SOME FIELD THEORETICAL STUDIES IN CONDENSED MATTER PHYSICS

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

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Conferences

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DEDICATION

Dedicated to my parents

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Contents

Summary

In this thesis we have presented studies on two problems in modern condensed matter physics using field theoretical methods.

Firstly, we have presented a one-loop renormalization group based study of perturbative Coulomb interaction in triple-Weyl semimetals. We found that the system is driven from its non-interacting fixed point to an anisotropic fixed point where the Coulomb interaction is screened only in the momentum direction in which the dispersion is linear. This gives logarithmic corrections to specific heat and longitudinal conductivities. We also presented a RG study of the effect of scalar Gaussian disorder in which we found that the system is perturbatively unstable to it and is likely driven to a diffusive metal phase. These findings are qualitatively similar to the double-Weyl semimetals, but with stronger (scaling) effects in the triple-Weyl case. In addition, we also gave a Boltzmann transport calculation for longitudinal conductivities and presented their scaling forms with temperature and chemical potential. Materials realizing the triple-Weyl semimetals have been elusive so far, but it would be interesting to compare our results with the experimental data once a material is synthesized.

Secondly, we studied bilayers and quasi-three-dimensional stacks formed from Jain's fractional quantum Hall states at $\nu = 2/5$ and $\nu = 2/3$, using their parton construction descriptions and argued for their relevance to the situation when the interlayer couplings between the layers is in the intermediate range. Using K-matrix descriptions of these theories, we calculated the toric ground state degeneracies and quasiparticle properties of the bilayered systems and argued that they are different from what one would have obtained for completely decoupled layers. In the quasi-3D scenario, we found that in the quasi-3D bulk a gapless gauge mode emerges whose origin is linked to the gauge interactions between the layers (generated through interlayer hopping). We also discuss that studying their surface transport is difficult due to the gapless gauge excitation in the bulk potentially destroying the energetic distinction between bulk and (gapless) boundary modes.

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Chapter 1

Introduction and structure of thesis

1.1 A preface

Quantum field theory has found many applications, both conceptually and methodologically, in many areas of condensed matter physics. In addition, many developments in field theory have been inspired from problems in condensed matter physics.

Diagrammatic Feynman methods, renormalization group theory, path integrals, abelian and non-abelian gauge theories, topological objects such as solitons and instantons, conformal field theory, topological field theories such as the Chern-Simons theory, non-linear sigma models, functional bosonization and many more topics in field theory have applications in modern condensed matter theory, and several books have been written covering these ideas and methods with the theme of "field theory in condensed matter physics", see for instance [1–4].

The common theme of this thesis is the use of some field theory methods in two problems in two topics within contemporary condensed matter. The two topics covered are -

A - A renormalization group study of the effects of Coulomb interactions and scalar disorder in the triple-Weyl semimetals.

B - A parton construction based study of bilayers, and quasi-three-dimensional limit, of Jain series fractional quantum Hall states.

Since the two topics covered are independent of each other, in this introductory chapter we give a very brief overview of the contents in the corresponding chapters.

1.1.1 Weyl semimetals

In many condensed matter systems in three dimensions, it can happen that a valence and a conduction band touch each other at isolated points (nodes) $\pm \boldsymbol{k}_0$ in the Brillouin zone. At these isolated points, the gap thus closes and these systems are thereby called as semimetals. When at least one of time or the inversion symmetry of the underlying system is broken, then at and very near these nodes, the effective low-energy Hamiltonian takes the form of a Weyl

equation (massless two-component Dirac equation), thus the name Weyl semimetal. That is, $H = v \pm \boldsymbol{\sigma}.\boldsymbol{k}$, where $\boldsymbol{\sigma}$ are the Pauli matrices denoting the pair of touching bands. These nodes in the momentum space carry unit (± 1) Berry monopole charge that are equal and opposite to each other in such a manner that the Brillouin zone as a whole carries no net monopole charge in order to obey the Nielsen-Ninomiya theorem [5, 6]. For example, when time reversal is obeyed but inversion is broken, nodes at $\pm\boldsymbol{k_1}$ have the same Berry monopole charge and thus the Brillouin zone has to have at least two other nodes at $\pm\boldsymbol{k_2}$ of opposite monopole charge, and thus the Brillouin zone has a total minimum of 4 nodes. On the other hand, when inversion symmetry is intact but time reversal is broken, nodes occuring at $\pm\boldsymbol{k_1}$ are of opposite monopole charge and thus the Brillouin zone has a minimum of 2 nodes [5, 6].

If additionally point group symmetries are present, [7] showed two more exotic possibilities, whose arguments we briefly review in Chapter 2. The two possibilities are the double-Weyl semimetals and the triple-Weyl semimetals. The former has quadratically dispersing bands in two directions and linearly dispersing band in the third, and its nodes have monopole charge ± 2. The latter has cubically dispersing bands in two directions and linearly dispersing band in the third, and its nodes have monopole charge ± 3. Expected material realizations of DWSM as yet are $HgCr_2Se_4$ and $SrSi_2$ [7–10], while TWSM awaits material realizations.

Semimetals, of whichever type, owing to their vanishing density of states at Fermi level can be readily affected by the long-range Coulomb interactions inherent in any system, as these long-range interactions may not be screened to a short-range interaction as usually happens in normal metals. In Chapter 2, after an introduction at the beginning with pointers to relevant studies in the literature on regular Weyl and other semimetals, we present a study of how perturbative Coulomb interactions effect the non-interacting triple-Weyl semimetals using renormalization group (RG) method at one-loop. The RG is perfectly suited for this purpose as the whole framework of the RG is to decide which sort of interactions are "relevant" or "irrelevant". A relevant one brings the earlier system at its noninteracting "fixed point" to a new "fixed point" where the interactions may produce a new phase, or a modified form of the original system such as, as in the case of triple-Weyl semimetals, an anitropically screened interacting semimetal. The effect of this on the analytical form of measurable quantities such as the specific heat is also presented. In addition, we also present a perturbative RG study of the weak scalar disorder in a triple-Weyl semimetal, finding that the system is perturbatively unstable to even a weak disorder. This study is based on [11]. We note that the published version [11] contains an error in the section about disorder and an erratum has been submitted to the journal. In this thesis we have presented the corrected version of the disorder section.

Before we end this section of this introductory chapter, let us present a brief review of Berry phases in band theory and Weyl semimetals.

1.1.1.1 Berry phases, Band theory and Weyl nodes

In solid state band theory, one deals with periodic lattices in which electrons reside under a potential function $V(\boldsymbol{r})$ which is translationally invariant under translation by lattice vectors $\boldsymbol{b}$. Thus, Bloch's theorem says that the eigenstates of the system can be labeled by the lattice momentum $\boldsymbol{k}$ along with a band index label n,

$$|\psi_{n\boldsymbol{k}}(\boldsymbol{r})\rangle = \exp(i\boldsymbol{k}.\boldsymbol{r})|\phi_{n\boldsymbol{k}}(\boldsymbol{r})\rangle \tag{1.1}$$

where $\phi_{n\boldsymbol{k}}(\boldsymbol{r})$ has inherits the lattice periodicity, and are called as the Bloch states. This also restricts the lattice momentum $\boldsymbol{k}$ to lie in the first Brillouin zone (BZ). In a periodic system, the BZ is also periodic and is geometrically often represented as a circle (in one dimensional BZ) or a torus (in two dimensional BZ), and generally as a d-torus in d-dimensions, where a d-torus $= S^1 \times S^1 \times ...dtimes$. The Bloch states are eigenstates of the Hamiltonian $H(\boldsymbol{k})$ and the corresponding eigenvalues $\epsilon_n(\boldsymbol{k})$ are also periodic in the momentum space with a period of the reciprocal lattice vector.

Berry phases [12] appear when a set of parameters in a Hamiltonian are varied adiabatically, such that when the parameters return to their original values, the eigenstates attain an extra phase factors, which are the Berry phases. In the above, $\boldsymbol{k}$ (within BZ) play the role of parameters over the BZ, and one round of traversing the BZ constitutes the action of $\boldsymbol{k}$ returning to their initial values. Thus, we illustrate the theory of Berry phase in this setting itself.

