Entanglement and correlation in many-electron systems

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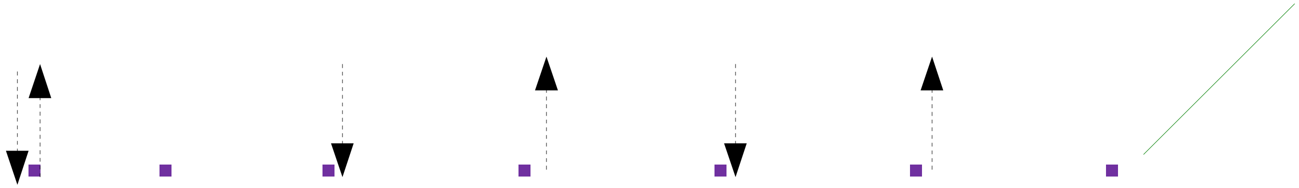
Many-Electron States: 4 states/site, $\langle n_{i\uparrow} n_{j\uparrow} \rangle$, $\langle n_{i\uparrow} n_{j\downarrow} \rangle$

A fixed number of sites, up spins, down spins



Distinct Labeled Spatial Part

Strongly-Correlated States: No Double Occupancy, 3 States/site, On-site Spin Correlations

Spin-Only States: No holes either, Two states/site, Qubits $\langle S_i^z \rangle, \langle S_i^z S_j^z \rangle, \langle S_i^+ S_j^- \rangle$

$$|\psi\rangle = \sum_{\{s_i\}} \phi(s_1, s_2 .. s_N) \, |s_1, s_2 .. s_N\rangle$$

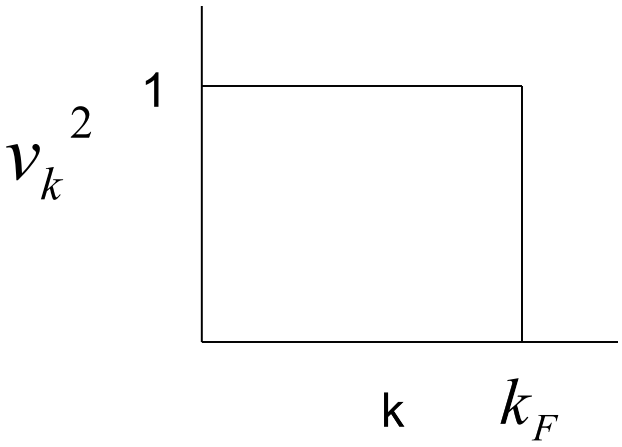
$$\Gamma(r) \approx m^2 + \frac{A}{r^p} + B e^{-\frac{r}{\xi}}$$

Diagonal LRO: Constant m nonzero

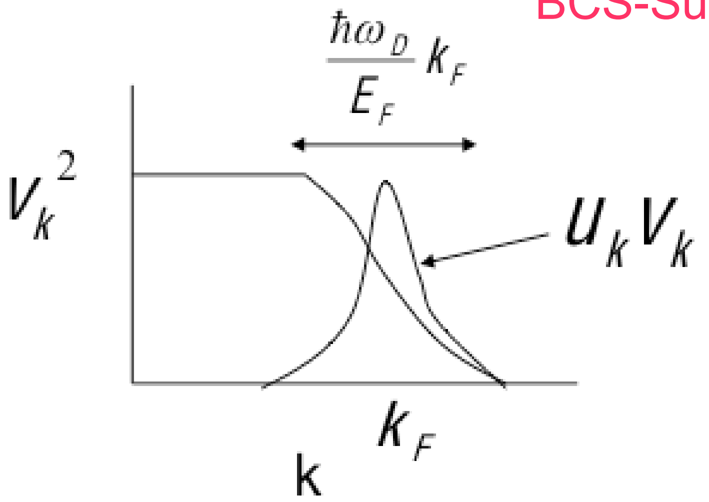
Long-ranged Correlations: A Nonzero

Many-eletcron State: $|\psi\rangle = \Pi_k \left(u_k + v_k c^*_{k\uparrow} c^*_{-k\downarrow}\right) |0\rangle$

Metallic/Uncorrelated



BCS-Superconductor



$$v_k^2 = \frac{1}{2}\left(1 - \frac{\epsilon_k - E_F}{\sqrt{(\epsilon_k - E_F)^2 + \Delta_k^2}}\right)$$

In the above range $\Delta_k = \Delta_0$

$$d = \frac{n^2}{4}\left(1 + \zeta^2\right)$$

Double Occupancy

$$\frac{n}{2}\zeta \equiv \left|\langle c_{i\uparrow} c_{i\downarrow}\rangle\right| \approx \frac{3\Delta_0}{2E_F} \sinh^{-1}\frac{\hbar\omega_D}{\Delta_0}$$

Order Parameter

Strongly-Correlated electron state: Gutzwiller Projection, Inhibit double occupancy

Start from a metallic 'Uncorrelated' State

$$|\psi_g\rangle = \Pi_i \left(1-(1-g)n_{i\uparrow}n_{i\downarrow}\right) |F\rangle$$

$$d(g) \le \frac{n^2}{4}$$

Projection operator for the site Hilbert Space:

g=0 states with Doubly-occupied
sites are projected out.

g=1 No Correlation, Metallic State, U=0: $d = n_\uparrow n_\downarrow$

g=0 Strong Correlation U=infinity : $d = 0$

$$d(g) = \frac{1}{N}\sum_{\mathrm{D}} D|\alpha_{\mathrm{D}}|^2 g^{2D} = \frac{1}{2N}\frac{\partial \ln\gamma}{\partial \ln g}$$

$$\alpha_D \equiv \langle D|P_D|g\rangle$$

$$\gamma(g) \equiv \langle g|g\rangle$$

1- D $\quad d(g) = \frac{1}{2}\frac{g^2}{(1-g^2)^2}(ng^2 - n - \ln[1-n-ng^2])$ $\quad n \le 1$

Metzner and Vollhardt (1988)

Plan

Quantum Entanglement: Subsystem Entropy, Quantum Correlations, Measures

Entanglement between spatial partitions: Correlated, Superconductor, Metal states

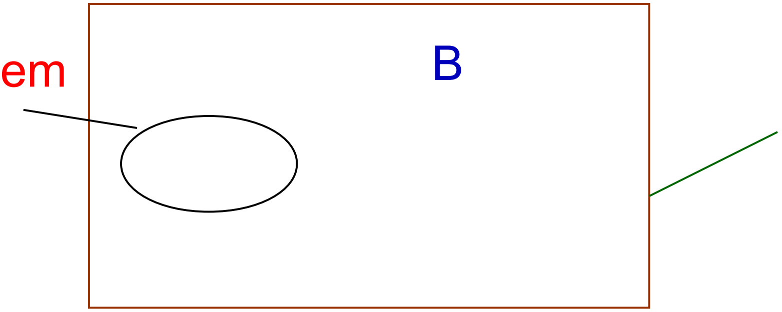
Multi-Species Entanglement: Macroscopic entanglement, Quantum Criticality
One-Dimensional Transverse-Ising model

