Quantum computation using two component Bose-Einstein condensates

Tim Byrnes[1], Alexey Pyrkov[1],

Kai Wen[2], Yoshihisa Yamamoto[1,2]

1) National Institute of Informatics, Tokyo, Japan

2) E. L. Ginzton Laboratory, Stanford University

National Institute of Informatics

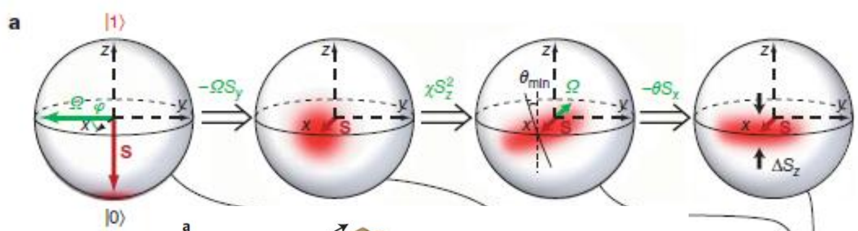
Supported by Navy/SPAWAR Grant N66001-09-1-2024

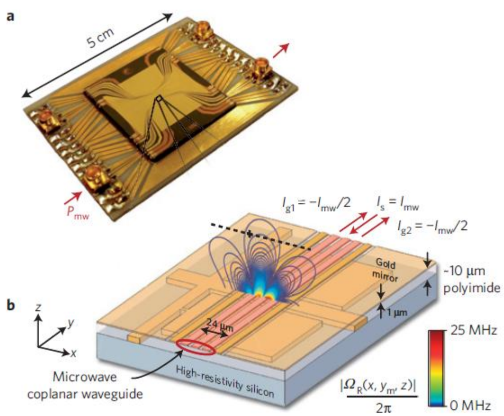
Coherent control of Bose-Einstein condensates (BEC)

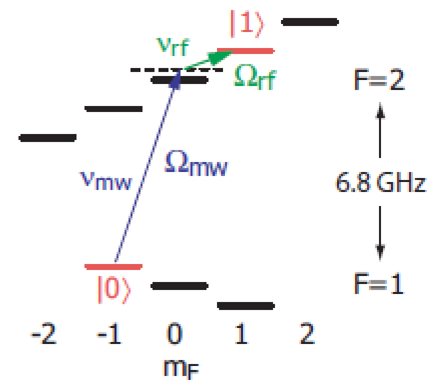
Recently experimental control of atomic BECs have advanced to a level that coherent control between two component BECs are possible.

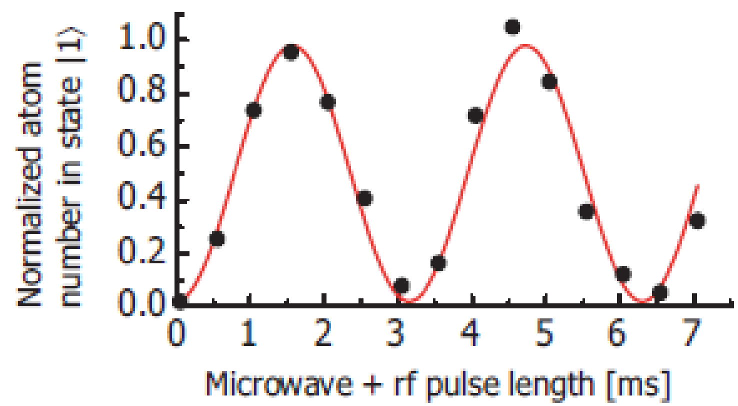
Coherent manipulation of Bose-Einstein condensates with state-dependent microwave potentials on an atom chip

Pascal Böhi[1], Max F. Riedel[1], Johannes Hoffrogge[1], Jakob Reichel[2], Theodor W. Hänsch[1] and Philipp Treutlein[1]*









Bohi et al. Nature Phys. 5, 592 (2009)

Riedel et al. Nature 464, 1170 (2010)

Treutlein et al. Fortschr. Phys. 54, 702 (2006)

Advantage/disadvantages of many boson systems

Advantages

- Usually to want see quantum effects we have to fight against other effects. Bosonic enhancement should make it easier to observe quantum effects

$$a|N\rangle = a\frac{1}{\sqrt{N!}}\left(a^{\dagger}\right)^{N}|0\rangle = \sqrt{N}|N\rangle$$

- Working with macroscopic, rather than microscopic objects (e.g. atoms, quantum dots, Nitrogen vacancies,…)

- Large energy scales usually means short time scales ($t \sim \hbar / E$), i.e. fast gates

Potential disadvantages

- Isn't decoherence enhanced for such macroscopic systems?

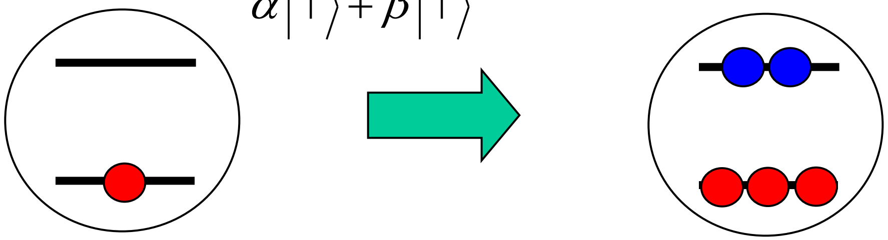
- For large N doesn't the system approach a classical limit? Aren't BECs "classical" in that sense?

- How to experimentally implement in practice?

Basic idea

Each qubit is mapped onto a bosonic qubit

$$\alpha\left|\uparrow\right\rangle + \beta\left|\uparrow\right\rangle$$



$$\left|\alpha,\beta\right\rangle\rangle \equiv \frac{1}{\sqrt{N!}}\left(\alpha a^{\dagger} + \beta b^{\dagger}\right)^{N}\left|0\right\rangle$$

The mapping takes advantage of the properties of Schwinger boson operators

$$S^{z} = a^{\dagger}a - b^{\dagger}b \qquad S^{x} = a^{\dagger}b + b^{\dagger}a \qquad S^{y} = -ia^{\dagger}b + ib^{\dagger}a$$

These obey exactly the same commutation relations as Pauli operators

$$[S^{i}, S^{j}] = 2i\varepsilon_{ijk}S^{k}$$

Same SU(2) algebra is obeyed.