Suppose we adiabatically evolve the system in time t, and track the eigenstates $\phi_{n\boldsymbol{k}(t)}$. Adiabaticity means that the state $\phi_{n\boldsymbol{k}(t)}$ is an instantaneous eigenstate of the Hamiltonian, but it can in general pickup a phase as per quantum mechanical principles. This is akin to a gauge redundancy in defining an (eigen)state. Thus, we can write the instantaneous eigenstate as,

$$|\psi_n(t)\rangle = \exp(i\theta_n(t)) \exp\big(-i\int_0^t dt' \epsilon_n(\boldsymbol{k}(t'))\big)|\phi_n(\boldsymbol{k}(t))\rangle. \tag{1.2}$$

Substituting this into the Schroedinger equation, $i\partial_t|\psi_n(t)\rangle = H(\boldsymbol{k}(t))|\psi_n(t)\rangle$, multiplying from the left by $\langle\psi_n(t)|$, some algebra yields the Berry phase,

$$\theta_n[\mathcal{C}] = i\int_{\mathcal{C}} d\boldsymbol{k}\langle\psi_n(t)|\frac{\partial}{\partial\boldsymbol{k}}|\psi_n(t)\rangle = \int_{\mathcal{C}} d\boldsymbol{k}.\mathcal{A}_n(\boldsymbol{k}). \tag{1.3}$$

The quantity $\mathcal{A}_n(\boldsymbol{k})$ is known as the Berry connection or the Berry vector potential, and like any vector potential or a connection field, it is gauge dependent. In the above, $\mathcal{C}$ is the closed contour in the parameter (momentum) space. If a gauge transformation is performed on the eigenstates, $\phi_n(\boldsymbol{k}) \to \exp(ig(\boldsymbol{k})\phi_n(\boldsymbol{k}(t))$, the Berry connection is transformed as,

$$\mathcal{A}_n(\boldsymbol{k}) \to \mathcal{A}_n(\boldsymbol{k}) - \frac{\partial}{\partial\boldsymbol{k}}g(\boldsymbol{k}). \tag{1.4}$$

Thus, we can define a gauge invariant quantity, the Berry curvature $\Omega_n(\boldsymbol{k})$ from the Berry

connection, which in three dimensions can be concisely written as $\Omega_n(\boldsymbol{k}) = \boldsymbol{\nabla} \times \mathcal{A}_n(\boldsymbol{k})$. From Stokes' theorem then, we can equivalently write $\theta_n[\mathcal{C}] = \int_{\mathcal{S}} d\boldsymbol{\mathcal{S}}.\Omega_n(\boldsymbol{k})$, where $\mathcal{S}$ is the surface enclosed by the contour $\mathcal{C}$.

In the parameter (momentum) space, the gauge invariant, and therefore physically meaningful, Berry curvature plays the role of a magnetic field. Clearly, the Berry phase picks up values in the multiples of 2π under varying the parameters (momenta) in closed contours. Moreover, the Berry phase depends only the contour $\mathcal{C}$ and is therefore a geometric property of the parameter space.

From the Berry curvature, one can define topological invariants by a global integration of the curvature, a most useful one among them being the Chern number [13]. When the parameter (momenta) space is a surface $\mathcal{S}$ without boundaries such as the sphere or the torus, as is the case with the BZ in our case, and since one can think of the Berry curvature as some abstract magnetic field, there can exist a non-vanishing "flux" through the surface $\mathcal{S}$. A powerful result is that this flux is quantized,

$$\frac{1}{2\pi}\int_{\mathcal{S}} \Omega_n(\boldsymbol{k}) d^2k = \mathfrak{C}_n, \tag{1.5}$$

where $\mathfrak{C}_n$ is the so-called Chern number (often also referred to in the context of Berry phases as the Berry monopole charge for its close resemblance to Dirac's monopole quantization) which is always an integer, and is directly seen from the above to also be gauge-invariant, and thus has physical meaning. Thus, every band n is labeled by an integer-valued topological invariant $\mathfrak{C}_n$.

Indeed, in the famous "TKNN" paper [14], the authors showed using the Kubo formula for the Hall conductivity and first order perturbation theory that the Hall conductivity σ_{xy} is directly related to the Berry curvature and thus to the Chern number,

$$\sigma_{xy} = \frac{e^2}{\hbar^2}\sum_n \frac{1}{4\pi^2}\int d^2k \Omega_n(\boldsymbol{k}), \tag{1.6}$$

where the integration is performed over energy levels below some Fermi energy. When the Fermi energy lies in a gap, the integral covers the entire BZ and then one has,

$$\sigma_{xy} = \frac{e^2}{h}\sum_n \mathfrak{C}_n. \tag{1.7}$$

Thus, the Hall conductivity is also a topological invariant, and this discovery marked the beginning of the field of topological phases in condensed matter physics.

Coming to regular Weyl semimetals, using the above expressions, one can readily calculate the Berry monopole charge associated to its Hamiltonian and find it to be ± 1, and likewise for double- and triple-Weyl semimetals to be ± 2 and ± 3 respectively. The Hall conductivity, owing to the above discussion, similarly also distinguishes, as a measurable quantity, between these semimetals.

1.1.2 Jain's FQH states

Fractional quantum Hall (FQH) effect is one of the most interesting and widely studied phenomena in modern condensed matter physics [15, 16]. Here we present a brief overview of the FQH physics. The idealized experimental system we have in mind is the two-dimensionally confined electron gas kept in sufficiently strong perpendicular magnetic field such that the spins of the electrons have been polarized and we can imagine the 2D quantum Hall strip to contain effectively spinless electrons strongly interacting with each other through microscopic Coulombic interactions while simultaneously also performing cyclotron motion in the plane due to the perpendicular magnetic field.

The interplay between the inter-particle interactions, planar confinement, effectively quenched kinetic energy due to the magnetic field transforming the system into a series of Landau levels, and the consequent cyclotron motion, makes the problem highly non-perturbative, as within a single Landau level (say the lowest one) the only energy scale is the Coulomb interaction scale e^2/ℓ_B, and there is no small parameter with which to construct a perturbation theory. From this setting emerges a unique and non-perturbative many-body state at certain filling factors that are called as the fractional quantum Hall states. We assume that the magnetic field is so strong that all the electrons of the system have been forced to reside only in the lowest Landau level, and the next Landau level is thus separated from the lowest one by a large cyclotron gap $\hbar\omega$ that is unachievable by the electrons in the lowest Landau level, that is we assume no Landau level mixing in this idealized scenario. The filling factor ν is defined as the ratio of the electron density to the magnetic flux density of the confined 2D layer. At these fillings, the Hall conductivity σ_{xy} is fractionally quantized as $\nu e^2/h$. The Hall resistance exhibit quantized plateaus at these fillings with respect to the magnetic field, where as the longitudinal resistance shows a deep minimum. Moreover, the FQH states are "incompressible", meaning the ground state is gapped from the excitations [15, 16].

1.1.2.1 Laughlin and Jain States

Let us review the two primary classes of FQH states observed in the experiments, the Laughlin and the Jain states. The Laughlin states [17] are states at filling factor $\nu = 1/m$, with m an odd integer (>1) for a fermionic system. The Jain states [18], also called composite fermion states, correspond to $\nu = m/(2pm \pm 1)$ with both m and p integers. Note that the Jain class of states subsume the Laughlin states.

The wavefunction of the fermionic Laughlin states take the form,

$$\Psi_{1/m} = \Pi_{i<j}(z_i - z_j)^m \exp(-\frac{1}{4\ell^2}\sum_i |z_i|^2), \tag{1.8}$$

where z_i denote the coordinates of the ith electron written as a complex number on the plane, the magnetic length $\ell = (\hbar c/eB)^{1/2}$, B being the applied perpendicular magnetic field. The oddity

of m ensures antisymmetry under fermionic exchange.

