
- Each isotropic line gives rise to $d-1$ lines parallel to it and thereby generates a striation of the phase space – decomposition of the set of d^2 phase points constituting the phase space space into d lines containing d points each. Each isotropic lines may therefore be chosen as the representative of the corresponding striation. As there are $d+1$ lines, one has $d+1$ striations.
- Any two non parallel lines intersect at exactly one point and there are exactly $d+1$ lines through a given phase point.

The situation is quite different in the ring case as we shall see later.

Notes on finite fields

The set

$$\mathbb{F}_{\mathfrak{p}} = \{0, 1, \cdots, \mathfrak{p} - 1\}$$

with addition and multiplication mod $\mathfrak{p}$ is a field. To go from $\mathbb{F}_{\mathfrak{p}}$ to $\mathbb{F}_{\mathfrak{p}^n}$, one considers

- all polynomials of degree $n - 1$ in x with coefficients in $\mathbb{F}_{\mathfrak{p}}$
- a monic polynomial of degree n in x with coefficients in $\mathbb{F}_{\mathfrak{p}}$ irreducible over $\mathbb{F}_{\mathfrak{p}}$ i.e does not become zero for any $x \in \mathbb{F}_{\mathfrak{p}}$

This set with addition modulo $\mathbb{F}_{\mathfrak{p}}$ and multiplication modulo the chosen irreducible polynomial constitute the field $\mathbb{F}_{\mathfrak{p}^n}$

An example

$\mathbb{F}_{2^2} = \{0, 1, x, 1 + x\};$ Irreducible Polynomial : $x^2 + x + 1$

A useful operation in finite fields is the trace operation:

$$\alpha \in \mathbb{F}_{\mathfrak{p}^n}. \quad \mathrm{tr}[\alpha] = \alpha + \alpha^{\mathfrak{p}} + \cdots + \alpha^{\mathfrak{p}^{\mathrm{n}-1}}.$$

It maps elements of $\mathbb{F}_{\mathfrak{p}^n}$ to $\mathbb{F}_{\mathfrak{p}}$. in a way such that

$$\begin{aligned} \mathrm{tr}[\alpha+\beta] &= \mathrm{tr}[\alpha] + \mathrm{tr}[\beta]; \quad && \alpha, \beta \in \mathbb{F}_{\mathfrak{p}^{\mathrm{n}}}; \\ \mathrm{tr}[\mathrm{a}\alpha] &= a\ \mathrm{tr}[\alpha]; && \alpha \in \mathbb{F}_{\mathfrak{p}^{\mathrm{n}}}, \mathrm{a} \in \mathbb{F}_{\mathfrak{p}}. \end{aligned}$$

and permits us to define a symplectic product between two phase points $\sigma = (q, p)$ and $\sigma' = (q', p')$as

$$< \sigma, \sigma' >= \mathsf{tr}[pq' - qp']$$

and hence an isotropic line λ as a set of phase points such that the symplectic product between any two vanishes.

Displacement operators

On the Hilbert space $\mathcal{H}$ of complex dimension $d = \mathfrak{p}^n$ we introduce the familiar Weyl operators

$$U(p) = \sum_{q \in \mathbb{F}_d} \omega^{\mathrm{tr}[qp]} |q\rangle\langle q|;$$

$$V(q) = \sum_{p \in \mathbb{F}_d} \omega^{\mathrm{tr}[qp]} |p)(p|,$$

where $\{|q >\}$ and $\{|p)\}$ denote the 'coordinate' and 'momentum' orthonormal bases related to each other by a discrete Fourier transform

$$|p) = \frac{1}{\sqrt{d}} \sum_{q \in \mathbb{F}_d} \omega^{\mathsf{tr}[qp]} |q >; \quad \omega = e^{2i\pi/\mathfrak{p}}$$

The Weyl operators obey the commutation relations relations

$$\begin{aligned} V(q)V(q') &= V(q+q'); \;\; U(p)U(p') = U(p+p'); \\ U(p)V(q) &= \omega^{\mathrm{tr}[pq]} V(q)U(p) = \tau^{2\mathrm{tr}[pq]} V(q)U(p). \end{aligned}$$

The displacement operators are defined in terms of the Weyl operators as follows:

$$\sigma = (q,p): \; D(\sigma) = \tau^{-\mathrm{tr}[\mathrm{qp}]} U(p) V(q) = \tau^{\mathrm{tr}[\mathrm{qp}]} V(q) U(p).$$

These operators have the following properties

$$\begin{aligned} \text{Unitary} &: \; D(\sigma)^\dagger D(\sigma) = \mathbb{I}; \\ \text{Trace Orthogonality} &: \; \mathrm{Tr}(D(\sigma')^\dagger D(\sigma)) = N\delta_{\sigma,\sigma'}; \\ \text{Composition} &: \; D(\sigma)D(\sigma') = \mu(\sigma,\sigma')D(\sigma+\sigma'). \end{aligned}$$

where

$$\mu(\sigma,\sigma') = \tau^{-\mathrm{tr}[\mathrm{qp}]-\mathrm{tr}[\mathrm{q'p'}]+\mathrm{tr}[(\mathrm{q+q'})(\mathrm{p+p'})]}\omega^{\mathrm{tr}[\mathrm{qp'}]}.$$