Conclusions

Entanglement:

$$\rho_{A,B} = |\psi\rangle\langle\psi|$$

Direct Product
or
Entangled



Subsystem
A

B

Total System in a
Pure state

Entropy = 0

State of
Subsystem

$$\rho_A \equiv Tr_B\, |\psi\rangle\langle\psi|$$

No Entanglement: ρ_A Pure Entropy $= 0$

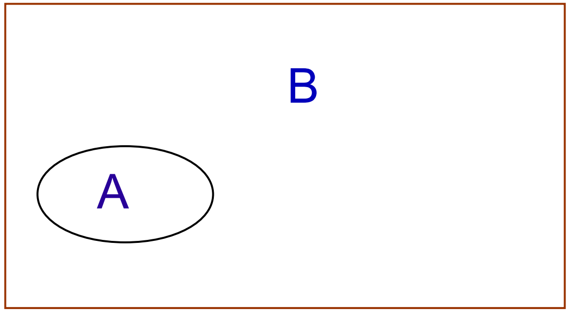
Entanglement: ρ_A Mixed Entropy $\neq 0$

Through Unitary Transform Entanglement can be generated
A can become mixed $\Longrightarrow$ Entanglement, Decoherence

Entanglement between Distinct Partitions

$\rho_{A,B} = |\psi\rangle\langle\psi|$

Spatial Partitions

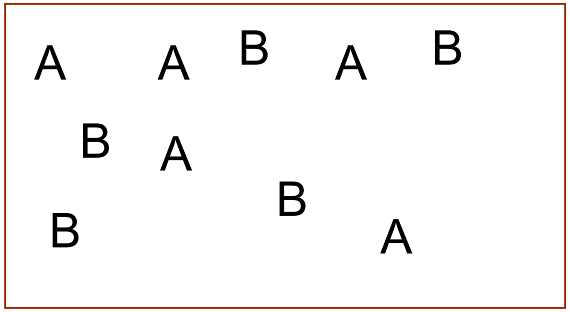


Many-dim Hilbert Space
For each Partition
Many ways to partition!

u denotes the state
of Partition A

$$\boxed{|\psi\rangle = \sum_{u,v} \psi(u,v)\ |u\rangle_A\ |v\rangle_B}$$

Two Species



Label Partition by A and B
Type of Particles
Both types access full space

u denotes the set of locations
occupied by A-type particles

Different Ways of Entanglement

Spin States of Distinct Spatial partitions

$$|\uparrow\rangle_1|\downarrow\rangle_2+\beta|\downarrow\rangle_1|\uparrow\rangle_2$$

Spatial parts 1 and 2 entangled

Spatial State of Distinct Spin partitions

$$|1\rangle_\uparrow|2\rangle_\downarrow+\beta|1\rangle_\uparrow|3\rangle_\downarrow+\gamma|2\rangle_\uparrow|3\rangle_\downarrow$$

Spin partitions entangled

Entangled Spins + Entangled Spatial Parts

$$|\uparrow_1\downarrow_2\rangle+\beta|\uparrow_1\downarrow_3\rangle+\gamma|\downarrow_2\downarrow_3\rangle$$

Two Different Sources of Entanglement

$$\boxed{|\psi\rangle = \sum_{u,v} \psi(u,v)\ |u\rangle_A\ |v\rangle_B}$$

Uncorrelated

$$\rightarrow \quad |x\rangle_A\ |y\rangle_B$$

Correlations Between A and B: Wave function not factorizable

Constraint over degrees of freedom: Ex. Number of down spins fixed

$$|\psi\rangle = \sum_{x,u,v} \psi_x(u,v)\ |x,u\rangle_A\ |x,v\rangle_B$$

Even if the amplitude factorizes in each sector

$$|\psi\rangle = \sum_x \lambda(x)\ |\tilde{x}\rangle_A\ |\tilde{x}\rangle_B$$

=> **Entanglement**

Schmidt Form

Spin States:

$$|\psi\rangle = \sum_{\{s_i\}} \phi(s_1, s_2 .. s_N)\ |s_1, s_2 .. s_N\rangle$$

ρ_1 Ent. Between 1 and Rest

ρ_2 Ent. Between 2 and Rest

ρ_{12} Ent. between (1,2) and Rest,

Between 1 and 2

Two-party Ent.
In a mixed State

Similarly $\rho_{1,2,3}$ contains Three-party Entanglement information

$$If\ \ [\rho_\psi, S^z] = 0, \qquad Implies \qquad [\rho_{1,2,3..n}, S_1^z + S_2^z + S_3^z + .. S_n^z] = 0$$

$$\rho_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \rho_{1,2} = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & x & z & 0 \\ 0 & z^* & y & 0 \\ 0 & 0 & 0 & v \end{pmatrix}$$

$\langle \frac{1}{2} - S_1^z \rangle$

$\langle S_1^- S_2^+ \rangle$

Basis states

$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

How Entangled Is a Spin State?

One-Qubit
Reduced ρ_i

Ave. Site purity
Global Ent.

$$\boxed{\varepsilon(\psi) = \frac{2}{N}\Sigma\, 1 - Tr\,\rho_i^2}$$

Larger in a State implies
better sharing of Ent.

Two-Qubit
Reduced ρ_{ij}

Concurrence of sites i and j
Bipartite entanglement

$$C_{ij}\ from\ [\rho_{ij}\,\rho_{ij}^T]$$

$$C_{ave} = \langle C_{ij} \rangle$$

$$E_{ij} = -Tr\,\rho_{ij}\ln\rho_{ij}$$

Entropy of the block: Ent. Between ij and rest

$$\varepsilon = 1 - 4\mathrm{m}^2$$

$$\frac{1}{2}C_{ij} = |\Gamma_{off-diag}| - \sqrt{(\frac{1}{4} + \Gamma_{diag})^2 - m^2}$$

Diagonal Ordering/Correlations Decrease Entanglement

Examples:

$$Bell\ State: \quad |00\rangle + e^{i\phi}|11\rangle \qquad \varepsilon = 1 \quad C = 1$$

$$GHZ\ State: \quad |000\rangle + |111\rangle \qquad \varepsilon = 1 \quad C = 0$$

$$W\ State: \quad |001\rangle + |010\rangle + |100\rangle \qquad \varepsilon = \frac{8}{9} \quad C = \frac{2}{3}$$

Shor's 9-qubit
Error-correcting Code

$$|0\rangle \equiv |000+111\rangle|000+111\rangle|000+111\rangle$$

$$|1\rangle \equiv |000-111\rangle|000-111\rangle|000-111\rangle$$

5-Qubit

$$|0\rangle \equiv |00000-(10100+11000+11110+11110+cycl)\rangle$$

$\varepsilon = 1$ For all states in code subspace

How Entangled are Many-Electron States?

$$n = n_{\uparrow} + n_{\downarrow}$$

$$d = \langle n_{i\uparrow} n_{i\downarrow} \rangle$$

$$|\psi\rangle = \sum_{\{A_i, B_i\}} \phi(A_1, B_1 \ldots A_N, B_N) |A_1, B_1 \ldots A_N, B_N\rangle$$

$$\rho = \begin{vmatrix} 1 - n + d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & n_{\uparrow} - d & 0 \\ 0 & 0 & 0 & n_{\downarrow} - d \end{vmatrix}$$

Global Entanglement

$$\varepsilon = \frac{4}{3} \sum 1 - Tr\,\rho_i^2$$

Max. Ent. Each Eigenvalue=1/4 $d = 1/4$ $n_{\uparrow} = n_{\downarrow} = 1/2$ $\varepsilon = 1$