The Jain series of states view the FQH of the electrons as an effective integral quantum Hall (IQH) effect of new emergent degrees of freedom called the composite fermions [16, 18]. In this picture, each electron is imagined to bind with itself two (or any even number in general) units of flux from the background magnetic field, and the resultant "composite fermion" object sees a reduced total magnetic field. In other words, if ρ is the electron density, ϕ_o is the flux quantum, and $2p$ denotes the even number of attached fluxes, then the effective reduced magnetic field seen by the composite fermions (CFs) is,

$$B^* = B - \rho\phi_o 2p. \tag{1.9}$$

The proposal of [18] says that when the CFs have an IQH effect of their own, that is, their effective filling factor is an integer m, where by definition of a filling factor, $m = \rho\phi_o/|B^*|$, and since the density of CFs is the same as the density of electrons, from the above equation it is directly seen that the electronic filling factor ν is simply,

$$\nu = m/(2pm \pm 1) \tag{1.10}$$

where the $\pm$ indicates whether the effective magnetic field experienced by the CFs B^* points in the same direction to B or opposite. We note that an important assumption is made in these arguments - that the quantum Hall gap does not close during this imaginative process of flux attachement.

The wavefunction for the states described in the CF framework can be written as,

$$\Psi_\nu = \mathcal{P}_{LLL}\Phi_{\pm m}\Pi_{i<j}(z_i - z_j)^{2p}\exp(-\frac{1}{4\ell^2}\sum_i |z_i|^2) \tag{1.11}$$

where $\mathcal{P}_{LLL}$ is formally an operator for projecting into the lowest Landau level the technicalities of which we do not get into here [16], and Φ_m is the Vandermonde determinant representing the fact that the CFs occupy m filled Landau levels, which for $m = 1$ takes the simple form,

$$\Phi_1 = \Pi_{i<j}(z_i - z_j)^m \exp(-\frac{1}{4\ell^2}\sum_i |z_i|^2) \tag{1.12}$$

For higher values of m, it takes substantially more complicated forms [16].

It is now clear from above that the Laughlin wavefunctions correspond to the $m = 1$ case of the CF wavefunctions, that is, Laughlin states are described by CFs exhibiting an integral quantum Hall effect of one (lowest) fully filled Landau level.

We would like to point out a few points in passing -

1 - Even though the above descriptive picture talks of attaching "flux" quanta to the electrons, the actual CF wavefunction above instead attaches vortices to the electrons [18], where a flux quanta refers to the phase of the complex number $(z_i - z_j)$ while a vortex is this complex number

of the Jastrow form itself. Indeed, the CF wavefunctions above take the Jastrow form and attach $2p$ vortices to the electrons. On the other hand, the field theoretic description of the CF physics in terms of CF Chern-Simons theory [19] attaches fluxes to the electrons at the mean field level and the "flux attachement" procedure is thus named from this theory. We do not get into the CF Chern-Simons description here and instead refer to its review in [1].

2 - The CF picture essentially reduces the problem of strongly interacting electrons in a fractionally filled Landau level to a problem of effectively non-interacting CFs occupying fully filled Landau levels. This lends the problem tractable to a large extent [16].

1.1.2.2 Effective Chern-Simons theory

In this subsection, we briefly describe how Chern-Simons (CS) theories [20] capture the universal low-energy picture of FQH physics. This particular view described below is often referred to as the hydrodynamic theory of FQH effect [21]. In the parton theory of FQHE, the CS theory enters in this manner indeed and Chapter 3 begins with reviewing the parton theory of the Jain states of relevance to the chapter. We consider just the basic case of an abelian CS terms, which correspond to abelian FQH states only.

The "hydrodynamic" description is an effective description of the quantum Hall bulk that is obtained only from considerations of universal physics of the low-energy. One begins with asking the general form for a gauge field $b_\mu(\boldsymbol{x})$ in two dimensions whose current is conserved, that is, $\partial_\mu J_\mu = 0$. This is identically true for

$$J_\mu(\boldsymbol{x}) = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda. \tag{1.13}$$

Under a local transformation $b\mu(\boldsymbol{x}) \to b_\mu(\boldsymbol{x}) + \partial_\mu\phi(\boldsymbol{x})$, $J_\mu \to J_\mu$, and so $b_\mu(\boldsymbol{x})$ is indeed a $U(1)$ gauge field. One then asks the general form for the action for this theory in two space dimensions that is time-reversal breaking and parity-breaking (as we want to describe a quantum Hall system), gauge invariant and local. The first natural guess of a Maxwell action is not correct, as it is invariant under time-reversal. Locality is demanded due to the fact that the QH bulk is an incompressible, that is gapped, system and thus the low-energy action must be local.

It is here that the Chern-Simons term enters into the picture. It not only satisfies all the above required conditions, but is more "relevant" in the low-energy than the Maxwell terms due to containing only one derivative instead of two (in the case of Maxwell terms). Thus the leading term in the effective theory is a CS term.

(Note that we are talking only about describing the QH bulk; for a QH system with boundary the CS bulk theory is not gauge-invariant in the bulk separately, but instead produces non-vanishing dynamics at the edges and gauge invariance is maintained only over the full system [21]. We do not get into these issues here.)

Thus, consider the Lagrangian,

$$\mathcal{L}[b_\mu] = \frac{m}{4\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda + eJ_\mu A_\mu, \tag{1.14}$$

where A_μ is the external electromagnetic (EM) field, e the charge of the incompressible fluid constituents described by the currents J_μ, and m is an integer.

It is readily verifiable from here that the theory above describes a QH bulk. To capture the Hall conductivity, we integrate out the b_μ field to obtain an effective Lagrangian for the EM field,

$$\mathcal{L}_{eff}[A_\mu] = -\frac{e^2}{4\pi m}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \tag{1.15}$$

From here, we have the induced current by definition,

$$eJ_\mu^{induced} = -\frac{\delta \mathcal{L}_{eff}}{\delta A_\mu} = \frac{e^2}{2\pi m}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \tag{1.16}$$

The Hall conductivity, in units of $\hbar = 1$ and $c = 1$), follows from the above,

$$\sigma_{xy} = \frac{1}{m}\frac{e^2}{2\pi}. \tag{1.17}$$

Thus, when $m \neq 1$, this describes an FQH system corresponding to the filling fraction $\nu = 1/m$, that is the Laughlin states. Likewise a description of this sort can be given for the Jain states, and the parton theory of the Jain states reviewed in Chapter 3 shall serve partly as an illustration in the same spirit as above, because in any parton description, the individual partons are put into effective QH theories with appropriate CS terms in the same spirit as above.

Note that if m is an even integer, the resultant theory describes a QH effect in a bosonic system (bosonic because the corresponding wavefunction, say the Laughlin type, would be symmetric under particle exchange). A composite boson CS theory of FQH effect was the first CS based theory of FQHE that was proposed in [22] and in a flux-attachement picture, a composite boson results from attaching odd number of flux quanta to the electron as opposed to the even number $2p$ of them in the composite fermion picture.

What about excitations? Suppose in the low-energy hydrodynamic picture, the excitations have smooth "worldlines" and can be represented by another current field j_μ and they minimally couple to the gauge fields b_μ. The total Lagrangian now is,

$$\mathcal{L}[b_\mu] = \frac{m}{4\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda + eJ_\mu A_\mu + j_\mu b_\mu. \tag{1.18}$$

Integrating the b_μ we have,

$$\mathcal{L}_{eff}[A_\mu, j_\mu] = -\frac{e^2}{4\pi m}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{e}{m} j_\mu A_\mu - \frac{\pi}{m} j_\mu \mathcal{B}_\mu. \tag{1.19}$$

Here, $\mathcal{B}_\mu$ is a field generated due to the fact that we can always write $j_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu \mathcal{B}_\lambda$. This term in the above action is responsible for giving the quasiparticle excitations fractional statistics angle $\theta = \pi/m$ through a subtle "worldline picture" derivation given by Polyakov [23] that we

do not review here and instead refer to the original article of Polyakov and to [1] for an indepth discussion of this.

