Separability of two qubit states

and Optimal state preparations

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Outline

Basic Notions: Separable and entangled states

- A bipartite state ρ^{AB} is separable if

$$\rho^{AB} = \mu_1 \rho_1^A \otimes \rho_1^B + \mu_2 \rho_2^A \otimes \rho_2^B + \cdots$$

where

$$\mu_1, \mu_2 \ldots \geq 0; \quad \mu_1 + \mu_2 + \cdots = 1$$

- States that are not separable are entangled; they exhibit nonlocality; violate Bell inequality.
- The most entangled states: Bell states or EPRB states. Example: $|01\rangle - |10\rangle$.
- Separable states: $|xy\rangle \equiv |x\rangle \otimes |y\rangle$; **1**

Outline

- A very well studied topic: Completely Positive Maps
  Our Aim: More modest and simpler

Criterion for separability

- Partial Transpose: Transpose taken on only one qubit
  $\rho_{x_1\mathbf{y_1};\mathbf{x_2}\mathbf{y_2}} \rightarrow \rho_{x_1\mathbf{y_2};\mathbf{x_2}\mathbf{y_1}}$
- Basic Observation:
  Partial Transpose of every separable state is also a valid state (Peres, 1996)

$$\rho^{T_2} = \mu_1\rho_1^A \otimes (\rho_1^B)^T + \mu_2\rho_2^A \otimes (\rho_2^B)^T + \cdots$$

- Positive Partial Transpose(PPT) is thus a necessary condition for separability
- **Note** : sufficient for $\frac{1}{2} \otimes \frac{1}{2}$ and $\frac{1}{2} \otimes 1$ systems. (Horedecki, 1996)

Measure of Separability: Negativity

- Negativity N: Perform a partial transpose operation: $\rho \to \rho^{T_2}$. Determine the eigenvalues If an eigenvalue is negative, multiply its magnitude by 2. That yields us negativity.
- $N = 0$ Otherwise.
- $N = 0$ for separable states; $N_{max} = 1$ for Bell states
- *Question*: Why flog a dead horse?

Some Interesting Questions

Examples:

- Find the set of all transformations that connect states with a given negativity
- Find all inequivalent separability expansions for a separable state
- Evolve optimal criteria for preparing a state with a given negativity

We address these questions here.
Separability expansions result acts as a tool to settle optimal criterion for preparation of states
Results restricted to two qubit states (2QS)

Outline

- Illustrative.
  Proof will only be outlined
  Special cases for illustration
  Hints for general arguments

  Start with purely tensor polarized states
  - invariably correlated
    - Classical versus Quantum
  - Has pure states as subsets

The setting: Tensor Polarized States (TPS)

Canonical form of TPS

- Recall: the most general state has the form

$$\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+\boldsymbol{a}.\boldsymbol{\sigma}\otimes\mathbf{1}+\mathbf{1}\otimes\boldsymbol{\sigma}.\boldsymbol{b}+X_{ij}\sigma_i\otimes\sigma_j\}$$

- Tensor polarized states (TPS):

$$\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+X_{ij}\sigma_i\otimes\sigma_j\}$$

- Local transformations $SU(2)\otimes SU(2)$ are the simplest gauge transformations vis-a-vis nonlocality.
  Canonical form of TPS:

$$\rho[\alpha,\beta,\gamma]=\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+\alpha\sigma_x\otimes\sigma_x+\beta\sigma_y\otimes\sigma_y+\gamma\sigma_z\otimes\sigma_z\}$$

Separable TPS: Examples

- Start with the state $\rho^{AB}[\alpha]$. It is trivially separable. For example:

$$\rho^{AB}[\alpha,0,0] = \frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+\alpha\sigma_x\otimes\sigma_x\} = \frac{1}{2}\rho_+^A\otimes\rho_+^B + \frac{1}{2}\rho_-^A\otimes\rho_-^B$$

  where

$$\rho_\pm^A = \frac{1}{2}\{1\pm\sqrt{\alpha}\sigma_x\}; \quad \rho_\pm^B = \frac{1}{2}\{1\pm\sqrt{\alpha}\sigma_x\}$$

  This expansion is not unique. In fact, there are many ways to distribute α between subsystems A and B.

