Symmetric Two Qubit Entanglers

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Outline:

- Definition of Entanglement
- Permutationally symmetric states
- Density matrix description-Fano statistical parameters
- Alternative representation of SU(n) basis
- Spin-1 systems
- Two qubit symmetric gates
- Special perfect entanglers
- LMG Hamiltonian
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- Considerable increase in experimental activity [1] aiming to create entangled quantum states which have potential applications in quantum information processing tasks.
- These entangled states are created by some physical operations involving the interaction between several systems.
- Analyzing these operations with regard to the possibility of creating maximally entangled states from an initial unentangled one and characterization of entangling capabilities or properties of quantum operators play an important role in quantum information theory.

[1] Fortschr Special issue, 2000 *Fortschr. Phys.* **48**, Nos. 9-11

Symmetric states

Symmetric N-particle states remain unchanged by permutations of individual particles.
Symmetric states offer elegant mathematical analysis as the dimension of the Hilbert space reduces drastically from 2^N to $(\mathrm{N}+1)$.
Such a Hilbert space is spanned by the eigen states $\{|j, m\rangle; -j \leq m \leq +j\}$ of angular momentum operators J^2 and J_z , where $j = \frac{N}{2}$.

A large number of experimentally relevant states possesses symmetry under particle exchange and this property allows us to significantly reduce the computational complexity.

If we have N two level atoms, each atom may be represented as a spin- $\frac{1}{2}$ system and theoretical analysis can be carried out in terms of collective spin operator $\vec{J} = \frac{1}{2}\Sigma_{\alpha=1}^{N}\vec{\sigma}_\alpha$. Here $\vec{\sigma}_\alpha$ denote the Pauli spin operator of the αth qubit.

Spherical tensor representation of density matrix[2]

The most general spin-j density matrix

$$\rho(\vec{J}) = \frac{Tr(\rho)}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} t_q^k \, \tau_q^{k^\dagger}(\vec{J}) \ , \tag{1}$$

τ_q^k (with $\tau_0^0 = I$,the identity operator) are irreducible tensor operators of rank 'k'.
τ_q^k satisfy the orthogonality relations

$$Tr(\tau_q^{k^\dagger} \tau_{q'}^{k'}) = (2j+1)\, \delta_{kk'} \delta_{qq'} \tag{2}$$

and

$$t_q^k = \frac{Tr(\rho\, \tau_q^k)}{Tr \rho} \tag{3}$$

[2] U Fano, Rev.Mod.Phys.**29,74(1957)**

ρ is Hermitian and $\tau_q^{k^\dagger} = (-1)^q \tau_{-q}^k$ and hence

$$t_q^{k^*} = (-1)^q \, t_{-q}^k \tag{4}$$

Spherical tensor parameters $t_q^k\,'s$ have simple transformation properties under co-ordinate rotation.

In the rotated frame $t_q^k\,'s$ are given by

$$(t_q^k)^R = \sum_{q'=-k}^{+k} D_{q'\,q}^k(\phi, \theta, \psi) \, t_{q'}^k \ , \tag{5}$$

$D_{q'\,q}^k(\phi, \theta, \psi)$ denote Wigner-D matrix,

(ϕ, θ, ψ) Euler angles

Weyl construction[3]

$\tau_q^{k\prime} s$ in terms of angular momentum operators J_x, J_y and J_z,

$$\tau_q^k(\vec{J}) = \mathcal{N}_{kj}\,(\vec{J}\cdot\vec{\bigtriangledown})^k\, r^k\, Y_q^k(\hat{r})\,, \tag{6}$$

where

$$\mathcal{N}_{kj} = \frac{2^k}{k!}\sqrt{\frac{4\pi(2j-k)!(2j+1)}{(2j+k+1)!}}, \tag{7}$$

are the normalization factors and $Y_q^k(\hat{r})$ are the spherical harmonics.