Strong Correlations U large d=0 Now only 3 states per site

Max. Ent. Each Eigenvalue=1/3 $n_{\uparrow} = n_{\downarrow} = 1/3$ $\varepsilon = 8/9$

Half-filled Case: No holes either
Max. Ent. Each Eigenvalue=1/2 $n_{\uparrow} = n_{\downarrow} = 1/2$ $\varepsilon = 2/3$

Gutzwiller Projection: Start from a metallic 'Uncorrelated' State

$$|\psi_g\rangle = \Pi_i \left(1-(1-g)n_{i\uparrow}n_{i\downarrow}\right) |F\rangle$$

Projection operator for the site Hilbert Space:

g=0 states with Doubly-occupied sites are projected out.

g=1 No Correlation, Metallic State, U=0: $d = n_\uparrow n_\downarrow$

g=0 Strong Correlation U=infinity : $d = 0$

$$d(g) = \frac{1}{N}\sum_{\mathrm{D}} D|\alpha_{\mathrm{D}}|^2 g^{2D} = \frac{1}{2N}\frac{\partial \ln\gamma}{\partial \ln g}$$

$$\alpha_D \equiv \langle D|P_D|g\rangle$$

$$\gamma(g) \equiv \langle g|g\rangle$$

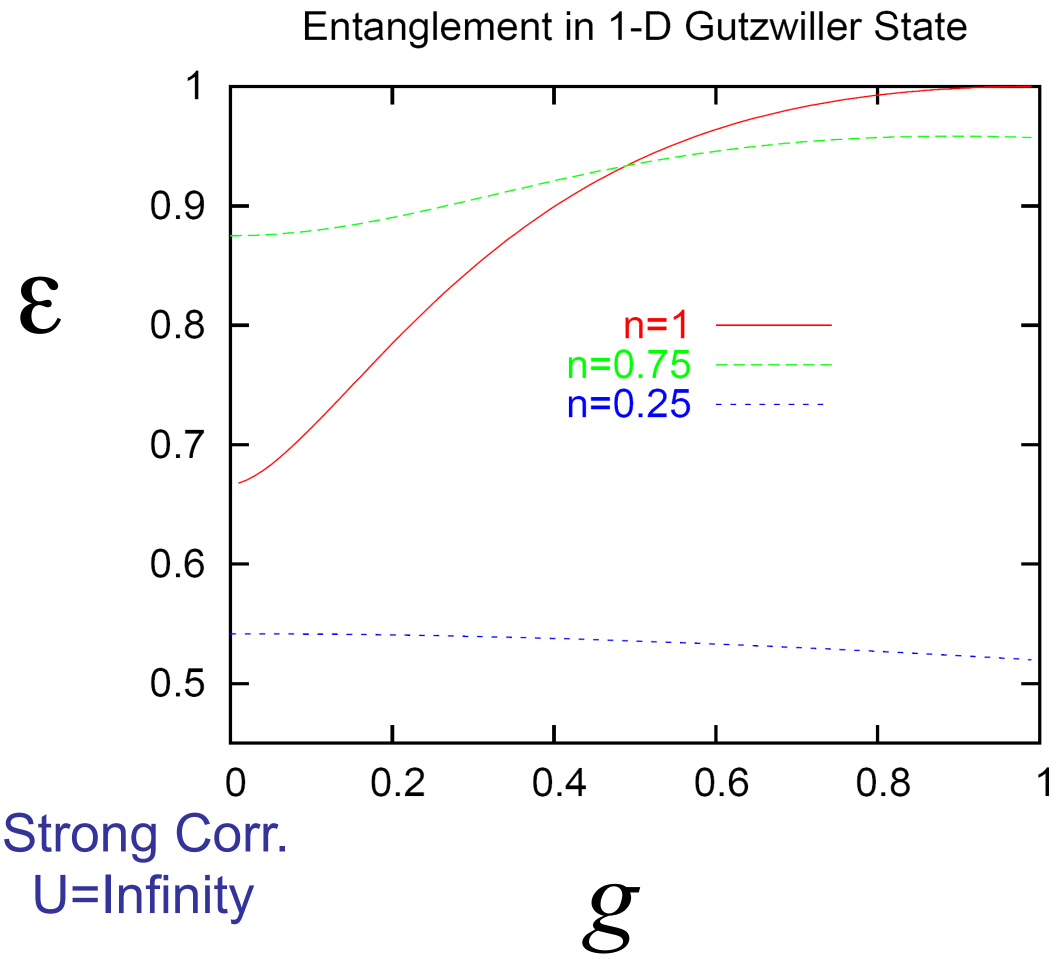
1- D $\quad d(g) = \frac{1}{2}\frac{g^2}{(1-g^2)^2}(ng^2 - n - \ln[1 - n - ng^2])$ $\quad n \leq 1$

Metzner and Vollhardt (1988)

Global Entanglement and electron correlation: d=1 Gutzwiller State

$$|\psi_g\rangle = \Pi_i \left(1-(1-g)n_{i\uparrow}n_{i\downarrow}\right)|F\rangle$$

VS, Phys. Lett. A374, 3151 (2010)



$$n_\uparrow = n_\downarrow = \frac{n}{2}$$

Dim. >1

Quite Complicated

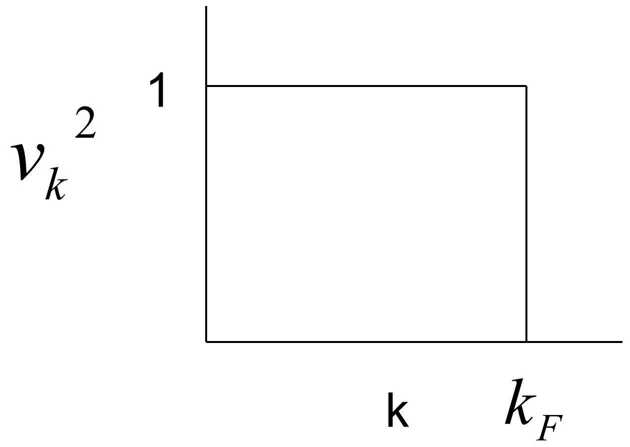
Combinatorial

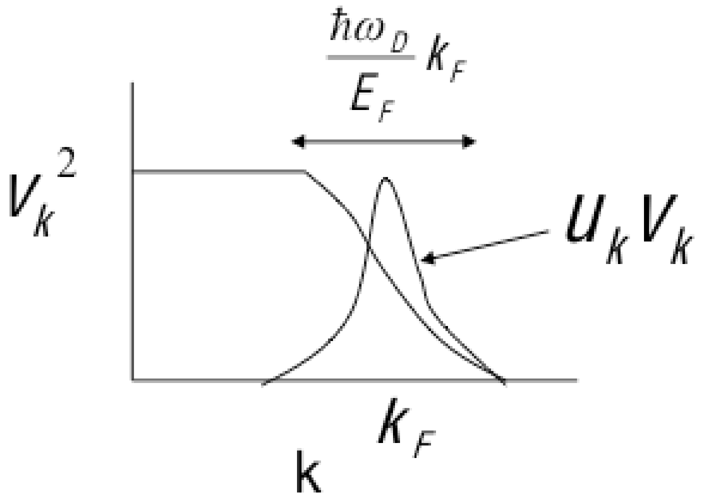
Strong Corr.
U=Infinity

Uncorrelated
No projection

Metallic Fermi State
BCS-Superconducting State

$$|\psi\rangle = \Pi_k \left(u_k + v_k c^*_{k\uparrow} c^*_{-k\downarrow}\right) |0\rangle$$





$$v_k^2 = \frac{1}{2}\left(1 - \frac{\epsilon_k - E_F}{\sqrt{(\epsilon_k - E_F)^2 + \Delta_k^2}}\right)$$