- Analogously expand $\rho[0,\beta,0]$ and $\rho[0,0,\gamma]$

Tool: Convexity of separable states

- Convexity: ρ_1 and ρ_2 are separable $\Rightarrow \lambda\rho_1 + (1-\lambda)\rho_2$ is also separable for $0 \leq \lambda \leq 1$.
- Hence, we have the construction for the states in the convex hull of the 6 points $\{\pm 1, 0, 0\}$, $\{0, \pm 1, 0\}$, $\{0, 0, \pm 1\}$.

Convexity for expansion of separable states

Depiction

- For state $\rho[\alpha,\beta,0]$;

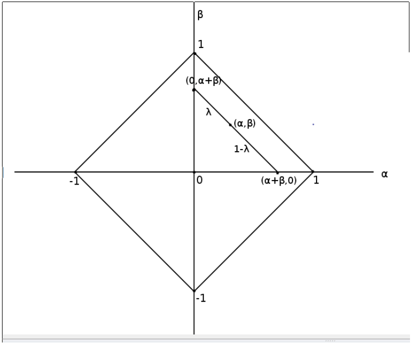

Figure: Using Convexity to construct an arbitrary state $\{\alpha,\beta\}$:
$\rho[\alpha,\beta] = \lambda\rho[\alpha+\beta,0] + (1-\lambda)\rho[0,\alpha+\beta]$

The constructed separable region

- The full constructed separable region in the parameter space $\alpha \times \beta \times \gamma$:

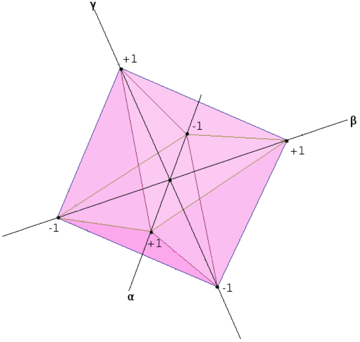

Figure: The octahedron: The convex hull of those 6 points

Comparison

Tensor polarized states: The separable region

- Compare with the region obtained from PPT criterion
- The constructed region coincides with the PPT determined region

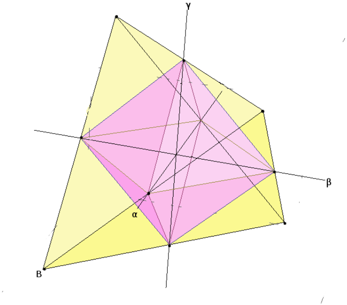

Figure: Yellow: Entangled; Pink: Separable

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1. Preliminaries
   - What do we know of separability?
2. Separability Construction
   - Strategy: Special to General
   - A few questions
3. Application: Optimal State Preparation
   - Separable states
   - Arbitrary States

- Question: Is this an accident?
- Can we evolve a general procedure, at least for 2QS?
- Can we use it for higher systems?
- At least narrow down the region of entangled states consistent with PPT criterion?

- Extension to two qubit states is possible.
- We need a few notions

The first notion: Uniform Separability

- Definition: A state is *Uniformly Separable* if it has a resolution of the form

$$\rho^{AB} = \mu_1 \rho_1^A \otimes \rho_1^B + \mu_2 \rho_2^A \otimes \rho_2^B + \cdots + \mu_n \rho_n^A \otimes \rho_n^B$$

  with $\mu_1 = \mu_2 = \cdots = \mu_n = \frac{1}{n}$; n is the number of terms in the expansion.
- $\mu_1, \mu_2, \cdots$ are rational $\implies$ the state is uniformly separable
- By density of rationals, every state can be closely approximated by uniformly separable states.
- **Redundancy is of no concern**. Will be useful later

The formulation

- Find a uniform separation of an arbitrary separable state:

$$\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+\boldsymbol{u}.\boldsymbol{\sigma}\otimes\mathbf{1}+\mathbf{1}\otimes\boldsymbol{\sigma}.\boldsymbol{v}+X_{ij}\sigma_i\otimes\sigma_j\}$$

$$=\frac{1}{n}\sum_{\nu=1}^{n}\frac{1}{2}\{\mathbf{1}+\boldsymbol{p}_\nu.\boldsymbol{\sigma}\}\otimes\frac{1}{2}\{\mathbf{1}+\boldsymbol{q}_\nu.\boldsymbol{\sigma}\}$$