In a group theoretical language, the bosonic spins form a spin N/2 representation, where N is the number of bosons.

Example

Qubit case

Initial state: $|t=0\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ Evolve with $H = \sigma^z$

Final state: $|t\rangle = e^{-i\sigma^z t}|t=0\rangle = \alpha e^{-it}|\uparrow\rangle + \beta e^{it}|\downarrow\rangle$

Bosonic qubit case

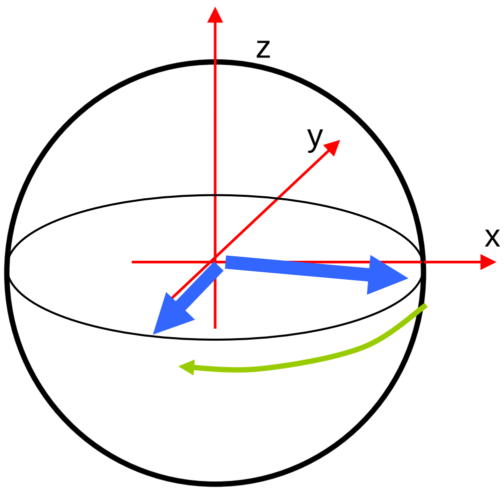
Initial state: $|t=0\rangle = |\alpha,\beta\rangle\rangle$ Evolve with $H = S^z = a^\dagger a - b^\dagger b$

Final state:

$$|t\rangle = e^{-iS^z t}|\alpha,\beta\rangle\rangle = \frac{1}{\sqrt{N!}}\sum_k \binom{N}{k}\left(\alpha a^\dagger e^{-it}\right)^k \left(\beta b^\dagger e^{it}\right)^{N-k}|0\rangle$$

$$= \frac{1}{\sqrt{N!}}\left(\alpha e^{-it}a^\dagger + \beta e^{it}b^\dagger\right)^N|0\rangle = |\alpha e^{-it}, \beta e^{it}\rangle\rangle$$

We can visualize the bosonic qubits as vectors in a Bloch sphere (N times bigger) with exactly the same rotation.



Average, variance, overlap

Expectation values

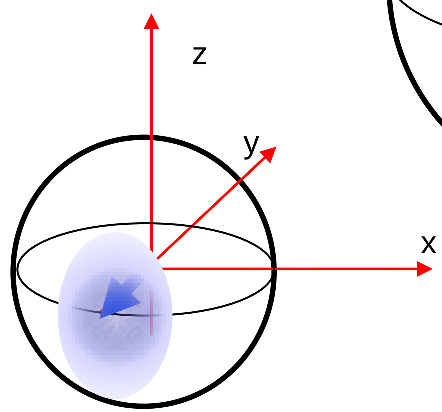
$$\langle S^x \rangle = N(\alpha^* \beta + \alpha \beta^*)$$

$$\langle S^y \rangle = N(-i\alpha^* \beta + i\alpha \beta^*)$$

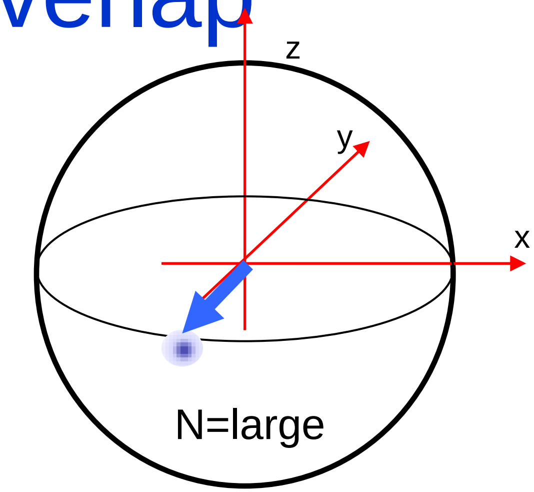
$$\langle S^z \rangle = N(|\alpha|^2 - |\beta|^2).$$

Variance

$$\frac{\langle (S^z)^2 \rangle - \langle S^z \rangle^2}{N^2} = \frac{4|\alpha\beta|^2}{N}$$



N=1



“classical”…. Or is it?

“Large Bloch spin” obeys pseudo-orthogonal properties

$$\alpha = \cos(\theta/2), \beta = \sin(\theta/2)e^{i\phi}$$

$$\langle\langle \alpha', \beta' | \alpha, \beta \rangle\rangle = \cos^N \left(\frac{\theta - \theta'}{2} \right) \approx \exp\left(-\frac{N(\theta - \theta')^2}{8} \right)$$

for $\phi = \phi'$

Two qubit gates

Let us consider the analogue of the maximally entangling operation

$$e^{-i\sigma_1^z\sigma_2^z\frac{\pi}{4}}(|\uparrow\rangle+|\downarrow\rangle)(|\uparrow\rangle+|\downarrow\rangle)=|+y\rangle|\uparrow\rangle+|-y\rangle|\downarrow\rangle$$

For now assume there is an interaction of the form

$$H=S_1^z S_2^z$$

$$e^{-iS_1^z S_2^z t}\left|\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}\right\rangle\!\!\Big\rangle\left|\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}\right\rangle\!\!\Big\rangle=\ldots..=\sum_{k_2}\sqrt{\binom{N}{k_2}}\left|\tfrac{1}{\sqrt{2}}e^{i(N-2k_2)t},\tfrac{1}{\sqrt{2}}e^{-i(N-2k_2)t}\right\rangle\!\!\Big\rangle|k_2\rangle$$

