Introduction to Basic Cryptography

RSA

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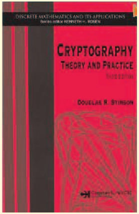
http://www.hri.res.in/~jaymehta/cryptographynotesCIMPA2010.pdf

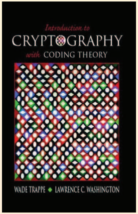
Outline

References

Cryptography - Theory and Practice
BY: **Douglas R. Stinson**



Introduction to Cryptography with Coding Theory
BY: **Wade Trappe and Lawrence Washington**



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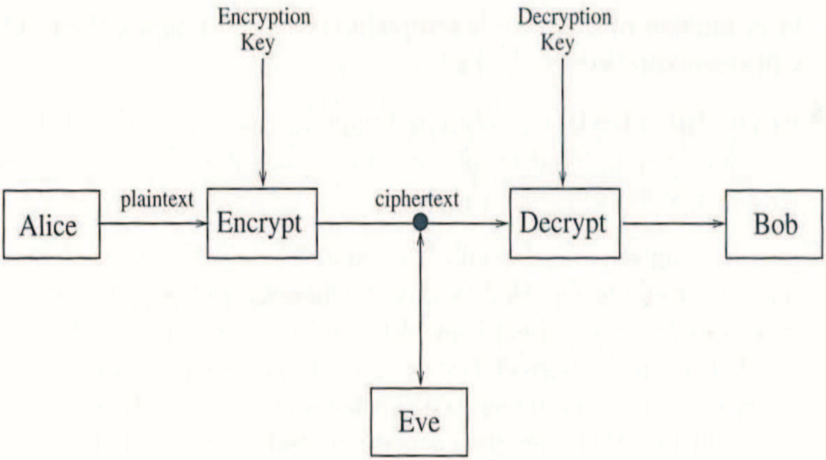

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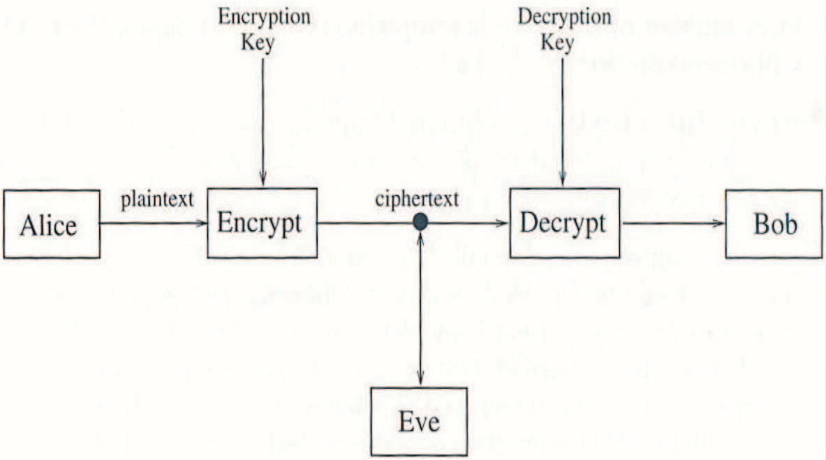
A third party, Eve, is a potential eavesdropper.
Alice wants to send a message to Bob, called '*Plaintext*'.



She encrypts it using a method prearranged with Bob.
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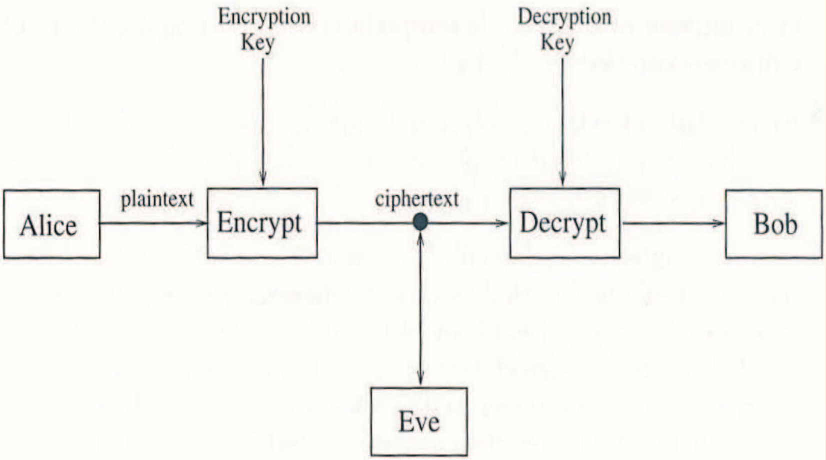

