Minimum Uncertainty for Entangled States

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Outline

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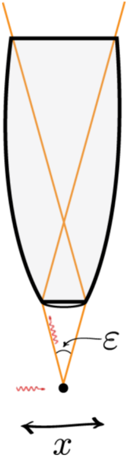
Heisenberg's Microscope

Optical resolution of the microscope

$$\Delta x = \frac{\lambda}{\sin \epsilon}$$

Momentum recoil on the electron = h/λ.
"the recoil cannot be exactly known, since the direction of the scattered photon is undetermined within the bundle of rays entering the microscope"

$$\Delta p_x \approx \frac{h}{\lambda} \sin \epsilon$$



$$\Delta x \Delta p_x \approx \left(\frac{\lambda}{\sin \epsilon}\right) \left(\frac{h}{\lambda} \sin \epsilon\right) = h.$$

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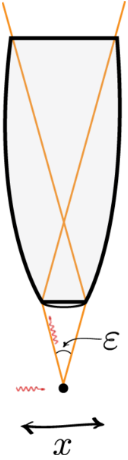

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Uncertainty inherent in Quantum Mechanics

The uncertainties in two observables **A** and **B**, are defined as

$$\Delta A = \sqrt{\langle\psi|\mathbf{A}^2|\psi\rangle - \langle\psi|\mathbf{A}|\psi\rangle^2}$$
$$\Delta B = \sqrt{\langle\psi|\mathbf{B}^2|\psi\rangle - \langle\psi|\mathbf{B}|\psi\rangle^2}$$

Uncertainty is an inherent property which depends on the state of the system. It is not a shortcoming of the measurement process.

- If $|\psi\rangle$ is an eigenstate of **A**,

$$\Delta A = 0$$

- Noncommuting observables cannot have common eigenstates.

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Uncertainty Relations

Heisenberg Uncertainty Relation (HUR) [1]

$$(\Delta X)^2(\Delta Y)^2 \geq \frac{1}{4}|\langle[\mathbf{X},\mathbf{Y}]\rangle|^2, \tag{1}$$

Schrödinger-Robertson inequality (SR) [2]

$$(\Delta X)^2(\Delta Y)^2 \geq \frac{1}{4}|\langle[\mathbf{X},\mathbf{Y}]\rangle|^2 + \frac{1}{4}|\langle\{\tilde{\mathbf{X}},\tilde{\mathbf{Y}}\}\rangle|^2 \tag{2}$$

where,

$$\tilde{\mathbf{X}} = \mathbf{X} - \langle\mathbf{X}\rangle, \quad \tilde{\mathbf{Y}} = \mathbf{Y} - \langle\mathbf{Y}\rangle$$

[1] W. Heisenberg, Zeitschrift für Physik **43**, 172-198 (1927).
H. P. Robertson, Phys. Rev. **34**, 163-164 (1929).

[2] E. Schrödinger, Ber. Kgl. Akad. Wiss. No. 296 (1930); H. P. Robertson, Phys. Rev. 35, 667A (1930).

Uncertainty and Entanglement

Question:
Does entanglement put a bound on the uncertainty relations?

System A ← entangled with → *System B*

$\Delta X_A \Delta Y_A \geq ?$ Let it be

Simplest bipartite entangled state

$$|\Psi\rangle = c_1|\psi_1\rangle_A|\alpha_1\rangle_B + c_2|\psi_2\rangle_A|\alpha_2\rangle_B$$

$|\psi_i\rangle_A$ → two states of system A,
$|\alpha_j\rangle_B$ → are two *orthonormal* states of system B.

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Uncertainty and Entanglement

The uncertainties in two observables $\mathbf{X}_A \otimes \mathbf{1}_B$ and $\mathbf{Y}_A \otimes \mathbf{1}_B$, in the entangled state $|\Psi\rangle$, are defined as

$$(\Delta X)^2_\Psi = \langle\Psi|\mathbf{X}^2|\Psi\rangle - \langle\Psi|\mathbf{X}|\Psi\rangle^2$$
$$(\Delta Y)^2_\Psi = \langle\Psi|\mathbf{Y}^2|\Psi\rangle - \langle\Psi|\mathbf{Y}|\Psi\rangle^2$$