However, let us show that the quasiparticle excitations carry fractional charge. The electromagnetic current is obtained from the above effective Lagrangian as,

$$eJ_{\mu}^{EM} = -\frac{\delta \mathcal{L}_{eff}}{\delta A_{\mu}} = \frac{e^2}{2\pi m}\epsilon_{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda} - \frac{e}{m}j_{\mu}. \tag{1.20}$$

If we assume that there is just one quasiparticle excitation at the origin, that is, $j_0 = \delta(\boldsymbol{x})$ and $\boldsymbol{j} = 0$, then the above equation becomes,

$$eJ_{0}^{EM} = \frac{e^2}{2\pi m}B - \frac{e}{m}\delta(\boldsymbol{x}), \tag{1.21}$$

where B is the external uniformly applied magnetic field. The second term in this equation clearly tells us that the quasiparticle excitation's charge is fractional at e/m.

A CS theory with coefficient $m > 1$ exhibits ground state degeneracy on torus $= m$ (see Appendix in Chapter 3 for a short demonstration). This is one of the ways in which FQH systems are "topologically ordered" [1, 2, 24, 25].

These results are readily generalizable to a multicomponent CS theory described by the K-matrix representation [21, 26], which will be used in Chapter 3 where we present a study of bilayers and quasi-three-dimensional stacks formed from Jain FQH states at $\nu = 2/5$ and $2/3$ using their parton descriptions and the consequent K-matrix theories, based on [27].

Chapter 2

Effects of Coulomb interaction and disorder on triple-Weyl semimetal

2.1 Introduction

As we reviewed in the introductory chapter, semimetals are systems with isolated band-touching points in the Brillouin zone which have attracted a lot of interest and activity in recent years. Near these band-touching points, quasiparticles typically obey a linearised dispersion. Prominent examples include Dirac semimetals in two dimensions such as graphene, and Weyl semimetals [5, 6, 28] in three dimensions. It is also possible to have band-touching points with non-linear dispersion in some or all directions, with the additional requirement of some lattice symmetry protecting them. Examples include parabolic (or quadratic band-touching) semimetals in 2D [29] and 3D [30], and double-Weyl and triple-Weyl semimetals [7].

Due to vanishing density of states (DOS) at Fermi level, semimetals in general behave differently under long-range Coulomb interactions compared to normal Fermi liquids where due to finite DOS Thomas-Fermi screening occurs and the long-range interactions are effectively reduced to short-range. In semimetals, vanishing DOS implies no such screening, and long-range interactions can drive a semimetallic system to a new non-trivial system such as, in case of 3D quadratic band-touching (QBT) systems for instance, the Abrikosov-Luttinger non-Fermi liquid [30–33] or a topological Mott insulator [34]. In case of linearly dispersing semimetals such as the undoped Dirac and Weyl semimetals, long-range interactions typically are irrelevant. For DWSM, due to its somewhat higher DOS compared to linearly-dispersing WSM, it has been shown that perturbative long-range Coulomb interaction drive the system to a new fixed point where anisotropic screening of the Coulomb interaction occurs and various measurable quantities such as the specific heat, conductivities, etc acquire logarithmic corrections [35, 36]. Short-range interactions are perturbatively irrelevant in semimetals in general, again due to vanishing DOS, however sufficiently strong short-range interactions can induce interesting gapped phases. In 2D

QBT models for instance they can result in nematic and quantum anomalous Hall states [29] and in Weyl semimetals resulting states can be charge and spin density waves and axionic insulators [37, 38], excitonic insulators [39], and superconductivity of both BCS and unconventional FFLO/PDW type [40–43].

The difference with respect to long-range Coulomb interaction in the behaviour of linearly dispersing semimetals compared to semimetals with higher power of dispersion stems from the fact that while they all have vanishing DOS, the latter type still has higher DOS than the former. In this chapter we will study the effects of long range Coulomb interaction in triple-Weyl semimetals, which in the low-energy limit have higher DOS $\propto \epsilon^{2/3}$ than DWSM with DOS $\propto \epsilon$ and regular WSM with DOS $\propto \epsilon^2$. We will see that similarly to DWSM, TWSM also screen the Coulomb interaction anisotropically and the screening effect is somewhat stronger compared to DWSM, and we will show logarithmic corrections to measurable quantities such as the specific heat and conductivities.

Subsequently, we will consider the case of weak scalar disorder in TWSM. The general subject of disorder in semimetals has been an active and widely debated area of research. More than 30 years ago, Fradkin [44, 45] studied the problem within a self-consistent approach and found that Dirac (and Weyl) semimetals are stable with respect to weak disorder. Subsequently, after the emergence of the field of topological semimetals, this problem was re-analysed using RG and other methods with the same conclusion, see for instance [46–54]. However, recently it was argued in [55] that even weak disorder can induce rare-regions effects in Dirac (and Weyl) semimetals (which are not accounted for in self-consistent Born apprimation and RG studies), thereby threatening their stability against disorder. Related issues have also been studied in [56, 57]. Moreover, in the presence of both disorder and Coulomb interactions, Dirac and Weyl semimetals can show non-Fermi liquid behaviour [58]. A good general review on disorder effects in various semimetals is [59].

Coming to multi-Weyl semimetals, for the case of DWSM, it is easy to argue that due to higher DOS than regular DSM/WSM, disorder must have stronger effects on DWSM and indeed, it was found that DWSM is unstable against weak scalar disorder in a one-loop RG analysis in [60, 61]. We will see that TWSM is also unstable against weak scalar disorder.

Let us first begin with reviewing the arguments for the existence in principle of multi-Weyl semimetals.

2.2 Generalities on multi-Weyl semimetals

In this section, we briefly review the general setting and arguments of [7] for the existence in principle of multi-Weyl semimetals, and we choose to use their notational conventions. Let $\mathcal{C}_n$ denote the rotation symmetry operator (about a chosen axis) of a three-dimensional lattice, where n can take values $(2, 3, 4, 6)$. Consider the operator $\mathcal{C}_m$, where m is a factor of n, the system obviously also remains invariant under $\mathcal{C}_m$ (we also assume translational symmetry). For

a tight-binding Hamiltonian $\mathcal{H}(\boldsymbol{k})$, we have then

$$\mathcal{C}_m \mathcal{H}(\boldsymbol{k}) \mathcal{C}_m^{-1} = \mathcal{H}(R_m \boldsymbol{k}), \tag{2.1}$$

where R_m is the corresponding rotation matrix about a chosen axis (which will be the z-axis in the discussion below). In general there exist lines in momentum space for which $R_m \boldsymbol{k} = \boldsymbol{k}$ holds for all $\boldsymbol{k}$ in the Brillouin zone (BZ), and for generic non-identity rotation matrices. On these lines then, the above equation directly implies,

$$[\mathcal{C}_m, \mathcal{H}(\boldsymbol{k})] = 0. \tag{2.2}$$

Thus, all bands on these lines can be labeled by the respective eigenvalue(s) of $\mathcal{C}_m$ operator. For semimetallic systems where the a valence band and a conduction band meet at $\boldsymbol{k_0}$ on a $\mathcal{C}_m$-invariant line, for a small in-plane deviation $\boldsymbol{q}$ around this point the Hamiltonian takes an effective form near this point as,

$$\mathcal{H}_{eff}(\boldsymbol{k_0} + \boldsymbol{q}) = f(\boldsymbol{q})\sigma_+ + f^*(\boldsymbol{q})\sigma_- + g(\boldsymbol{q})\sigma_z. \tag{2.3}$$

Here, $\sigma_\pm = \sigma_x \pm i\sigma_y$, f and g are complex and real functions respectively, and the conduction and valence bands correspond to the basis $(1,0)^T$ and $(0,1)^T$ respectively. It is readily seen that in this basis, the matrix representation of $\mathcal{C}_m$ is diagonal, and we denote its diagonal elements as u_c and u_v where the subscripts correspond to conduction and valence bands respectively.