[3] Rose M E 1957 *Elementary theory of Angular momentum*(Wiley,Newyork)

Linearly independent, traceless (except $(T^0)^0_0$), orthonormal Hermitian basis matrices:

$(T^\alpha)^k_q$, where $\alpha = +, -, 0$, $k = 1...2j$, and q $=$ 1 to $+$k

$$(T^+)^k_q = \frac{\tau^k_q + (\tau^k_q)^\dagger}{\sqrt{2(2j+1)}} \ , \tag{8}$$

$$(T^-)^k_q = \frac{i(\tau^k_q - (\tau^k_q)^\dagger)}{\sqrt{2(2j+1)}} \ , \tag{9}$$

and

$$(T^0)^k_0 = \frac{\tau^k_0}{\sqrt{2j+1}} \ . \tag{10}$$

These matrices satisfy the relation $Tr((T^\alpha)^k_q (T^\beta)^{k'}_{q'}) = \delta_{\alpha\beta}\delta_{kk'}\delta_{qq'}$.

New representation of the most general density matrix

$$\rho = (r^0)^0_0 (T^0)^0_0 + \sum_{k=1....2j} (r^0)^k_0 (T^0)^k_0 + \sum_{\alpha=+,-} \sum_{k=1...2j} \sum_{q=1...k} (r^\alpha)^k_q (T^\alpha)^k_q \tag{11}$$

Apart from $(T^0)^0_0$ which is proportional to identity matrix, there are 2j diagonal matrices namely $(T^0)^k_0$, k $=$ 1...2j and the rest are off diagonal.

In the particular case of two qubit symmetric subspace, our set of basis matrices in $|1m\rangle$ basis where m = 1, 0, -1 are

$$M_0 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$M_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \; M_5 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$M_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \ M_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ ,$$

$$M_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ .$$

The above matrices are normalized i.e., $Tr(M_k M_{k'}) = 2\,\delta_{kk'}$ and $M_1, ..., M_7$ have eigen values 1, 0, -1.

Spin-1 systems

The most general spin-1 Hamiltonian is

$$\mathcal{H}(t) = \frac{1}{2} \sum_{i=0}^{8} h_k(t) M_k \quad . \tag{12}$$

Here M_k's in terms of angular momentum operators J_x,J_y,J_z are

$$M_1 = -(J_x)\ ,\ M_2 = (J_y)\,,\ M_3 = (J_z)\,,$$

$$M_4 = -(J_x J_y + J_y J_x)\ ,\ M_5 = (J_y J_z + J_z J_y)\ ,$$

$$M_6 = -(J_x J_z + J_z J_x)\ ,\ M_7 = (J_x^2 - J_y^2)\ ,\ M_8 = (3J_z^2 - 2)\,.$$

The expansion co-efficients $h_k = Tr(\mathcal{H} M_k)$ are real and hence they constitute an experimentally measurable set of parameters.

Two qubit symmetric gates

Time evolution of the operators M_k's provide various symmetric logic gates for quantum computation The closed form expression for $e^{iM_k\theta}$ is

$$B_k = e^{iM_k\theta} = I + (cos\theta - 1)M_k^2 + isin\theta M_k. \qquad (13)$$

Here k $=$ 1....7 and I is a 3×3 unit matrix.

Explicit forms of the gates B_k's in the symmetric subspace:

$$B_1 = \begin{pmatrix} cos^2\frac{\theta}{2} & \frac{-isin\theta}{\sqrt{2}} & -sin^2\frac{\theta}{2} \\ \frac{-isin\theta}{\sqrt{2}} & cos\theta & \frac{-isin\theta}{\sqrt{2}} \\ -sin^2\frac{\theta}{2} & \frac{-isin\theta}{\sqrt{2}} & cos^2\frac{\theta}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} cos^2\frac{\theta}{2} & \frac{sin\theta}{\sqrt{2}} & sin^2\frac{\theta}{2} \\ \frac{-sin\theta}{\sqrt{2}} & cos\theta & \frac{sin\theta}{\sqrt{2}} \\ sin^2\frac{\theta}{2} & \frac{-sin\theta}{\sqrt{2}} & cos^2\frac{\theta}{2} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}, B_4 = \begin{pmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{pmatrix},$$