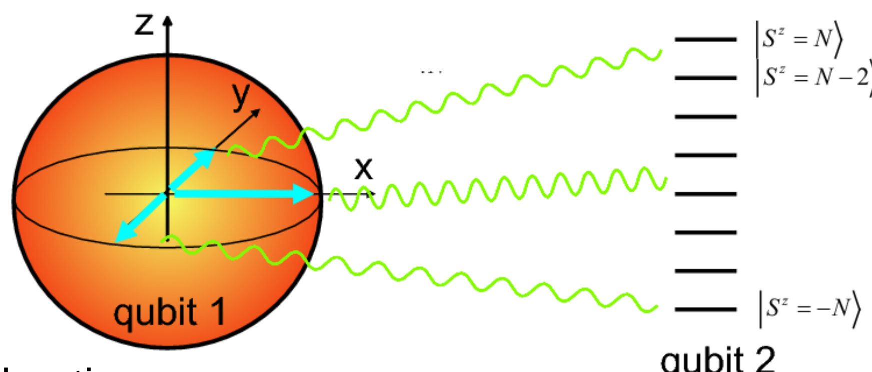
+x eigenstates

+/-y eigenstate

+/-z eigenstate

After a time

$$|k\rangle\equiv\frac{1}{\sqrt{k!(N-k)!}}\left(a^\dagger\right)^k\left(b^\dagger\right)^{N-k}|0\rangle \qquad t=\frac{\pi}{4N}$$



Can get corresponding entangled state with a time shortened by a factor N

Entropy

For bipartite entanglement we have a good measurement of entanglement

$$S(\rho) = -Tr(\rho \log \rho) = -\sum_i \lambda_i \log \lambda_i$$

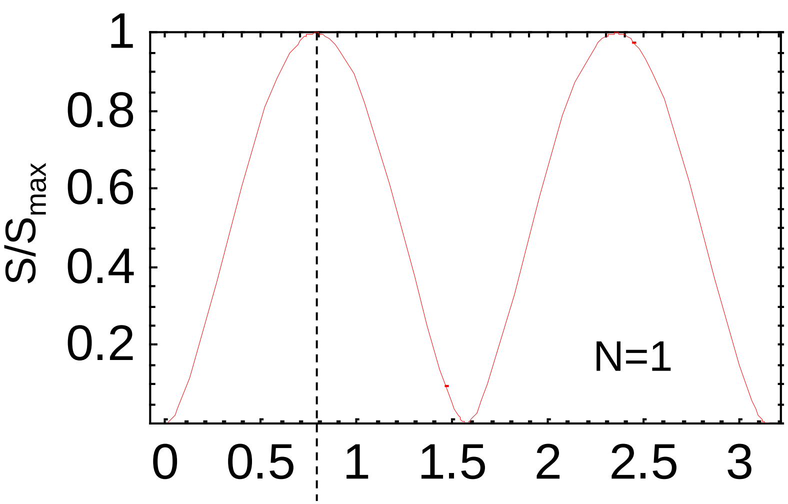
$\lambda_i =$ eigenvalues of density matrix

$$\rho = Tr_2(\rho_{tot})$$

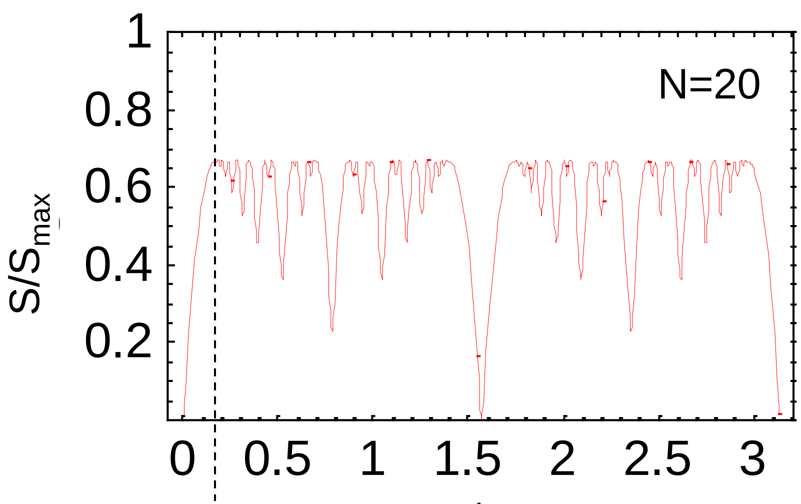
Using the expression on the previous page, after some algebra

$$\rho = \frac{1}{2^N} \sum_{n,m} \sqrt{\binom{N}{n}\binom{N}{m}} \cos^N 2(m-n)t \,|n\rangle\langle m|$$