An eigenvalue of $\mathcal{C}_m$ is of the general form $\alpha_p = \exp(2\pi i p/m)$, with $p = 0, 1, , , m-1$. Let then $u_c = \alpha_p$ and $u_v = \alpha_r$. When $p \neq r$, the matrix representation of $\mathcal{C}_m$ is simply,

$$rep(\mathcal{C}_m) = \exp(i\pi \frac{p-r}{m} \sigma_z). \tag{2.4}$$

The general form of the effective Hamiltonian written above transforms under $\mathcal{C}_m$ as,

$$\mathcal{C}_m \mathcal{H}_{eff}(\boldsymbol{q}) \mathcal{C}_m^{-1} = g(\boldsymbol{q})\sigma_z + f(\boldsymbol{q}) \exp(-2\pi i \frac{p-r}{m})\sigma_+ + h.c. \tag{2.5}$$

whereas the momenta basis $q_\pm = q_x \pm i q_y$ transforms under the rotation as,

$$R_m(q_+, q_-) = (q_+ \exp(\frac{2\pi i}{m}), q_- \exp(-\frac{2\pi i}{m})). \tag{2.6}$$

From these, we have the following constraint equations on the form of f and g for a given (p, q),

$$f(q_+, q_-) \exp(-2\pi i \frac{p-r}{m}) = f(q_+ \exp(\frac{2\pi i}{m}), q_- \exp(\frac{-2\pi i}{m})), \tag{2.7}$$

$$G(q_+, q_-) = g(q_+ \exp(\frac{2\pi i}{m}), q_- \exp(\frac{-2\pi i}{m})). \tag{2.8}$$

The essential message here is that in a lattice the rotational symmetry is discrete of a finite order (m here) and thus total angular momentum is conserved only modulo m. So when $p \neq r$, we have $u_c/u_v = \exp(2\pi i(p-r)/m)$, implying that the total angular momentum between the valence and conduction bands differs by $p-r$. To conserve the total angular momentum modulo m, the off-diagonal term in the effective Hamiltonian must have the momentum dependence of the form q_-^{p-r} or q_+^{m-p+r}.

These considerations gives the classification of Weyl-semimetals protected by the lattice rotation symmetry as described in greater detail in, and listed in, [7]. The double-Weyl semimetal corresponds to the case of $\mathcal{C}_4$ symmetry and $u_c/u_v = -1$, for which the effective Hamiltonian takes the form, in terms of general complex parameters a, b, c in which a may be equal to b,

$$\mathcal{H}_{eff}(\boldsymbol{q}) = (aq_+^2 + bq_-^2)\sigma_+ + c\sigma_z + h.c., \tag{2.9}$$

and a similar form is obtained when the symmetry of the lattice is $\mathcal{C}_6$ with $u_c/u_v = -\exp(\pm 2\pi i/3)$.

The triple-Weyl semimetal is obtained only in the case of lattice symmetry being $\mathcal{C}_6$ with $u_c/u_v = -1$, and the effective Hamiltonian is of the form,

$$\mathcal{H}_{eff}(\boldsymbol{q}) = (aq_+^3 + bq_-^3)\sigma_+ + c\sigma_z + h.c., \tag{2.10}$$

No higher order Weyl nodes are obtained from this analysis of [7]. From these effective Hamiltonians, the Berry monopole charges can be directly evaluated and found to be ± 2 and ± 3 respectively for the DWSM and TWSM.

2.3 TWSM with Coulomb interaction

As reviewed above, the effective Hamiltonian of a triple-Weyl semimetal takes the form $\mathcal{H} = \vec{d}_q.\vec{\sigma}$, where,

$$\vec{d}_q \approx [\frac{q_x^3 - 3q_xq_y^2}{m}, \frac{q_y^3 - 3q_yq_x^2}{m}, v_zq_z]. \tag{2.11}$$

This Hamiltonian has been written after expanding the $q_\pm$ in the previous notations, and the parameters being $a = b = 1/m$ and $c = v_z$.

Let the two nodal points be at $\pm Q$. We can write the fermion operators near the nodes as,

$$\begin{aligned}
\Psi(\boldsymbol{r}) &= \int \frac{d^3k}{(2\pi)^3}\psi(\boldsymbol{k})e^{i\boldsymbol{k}.\boldsymbol{r}} \\
&= \int \frac{d^3k}{(2\pi)^3}\psi_Q(\boldsymbol{k})e^{i(\boldsymbol{k}+\boldsymbol{Q}).\boldsymbol{r}} + \int \frac{d^3k}{(2\pi)^3}\psi_{-Q}(\boldsymbol{k})e^{i(\boldsymbol{k}-\boldsymbol{Q}).\boldsymbol{r}} \\
&= \Psi_Q(\boldsymbol{r})e^{i\boldsymbol{Q}.\boldsymbol{r}} + \Psi_{-Q}(\boldsymbol{r})e^{-i\boldsymbol{Q}.\boldsymbol{r}}
\end{aligned} \tag{2.12}$$

where $\Psi_{\pm Q}(\boldsymbol{k}) = \Psi(\boldsymbol{k} \pm \boldsymbol{Q})$. The charge density can then be written in terms of the low energy

fermions $\Psi_{\pm Q}$ as,

$$\rho(\boldsymbol{r}) = \rho_Q(\boldsymbol{r}) + \rho_{-Q}(\boldsymbol{r}) + (\Psi^\dagger_{-Q}(\boldsymbol{r}\Psi_Q(\boldsymbol{r})e^{i2\boldsymbol{Q}.\boldsymbol{r}} + hc), \tag{2.13}$$

where $\rho_{\pm Q}(\boldsymbol{r}) = \Psi^\dagger_{\pm Q}(\boldsymbol{r}\Psi_{\pm Q}(\boldsymbol{r})$. The additional finite momenta terms above may be neglected due to the fast oscillating factor. Next we consider instantaneous Coulomb interaction $V(r) = 1/r$ (in Fourier space this is $V(q) = 1/q^2$), with its corresponding action,

$$\begin{aligned} S_{int} &= \frac{g^2}{2}\int d^3r d^3r' \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r'})}{4\pi|r-r'|} \\ &= \frac{g^2}{2}\int d^3r d^3r' \frac{[\rho_Q(\boldsymbol{r}) + \rho_{-Q}(\boldsymbol{r})][\rho_Q(\boldsymbol{r'}) + \rho_{-Q}(\boldsymbol{r'})]}{4\pi|r-r'|}. \end{aligned} \tag{2.14}$$

By Hubbard-Stratonovich transformation, we can decouple the four-fermion interaction by introducing a bosonic scalar field ϕ. The effective Euclidean Lagrangian for triple-Weyl semimetals in the presence of Coulomb interaction then has the final form,

$$\begin{aligned} L = \quad & \Psi^\dagger(\partial_\tau - ig\phi + H(-i\vec{\nabla}))\Psi \\ & + \frac{1}{2}\left(\frac{1}{\sqrt{\eta}}((\partial_x\phi)^2 + (\partial_y\phi)^2) + \sqrt{\eta}(\partial_z\phi)^2\right), \end{aligned} \tag{2.15}$$

where ϕ denotes the Coulomb interaction scalar field, Ψ the nodal fermions, g is the Coulomb coupling ($\sim e/\epsilon$ where ϵ is the dielectric constant of the material) and η is an anisotropic parameter introduced due to the inherent anisotropic nature of the fermionic dispersion. Since we are studying the problem of long-range interaction, or equivalently, since we are interested in low momentum scales, we ignore the possible coupling between the two nodes which occurs at a much larger and finite momentum scale. We also remark that we are considering only a Thomas-Fermi like screening, that is, no dynamical screening (Lindhard screening) is being take into account. All our calculations below are for instantaneous Coulomb interaction, which has no dynamic component.

2.4 Renormalization group analysis

We will adopt the RG scheme of integrating $q_\perp$ in a shell of momentum $(\Lambda e^{-\ell}, \Lambda)$ while no such momentum shell integration over q_z. Here, Λ is some cutoff parameter. This procedure is similar to the one adopted for the case of DWSM in [35, 36]. Defining $b = e^l, l \ll 1$, we write the scaling dimensions as

$$\tau = b^z\tau_R, x = bx_R, y = by_R, z = b^{z_1}z_R, \tag{2.16}$$

$$v_z = Z^{-1}_{v_z}v_{zR}, m = Z_m m_R, \tag{2.17}$$

$$\eta = Z^{-1}_\eta \eta_R, g = Z^{-1/2}_g g_R, \tag{2.18}$$

$$\Psi = Z_{\Psi}^{-1/2}\Psi_R, \phi = Z_{\phi}^{-1/2}\phi_R. \tag{2.19}$$

At tree level, we have $[v_z] = z - z_1$, $[m] = 3 - z$, $[\eta] = 2(1 - z_1)$, $[g] = \frac{z-1}{2}$.