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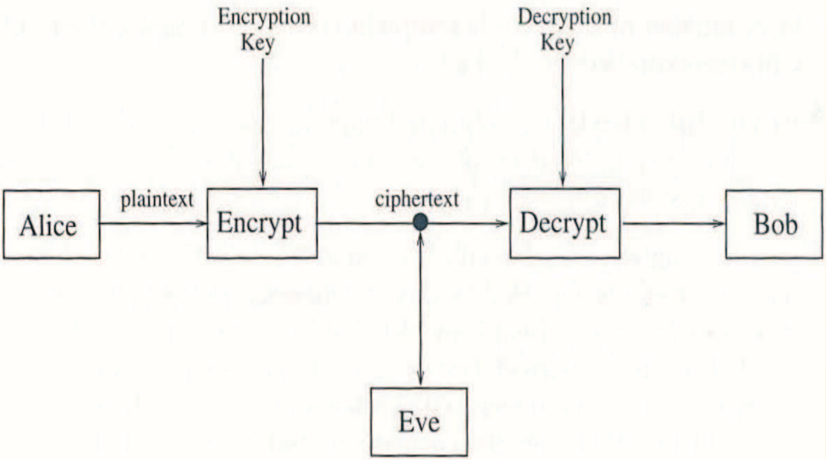
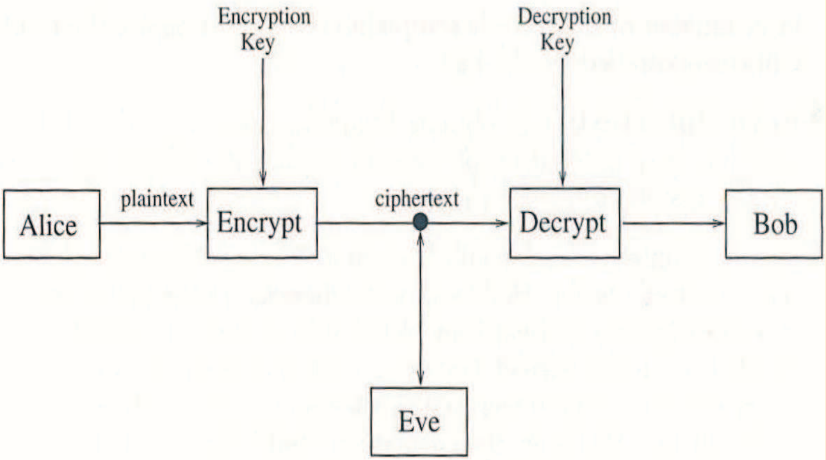
The encrypted message is called '*Ciphertext*'.



Bob receives the 'ciphertext' and changes it to the 'plaintext' by using a decryption key.



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Eve is as bad as the situation allows.

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Disadvantages :

- Needs secure channel for key exchange.
- Too many keys - Sharing a new key with every different party creates problem in managing and ensuring security.
- Origin and Authenticity of message cannot be guaranteed.

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This problem has a solution, called 'PKC', where encryption key is public, but it is computationally infeasible to find the decryption key without information which is known to Bob only.

The most popular implementation is RSA, based on difficulty of factoring large integers. Other versions are due to ElGamal (based on DLP), etc.

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In the context of encryption we want e_k to be injective one way function so that decryption can be performed. Unfortunately there aren't many functions which can be considered 'one way'.

Example: $n = pq$; b a positive integer. Then

$$f : \mathbb{Z}_n \to \mathbb{Z}_n;$$
$$f(x) \equiv x^b \pmod{n}.$$

(if $\gcd(b, \phi(n)) = 1$, this is RSA encryption function).

While construction PKC, we don't want e_k to be one way from Bob's point of view, because he should be able to decrypt messages efficiently that he receives.

Thus, it is necessary that Bob possesses a "trapdoor" which consists of secret information that permits easy inverse of e_k.