- Given $\boldsymbol{u},\boldsymbol{v}$ and X, we are to find vectors $\boldsymbol{p}_\nu$ and $\boldsymbol{q}_\nu$

The matrices P, Q

- Arrange $\boldsymbol{p}_\nu$ and $\boldsymbol{q}_\nu$ as column vectors of $3 \times n$ real matrices P and Q.

$$P = \begin{pmatrix} p_1^x & p_2^x & p_3^x & \cdots & p_n^x \\ p_1^y & p_2^y & p_3^y & \cdots & p_n^y \\ p_1^z & p_2^z & p_3^z & \cdots & p_n^z \end{pmatrix}; \quad Q = \begin{pmatrix} q_1^x & q_2^x & q_3^x & \cdots & q_n^x \\ q_1^y & q_2^y & q_3^y & \cdots & q_n^y \\ q_1^z & q_2^z & q_3^z & \cdots & q_n^z \end{pmatrix}$$

The Conditions

- Conditions to be satisfied by P and Q:

$$PQ^T = nX; \quad P\boldsymbol{a}_n = n\boldsymbol{u}; \quad Q\boldsymbol{a}_n = n\boldsymbol{v}; \quad \boldsymbol{a}_n = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

  $\boldsymbol{a}_n$ is the n dimensional vector with all entries equal to 1.
- Recall: X is the tensor polarization; $\boldsymbol{u}$, v are the polarizations of qubits A, B.
- We will demonstrate the method to solve the above equations through a chain of examples

Illustration

The simplest nontrivial example

- Consider the special state

$$\rho[\alpha,\beta,u] = \frac{1}{4}\{\mathbf{1}\otimes\mathbf{1} + u\sigma_z\otimes\mathbf{1} + \alpha\sigma_x\otimes\sigma_x + \beta\sigma_y\otimes\sigma_y\}$$

- Conditions on P, Q:

$$PQ^T = n\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P\boldsymbol{a}_n = n\begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}; \quad Q\boldsymbol{a}_n = 0$$

- Rank and nullity requirements on P and Q $\implies n \geq 3$. Choose n=4 (works for the most general case as well).

Uniqueness of P, Q

We need two notions to settle this question.

- 1. *Left Elementary Transformations*,
  2. *Right Orthogonal Transformations*
- Left E, F be nonsingular 3×3 matrices. Let us perform the elementary transformations

$$P' = EP; \quad Q' = FQ$$

Then the conditions on P, Q change to

$$P' Q'^{T} = n\, E \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} F^{T}$$

$$P' \boldsymbol{a}_n = n\, E \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}; \quad Q' \boldsymbol{a}_n = 0$$

- Generates a class of separable states

Left Elementary transformations: contd

- E, F have a dual role to play
  (i) Generating a family of separable states from a given separable state:
  $P'Q'^T \neq PQ^T$; or E does not leave $\boldsymbol{u}$ invariant.
  (ii) Generate inequivalent separability expansions for a given state:
  $P'Q'^T = PQ^T$; and E stabilizes $\boldsymbol{u}$.
  This set of inequivalent resoluutions is not exhaustive
- *An Observation*: If E, F are chosen to be orthogonal, they reduce to local transformation (trivial gauge transformations). Not of Interest.
- Notice that E and F are much more general than orthogonal transformations. The only invariant is the rank of X.

A simple Example for generating inequivalent expansions: scaling

- Consider

$$n\begin{pmatrix}\gamma & 0 & 0\\ 0 & \gamma' & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\gamma^{-1} & 0 & 0\\ 0 & \gamma'^{-1} & 0\\ 0 & 0 & 1\end{pmatrix}$$

Inequivalent Expansions: Right Orthogonal Transformations

- Consider the set of transformations

$$P' = PO; \quad Q' = QO$$

- O are $\mathrm{n} \times \mathrm{n}$ orthogonal matrices which stabilize a_n: $Oa_n = a_n$.
- They have further stabilizing properties. They leave the following invariant:

$$P'Q'^T = PQ^T$$

$$P'a_n = Pa_n; \quad Q'a_n = Qa_n$$

- Consequence: The group generates a family of inequivalent (almost always) expansions for a given separable state.