From the Lagrangian of a triple-Weyl semimetal with long-range Coulomb interaction given in the main text, we have the fermion and boson Green's functions as

$$D_o(q_0, \vec{q}) = \frac{\sqrt{\eta}}{q_\perp^2 + \eta q_z^2}, \tag{2.20}$$

$$G_o(q_0, \vec{q}) = \frac{iq_0 + \vec{d}_q.\vec{\sigma}}{q_0^2 + E_q^2}, \tag{2.21}$$

where $q_\perp^2 = q_x^2 + q_y^2$ and $E_q^2 = \frac{q_\perp^6}{m^2} + v_z^2 q_z^2$.

The static boson self-energy at one-loop is

$$\begin{aligned} \Pi(\vec{p}) \quad &= g^2 \int \frac{d^4q}{(2\pi)^4} tr(G_0(q)G_0(p+q)) \\ &= \frac{2g^2}{(2\pi)^4} \int dq_0 dq_z \int' dq_x dq_y \frac{-q_0^2 + \vec{d}_q.\vec{d_{p+q}}}{(q_0^2 + E_q^2)(q_0^2 + E_{p+q}^2)}. \end{aligned} \tag{2.22}$$

where the prime on the integral denotes that momentum-shell integration will be performed on the corresponding variables. Recall that the RG scheme we have adopted is to integrate $q_\perp$ in a shell of momentum $(\Lambda e^{-\ell}, \Lambda)$.

After doing the frequency integral, expanding and evaluating the momentum integrals to quadratic order in the external momenta (as they are the slow modes), we get

$$\Pi(\vec{p}) = \frac{-3g^2 p_\perp^2 \ell}{2\pi^2 v_z} - \frac{g^2 v_z m^2 p_z^2 \ell}{6\pi^2 \Lambda^4}. \tag{2.23}$$

The corrected one-loop boson propagator then is $D^{-1}(p) = D_o^{-1}(p) - \Pi(p)$. In terms of the two dimensionless parameters

$$\alpha = \frac{m^2 g^2 v_z}{\sqrt{\eta}\Lambda^4} \text{ and } \beta = \frac{m^2 g^4}{\Lambda^4}, \tag{2.24}$$

we have the corrected bosonic propagator as

$$D^{-1}(\vec{p}) = \frac{1}{2\sqrt{\eta}} p_\perp^2 \left(1 + \frac{3\beta}{\pi^2\alpha}\ell\right) + \frac{\sqrt{\eta}}{2} p_z^2 \left(1 + \frac{\alpha}{3\pi^2}\ell\right). \tag{2.25}$$

The static fermion self-energy at one-loop is

$$\Sigma(\vec{p}) = -g^2 \int \frac{d^4q}{(2\pi)^4} G_o(q) D_o(p-q). \tag{2.26}$$

After the frequency integral, we get

$$\Sigma(\vec{p}) = -\frac{g^2}{16\pi^3}\int' d^3q \frac{\vec{d}.\vec{\sigma}}{E_q} I(q), \tag{2.27}$$

where

$$I(q) = \frac{\sqrt{\eta}}{(p_x-q_x)^2+(p_y-q_y)^2+\eta(p_z-q_z)^2}. \tag{2.28}$$

As before, the prime on the integral denotes momentum-shell integration declared earlier. The correction to the σ_x term can be evaluated as

$$\begin{aligned}\Sigma_x &= \frac{tr(\sigma_x(\partial^3_{p_x} - 3\partial_{p_x}\partial^2_{p_y})\Sigma(\vec{p}))}{tr(\sigma_x\sigma_x)}|_{\vec{p}=0} \\ &\approx -\frac{6c_1 g^2\sqrt{\eta}\ell}{mv_z\pi^2},\end{aligned}$$

where $c_1 \approx 3.8$. By symmetry of the dispersion in the (q_x, q_y) plane, Σ_y has the same value as Σ_x. Similarly, the correction to the σ_z term can be evaluated as

$$\begin{aligned}\Sigma_z &= \frac{tr(\sigma_z\partial_{p_z}\Sigma(\vec{p})}{tr(\sigma_z\sigma_z)}|_{\vec{p}=0} \\ &\approx -\frac{c_2 g^2\sqrt{\eta}\ell}{2\pi^2},\end{aligned}$$

where $c_2 \approx 0.5$. Finally we have the one-loop corrected fermionic part of the Lagrangian in terms of the dimensionless parameters as

$$\begin{aligned}&= d_x\sigma_x\left(1+\frac{6c_1\beta\ell}{\alpha\pi^2}\right) + d_y\sigma_y\left(1+\frac{6c_1\beta\ell}{\pi^2\alpha}\right) \\ &+d_z\sigma_z\left(1+\frac{c_2\beta\ell}{2\alpha\pi^2}\right).\end{aligned} \tag{2.29}$$

The scaling dimensions at one-loop are

$$Z_{v_z} = e^{\ell(z-z_1)}\left(1+\frac{c_2\beta\ell}{2\alpha\pi^2}\right), \tag{2.30}$$

$$Z_\eta = e^{2\ell(1-z_1)}\left(1+\frac{\alpha\ell}{3\pi^2}\right)\left(1+\frac{3\beta\ell}{\alpha\pi^2}\right)^{-1}, \tag{2.31}$$

$$Z_{g^2} = e^{\ell(z-1)}\left(1+\frac{3\beta\ell}{\alpha\pi^2}\right)^{-\frac{1}{2}}\left(1+\frac{\alpha\ell}{3\pi^2}\right)^{-\frac{1}{2}}, \tag{2.32}$$

$$Z_m = e^{-\ell(z-3)}\left(1+\frac{6c_1\beta\ell}{\alpha\pi^2}\right)^{-1}, \tag{2.33}$$

$$Z_\phi = e^{\ell(1+z)}\left(1+\frac{3\beta\ell}{\alpha\pi^2}\right)^{\frac{1}{2}}\left(1+\frac{\alpha\ell}{3\pi^2}\right)^{\frac{1}{2}}, \tag{2.34}$$

$$Z_\Psi = e^{\ell(2+z_1)}. \tag{2.35}$$

From the above we get the RG equations as

$$\frac{d\ln\eta}{d\ell} = 2(1-z_1) + \frac{\alpha}{3\pi^2} - \frac{3\beta}{\alpha\pi^2}, \tag{2.36}$$
$$\frac{d\ln m^{-1}}{d\ell} = z - 3 + \frac{6c_1\beta}{\alpha\pi^2}, \tag{2.37}$$
$$\frac{d\ln v_z}{d\ell} = z - z_1 + \frac{c_2\beta}{2\alpha\pi^2}, \tag{2.38}$$
$$\frac{d\ln g^2}{d\ell} = z - 1 - \frac{3\beta}{2\alpha\pi^2} - \frac{\alpha}{6\pi^2}. \tag{2.39}$$

From the above equations we can determine the RG equations for the two dimensionless parameters as

$$\frac{d\ln\beta}{d\ell} = 2z - 2 - \frac{\alpha}{3\pi^2} - \frac{3\beta}{\alpha\pi^2}, \tag{2.40}$$
$$\frac{d\ln\alpha}{d\ell} = z + z_1 - 2 - \frac{\alpha}{3\pi^2}. \tag{2.41}$$

together with the RG equations for m and v_z above.

The RG flow plot is shown in the figure below, from which we see that the fixed point at $(\beta,\alpha) = (0,0)$ is unstable and the stable fixed point lies at $(\beta,\alpha) = (0, 12\pi^2)$.