$$B_5 = \begin{pmatrix} cos^2\frac{\theta}{2} & \frac{sin\theta}{\sqrt{2}} & -sin^2\frac{\theta}{2} \\ \frac{-sin\theta}{\sqrt{2}} & cos\theta & \frac{-sin\theta}{\sqrt{2}} \\ -sin^2\frac{\theta}{2} & \frac{sin\theta}{\sqrt{2}} & cos^2\frac{\theta}{2} \end{pmatrix}, B_6 = \begin{pmatrix} cos^2\frac{\theta}{2} & \frac{-isin\theta}{\sqrt{2}} & sin^2\frac{\theta}{2} \\ \frac{-isin\theta}{\sqrt{2}} & cos\theta & \frac{isin\theta}{\sqrt{2}} \\ sin^2\frac{\theta}{2} & \frac{isin\theta}{\sqrt{2}} & cos^2\frac{\theta}{2} \end{pmatrix},$$

$$B_7 = \begin{pmatrix} cos\theta & 0 & isin\theta \\ 0 & 1 & 0 \\ isin\theta & 0 & cos\theta \end{pmatrix}, B_8 = \begin{pmatrix} e^{\frac{i\theta}{\sqrt{3}}} & 0 & 0 \\ 0 & e^{\frac{-2i\theta}{\sqrt{3}}} & 0 \\ 0 & 0 & e^{\frac{i\theta}{\sqrt{3}}} \end{pmatrix}.$$

- Useful property of a two qubit symmetric gate is its ability to produce a maximally entangled state from an unentangled one.
- perfect entanglers are those unitary operators that can generate maximally entangled states from some suitably chosen separable states.

- These two qubit symmetric gates are capable of producing entanglement, quantifying their entangling capability is very important. Makhlin [4] has analyzed nonlocal properties of two-qubit gates and also studied some basic properties of perfect entanglers.
- perfect entanglers are defined as the unitary operators that can generate maximally entangled states from some suitably chosen separable states.
- The entangling capability of a unitary quantum gate can be quantified by its entangling power $e_p(U)$[5].
- Balakrishnan et al.[6] have derived $e_p(U)$ in terms of local invariant G_1.
- entangling power of two qubit symmetric gates is

$$e_p(B) = \frac{2}{9}(1 - |G_1|).$$

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[4] Makhlin Y 2002 *Quant. Inf. Proc.* **1**, 243

[5] Paolo Zanardi, Christof Zalka and Lara Faoro 2000 *Phys Rev.* **A 62**, 030301(R)

[6] Balakrishnan S and Sankaranarayanan R 2010 *Phys. Rev.* **A 82**, 034301

- The local invariant G_1[7] in terms of symmetric, unitary matrix m is

$$G_1 = \frac{tr^2 m}{16 det[B]}.$$

- $m = B_B^T B_B$, where
- $B_B = UBU^\dagger$. U is a transformation matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -\sqrt{2}i & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ -i & 0 & i & 0 \end{pmatrix}$$

connecting the angular momentum basis $|11\rangle$, $|10\rangle$, $|1-1\rangle$, $|00\rangle$ to the Bell basis.

- A perfect entangler has the range[8] $\frac{1}{6} \leq e_p \leq \frac{2}{9}$.

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[7] Makhlin Y 2002 *Quant. Inf. Proc.* **1**, 243 .

[8] Balakrishnan S and Sankaranarayanan R 2010 *Phys. Rev.* **A 82**, 034301.

- B_1, B_2, B_3 do not produce entanglement, they represent rotations which is a localhspac unitary transformation($|G_1| = 1$, $e_p = 0$)
- For the gates B_4, B_5, B_6 and B_7, $|G_1| = Cos^4(\theta)$.
- $0 \leq G_1 \leq 1$ for $0 \leq \theta \leq \frac{\pi}{2}$, it is clear that $0 \leq e_p(B_B)_k \leq \frac{2}{9}$ (k = 4...7).
- The above mentioned gates are perfect entanglers for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$.
- The gate B_8 will have maximum entangling power i.e., $e_p = 2/9$ when $\theta = \sqrt{3}\frac{\pi}{2}$.