In the above range $\Delta_k = \Delta_0$

$$d = \frac{n^2}{4}\left(1 + \zeta^2\right)$$

Double Occupancy

$$\frac{n}{2}\zeta \equiv |\langle c_{i\uparrow} c_{i\downarrow}\rangle| \approx \frac{3\Delta_0}{2E_F}\sinh^{-1}\frac{\hbar\omega_D}{\Delta_0}$$

Order Parameter

Entanglement in the BCS Superconducting State

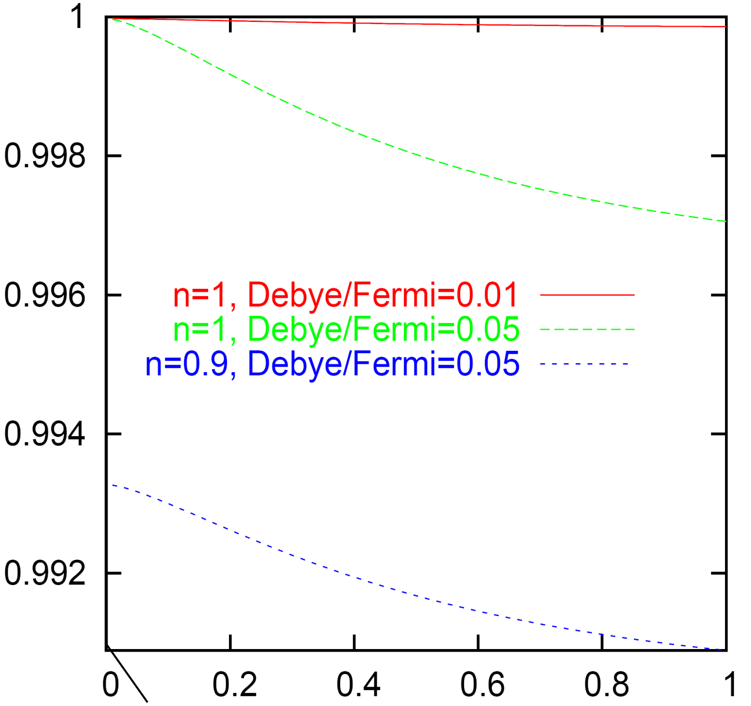
VS (2010)

$$\varepsilon = \frac{8}{3} d(2n - 2d - 1) + \frac{8}{3} n - \frac{1}{2} n^2$$

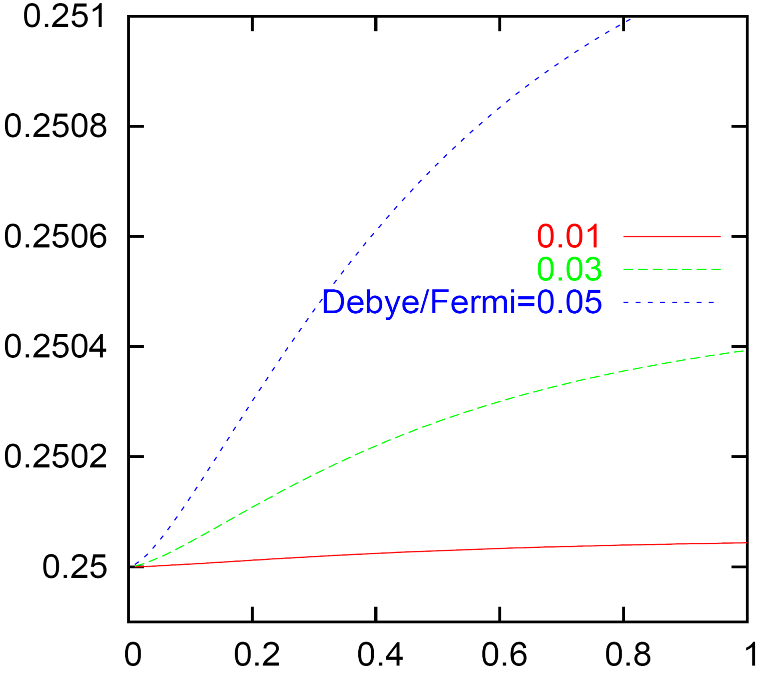
$$\zeta = \frac{3\Delta_0}{nE_F}\sinh^{-1}\frac{\hbar\omega_D}{\Delta_0}$$

$$d = \frac{n^2}{4}(1+\zeta^2)$$

ε



d



Uncorrelated
Matallic State

$\Delta_0/\hbar\omega_D$

Multi-Species Entanglement

VS, Quant. Inform. Comp. (2011)

Let u (v) denote the set of sites occupied by A (B) particles

$$|\psi\rangle = \sum_{u,v} \psi(u,v)\ |u\rangle_A\ |v\rangle_B$$

Strong exclusion: B occupies only sites unoccupied by A

Half filling: Total number of particles equals the number of

Now, v stands for the complement of the set u

$$|\psi\rangle = \sum_{u} \psi(u)\ |u\rangle_A\ |u\rangle_B$$

Schmidt numbers: Eigenvalues of reduced density matrix

$$S_A = -\sum_u |\psi(u)|^2 \log |\psi(u)|^2 \qquad \varepsilon_{A,B} = \lim_{N\to\infty} \frac{S_A}{N}$$

Example:Multi-Species Entanglement

$$|\uparrow\rangle_A|\uparrow\rangle_B+|\uparrow\rangle_A|\downarrow\rangle_B \qquad =|A,B\rangle_\uparrow\ |0\rangle_\downarrow\ +\ |A\rangle_\uparrow\ |B\rangle_\downarrow$$

Sites A and B unentangled

Up and Down Spins Entangled

Multi-Species Entanglement!

$$|\psi\rangle \ = \ |\uparrow\rangle+|\downarrow\rangle \ = \ |O\rangle_\uparrow\ |U\rangle_\downarrow+|U\rangle_\uparrow\ |O\rangle_\downarrow$$

Entanglement for a single Qubit

$$|\psi\rangle \ = \prod \ |\uparrow\rangle_i+|\downarrow\rangle_i$$

No entanglement in any spatial partitionng

Macroscopic Entanglement betwen Up and Down Spins: N log 2

Spin Systems: States with a given number Spins $N, \quad N_{\uparrow}. \quad N_{\downarrow}$

$$|\psi\rangle = \sum_{\{s_i\}} \phi(s_1, s_2..s_N)\ |s_1, s_2..s_N\rangle$$

Macroscopic System with
Thermodynamic densities

$$n_{\uparrow} = \lim_{N\rightarrow\infty} \frac{N_{\uparrow}}{N} \qquad n_{\downarrow} = \lim_{N\rightarrow\infty} \frac{N_{\downarrow}}{N}$$

$$|\phi[s_i]|^2 \ Eigenvalue\, of\ \rho_{\uparrow}$$

Macroscopic Entangment
Betwen up and down spins

$$\varepsilon_{\uparrow,\downarrow} = \lim_{N\rightarrow\infty} \frac{S_{\uparrow}}{N}$$

We will see Multi-Species Entanglement has a Thermodynamic Limit
Can display a singularity that are Associated with Quantum Phase Transitions.