Explicit solution: Inspection and Construction

Recall that we started with the state

$$\rho[\alpha,\beta,u] = \frac{1}{4}\{\mathbf{1}\otimes\mathbf{1} + u\sigma_z\otimes\mathbf{1} + \alpha\sigma_x\otimes\sigma_x + \beta\sigma_y\otimes\sigma_y\}$$

- We attempt an expansion with four terms. Write $a_4 = 2(\boldsymbol{a}\otimes\boldsymbol{a})^T$. We employ the eotation

$$\boldsymbol{a} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}; \quad \boldsymbol{b} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}$$

- Simple algebra gives a solution for P, Q:

$$P = 2\begin{pmatrix} \sqrt{\alpha}\boldsymbol{a}\otimes\boldsymbol{b} \\ \sqrt{\beta}\boldsymbol{b}\otimes\boldsymbol{a} \\ u\boldsymbol{a}\otimes\boldsymbol{a} \end{pmatrix}; \quad Q = 2\begin{pmatrix} \sqrt{\alpha}\boldsymbol{a}\otimes\boldsymbol{b} \\ \sqrt{\beta}\boldsymbol{b}\otimes\boldsymbol{a} \\ 0 \end{pmatrix}$$

- α and β can be redistributed between P and Q. That is a left elementary transformation. This has an important role.

Soln contd: A further constraint

- The solutions obtained above must represent valid states in rhs.

$$|\boldsymbol{p}_\nu| \leq 1; \; |\boldsymbol{q}_\nu| \leq 1$$

- This imposes additional constraints: We choose the Left elementary transformations to maximize the range.
- Example: Suppose that $\alpha, \; \beta \geq 0$. The optimal choice is

$$P = 2\begin{pmatrix} \sqrt{\alpha(\alpha+\beta)}\boldsymbol{a}\otimes\boldsymbol{b} \\ \sqrt{\beta(\alpha+\beta)}\boldsymbol{b}\otimes\boldsymbol{a} \\ u\boldsymbol{a}\otimes\boldsymbol{a} \end{pmatrix}; \quad Q = 2\begin{pmatrix} \sqrt{\frac{\alpha}{\alpha+\beta}}\boldsymbol{a}\otimes\boldsymbol{b} \\ \sqrt{\frac{\beta}{\alpha+\beta}}\boldsymbol{b}\otimes\boldsymbol{a} \\ 0 \end{pmatrix}$$

Similar expressions (involving sign changes) cover the full separability range range $u^2 + (|\alpha| + |\beta|)^2 \leq 1$

Class of solutions: Invariants

- The right orthogonal transformation O provides us a class of solutions $\{PO, QO|\ O \in O(4),\ O\boldsymbol{a} \otimes \boldsymbol{a} = \boldsymbol{a} \otimes \boldsymbol{a}\}$ from a single solution P, Q.
- The invariants are $\sum_{\nu=1}^{n} |\boldsymbol{p}_\nu|^2$, $\sum_{\nu=1}^{n} |\boldsymbol{q}_\nu|^2$, and $\sum(\boldsymbol{p}_\nu.\boldsymbol{p}_\mu)^2$, $\sum(\boldsymbol{q}_\nu.\boldsymbol{q}_\mu)^2$.

Generalization to a larger class

- We have constructed a resolution for the class

$$\rho[\alpha,\beta,u] = \frac{1}{4}\{\mathbf{1}\otimes\mathbf{1} + u\sigma_z\otimes\mathbf{1} + \alpha\sigma_x\otimes\sigma_x + \beta\sigma_y\otimes\sigma_y\}$$