Example: the direct product state $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, of two spinors in the qubit basis.

$$|\psi_{12}\rangle = \begin{pmatrix} cos\frac{\alpha_1}{2} \\ sin\frac{\alpha_1}{2}e^{i\phi_1} \end{pmatrix} \otimes \begin{pmatrix} cos\frac{\alpha_2}{2} \\ sin\frac{\alpha_2}{2}e^{i\phi_2} \end{pmatrix}$$

$$= \begin{pmatrix} cos\frac{\alpha_1}{2}cos\frac{\alpha_2}{2} \\ cos\frac{\alpha_1}{2}sin\frac{\alpha_2}{2}e^{i\phi_2} \\ sin\frac{\alpha_1}{2}cos\frac{\alpha_2}{2}e^{i\phi_1} \\ sin\frac{\alpha_1}{2}sin\frac{\alpha_2}{2}e^{i(\phi_1+\phi_2)} \end{pmatrix},$$

$0 \leq \alpha_{1,2} \leq \pi$, $0 \leq \phi_{1,2} < 2\pi$.

A separable state in the symmetric subspace in $|1m\rangle$ basis ($m = 1, 0, -1$) will have the form

$$|\psi_{12}\rangle_{sym} = \begin{pmatrix} cos^2\frac{\alpha}{2} \\ \sqrt{2} sin\frac{\alpha}{2} cos\frac{\alpha}{2} e^{i\phi} \\ sin^2\frac{\alpha}{2} e^{2i\phi} \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha$ and $\phi_1 = \phi_2 = \phi$.
For a pure state of two qubits

$$|\,\psi\rangle = a\,|\uparrow\uparrow\rangle + b\,|\uparrow\downarrow\rangle + c\,|\downarrow\uparrow\rangle + d\,|\downarrow\downarrow\rangle,$$

the expression for concurrence[9] is

$$C(\psi) = 2|ad - bc|\,.$$

For a maximally entangled quantum state concurrence $C = 1$. Under the action of the gates B_4, B_7 and B_8 (with e_p being maximum i.e., 2/9), $|\psi_{12}\rangle_{sym}$ will become maximally entangled state when $\alpha = \frac{\pi}{2}$.

[9] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245

i.e.,

$$B_4|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=\frac{\pi}{2}} \begin{pmatrix} -\frac{1}{2}e^{2i\phi} \\ \frac{1}{\sqrt{2}}e^{i\phi} \\ \frac{1}{2} \end{pmatrix}, B_7|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=\frac{\pi}{2}} \begin{pmatrix} \frac{i}{2}e^{2i\phi} \\ \frac{1}{\sqrt{2}}e^{i\phi} \\ \frac{i}{2} \end{pmatrix},$$

$$B_8|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=\frac{\pi}{2}} \begin{pmatrix} \frac{i}{2} \\ -\frac{1}{\sqrt{2}}e^{i\phi} \\ \frac{i}{2}e^{2i\phi} \end{pmatrix}.$$

Similarly, the gates B_5, B_6 acting on the symmetric separable state transform it into maximally entangled one when $\alpha = 0, \pi$. For eg:

$$B_5|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=0} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}, B_6|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=0} \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}.$$

Special perfect entanglers

- Some of the perfect entanglers have the unique property of maximally entangling a complete set of orthonormal product vectors.
- Such operators for which $e_p = \frac{2}{9}$ belong to a well known family of special perfect entanglers[10].

[10] Rezakhani A T 2004 *Phys. Rev.* **A 70**, 052313.

When $e_p = \frac{2}{9}$, $B_4, ...B_8$ in the qubit basis are given by

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; B_5 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$B_6 = \frac{1}{2}\begin{pmatrix} 1 & -i & -i & 1 \\ -i & 1 & -1 & i \\ -i & -1 & 1 & i \\ 1 & i & i & 1 \end{pmatrix}, \; B_7 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$B_8 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

Following Rezakhani, the most general separable basis (upto general phase factors for each vector) is

$$|\psi_1\rangle = (a|\uparrow\rangle + b|\downarrow\rangle) \otimes (c|\uparrow\rangle + d|\downarrow\rangle)\,,$$

$$|\psi_2\rangle = (-b^*|\uparrow\rangle + a^*|\downarrow\rangle) \otimes (c|\uparrow\rangle + d|\downarrow\rangle)\,,$$

$$|\psi_3\rangle = (e|\uparrow\rangle + f|\downarrow\rangle) \otimes (-d^*|\uparrow\rangle + c^*|\downarrow\rangle)\,,$$

$$|\psi_4\rangle = (-f^*|\uparrow\rangle + e^*|\downarrow\rangle) \otimes (-d^*|\uparrow\rangle + c^*|\downarrow\rangle)\,,$$

where $|a|^2 + |b|^2 = |c|^2 + |d|^2 = |e|^2 + |f|^2 = 1$.