In Contrast, entropy of Spatial Partition does not scale with System Size!

Block entropy per site does not tend to a limit

Transverse Ising Model: Quantum Phase Transition in the ground state

$$\mathcal{H} = -J\sum_i \sigma_i^x \sigma_{i+1}^x - h\sum_i \sigma_i^z$$ ← Pauli operators

$h=0$ $|GS\rangle = \Pi|\uparrow + \downarrow\rangle \quad J > 0$

$= \Pi|\uparrow + \downarrow\rangle|\uparrow - \downarrow\rangle \quad J < 0$

Ordered Phase

$h \neq 0$ Fluctuations similar to thermal (incoherent) excitations

$h = J$ Quantum critical point, Order parameter vanishes

Nearest-Neighbor Concurrence

$$\frac{dC(1)}{d\lambda} = \frac{8}{3\pi^2}\ln|\lambda - \lambda_c| + \text{const}$$

$\lambda = |J/h|$

Osterloh et al, Nature, 416, 608 (2002)

Transverse Ising....

$$\mathcal{H} = -J\sum_i \sigma_i^x \sigma_{i+1}^x - h\sum_i \sigma_i^z$$

Spin Operators acting at different sites behave like bose operators $[\sigma_i^x, \sigma_j^y] = 0$

Spin Operators acting at one site behave like fermi operators $\{\sigma_l^x, \sigma_l^y\} = 2i\sigma_l^z$

Jordan-Wigner Transform to Fermi Operators, but nonlocal phase-factor tags

Impose Periodic Boundary Conditions, Even and Odd Sectors have different Effective Fermion Hamiltonians, States with either Even or Odd number of particles

Nearest-Neighbor interaction absorbs phases, effectively a quadratic Hamiltonian

Fourier transform, followed by a Bogoliubov transform, yields all eigenstates

Spin-Spin Correlation functions easy to find, though entropy/entanglement tough

Lieb Schultz Mattis, Ann. Phys (1961) Pfeuty, Ann. Phys (1970)

Exact Solution: Jordan-Wigner Fermions Fermion Occupation--> Up Spin

$$\boxed{\sigma_l^z = 2n_l - 1, \quad \sigma_l^+ = \mathrm{e}^{i\pi \sum_{j=1}^{l-1} n_j} c_l^\dagger}$$

$$c_q = \frac{1}{\sqrt{N}} \sum c_l \ \mathrm{e}^{-iql}$$

Modes

$$q = \pm\frac{\pi}{N}, \pm\frac{3\pi}{N} \ldots \pm\frac{(N-1)\pi}{N}$$

Ground State

$$\boxed{|G\rangle = \prod_{q>0} (a_q|0\rangle + b_q|\phi_q\rangle)}$$

$$|\phi_q\rangle \equiv c_q^\dagger c_{-q}^\dagger |0\rangle$$

$$\boxed{|a_q|^2 = \frac{1}{2}(1 - \frac{h + J\cos q}{\sqrt{h^2 + J^2 + 2Jh\cos q}})}$$

Label components of Superposition by u (set of q values occupied by Up Spins)
Corresponding amplitude is product of b's and a's (Schmidt Numbers)

$$\rho_\uparrow = \prod a_q^2|0\rangle\langle 0| + \sum_q b_q^2 \prod_{q' \neq q} a_{q'}^2 |0, 0..\phi_q..0\rangle\langle 0, 0..\phi_q..0| + ..$$

Eigenvalues of $\rho_{\uparrow}$ $\quad \lambda_0 = \prod |b_{q_i}|^2 \quad \lambda_1 = \lambda_0 \dfrac{|a_{Max}|^2}{1 - |a_{Max}|^2}$

$$\lambda(\eta_{1,} \eta_{2..}) = \Pi \; |a_{q_i}|^{2\eta_i} \; |b_{q_i}|^{2(1-\eta_i)}$$

$$S_{\uparrow} = -\sum_i |a_{q_i}|^2 \log |a_{q_i}|^2 + |b_{q_i}|^2 \log |b_{q_i}|^2$$

Entropy

$$S_{\uparrow} = \sum_i H(p_i) \quad H(p) = -p \log p - (1-p) \log (1-p)$$

Shannon Binary Entropy

$$\varepsilon(x) = \frac{1}{N} \sum_{i=1}^{N/2} H(p_i) = \int_0^1 dp \; g(p,x) \log \frac{1-p}{p}$$

$$g(p,x) = \frac{1}{N} \sum_i \theta(p - p_i(x))$$

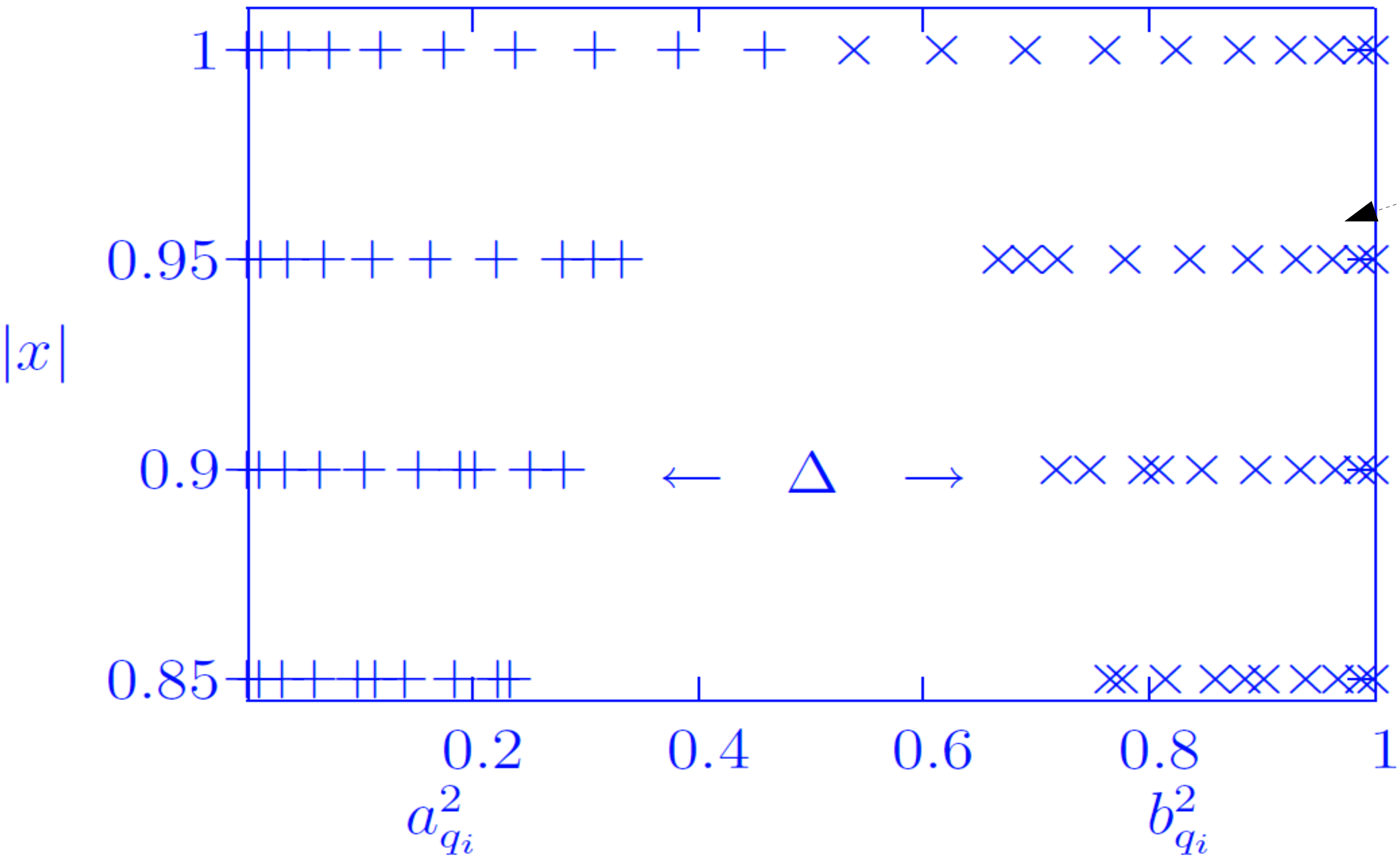
Integrated Density
Of Eigenvalues $\quad x = J/h$