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- In 1970, James Ellis discovered 'PKC'.
- In 1973, Clifford Cocks had written an internal document describing a version of RSA algorithm in which the encryption exponent e was same as the modulus n.

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and sends c to Bob.
As Bob knows p and q, he can find the decryption exponent d with

$$de \equiv 1\big(\bmod (p-1)(q-1)\big)$$

and calculates

$$m \equiv c^d \pmod n$$

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The RSA Algorithm

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2. Bob chooses e with $\big(e, (p-1)(q-1)\big) = 1$.
3. Bob computes d with $de \equiv 1\big(\mathrm{mod}(p-1)(q-1)\big)$.
4. Bob makes n and e public and keeps p, q, d secret.
5. Alice encrypts m as $c \equiv m^e(\mathrm{mod} n)$ and sends c to Bob.
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- The security of RSA is based on the belief that the encryption formula $\boxed{e_k(m) = m^e \bmod n}$ is a one-way function. The trapdoor that allows Bob to decrypt a Ciphertext is the knowledge of factorization $n = pq$.

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When Bob receives 5761 he uses d to compute

$$5761^{6597} \bmod 11413 = 9726$$

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Exercise : Show that $(x^d)^e \equiv x(\text{mod } n)$ if $x \in \mathbb{Z}_n$

(**Hint:** Use the fact that $x_1 \equiv x_2(\mod pq)$ if and only if $x_1 \equiv x_2(\mod p)$ and $x_1 \equiv x_2(\mod q)$).

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- Step 1 will be discussed next.
- Step 2, 3, 4 can be done in time $O((\log n)^2)$.

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  Exercise : The ciphertext 5859 was obtained from the RSA algorithm using $n = 11413$ and $e = 7467$. Using the factorization $11413 = 101 \times 113$, find the plaintext.

Connection with Factoring

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Current factoring algorithm are able to factor numbers having upto 512 bits in their binary representation. It is generally recommended, one should choose each of p and q to be 512-bit prime, then n would be a 1024-bit modulus.
Factoring a number of this size is well beyond the capacity of the best current algorithm.

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The other pertinent question is how many random integers (of a specified size) will need to be tested until we find one that is prime.

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For a 1024 bit modulus $n = pq$; p and q will be 512 bit primes. A random 512 bit integer will be prime with probability approx.

$$\frac{1}{\ln 2^{512}} \approx \frac{1}{355}$$

i.e. on average, given 355 random 512 bit integers p, one of them will be prime (restricting to odd integers, probability doubles to about $\frac{2}{355}$).

$$\text{Set } \pi(N) \rightarrow \# \text{ of primes} \leq N.$$
$$\text{PNT} \rightarrow \pi(N) \sim \frac{N}{\log\ N}.$$

Hence, if an integer p is chosen at random between 1 and N, then probability that it is prime is about $\frac{1}{\log\ N}$.

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i.e. on average, given 355 random 512 bit integers p, one of them will be prime (restricting to odd integers, probability doubles to about $\frac{2}{355}$).

So, we can generate sufficiently large random numbers that are "probably prime" and hence parameter generation for the RSA Cryptosystem is indeed practical.

Definition:

Let p be an odd prime and $a \in \mathbb{Z}$;

- a is said to be quadratic residue modulo p if $a \not\equiv 0 \pmod{p}$ and the congruence $y^2 \equiv a \pmod{p}$ has a solution $y \in \mathbb{Z}_p$.
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Example:
In $\mathbb{Z}_{11}$,
$1, 3, 4, 5, 9$ are quadratic residue modulo 11.
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Euler's Criterion :

Let p be an odd prime. Then a is a quadratic residue mod p if and only if

$$a^{\frac{(p-1)}{2}} \equiv 1 \pmod{p}.$$

Legendre and Jacobi Symbols :

Legendre Symbol $\left(\frac{a}{p}\right)$:

Suppose p is an odd prime. For any integer a, define symbol $\left(\frac{a}{p}\right)$ as:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod{p} \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p \end{cases}$$

Therefore, $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$.

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Therefore, $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$.

Jacobi Symbol $\left(\frac{a}{n}\right)$:

Suppose n is an odd positive integer, and $n = \prod_{i=1}^{k} {p_i}^{e_i}$.