For a generic observable **O**

$$(\Delta O)^2_\Psi = |c_1|^2(\Delta O)^2_1 + |c_2|^2(\Delta O)^2_2 + |c_1|^2|c_2|^2(\langle\mathbf{O}\rangle_1 - \langle\mathbf{O}\rangle_2)^2$$

where

$$(\Delta O)^2_1 = \langle\psi_1|\mathbf{O}^2|\psi_1\rangle - \langle\psi_1|\mathbf{O}|\psi_1\rangle^2$$
$$(\Delta O)^2_2 = \langle\psi_2|\mathbf{O}^2|\psi_2\rangle - \langle\psi_2|\mathbf{O}|\psi_2\rangle^2$$

Uncertainty and Entanglement

The product of uncertainties can be worked out to be

$$\begin{aligned}(\Delta X)^2_\Psi(\Delta Y)^2_\Psi &= |c_1|^4(\Delta X)^2_1(\Delta Y)^2_1 + |c_2|^4(\Delta X)^2_2(\Delta Y)^2_2 + 2|c_1|^2|c_2|^2(\Delta X)_1(\Delta Y)_1 \\ &(\Delta X)_2(\Delta Y)_2 + |c_1|^4|c_2|^4(\langle\mathbf{Y}\rangle_1 - \langle\mathbf{Y}\rangle_2)^2(\langle\mathbf{X}\rangle_1 - \langle\mathbf{X}\rangle_2)^2 + |c_1|^2|c_2|^2 \\ &\{(\Delta X)_1(\Delta Y)_2 - (\Delta X)_2(\Delta Y)_1\}^2 + |c_1|^2|c_2|^2\{\sum_i |c_i|^2(\Delta X)^2_i \cdot \\ &(\langle\mathbf{Y}\rangle_1 - \langle\mathbf{Y}\rangle_2)^2 + \sum_i |c_i|^2(\Delta Y)^2_i \cdot (\langle\mathbf{X}\rangle_1 - \langle\mathbf{X}\rangle_2)^2\}\end{aligned}$$

Necessary conditions for this to reach its minimum value are

1. $\langle\mathbf{X}\rangle_1 = \langle\mathbf{X}\rangle_2$, $\langle\mathbf{Y}\rangle_1 = \langle\mathbf{Y}\rangle_2$
2. $|\psi\rangle_1$, $|\psi\rangle_2$ be minimum uncertainty states themselves
3. $(\Delta X)_1/(\Delta Y)_1 = (\Delta X)_2/(\Delta Y)_2$.

Outline

Angular Momentum Operators

HUR for $\mathbf{J}_x$ and $\mathbf{J}_y$

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle i\hbar\mathbf{J}_z\rangle|^2$$

State: $|\Psi\rangle = c_1|m_1\rangle|\alpha_1\rangle + c_2|m_2\rangle|\alpha_2\rangle$

The uncertainties for $|m\rangle$ are given by
$(\Delta J_x)^2 = (\Delta J_y)^2 = \frac{\hbar^2}{2}(j(j+1) - m^2)$ minimum for $m = \pm j$

So $|\Psi\rangle$ can be entangled only if $m_1 = +j$ and $m_2 = -j$, or vice-versa.

But, $(\Delta J_x)^2_\Psi \cdot (\Delta J_y)^2_\Psi = \frac{j^2\hbar^2}{4}$ whereas $|\langle \mathbf{J}_z\rangle_\Psi|^2 = j^2\hbar^2(|c_1|^2 - |c_2|^2)^2$.

$$(\Delta J_x)^2(\Delta J_y)^2 > \frac{1}{4}|\langle i\hbar\mathbf{J}_z\rangle|^2$$

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Position-Momentum

The HUR has the form $(\Delta X)^2(\Delta P)^2 \geq \frac{\hbar^2}{4}$

Entangled state made up of two Gaussian states entangled with two orthogonal states of another system.

$$\langle x|\psi_i\rangle = \frac{1}{(2\pi\sigma_i^2)^{1/4}} e^{ip_i x/\hbar} \exp\left(-\frac{(x-x_i)^2}{4\sigma_i^2}\right) \quad (3)$$

$\langle \mathbf{X}\rangle_i = x_i$, $\langle \mathbf{P}\rangle_i = p_i$, $(\Delta X)_i = \sigma_i$, $(\Delta p)_i = \frac{\hbar}{2\sigma_i}$.