Mode eigenvalues as x=J/h is varied

Macroscopic limit, Large N

Gap between Two bands stays

Within one Band continuous

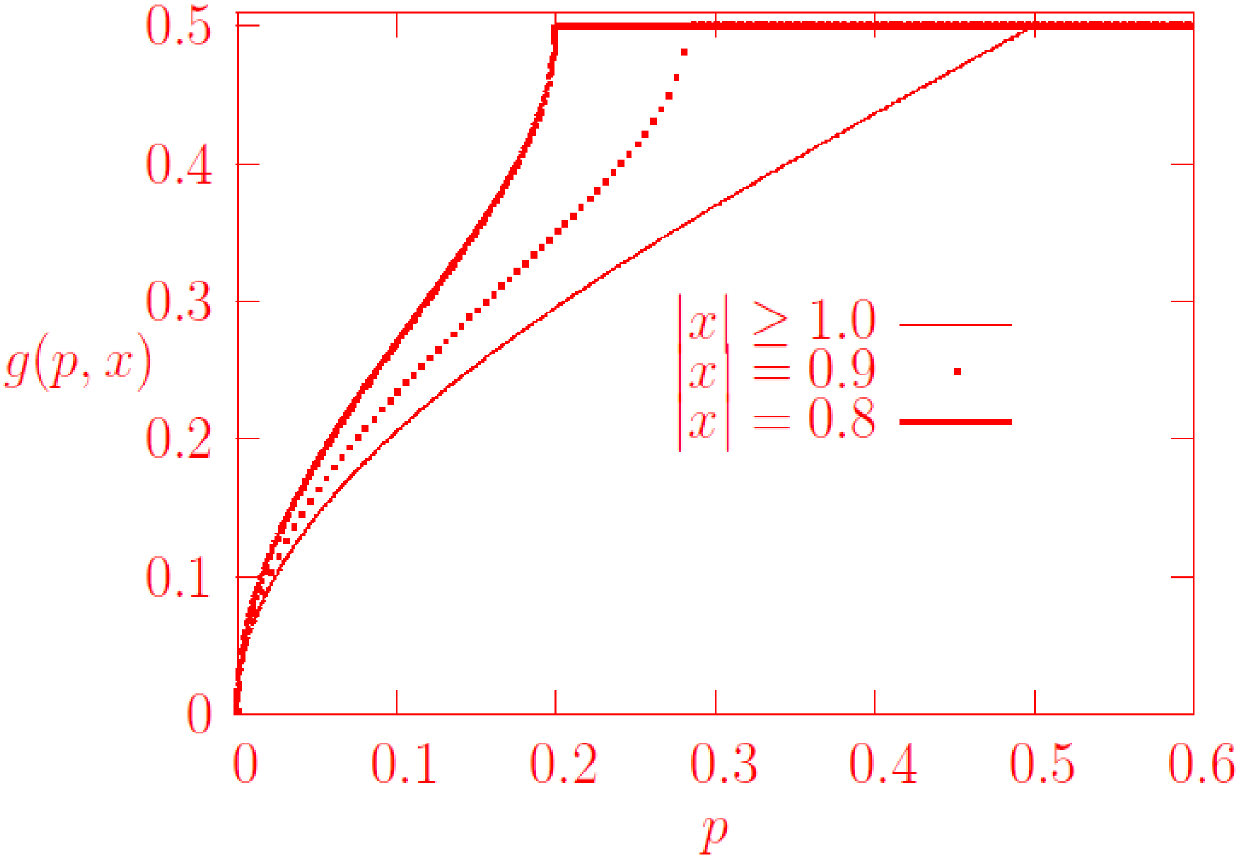


Location

N=20

$$\Delta \approx (1-|x|) \quad near \quad |x| \approx 1$$

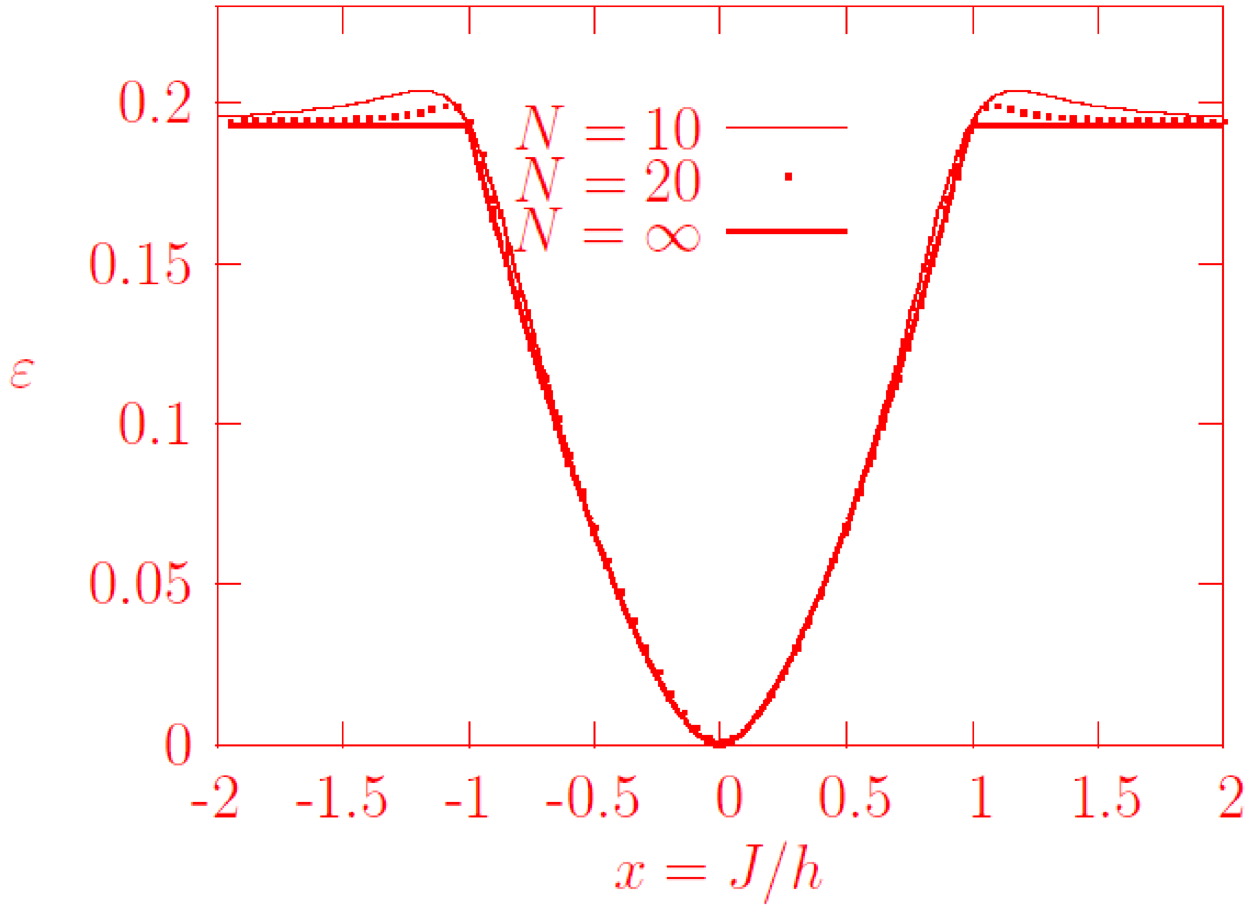
Integrated Eigenvalue Density



Counting only lower
Of the two eigenvalues
For every mode

Contribution for entropy
From the other eigenvalue
Taken care of by
Shannon Binary Entropy

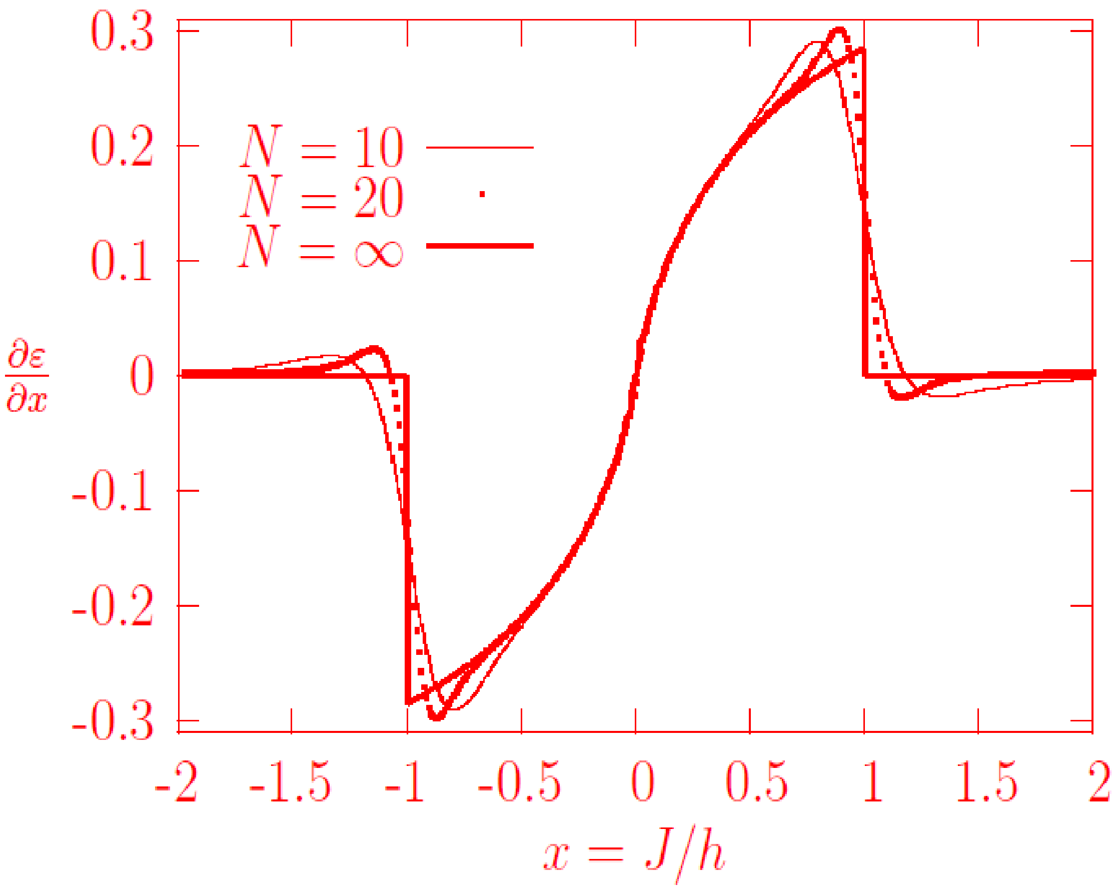
Multi-Species Entanglement in Transverse Ising Model



$$\varepsilon(x=1)=\varepsilon(x=\infty)$$

$$\varepsilon(x \to \infty) = \frac{\log 2}{2} - \frac{1}{2}\sum_{\text{even } n} \frac{1}{n(n-1)2^n}\, {}^{n}C_{\frac{n}{2}} \approx 0.19$$

Jump-discontinuity in the derivative of Entanglement



Second Derivative
Diverges at |x|=1

One can see
Signature for
Small Sizes!

$$\delta\varepsilon'(x=1) = \frac{1}{2\pi}\int_0^1 d\zeta \, \log\frac{1+\zeta}{1-\zeta}\,\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$= \frac{1}{4}\sum_{\text{odd } n}\frac{1}{2^n n^2}\;{}^{n+1}C_{\frac{n+1}{2}} \approx 0.28$$

Conclusions

Entanglement useful: Measures aplenty tracking how entangled is a state

Different Sources of Entanglement: Correlations between partitions, Constraints