When $\mathfrak{p}$ is odd, so that $\mathbb{F}_{\mathfrak{p}}$ contains 2 as an element, using the properties of $\mathrm{tr}[\cdot]$ the R.H.S. simplifies to

$$\mu(\sigma,\sigma') = \omega^{\langle\sigma,\sigma'\rangle}$$

However, when $\mathfrak{p} = 2$, the expression for $\mu(\sigma,\sigma')$ should be kept as such

Phase space description of a qubit: Quantum Nets

Phase space: 2×2 lattice with q and p taking values in $\mathbb{F}_2 = \{0, 1\}$

$$\Gamma_0 \qquad \begin{array}{ccc} \uparrow & \times & \times \\ p & \times & \times \\ & q & \longrightarrow \end{array}$$

Striations		MUBS	
$\begin{array}{cc} \bullet & \star \\ \bullet & \star \end{array}$	Vertical	$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$	Coordinate Basis
$\begin{array}{cc} \star & \bullet \\ \bullet & \star \end{array}$	$\vdots$	$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$	$\vdots$
$\begin{array}{cc} \star & \star \\ \bullet & \bullet \end{array}$	Horizontal	$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$	Momentum Basis

Quantum net construction :

striation $\longrightarrow$ MUB

isotropic lineλ $\longrightarrow$ the projector P_λ corresponding to a MUB vector

Assignment of projectors to the lines parallel to the isotropic line done consistent with translational covariance. Each such assignment is a quantum net
We thus have 2^3 (in general d^{d+1}) possible quantum nets. A quantum net captures the geometric properties of the lines to the Hilbert space level via the MUBs in the following sense:

Overlap between two lines λ and λ' =No. of common points$/d$

$$\longrightarrow \mathrm{Tr}[P_\lambda P_{\lambda'}]$$

For a given quantum net one then defines the phase point opertors as

$$\hat{W}(\sigma) = \sum_{\lambda, \sigma\in\lambda} P_\lambda - I$$

These have the following properties:

$$\mathsf{Tr}(\hat{W}(\sigma)) = 1; \mathsf{Tr}(\hat{W}(\sigma)\hat{W}(\sigma')) = N\delta_{\sigma,\sigma'}; \frac{1}{d}\sum_{\sigma\in\lambda}\hat{W}(\sigma) = P_\lambda$$

There are d^{d-1} ways of defining sets of d^2 such operators and hence that many Wigner distributions. In particular, for $d = 2^n$ there are $2^{n(2^n-1)}$ possible definitions of Wigner distributions. Some interesting open questions: How many of these d^{d-1} possibilities are spectrally distinct? What are the extremal values of the spectra? These questions are of interest in the conext of an application of quantum nets to Quantum Random Access Codes as discussed in the work of A.Casaccino, E. F. Galvão and S. Severini Phys Rev A **78**, 022310 (2008)

Quantum nets for based on $\mathbb{F}_{2^n}$: an algebraic approach

In our 'square root' approach to Wigner distributions which takes the traciality property as the starting point one is led to phase point operators having the following structure;

$$\hat{W}(\sigma) = \frac{1}{d}\sum_{\sigma'} \omega^{\langle \sigma, \sigma' \rangle} S(\sigma') D(\sigma')$$

where $S(\sigma)$ take values ± 1 and satisfy the symmetry condtion

$$S(\sigma) = S(-\sigma)$$

The phase point opertors so defined are hermitian and trace orthogonal:

$$\mathrm{Tr}(\hat{W}(\sigma')\hat{W}(\sigma)) = d\delta_{\sigma,\sigma'}.$$

The standard $q - p$ marginals conditions requires

$$S(0, p) = S(q, 0) = 1.$$

Further, requiring that the average of the phase point operator along any istropic line yields a rank one projector yields the following conditions on $S(\sigma)$:

$$S(\sigma)S(\sigma')\mu(\sigma,\sigma') = S(\sigma+\sigma'); \sigma,\sigma' \in \lambda.$$

For $d = 2^n$ one can count the number of free signs and these turn out to be $2^{n(2^n-1)}$ exactly the same as in the quantum net construction. The advantage of this approach is that it does not require explicit knowledge of the MUBS. The construction is entirely algebraic – each assignment of MUB projector to lines in the quantum net corresponds a specific choice for the free signs.

Quantum nets for based on $\mathbb{Z}_{2^n}$: an algebraic approach

Our algebraic approach works here as well. There are $2^{n+1} - 1$ isotropic lines to deal with. Of course, the lines on the phase space no longer have the nice properties as in the field case–two non parallel lines may have more than one point in common etc. The marginals conditions can no longer be implemented on all of them but on specific subsets thereof. When this is done, what remains true is the relation

Overlap between two lines λ and λ' =No. of common points$/d$

$$\longrightarrow \text{Tr}[P_\lambda P_{\lambda'}]$$

Same questions as in the field case still remain: How many of these various possibilities for the Wigner distributions are spectrally distinct? What are the extremal values of the spectra?