The minimum uncertainty conditions (1-3) yield:
$x_1 = x_2$, $p_1 = p_2$ and $\sigma_1 = \sigma_2$.
$\Rightarrow$ the Gaussians are identical $\Rightarrow$ the state is disentangled.

EPR-like state

Two entangled particles A and B:

$$\Psi(x_A, x_B) = C\int_{-\infty}^{\infty} dp\; e^{-ipx_B/\hbar} e^{ipx_A/\hbar} e^{-p^2/4\hbar^2\sigma^2} e^{-\frac{(x_A+x_B)^2}{16\Omega^2}}$$

In the limit $\sigma \to \infty$, $\Omega \to \infty$, the state reduces to the EPR state.

The uncertainties in position and momentum of particle A (say)

$$\Delta X_A = \sqrt{\Omega^2 + 1/16\sigma^2}, \quad \Delta P_A = \hbar\sqrt{\sigma^2 + \frac{1}{16\Omega^2}}.$$

$\Delta X_A \Delta P_A = \hbar/2$ is obtained only if $\Omega = 1/4\sigma$.
In that case, the state becomes

$$\Psi(x_A, x_B) = \frac{1}{\sqrt{\pi/4\sigma^2}} e^{-(x_A^2+x_B^2)2\sigma^2}$$

Disentangled!

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Can the equality in the uncertainty relations be acheived for entangled states?

B.G. Englert's counter-example [3]
Consider the hermitian observables ($\hbar = 1$)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

together with the mixed-state density operator

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For these we have $(\Delta A)^2 = (\Delta B)^2 = \frac{1}{2}\left|\langle \mathrm{i}[A,B]\rangle\right| = \frac{1}{2}$

[3] arXiv:1108.1106 [quant-ph]

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General Analysis

Finite-dimensional Hilbert spaces

Entangled state $|\Psi\rangle$ admits a Schmidt decomposition

$$|\Psi\rangle = \sum_{i=1}^{s} c_i |a_i\rangle_A \, |b_i\rangle_B$$

$|a_i\rangle_A, |b_i\rangle_B \quad \rightarrow$ orthonormal basis vectors in $\mathcal{H}_A, \mathcal{H}_B$ respectively.
$s \leq d_A \quad \rightarrow$ Schmidt rank.
Consider the operators

$$\tilde{\mathbf{X}}_A = \mathbf{X}_A - \langle \mathbf{X}_A \rangle_\Psi, \qquad \tilde{\mathbf{Y}}_A = \mathbf{Y}_A - \langle \mathbf{Y}_A \rangle_\Psi$$

and the states

$$(\mathbf{X}_A - \langle \mathbf{X}_A \rangle_\Psi) \, |\Psi\rangle \qquad (\mathbf{Y}_A - \langle \mathbf{Y}_A \rangle_\Psi) \, |\Psi\rangle$$

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General Analysis

Schwarz inequality of the states $(\mathbf{X}_A - \langle \mathbf{X}_A \rangle_\Psi)\,|\Psi\rangle,\ (\mathbf{Y}_A - \langle \mathbf{Y}_A \rangle_\Psi)\,|\Psi\rangle$ gives

$$\langle \tilde{\mathbf{X}}_A^2 \rangle_\Psi \cdot \langle \tilde{\mathbf{Y}}_A^2 \rangle_\Psi \geq \frac{1}{4} |\langle [\mathbf{X}_A, \mathbf{Y}_A] \rangle_\Psi|^2 + \frac{1}{4} |\langle \{\tilde{\mathbf{X}}_A, \tilde{\mathbf{Y}}_A\} \rangle_\Psi|^2$$

Note: $\langle \tilde{\mathbf{X}}_A^2 \rangle_\Psi \cdot \langle \tilde{\mathbf{Y}}_A^2 \rangle_\Psi = (\Delta X_A)_\Psi^2 \cdot (\Delta Y_A)_\Psi^2$.
This is nothing but Schrödinger-Robertson inequality!