Different ways of entanglement: Multipartite, Spatial partitions, Multi Species

Multi-Species Entanglement: Macroscopic Entanglement Scales with Size

Exists in the Thermodynamic Limit, Can Display Singularities

Transverse-field Ising Model: Macroscopic Entanglement between $\uparrow, \downarrow$

Capable of tracking quantum phase transition and critical behaviour.

Thank You

For Your Attention

Superconducting State...

$$|\psi\rangle = \prod_k \left(u_k + v_k\, c^*{}_{k\uparrow} c_{-k\downarrow}\right)|0\rangle$$

$u^2 + v^2 = 1$ **every k**

$$\langle n_{k\uparrow}\rangle = \langle 0|(u_k + v_k c_{-k\downarrow} c_{k\uparrow}) n_{k\uparrow} (u_k + v_k c^*{}_{k\uparrow} c^*{}_{-k\downarrow})|0\rangle \quad = v_k^2$$

$$\langle c_{-k\downarrow} c_{k\downarrow}\rangle = u_k v_k$$

Pairing stabilized by attractive interaction

$$D(E) \equiv \frac{1}{\Omega}\sum_k \delta(E - \varepsilon_k)$$

$$V(q) = -V_0 \quad |\varepsilon_q - E_F| < \hbar\omega_D$$

$$\Rightarrow \Delta_k = \Delta_0$$

$$n_\uparrow = \Sigma n_{k\uparrow}$$

$$\frac{n}{2}\varsigma \equiv \langle c_{i\downarrow} c_{i\uparrow}\rangle$$

Off-diag element
Of Density matrix

$$n_\uparrow = \frac{n}{2} = \int d\varepsilon\, D(\varepsilon) v^2(\varepsilon)$$