- We can use convexity to extend it further

$$\rho[\alpha,\beta,u,\gamma] \equiv \frac{1}{4}\{\mathbf{1}\otimes\mathbf{1} + u\sigma_z\otimes\mathbf{1} + \alpha\sigma_x\otimes\sigma_x + \beta\sigma_y\otimes\sigma_y + \gamma\sigma_z\otimes\sigma_z\}$$

$$\begin{aligned}\rho[\alpha,\beta,u,\gamma] &= |\gamma|\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1} \pm \sigma_z\otimes\sigma_z\} \\ &+ (1-|\gamma|)\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1} + \frac{u}{1-|\gamma|}\sigma_z\otimes\mathbf{1} + \frac{\alpha}{1-|\gamma|}\sigma_x\otimes\sigma_x \\ &+ \frac{\beta}{1-|\gamma|}\sigma_y\otimes\sigma_y\}\end{aligned}$$

The next step: Further convexity

- We possess resolution for states of the form
$$\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+u\sigma_z\otimes\mathbf{1}+\alpha\sigma_x\otimes\sigma_x+\beta\sigma_y\otimes\sigma_y+\gamma\sigma_z\otimes\sigma_z\}$$
and
$$\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+v\mathbf{1}\otimes\sigma_z+\alpha\sigma_x\otimes\sigma_x+\beta\sigma_y\otimes\sigma_y+\gamma\sigma_z\otimes\sigma_z\}$$
- Label the convex combined state $\rho[u,v,\alpha,\beta,\gamma]$.

$$\begin{aligned}
\rho[u,v,\alpha,\beta,\gamma] &= \frac{|u|}{|u|+|v|}\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}\pm(|u|+|v|)\sigma_z\otimes\mathbf{1} \\
&+ \alpha\sigma_x\otimes\sigma_x+\beta\sigma_y\otimes\sigma_y+\gamma\sigma_z\otimes\sigma_z\} \\
&+ \frac{|v|}{|u|+|v|}\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}\pm(|u|+|v|)\mathbf{1}\otimes\sigma_z \\
&+ \alpha\sigma_x\otimes\sigma_x+\beta\sigma_y\otimes\sigma_y+\gamma\sigma_z\otimes\sigma_z\}
\end{aligned}$$

The final step: Independent Local Transformations (ILT)

We now introduce the third important notion:

- Consider any separable state:

$$\rho^{AB} = \mu_1 \rho_1^A \otimes \rho_1^B + \mu_2 \rho_2^A \otimes \rho_2^B + \cdots$$

- **Key Idea**: Independent local $SU(2) \times SU(2)$ transformations on the constituent subsystems generate new separable states:

$$\mu_1 (U_1 \otimes V_1)^\dagger \rho_1^A \otimes \rho_1^B (U_1 \otimes V_1) + \mu_2 (U_2 \otimes V_2)^\dagger \rho_2^A \otimes \rho_2^B (U_2 \otimes V_2) + \cdots$$

$$U_i, V_i \in SU(2)$$

- **Key result** : All separable states are connected to each other by ILT.

ILT
Final Generalization

- Consider the convex combination of states

$$\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+u\sigma_z\otimes\mathbf{1}+\alpha\sigma_x\otimes\sigma_x+\beta\sigma_y\otimes\sigma_y+\gamma\sigma_z\otimes\sigma_z\}$$

and

$$\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+v\mathbf{1}\otimes\sigma_z+\alpha\sigma_x\otimes\sigma_x+\beta\sigma_y\otimes\sigma_y+\gamma\sigma_z\otimes\sigma_z\}$$

.

- **Observation**: Operation of ILT on the convex combinations generates almost all the states.
- **Query: What are the exceptional states?**
- *Answer*: The family of pure separable states

Outline

Entropy of preparation

For separable states

- For a separable state, consider the quantity:

$$S_{res} = \mu_1 S[\rho_1^A \otimes \rho_1^B] + \mu_2 S[\rho_2^A \otimes \rho_2^B] + \cdots$$

  where S is the shannon entropy.

- S_{res} depends on the resolution, but is bounded by $S[\rho]$. Hence we can maximize it over all possible resolutions:

$$S_p[\rho] = \max_{res} S_{res}$$

- *An aside*: Equivalently, we use linear entropy $(= 1 - Tr(\rho^2))$ in place of shannon entropy.