Let a be an integer, then

$$\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{e_i}.$$

The Solovay-Strassen Algorithm (n) :

Choose a random integer a such that $1 \leq a \leq n-1$

$x \leftarrow \left(\frac{a}{n}\right)$

if $x = 0$ **then**

  return ("n is composite")

$y \leftarrow a^{\frac{(n-1)}{2}} \pmod{n}$

if $x = y \pmod{n}$ **then**

  return ("n is prime")

else

  return ("n is composite")

It is a yes - biased Monte Carlo algorithm with error probability at the most $\frac{1}{2}$.

REMARKS on Solovay-Strassen Algorithm:

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Suppose $n > 1$ is odd. If n is prime then $\left(\frac{a}{n}\right) \equiv a^{(n-1)/2} \pmod{n}$ for any a. But, if n is composite, it may or may not be the case. If this is the case then a is called *Euler pseudo-prime* to the base n.

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Example: 10 is an Euler pseudo-prime to the base 91, since

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However, it can be shown that, for any odd composite n, at most half of the integers a such that $1 \leq a \leq n-1$ are *Euler pseudo - primes* to the base n.
Hence, error probability of Solovay-Strassen Algorithm is atmost $\frac{1}{2}$.
(The next exercise will prove this error probability).

Exercise

Define $G(n) = \{a \ : \ a \in \mathbb{Z}_n^*, \ \left(\frac{a}{n}\right) \equiv a^{(n-1)/2} \mod n\}$.

- Show that $G(n)$ is a subgroup of $\mathbb{Z}_n^*$. Thus, if $G(n) \neq \mathbb{Z}_n^*$,
$$|G(n)| \leq \frac{|\mathbb{Z}_n^*|}{2} \leq \frac{n-1}{2}.$$
- If $n = p^k q$ where p and q are odd, p is prime, $k \geq 2$ and $\gcd(p,q) = 1$. Let $a = 1 + p^{(k-1)q}$. Show that $\left(\frac{a}{n}\right) \not\equiv a^{(n-1)/2} \mod n\}$.
- If $n = p_1 p_2, \dots p_s$ where p_i's are distinct odd primes. Suppose $a \equiv u \mod p_1$ and $a \equiv 1 \mod p_2 \dots p_s$ where u is a quadratic non-residue $\mod p_1$.
Then show that $\left(\frac{a}{n}\right) \equiv -1 \mod n$ but $a^{(n-1)/2} \equiv 1 \mod p_2 \dots p_s$.
So, $a^{(n-1)/2} \not\equiv 1 \mod n$
- If n is odd and composite $|G(n)| \leq \frac{n-1}{2}$.
- Conclude : The error probability of the Solovay-Strassen Primality test is atmost $\frac{1}{2}$.

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2. If n is positive,
$$\left(\frac{2}{n}\right) = \begin{cases} 1 & \text{if } n \equiv \pm 1 (\text{mod } 8) \\ -1 & \text{if } n \equiv \pm 3 (\text{mod } 8) \end{cases}$$

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3. $\left(\frac{m_1 m_2}{n}\right) = \left(\frac{m_1}{n}\right)\left(\frac{m_2}{n}\right)$. In particular, if $m = 2^k t$; t odd
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4. Suppose m and n are two positive, odd integers.
$$\left(\frac{m}{n}\right) = \begin{cases} -\left(\frac{n}{m}\right) & \text{if } m \equiv n \equiv 3 (\bmod\ 4) \\ \left(\frac{n}{m}\right) & \text{otherwise} \end{cases}$$

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  $\big($In fact, it can be shown that it is $O((\log n)^2)\big)$.

- Suppose that we have generated a number n and tested it for primality using the Solovay-Stressan algorithm.
  If we have run the algorithm m times, what is our confidence that n is prime?

  It is $1 - 2^{-m}$.

Miler-Rabin Primality Test

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Moreover, if a computer can handle 10^9 primes/ sec., the calculation would take 10^{81} years.

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Clearly better methods are needed.

Basic Principle: Let n be an integer and suppose there exists integers x and y with $x^2 \equiv y^2 \pmod{n}$, but $x \not\equiv \pm y (\text{mod } n)$. Then n is composite. Moreover, $(x - y, n)$ gives a non-trivial factor of n.