We see that at this fixed point, from eq (3.5) above, the corrected bosonic propagator goes as

$$D^{-1}(0,\vec{p}) \approx \frac{1}{2\eta}p_\perp^2 + \frac{\eta}{2}p_z^2(1+4\ell), \tag{2.42}$$

implying the screening of the bosonic mode (and hence Coulomb interaction) only in the p_z direction. We expect the exact momentum dependence of the screened part to go as $p_z^{2/3}$ from scaling arguments as well. This screening is somewhat larger than in double-Weyl semimetals due to having more density of states at Fermi level. In real space, the anisotropic screening of Coulomb potential takes the form $V(r_\perp, r_z = 0) \sim r_\perp^{-2}$, and $V(r_\perp = 0, r_z) \sim r_z^{-7/3}$. At the stable fixed point, $z = z_1 = 3$, and $\frac{g^2}{\sqrt{\eta}}$ approaches a fixed value while $g^2\sqrt{\eta}$ goes to zero, and therefore $\eta \sim \gamma/\alpha \to 0$. While the Coulomb interaction is now irrelevant at the new fixed point, it becomes infinitely anisotropically screened.

2.5 Effect on measurable quantities

In this section we will see that measurable quantities such as the specific heat and longitudinal conductivities acquire a logarithmic correction due to long-range Coulomb interaction.

For convenience we can focus on the line $\alpha = 12\pi^2$ for general γ, following the approach used in [62]. At this point it is convenient to define a parameter $\gamma = \beta/\alpha$. The RG flow for γ

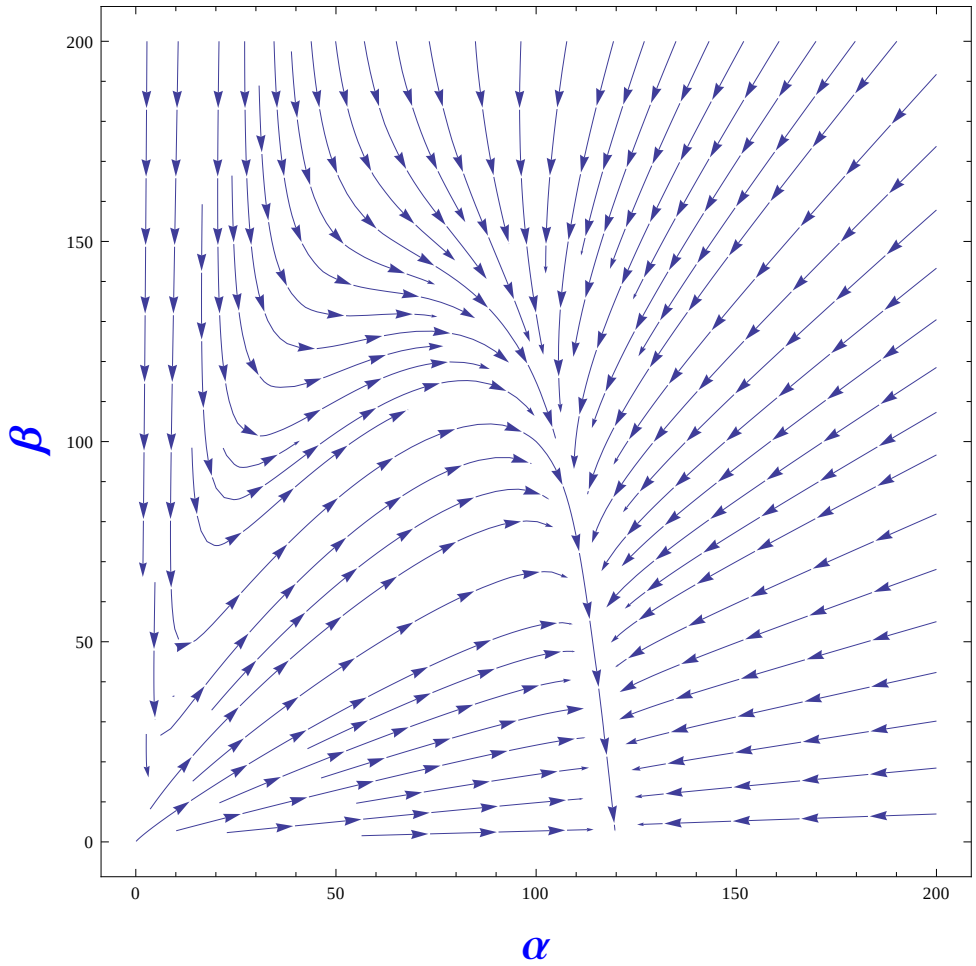

Figure 2.1: RG plot in the parameter space of the two dimensionless parameters defined in the main text. The stable fixed point lies at $\beta = 0$,$\alpha = 12\pi^2$.

along this path is

$$\frac{d\gamma}{d\ell} = -\frac{\gamma^2}{\pi^2}\left(3 + \frac{c_2}{2}\right), \tag{2.43}$$

from which we get the solution

$$\gamma = \frac{\gamma_o}{1 + \gamma_o\left(\frac{6+c_2}{2\pi^2}\right)\ell}. \tag{2.44}$$

Consider the scaling of free energy density F. It enters the action as $\int d^3x d\tau F$, so its scaling is of the form $F = e^{-\ell(2+z+z_1)}F_R$. From the definition of specific heat, $C = -T\frac{\partial^2 F}{\partial T^2}$, we see that it scales as $C = e^{-\ell(2+z_1)}C_R$, where we have used $\ell = \frac{1}{3}\ln\frac{T_o}{T}$. From this the RG equation for specific heat follows

$$\frac{dC}{d\ell} = -(2+z_1)C = -(5 - \frac{6c_1\gamma}{\pi^2} + \frac{c_2\gamma}{2\pi^2})C, \tag{2.45}$$

where we have used eq (2.37). Solving this equation together with eq.(2.41) gives

$$\begin{aligned} C \quad &\sim T^{\frac{5}{3}}\left(1 + \gamma_o\left(\frac{6+c_2}{6\pi^2}\right)\ln\frac{T_o}{T}\right)^{\frac{c_2-12c_1}{c_2+6}} \\ &\sim T^{\frac{5}{3}}\left(1 + \gamma_o\left(\frac{c_2-12c_1}{6\pi^2}\right)\ln\frac{T_o}{T}\right). \end{aligned} \tag{2.46}$$

where we have used the non-interacting specific heat $C \sim T^{\frac{5}{3}}$ which is due to DOS $\propto \epsilon^{2/3}$. We see that the specific heat receives a logarithmic correction.

We note that such logarithmic corrections to measurable quantities is typical of marginal Fermi liquids (MFL) [63]. By this analogy, we can consider this new fixed point as a marginal Fermi liquid phase. However, an important and subtle distinction needs to be made at this point. Our system is inherently a semimetal (and therefore, vanishing density of states at Fermi level), while a Fermi liquid (marginal or otherwise) is understood as a state of fermions with a Fermi surface (and therefore a finite density of states). Therefore, in our view, the logarithmic corrections to measurable quantities should at best be considered as a MFL-like behaviour.

2.6 Model with disorder

We will now present a short study of the effect of weak (Gaussian) scalar disorder on TWSM. To the non-interacting action of the TWSM, we add the following disorder term,

$$S_{dis} = \int d^4x V(x)\Psi^\dagger\sigma_0\Psi, \tag{2.47}$$

where $V(x)$ is a white-noise Gaussian disorder potential with zero mean,

$$< V(x)V(x') >= \lambda\delta^3(x - x'). \tag{2.48}$$

After applying the standard replica-averaging (labeled by indices i and j), the disorder term generates a four-fermion interaction term

$$S_{dis} = -\frac{\lambda}{2}\int d^3x d\tau d\tau^{'}(\Psi_i^{\dagger}\sigma_0\Psi_i)_{\vec{x},\tau}(\Psi_j^{\dagger}\sigma_0\Psi_j)_{\vec{x'},\tau}. \tag{2.49}$$

We can choose to carry out the RG calculations either on the four-fermion terms or on the disorder vertex terms. Making the latter choice, a simple one-loop calculation at zero external frequency (static disorder) for the disorder vertex obtains the following RG equation,

$$\frac{d\lambda}{d\ell} = \frac{m\lambda}{4\pi v_z \Lambda^2}. \tag{2.50}$$

It is clear from the above equation that $\lambda = 0$ is the only fixed point and it is unstable, which means that the TWSM is unstable to weak scalar disorder in general and likely flows to a diffusive metal phase, as argued in [64] in which the case of vector disorder has also been considered in detail. We note that the normal Weyl semimetal also flows to a diffusive metal phase but only when the disorder strength is strong enough [59].