Normalized to maximally entangled state $S_{\max} = \log(N+1)$

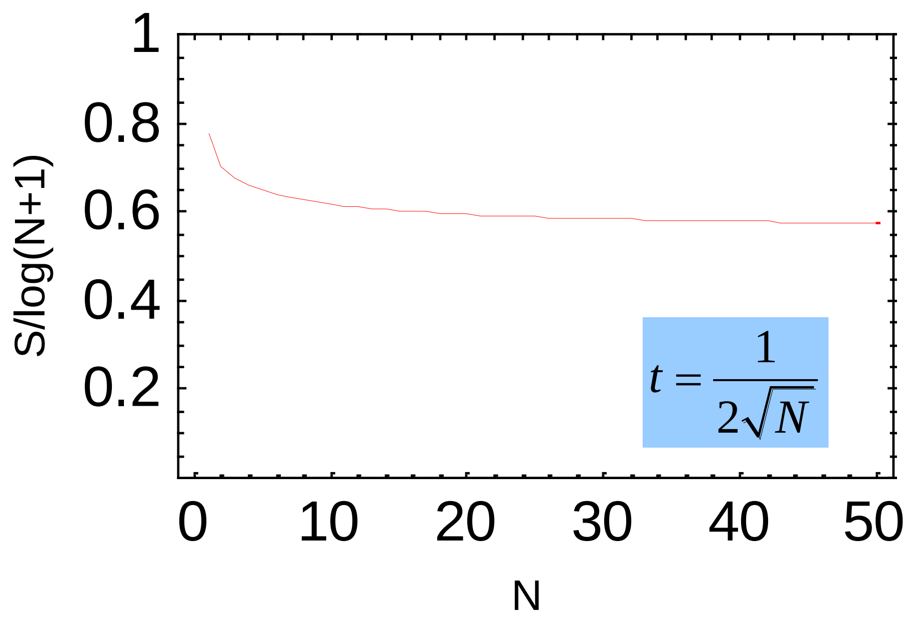


$t = \pi/4$



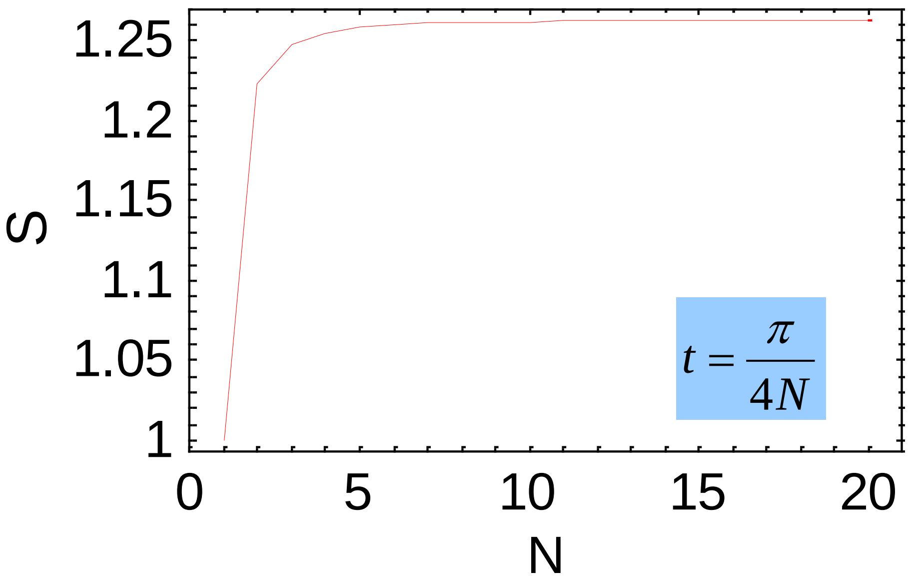
$t = \frac{1}{2\sqrt{N}}$

Particle number dependence



$$S \approx \frac{1}{2}\log(N+1)$$

The total entanglement keeps growing with N



$$S \approx 1.25$$

At times of order 1/N, there is about the same entaglment as for one qubit

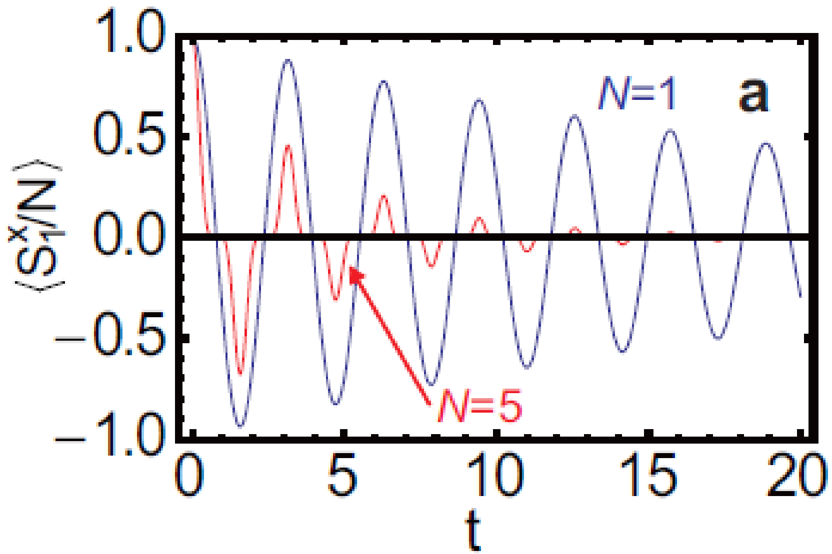
Entanglement doesn't disappear even for large N!

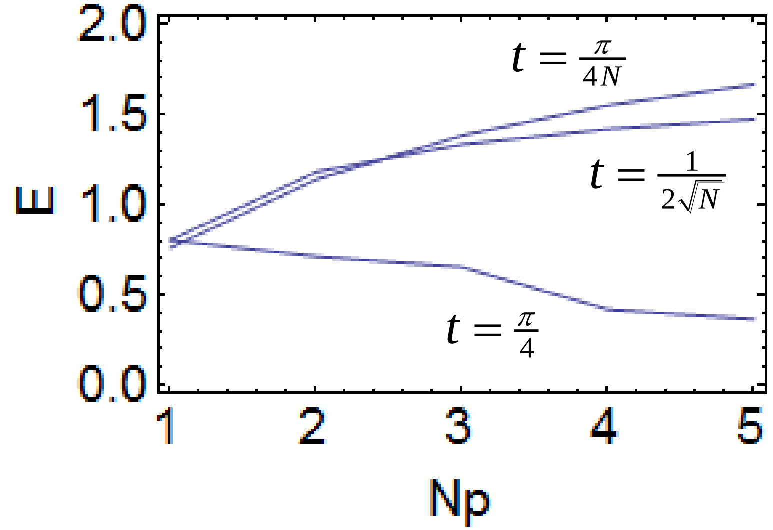
Entanglement with decoherence

$$\frac{d\rho}{dt} = i[\rho, H_{xx}] - \frac{\Gamma_z}{2}\sum_{n=1}^{2}[(S_n^z)^2\rho - 2S_n^z\rho S_n^z + \rho(S_n^z)^2]$$

Initial state

$$|t=0\rangle = |1,0\rangle\rangle|1,0\rangle\rangle \qquad H_{xx} = \hbar\Omega_2 S_1^x S_2^x$$

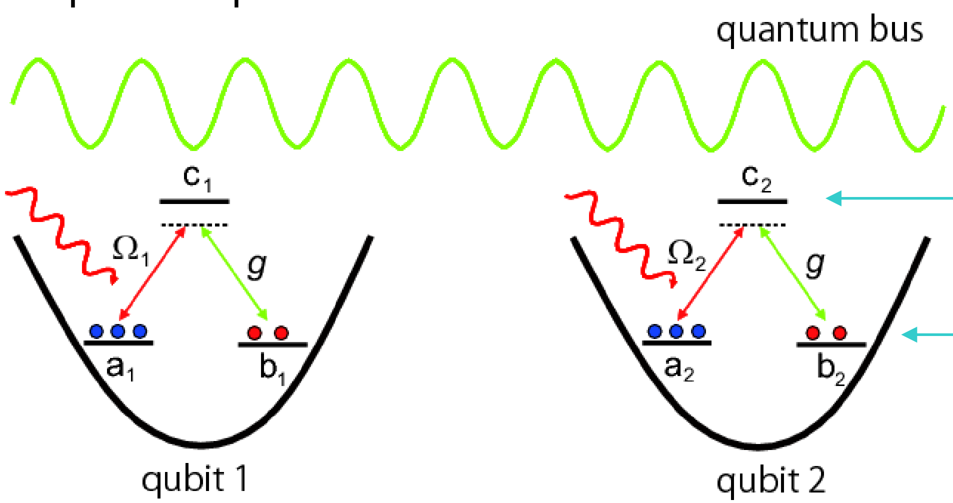




For highly entangled states, fidelity falls off quickly. For short gate times $t = \pi / 4N$ the entanglement increases thanks to reduced gate time