Minimum uncertainty

The equality holds if the two state-vectors $\tilde{\mathbf{X}}_A|\Psi\rangle$ and $\tilde{\mathbf{Y}}_A|\Psi\rangle$ are parallel:

$$(\mathbf{X}_A - \langle \mathbf{X}_A \rangle_\Psi)|\Psi\rangle + \Gamma\,(\mathbf{Y}_A - \langle \mathbf{Y}_A \rangle_\Psi)|\Psi\rangle = 0$$

Γ imaginary $\rightarrow$ HUR equality
Γ complex $\rightarrow$ SR equality

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Minimum uncertainty for the entangled state

$$\sum_{i=1}^{s} c_i \, \{(\mathbf{X}_A - \langle \mathbf{X}_A \rangle_\Psi) + \Gamma \, (\mathbf{Y}_A - \langle \mathbf{Y}_A \rangle_\Psi)\} \, |a_i\rangle_A |b_i\rangle_B = 0$$

This can be satisfied only if

$$\{(\mathbf{X}_A - \langle \mathbf{X}_A \rangle_\Psi) + \Gamma \, (\mathbf{Y}_A - \langle \mathbf{Y}_A \rangle_\Psi)\} \, |a_i\rangle_A = 0$$

for every i. Following conclusions can be drawn from the above:

1. $\langle X_A \rangle_i = \langle X_A \rangle_\Psi, \qquad \langle Y_A \rangle_i = \langle Y_A \rangle_\Psi$
2. $(\Delta X_A)_i^2 (\Delta Y_A)_i^2 = \frac{1}{4} |\langle [X_A, Y_A] \rangle_i|^2$
3. $(\Delta X_A)_i^2 / (\Delta Y_A)_i^2 = -2\Gamma^2$

These constitute a generalization of conditions (1-3) spelt out earlier.

More conclusions

- Operators $(\mathbf{X}_A - \langle \mathbf{X}_A \rangle_\Psi)$ and $(\mathbf{Y}_A - \langle \mathbf{Y}_A \rangle_\Psi)$ are zero in the subspace spanned by $|a_1\rangle,\ |a_2\rangle \ldots\ |a_s\rangle$.
  If $s = d_A$ (maximal Schmidt rank)
  it implies that **all** the states $a_i\rangle$ are simultaneous **degenerate** eigenstates of $\mathbf{X}_A,\ \mathbf{Y}_A$. (Impossible for non-commuting $\mathbf{X}_A,\ \mathbf{Y}_A$)

  For entangled states with maximal Schmidt rank, equality in HUR and SR cannot be reached!

- For $s < d_A$, equality in HUR and SR can be reached for some entangled states.

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Infinite-d Hilbert space: Position-Momentum

Condition for equality in Heisenberg Uncertainty Relation

$$\{(\mathbf{P}_A - \langle \mathbf{P}_A \rangle_\Psi) + \Gamma\,(\mathbf{Q}_A - \langle \mathbf{Q}_A \rangle_\Psi)\}\,|a_i\rangle_A = 0$$

In the position representation:

$$\left\{-i\hbar\frac{d}{dq} + i\Gamma_I q - (\langle \mathbf{P} \rangle_\Psi + i\Gamma_I \langle \mathbf{Q} \rangle_\Psi)\right\}\psi_{a_i}(q) = 0$$

The solution is

$$\psi_{a_i}(q) = Ce^{i\langle P \rangle_\Psi q/\hbar} \exp\left(-\frac{|\Gamma|(q - \langle Q \rangle_\Psi)^2}{2}\right)$$

HUR equality can be acheived only when all the $\psi_{a_i}(q)$ are Gaussian states with same centre and same average momentum. Such a state is **not entangled**

For position and momentum, equality in Heisenberg uncertainty relation can never be reached for any entangled state!

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Summary

- Equality in the uncertainty relations cannot be reached for entangled states with maximal Schmidt rank for any two observables.
- For some entangled states with non-maximal Schmidt rank, the minimum uncertainty equality can be reached.
- For position and momentum, no entangled state can acheive minimum uncertainty equality.
- Entanglement does put some restrictions on the uncertainty relations.

N.D. Hari Dass, Tabish Qureshi, Aditi Sheel
Minimum Uncertainty and Entanglement
arXiv:1107.5929 [quant-ph]

Thank You!