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Example: Since $12^2 \equiv 2^2 \pmod{35}$, but $12 \not\equiv 2 \pmod{35}$. 35 is composite and $(12-2, 35) = 5$ is a non-trivial factor of 35.

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We show that 35 is not prime. By successive squaring, we find

$$\begin{array}{lll} 2^4 & \equiv & 16 \\ 2^8 & \equiv & 256 \quad \equiv 11 \\ 2^{16} & \equiv & 121 \quad \equiv 16 \\ 2^{32} & \equiv & 256 \quad \equiv 11 \end{array}$$

Therefore, $2^{34} \equiv 2^{32}.2^2 \equiv 11.4 \equiv 9 \not\equiv 1 (\text{mod } 35)$. So, it is not a prime.

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Therefore, $2^{34} \equiv 2^{32}.2^2 \equiv 11.4 \equiv 9 \not\equiv 1(\text{mod } 35)$. So, it is not a prime.

So, we have proved that 35 is composite without finding its factors. This method generalizes as : Miller-Rabin Primality Test.

Miler-Rabin Primality Test

Let $n > 1$ be an odd integer. Write

$$n - 1 \equiv 2^k m \text{ with } m \text{ odd.}$$

Choose a random integer a with $1 < a < n - 1$.
Compute $b_0 \equiv a^m (\text{mod } n)$.

if $b_0 \equiv \pm 1 (\text{mod } n)$, **then**

stop and declare that n is probably prime.

otherwise

let $b_1 \equiv {b_0}^2 (\text{mod } n)$.

if $b_1 \equiv 1 (\text{mod } n)$, **then**

n is composite and $(b_0 - 1, n)$ is a factor of n.

if $b_1 \equiv -1 (\text{mod } n)$, **then**

stop and declare that n is probably a prime.

Miler-Rabin Primality Test

otherwise

let $b_2 \equiv {b_1}^2 (\text{mod } n)$.

if $b_2 \equiv 1(\text{mod } n)$, **then**

n is composite and $(b_1 - 1, n)$ is a factor of n.

if $b_2 \equiv -1(\text{mod } n)$, **then**

stop and declare that n is probably a prime.

otherwise

let $b_3 \equiv {b_2}^2 (\text{mod } n)$.

Continue in this way until stopping or reaching b_{k-1}.

If $b_{k-1} \not\equiv -1(\text{mod } n)$, then n is composite.

The Miller-Rabin Algorithm (mod n) :

Write $n-1 = 2^k m$, where m is odd.

Choose a random integer a, $1 \leq a \leq n-1$.

Compute $b = a^m \pmod{n}$.

if $b \equiv 1 \pmod{n}$ **then**

  return ("n is prime") and **quit**

for $i = 0$ **to** $k-1$

$$\textbf{do} \begin{cases} \textbf{if} & b \equiv -1 \pmod{n} \\ \textbf{then} & \text{return ("}n\text{ is prime")} \\ \textbf{else} & b \equiv b^2 \pmod{n} \end{cases}$$

return ("n is composite")

Example

Let $n = 561$. Then $n - 1 = 560 = 16.35$; so $2^k = 2^4$ and $m = 35$. Let $a = 2$. Then

$$\begin{aligned} b_0 &\equiv 2^{35} &\equiv 263 \pmod{561} \\ b_1 &\equiv {b_0}^2 &\equiv 166 \pmod{561} \\ b_2 &\equiv {b_1}^2 &\equiv 67 \pmod{561} \\ b_3 &\equiv {b_2}^2 &\equiv 1 \pmod{561} \end{aligned}$$

as $b_3 \equiv 1 \pmod{561}$, we conclude that 561 is composite. Moreover $(b_2 - 1, 561) = 33$, is a non-trivial factor of 561.

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So, if n is prime, $b_{k-1} \equiv \pm 1 (\mathrm{mod}\ n)$, all other choices means n is composite.

Moreover, if $b_{k-1} = 1$, then, if we didn't stop at an earlier step,

$b_{k-2}^2 \equiv 1^2 (\mathrm{mod}\ n)$ with $b_{k-2} \not\equiv \pm 1 (\mathrm{mod}\ n)$.

$\Rightarrow n$ is composite.