Implementation: optical cavity QED

One possible implementation of a two qubit interaction is cavity QED. BECs strongly coupled to optical cavities have been achieved in Colombe et al., Nature 450, 272 (2007)



intermediate state (c)

$$e.g. \left| F' = 2, m_F' = 0 \right\rangle$$

qubit states (a,b)

$$e.g. \left| F = 1, m_F = -1 \right\rangle, \left| F = 2, m_F = 1 \right\rangle$$

$$H = \frac{1}{2}\omega_0 \sum_{n=1,2} F_n^z + \omega p^\dagger p + g \sum_{n=1,2} \left[F_n^+ p + p^\dagger F_n^- \right]$$

F are boson spin operators involving levels b and c, e.g. $F^z = c^\dagger c - b^\dagger b$

By using standard adiabatic elimination of the cavity photon and intermediate state

$$H_{\text{int}} = \frac{g^2 \Omega^2}{\Delta^3} (S_1^+ + S_2^+)(S_1^- + S_2^-) + H.c.$$

$$S^+ = a^\dagger b$$

Order of the interaction is $\sim O(N^2)$ giving fast two quibt gates

Universality

Arguments relating to universality (any one and two qubit gate can create any unitary) can also be used here

Lloyd Phys. Rev. Lett. 75, 346 (1995)

$$\lim_{n\to\infty}\left(e^{-iB\sqrt{t/n}}e^{-iA\sqrt{t/n}}e^{iB\sqrt{t/n}}e^{iA\sqrt{t/n}}\right)^n = e^{[A,B]t}$$

i.e. given available gates A and B, we can then create C=i[A,B]

The available gates are

$$A = S_i^z, S_i^x \qquad\qquad B = (S_1^+ + S_2^+)(S_1^- + S_2^-) + H.c.$$

This forms a universal set of gates that can produce any state

Decoherence for state storage

Isn't having a large number of particles in the BEC going to enhance decoherence?.

Dephasing

Since we are encoding a single qubit, the only information that we are interested in is

$$\langle S^x \rangle \qquad \langle S^y \rangle \qquad \langle S^z \rangle \qquad \frac{d\rho}{dt} = -\Gamma\left(\left(S_i^z\right)^2 \rho - 2S_i^z \rho S_i^z + \rho\left(S_i^z\right)^2\right)$$

$$\frac{d\langle S^z \rangle}{dt} = 0 \qquad \frac{d\langle S^x \rangle}{dt} = -4\Gamma\langle S^x \rangle$$

$$\langle S^x \rangle \approx const.\exp[-4\Gamma t]$$

NOT enhanced by bosonic factor

Particle loss

$$\frac{d\rho}{dt} = -\Gamma\left(aa^\dagger \rho - 2a^\dagger \rho a + \rho aa^\dagger\right) - \Gamma\left(bb^\dagger \rho - 2b^\dagger \rho b + \rho bb^\dagger\right)$$

For any i=x,y,z

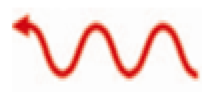
$$\frac{d\langle S^i \rangle}{dt} = -4\Gamma\langle S^i \rangle$$

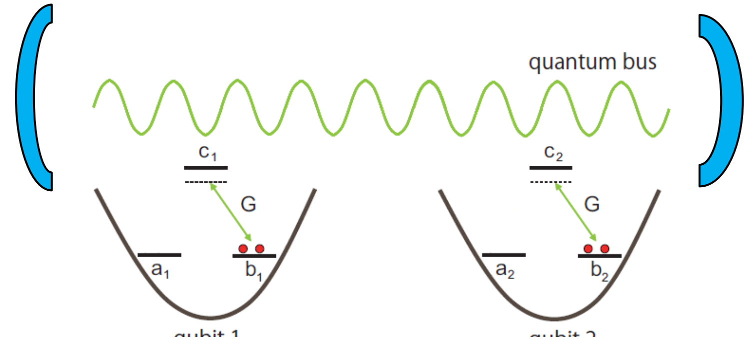
NOT enhanced by bosonic factor. Same result can be generalized to any number of qubits.

Other decoherence effects

Cavity decay

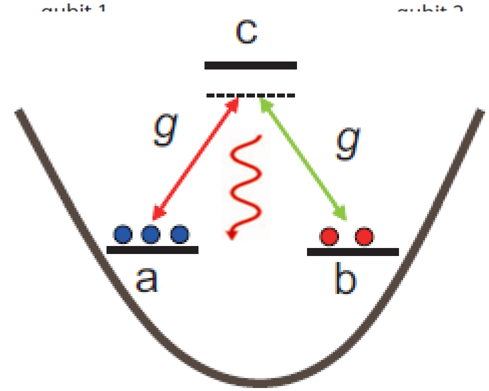
$$\Gamma_c^{\text{eff}} \approx \frac{G^2 \Gamma_c}{\Delta^2}$$





Spontaneous emission

$$\Gamma_s^{\text{eff}} \approx \frac{g^2 \Gamma_s (N+1)}{\Delta^2}$$



Using experimental parameters in in Colombe et al., Nature 450, 272 (2007) give

$$\Gamma_s^{\text{eff}}/h \sim 0.3\text{MHz}$$ (spontaneous emission)

$$\Gamma_c^{\text{eff}}/h \sim 5\text{MHz}$$ (cavity decay)

$$1/t_{CNOT} \approx N\Omega_2^{\text{eff}}/2\pi \sim 200\text{MHz}$$ (inverse 2 qubit gate time)