When the gates B_4, B_7 and B_8 as perfect entanglers act on the state - say $|\psi_1\rangle$, we obtain

$$[B_{4,7,8}]|\psi_1\rangle = -bd\,|\uparrow\uparrow\rangle + ad\,|\uparrow\downarrow\rangle + bc\,|\downarrow\uparrow\rangle + ac\,|\downarrow\downarrow\rangle.$$

- This state is maximally entangled if its concurrence, $\mathrm{C} = 4|abcd| = 1$.
- The above said two qubit symmetric gates transform the orthonormal states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ and $|\psi_4\rangle$ into maximally entangled ones if

$$|abcd| = |cdef| = \frac{1}{4}.$$

- Similarly, for the gates B_5 and B_6,

$$|(a^2+b^2)(c^2+d^2)| = |(e^2+f^2)(c^2+d^2)| = 1.$$

Lipkin-Meshkov-Glick interaction Hamiltonian[11,12]

$$\mathcal{H}_L = \mathcal{G}_1(J_+^2 + J_-^2) + \mathcal{G}_2(J_+J_- + J_-J_+) \ . \tag{14}$$

$\mathcal{G}_1$ and $\mathcal{G}_2$ are the coupling constants. In terms of our operators $M_k\,'s$,

$$\mathcal{H}_L = \mathcal{G}_1' M_7 + \mathcal{G}_2'(\sqrt{8}M_0 - M_8) \ , \tag{15}$$

$\mathcal{G}_1' = 2\mathcal{G}_1$ and $\mathcal{G}_2' = \frac{2}{\sqrt{3}}\mathcal{G}_2$.

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[11] Lipkin etal. 1965 *Nucl. Phys.* **62** 188

[12] Pathak P K Deb R N Nayak N and Dutta-Roy B 2008 *J. Phys. A: Math. Theor.* **41** 145302

$[M_7, M_8] = 0$, we have

$$e^{iH_L t} = B_L = \left(\begin{array}{ccc} e^{\sqrt{3}\, i\beta} cos\xi & 0 & ie^{\sqrt{3}\, i\beta} sin\xi \\ 0 & e^{2\sqrt{3}\, i\beta} & 0 \\ ie^{\sqrt{3}\, i\beta} cos\xi & 0 & e^{\sqrt{3}\, i\beta} cos\xi \end{array} \right),$$

in spin-1 subspace.

- Here $\xi = \mathcal{G}_1' t$ and $\beta = \mathcal{G}_2' t$
- $e_p = \frac{2}{9}$ for $2\mathcal{G}_2 t = \frac{\pi}{2} + 2\mathcal{G}_1 t$. Under the action of this gate (with $e_p = \frac{2}{9}$), the separable state $|\uparrow\uparrow\rangle(|\downarrow\downarrow\rangle)$ becomes entangled for all values of t except when
- t $= \frac{n\pi}{4\mathcal{G}_1}$; n=0,1,2......
- maximally entangled when $4\mathcal{G}_1 t = (2n+1)\frac{\pi}{2}$.

eg.,

$$B_L|\psi_{12}\rangle_{sym} \xrightarrow{\alpha=0} cos(2\mathcal{G}_1 t)\,|\uparrow\uparrow\rangle + isin(2\mathcal{G}_1 t)\,|\downarrow\downarrow\rangle\ .$$

Conclusion

- Constructed traceless,Hermitian and linearly independent set of basis matrices of SU(n)
- Considered unitary evolutions of two spin-1/2 states in angular momentum subspace (j=1) and constructed physically realizable logic gates using (2j+1) dimensional representation of the above set of basis matrices.
- Entangling properties of these gates have been studied in terms of their entangling power e_p.
- These logic gates are obtained by the exponentiation of the quadratic form of angular momentum operators $\mathbf{J_x}, \mathbf{J_y}, \mathbf{J_z}$.
- Entangling properties of Lipkin-Meshkov-Glick Hamiltonian is studied in spin-1 subspace.


Thank You