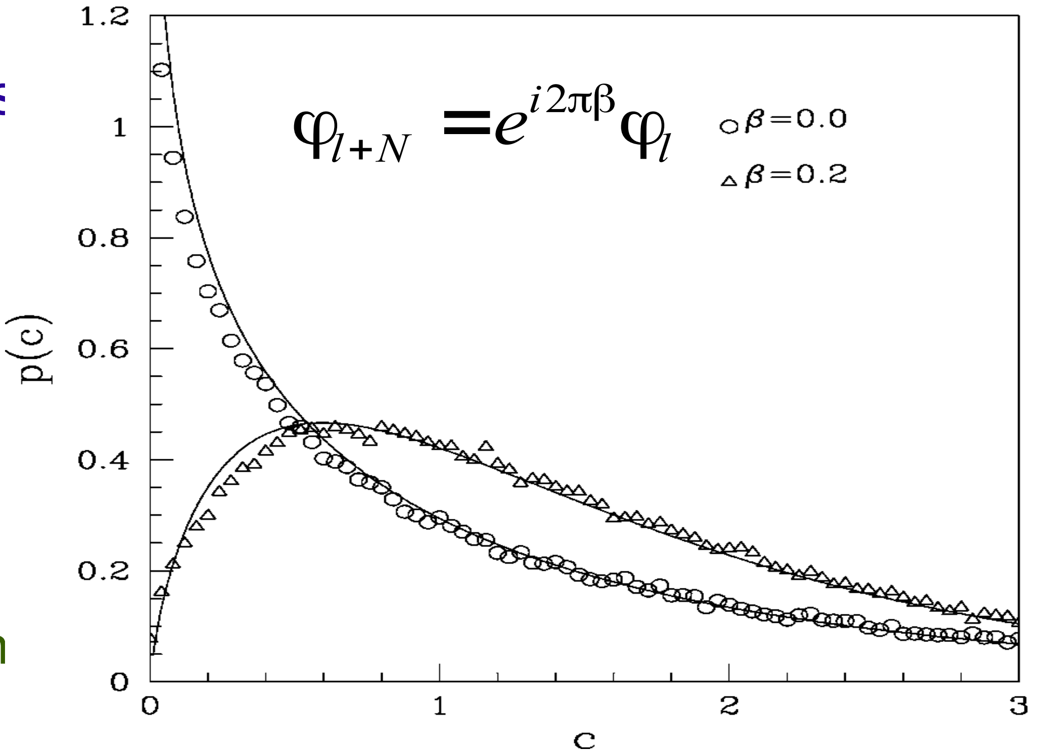
$$\frac{n}{2}\varsigma = \int d\varepsilon\, D(\varepsilon)\, u(\varepsilon) v(\varepsilon)$$

Pair-Entanglement Distributions

$$|\psi\rangle = \sum_i \varphi_i \left| \uparrow\uparrow \cdot\cdot \uparrow \underset{i}{\downarrow} \uparrow \cdot\cdot \right\rangle$$

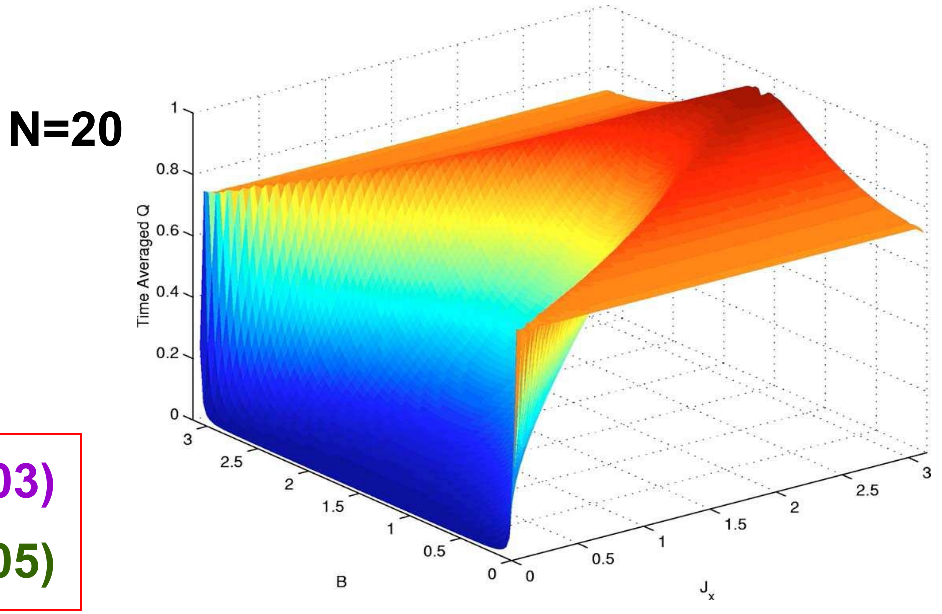
$$p(c) = \begin{cases} (1/\pi) K_0(c/2) & \text{(GOE)} \\ c K_0(c) & \text{(GUE)} \end{cases}$$

TR Breaking --> More Ent.Sharin
Ave. Ent. Related to Localization



Kicked-Transverse Ising Chain

Global Entanglement Generation



AL & VS, Phys. Rev. A65, 052304 (2003)

AL & VS, Phys. Rev. A71, 062324 (2005)

Heisenberg Antiferromagnet GS: S=0 Spatially Uniform

VS (2004)

$1-D\ \ Chain \quad C_{1,2}=0.398, \quad C_{1,l}=0 \ \ l>2 \quad C_{ave}=\frac{0.796}{N-1}$

$2-D\ \ Square \quad C_{1,2}=0.16, \quad C_{1,l}=0 \ \ l>2 \quad C_{ave}=\frac{0.64}{N-1}$

Kagome and Triangular: $C_{ij}=0$ No pair concurrences!

Average Site Mixedness $\varepsilon=1$ for all lattices!

One-Magnon State $\left|\psi\right\rangle=\Sigma\ \ \phi_l\ \left|l\right\rangle$ ← Location of Down Spin

$C_{ave}=\frac{2}{N}(\frac{2}{\pi})$ **Time Reversal Invariant**

AL,VS (2003)

$=\frac{2}{N}(\frac{\pi}{4})$ **No Time Reversal**

Entropy: Independent mode contributions: $p_i = |a_{q_i}|^2 \quad x = J/h$

$$\varepsilon(x) = \frac{1}{N}\sum_{i=1}^{N/2} H(p_i) = \int_0^1 dp\; g(p,x) \log\frac{1-p}{p}$$

$$g(p,x) = \frac{1}{N}\sum_i \theta(p - p_i(x))$$

Integrated Eigenvalue Density

$$\boxed{\varepsilon(x) = \frac{1}{4\pi}\int_0^1 d\zeta \log\frac{1+\zeta}{1-\zeta}(\Phi_- - \Phi_+)\theta(x^2+\zeta^2-1)}$$

$$\cos\Phi_\pm = \frac{-1+\zeta^2 \pm |\zeta|\mathrm{Sgn}(x)\sqrt{\zeta^2+x^2-1)}}{x}$$

$$\frac{\partial\Phi_-}{\partial x} = -\frac{1-x}{|1-x|}\frac{\sqrt{1-\zeta^2}}{\zeta} \qquad \frac{\partial^2\Phi_-}{\partial x^2} \approx \frac{1}{|x-1|}\frac{\sqrt{1-\zeta^2}}{\zeta}$$