Quantum algorithms: Grover's alogorithm

Use the "Hamiltonian" version of Grover's algorithm.
For qubits

$$H_G = |X\rangle\langle X| + |ANS\rangle\langle ANS|$$

e.g. $|ANS\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$

$$|X\rangle = \left(|\uparrow\rangle+|\downarrow\rangle\right)\left(|\uparrow\rangle+|\downarrow\rangle\right)\ldots\left(|\uparrow\rangle+|\downarrow\rangle\right)$$

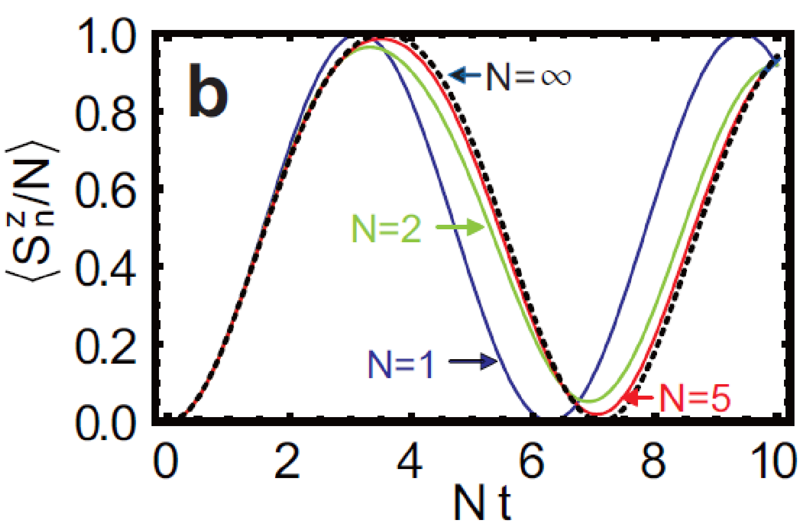
Using the mapping procedure the bosonic version of the Hamiltonian is

$$H_G = N^2 \prod_n \frac{1}{2}\left(1+\frac{S_n^x}{N}\right) + N^2 \prod_n \frac{1}{2}\left(1+\frac{S_n^z}{N}\right) \Rightarrow T_G = \frac{\pi}{\omega} \propto \frac{\sqrt{2^M}}{N}$$

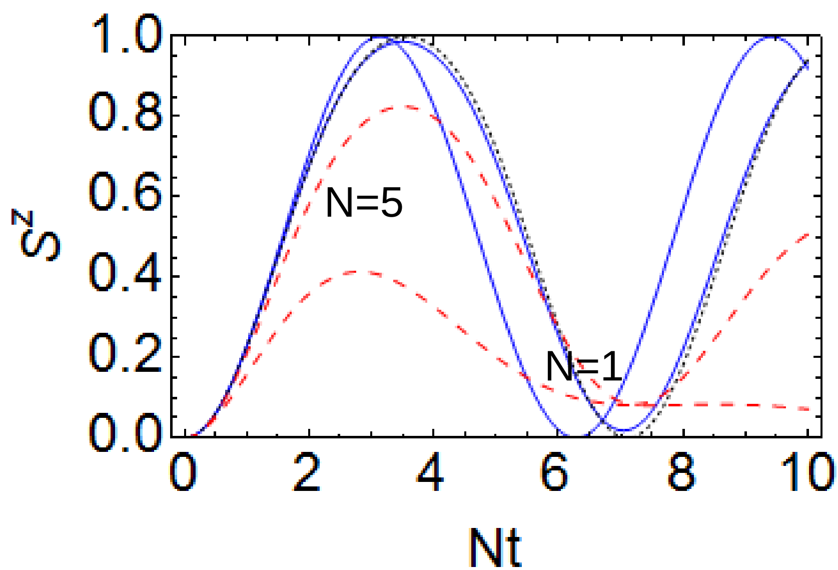
Same quadratic speedup with BEC qubits!

M=2 site

No decoherence



With dephasing (red lines)



Conclusions

- Using bosonic qubits seem to be able to perform quantum computation. The mapping appears to be straightforward, due to the preservation of the spin commutation relations.
- Can obtain factor N speedup of two qubit gate from bosonic enhancement.
- Decoherence for state storage is not necessarily enhanced by N. For highly entangled states, there can be an enhancement.
- Protocols that use non-unitary evolution (e.g. measurement) are a bit more tricky but possible. e.g. Teleportation (See poster of Alexey Pyrkov)

See also Byrnes, Wen, Yamamoto arxiv: 1103.5512

Mapping from qubits to bosonic qubits

Quantum protocols seem to be able to be mapped straightforwardly according to the prescription

1) Find the sequence of Hamiltonians that need to be implemented
2) Make the replacement

$$\sigma_n^\alpha \to N S_n^\alpha \qquad \sigma_n^\alpha \sigma_m^\beta \to S_n^\alpha S_m^\beta$$

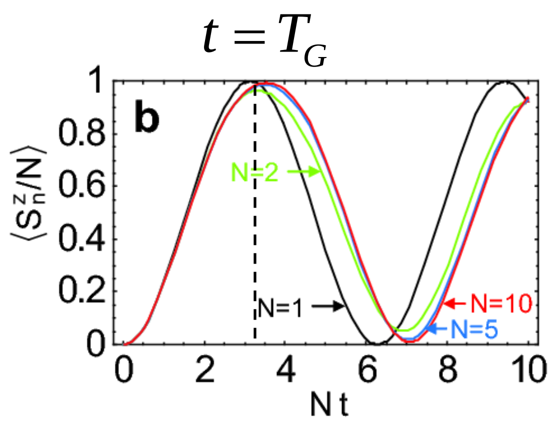
$$\alpha = x, y, z \qquad n = \text{site number}$$

3) Evolve the Hamiltonian for a reduced time of $t \to t / N$

Scaling with number of sites

We can estimate the period of the oscillations by assuming that they follow a generic form

$$\left\langle S_n^z / N \right\rangle \sim \sin^2 \omega t$$



Then

$$\frac{d^2 \left\langle S_n^z / N \right\rangle}{dt^2} \sim 2\omega^2 \cos 2\omega t$$