Example: Isotropic states

- States $\frac{1}{4}\{\mathbf{1}\otimes\mathbf{1}+\alpha\boldsymbol{\sigma}^A.\boldsymbol{\sigma}^B\}$are isotropic.
- Range of α:

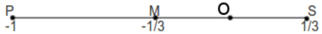

Figure: Isotropic states: MS: Separable; PM: Entangled; B: bell state; O: unpolarized state

Entropy of preparation
Isotropic states

- Comparing S and S_p for isotropic separable states:

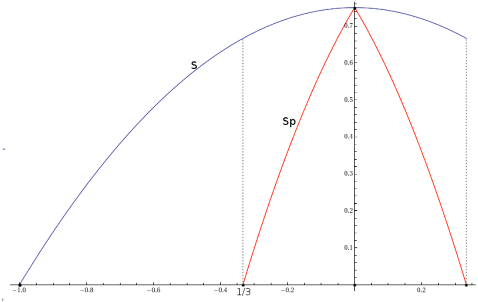

Figure: S and S_p for separable range.

- S_p drops to zero just at the boundary of separable and entangled states.

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Entopy of preparation
For Arbitrary states

- Arbitrary state preparation:

$$\rho = \lambda \Pi_1 + (1-\lambda)\rho_s$$

where, Π_1 is pure and entangled; ρ_s is separable.

- Π_1: Zero entropy; ρ_s: Zero entanglement
- Extending the definition of entropy of preparation

$$S_p^{\lambda}[\rho] = (1-\lambda)S_p[\rho_s]$$

Optimal Preparation
Isotropic States

- The decomposition $\rho = \lambda \Pi_1 + (1-\lambda)\rho_s$ is not unique; consider the isotropic states for instance:

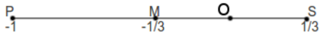

Figure: Isotropic states: MS: Separable; PM: Entangled; B: bell state; O: unpolarized state

- To prepare a state in the region PM, choosing the bell state(P) for Π_1, ρ_s can be chosen to be any point between M and S.
- Hence we need a further extremization to define an optimal preparation.

Free Entanglemet

- In the spirit of Gibbs' Free energy, define *Free Entanglement*

$$\mathscr{F} = \lambda N[\Pi_1] - N[\rho] S_p^{\lambda}[\rho] = N[\Pi_1] - N(\rho) S_p[\rho_s](1-\lambda)$$

  where $N[\rho]$ is the negativity of ρ.

- This quantity depends on the choice of the decomposition.

Optimal preparation
An Example

- Our Proposal: Minimize $\mathscr{F}$ over all possible decompositions
- Consider the states: $\rho[x,y] = \frac{1}{4}\{\mathbf{1}\otimes\mathbf{1} + x(\sigma_z\otimes\mathbf{1} - \mathbf{1}\otimes\sigma_z) - y(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y) - \sigma_z\otimes\sigma_z\}$

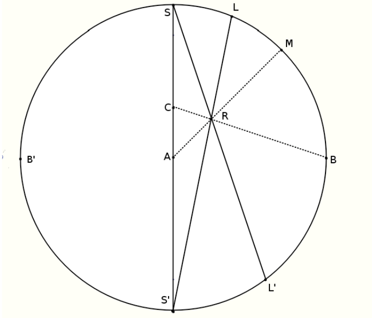

Figure: State space

Illustrating the optimal preparation

- The state space and separable subspace

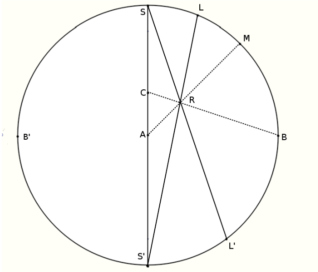

Figure: Circumference: pure states; Diameter SS': separable; B and B': bell states; S and S': pure separable states

- $S_p = 0$ on SS'; R is the state to be prepared; Choices of decomposition: AM, CB, SL', S'L. Minimum $\mathscr{F}$ at S'L

Summary

- We develop a new method for explicit construction of the separation for a separable two qubit state.
- We use the obtained separation to find optimal preparation for a given state, cost of preparation expressed as free entanglement.

- Outlook
    - Extend the construction procedure to higher spin bipartite systems, where no sufficient condition for separability is known.

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