The second derivative at t=0 is

$$P_\alpha = \prod_n \frac{1}{2}\left(1 + \frac{S_n^\alpha}{N}\right)$$

$$H_G = N^2 \left(P^x + P^z\right)$$

$$\frac{d^2 \left\langle S_n^z / N \right\rangle}{dt^2} = -\left\langle \left[H_G, \left[H_G, S_n^z \right] \right] \right\rangle / N$$

$$= -N^3 \left\langle S_n^z P_x^2 + S_n^z P_x P_z - P_x S_n^z P_z - 2P_x S_n^z P_x - P_z S_n^z P_x + P_z P_x S_n^z + P_x^2 S_n^z \right\rangle$$

$$= \frac{2N^2}{2^M}$$

$$\Rightarrow \quad T_G = \frac{\pi}{\omega} \propto \frac{\sqrt{2^M}}{N}$$

Grover scaling with additional N fold bosonic speedup

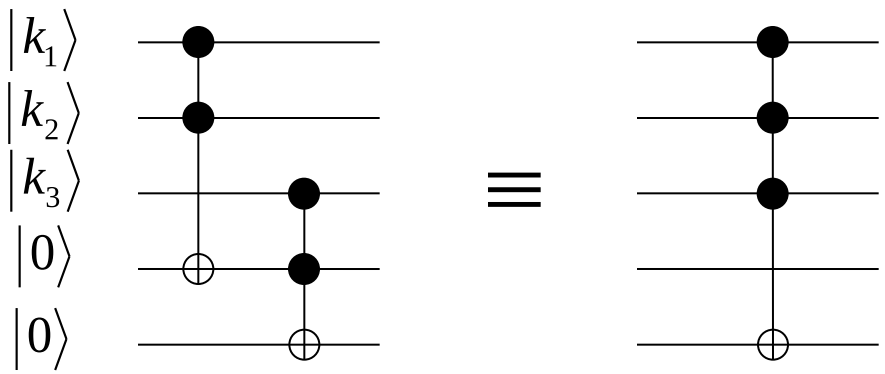
How to implement Grover Hamiltonian?

$$H_G = N^2 \prod_n \frac{1}{2}\left(1+\frac{S_n^x}{N}\right) + N^2 \prod_n \frac{1}{2}\left(1+\frac{S_n^z}{N}\right)$$

Breaking the Grover Hamiltonian into gates requires an exponential number of terms, so we cannot literally perform this.

We can adapt the gate decomposition methods to the bosonic qubit case

Example

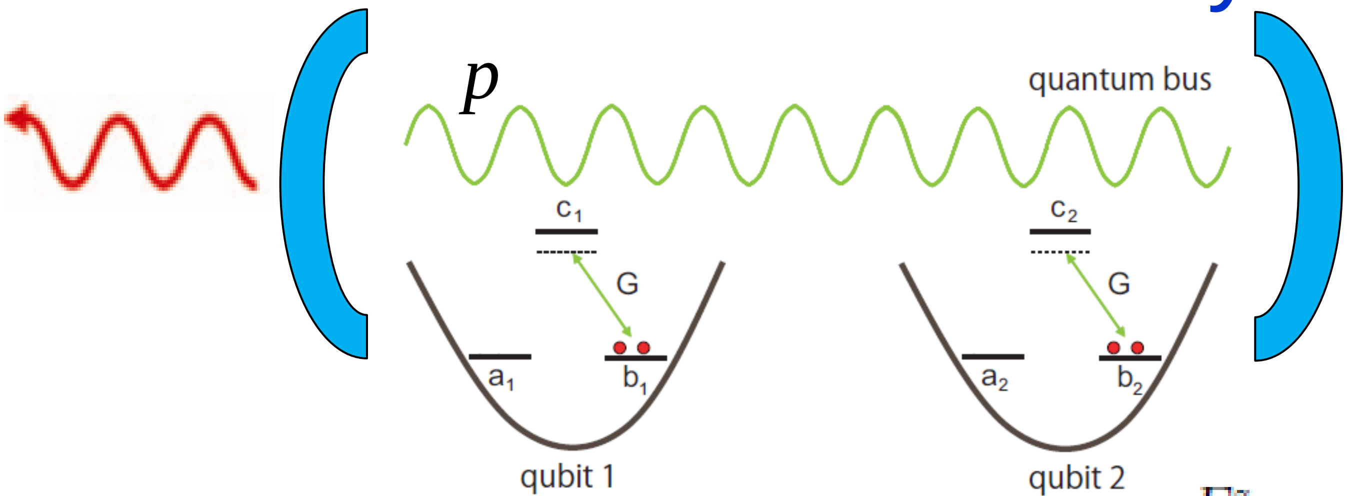


(sketch)

$$|k_1\rangle|k_2\rangle|k_3\rangle|1,0\rangle\rangle|1,0\rangle\rangle \to |k_1\rangle|k_2\rangle|k_3\rangle|\cos k_1 k_2 t, i \sin k_1 k_2 t\rangle\rangle|1,0\rangle\rangle$$

$$\to |k_1\rangle|k_2\rangle|k_3\rangle\left|k_4 = \frac{k_1 k_2}{N}\right\rangle\left|\cos\frac{k_1 k_2 k_3 t}{N}, i \sin\frac{k_1 k_2 k_3 t}{N}\right\rangle\rangle \qquad k_i = 1+\frac{S_i}{N}$$

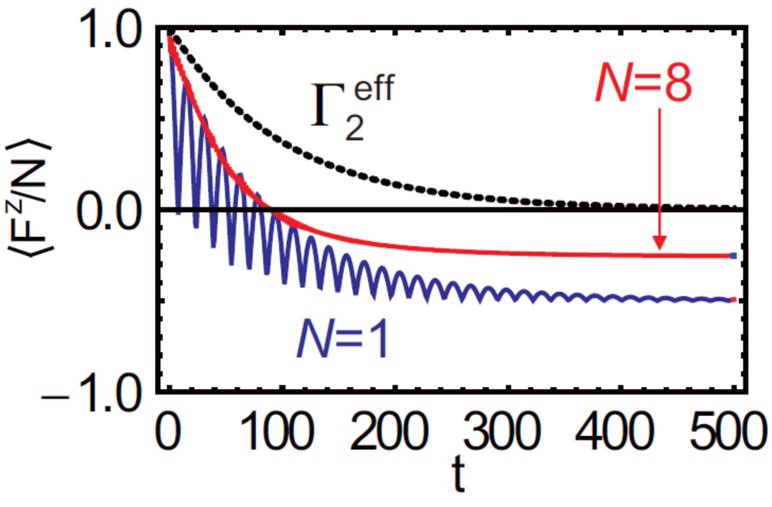
Decoherence from cavity decay



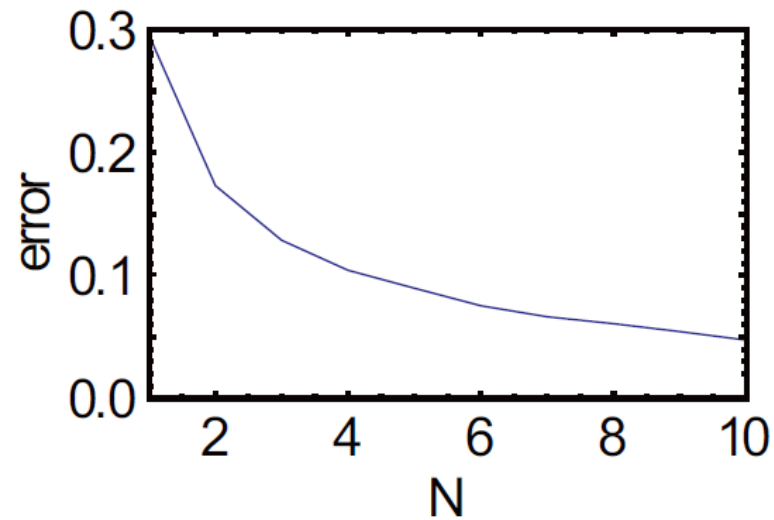
$$F^z = c^\dagger c - b^\dagger b,\ F^+ = c^\dagger b$$

$$H_2 = \frac{\hbar\omega_0}{2}\sum_{n=1,2} F_n^z + \hbar\omega p^\dagger p + G\sum_{n=1,2}\left[F_n^- p^\dagger + F_n^+ p\right]$$

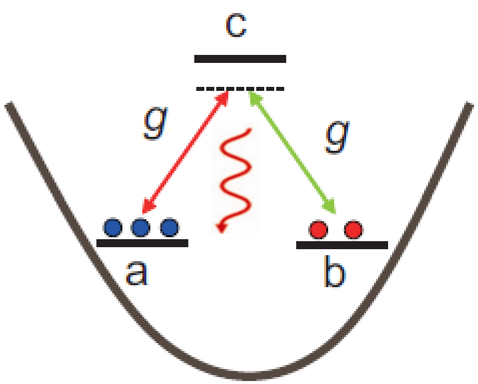
$$\frac{d\rho}{dt} = i[\rho, H_2] - \frac{\Gamma_c}{2}\left[p^\dagger p\rho - 2p^\dagger\rho p + \rho p^\dagger p\right]$$



$$\Gamma_c^{\mathrm{eff}} \approx \frac{G^2\Gamma_c}{\Delta^2}$$

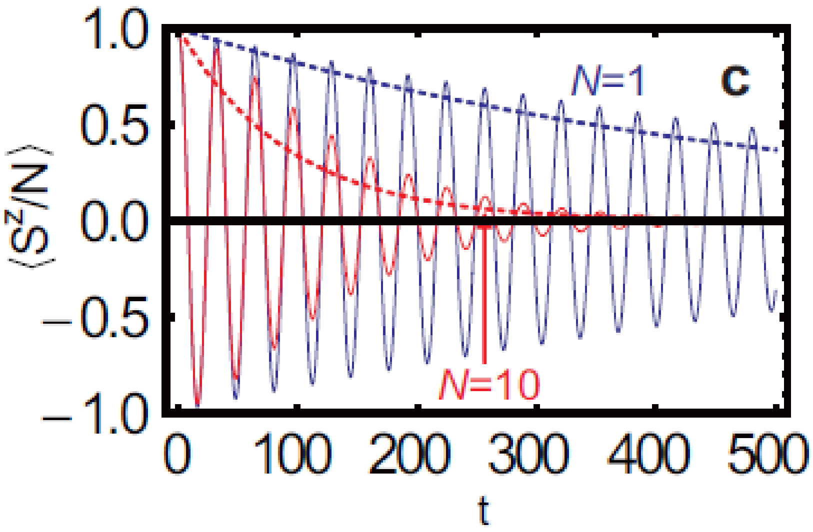


Decoherence from spontaneous emission



$$H_1 = \Delta c^\dagger c + g(a^\dagger c + c^\dagger a) + g(b^\dagger c + c^\dagger b)$$

$$\frac{d\rho}{dt} = i[\rho, H_1] - \frac{\Gamma_s}{2}\left[c^\dagger a a^\dagger c\rho - 2a^\dagger c\rho c^\dagger a + \rho c^\dagger a a^\dagger c\right] - \frac{\Gamma_s}{2}\left[c^\dagger b b^\dagger c\rho - 2b^\dagger c\rho c^\dagger b + \rho c^\dagger b b^\dagger c\right]$$



$$\Gamma_s^{\mathrm{eff}} \approx \frac{g^2 \Gamma_s (N+1)}{\Delta^2}$$

Using excited states scales rather badly with N

What about higher order spins operators?

Consider the Schrodinger cat state $$\left|cat\right\rangle = \frac{1}{\sqrt{2}}\left(\left|S^z = -N\right\rangle + \left|S^z = N\right\rangle\right)$$

Substituting into the dephasing master equation we find

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{-2\Gamma N^2} \\ e^{-2\Gamma N^2} & 1 \end{pmatrix}$$

In this case off diagonal elements do degrade very quickly.

We can expect high order operators (e.g $\left\langle (S^{x})^{M} \right\rangle$) to degrade quickly. However, if the task is to perform the qubit mapping, these are unnecessary!