2.7 Some Boltzmannian transport properties

Here we present a Boltzmannian calculation of longitudinal conductivities of the non-interacting TWSM.

The Boltzmann equation for the distribution function reads as [65],

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_r f + e\vec{E}\cdot\vec{\nabla}_k f = I_{coll}. \tag{2.51}$$

In the standard relaxation time approximation,

$$I_{coll} = -\frac{f(\boldsymbol{k},\boldsymbol{r},t) - f_{\text{eq}}}{\tau}, \tag{2.52}$$

where τ is a scattering time scale (lifetime of the carriers).

Assuming the following steady-state solution valid in the linear response regime [65],

$$f = f_{eq} + \tau(\epsilon(k))\left(-\frac{\partial f_{eq}}{\partial\epsilon}\right)\vec{v}\cdot\left(-e\vec{E} + \frac{\epsilon(k)-\mu}{T}(-\vec{\nabla}T)\right), \tag{2.53}$$

the thermoelectric response are obtained from (in the notation of [65])

$$J_\alpha = L^{11}_{\alpha\beta}E_\beta + L^{12}_{\alpha\beta}(-\vec{\nabla}_\beta T), \tag{2.54}$$

$$J_{q,\alpha} = L^{21}_{\alpha\beta}E_\beta + L^{22}_{\alpha\beta}(-\vec{\nabla}_\beta T), \tag{2.55}$$

where the Ls are the required transport coefficients, with $L^{11}_{\alpha\alpha} = \sigma_{\alpha\alpha} = \mathcal{L}^0_\alpha$, $L^{21}_{\alpha\alpha} = TL^{12}_{\alpha\alpha} = \frac{-\mathcal{L}^1_\alpha}{e}$

and $L^{22}_{\alpha\alpha} = \frac{\mathcal{L}^2_{\alpha\alpha}}{e^2T}$. The thermal conductivities are

$$\kappa_{\alpha\beta} = L^{22}_{\alpha\beta} - L^{21}_{\alpha\gamma}\left(L^{11}_{\gamma\rho}\right)^{-1} L^{12}_{\rho\beta}. \tag{2.56}$$

The explicit forms of the L coefficients are

$$\begin{aligned}\mathcal{L}^n_\alpha &= e^2 \sum_{s=\pm} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tau\left(\epsilon_{s\mathbf{k}}\right)\left(-\frac{\partial f}{\partial \epsilon_{s\mathbf{k}}}\right)\left(\frac{1}{\hbar}\frac{\partial \epsilon_{s\mathbf{k}}}{\partial k_\alpha}\right)^2 \times \\ &\left(\epsilon_{s\mathbf{k}} - \mu\right)^n.\end{aligned} \tag{2.57}$$

For TWSM, we have

$$\epsilon = \sqrt{v_z^2 q_z^2 + \frac{q_\perp^6}{m^2}}. \tag{2.58}$$

Converting the integrals over momenta to integral over energy via the density of states, and using the above formulas for the conductivity, one gets

$$\sigma_{xx} = \sigma_{yy} = \frac{15e^2\tau\beta}{64\pi h^3 v_z}\int d\epsilon \epsilon^2 I(\epsilon), \tag{2.59}$$

and

$$\sigma_{zz} = \frac{5e^2\tau\beta v_z m^{2/3}}{24\pi h^3}\int d\epsilon \epsilon^{2/3} I(\epsilon), \tag{2.60}$$

where

$$I(\epsilon) = (sech^2(\frac{\beta(\epsilon+\mu)}{2}) + sech^2(\frac{\beta(\epsilon-\mu)}{2}). \tag{2.61}$$

The full expression of these integrals are complicated but the important leading order terms in the quantum limit $\mu \gg T$ are obtained as

$$\sigma_{xx} = \sigma_{yy} \approx \frac{15e^2\tau}{64\pi h^3 v_z}(\left(\mu^2 + \frac{\pi^2}{3}\left(k_B T\right)^2\right)), \tag{2.62}$$

$$\sigma_{zz} \approx \frac{5e^2 m^{2/3} v_z}{24\pi h^3}\tau\mu^{2/3}. \tag{2.63}$$

Likewise, the thermal conductivities are obtained as

$$\kappa_{xx} = \kappa_{yy} \approx \frac{5\mu^2\tau T}{18h^3 v_z}(1 + \frac{\pi^2 T^2}{9\mu^2}), \tag{2.64}$$

$$\kappa_{zz} \approx \frac{6\pi v_z m^{2/3}\mu T\tau}{35h^3}. \tag{2.65}$$

2.8 Summary

Finally, we comment on the possible fate of TWSM in the simultaneous presence of both Coulomb interactions and disorder, as is typically the case in real materials. In this situation, combining the separate effects of Coulomb interactions and Gaussian scalar disorder, we would expect the system to be perturbatively unstable to the simultaneous presence of both Coulomb interactions and disorder. More precisely, since the effect of disorder is more drastic, we would expect the system to flow to a diffusive metal phase [64]. However, as with the case of normal Weyl semimetals [58], if the bare Coulomb interaction strength is sufficiently strong, we can expect a stable non-Fermi liquid phase in the simultaneous presence of disorder. Indeed, recently [64] has studied this problem and claim to have found this result for certain generic classes of disorder potentials. We do not review their results here and instead refer to their original article.

To summarise, in this chapter we have presented a study of the effects of long-range Coulomb interactions on triple-Weyl semimetals. Similar to double-Weyl semimetals, we have found that the Coulomb interaction drives the noninteracting system to a new fixed point where it is anisotropically screened in the $z-$direction. The screening is somewhat stronger than in DWSM due to higher (but still vanishing) DOS compared to DWSM. We have also found that measurable quantities such as the specific heat receives logarithmic corrections due to the anisotropic screening of Coulomb interaction. Similar studies have also been done in [66, 67], and our results are in agreement although some of the technical details differ due to somewhat different methods used. We have also shown that TWSM are unstable against perturbatively weak scalar disorder, which also can be attributed to its somewhat larger DOS.

Chapter 3

Bilayers and multilayers of Jain fractional quantum Hall states

3.1 Introduction

In the introductory chapter, we reviewed some aspects of the fractional quantum Hall (FQH) effect. While the essential physics of the effect is strictly two-dimensional, much research has been done to study the consequences of the third spatial direction on FQH systems. Naturally, the main method for such purposes is to construct multilayer systems whose individual layers are FQH liquids, and then exploring the resultant phase or phases as a function of the separation between the layers (which effectively controls the interactions or hopping of electrons between the layers).

A panoply of emergent phases can result from the coupling between the FQH layers. Couplings can come from either interlayer Coulomb interactions or interlayer electron hopping, both of which depend on the interlayer separation d, and their overall effect depends on the ratio of the interlayer separation to the intrinsic magnetic length scale of the FQH layers $\ell_B \propto 1/\sqrt{B}$, where B is the magnetic field. When $d/\ell_B \gg 1$, such that barely any interlayer coupling exists, the result is essentially a system of decoupled FQH layers, such as those studied in [68] for the case of $\nu = 1/3$ Laughlin states in each layer. On the other hand, when $d/\ell_B \ll 1$ in a bilayer, the individual layers may lose their FQH identity and effectively fuse together into a new, generally non-FQH phase such as exciton superfluid states formed from pairing between particles of one layer with holes of the other [69, 70], or interlayer paired composite fermion condensates [71–75] for the case when each layer is the half-filled Halperin-Lee-Read state [76]. The latter phases may also emerge when $d < \ell_B$ (but not $\ll \ell_B$), while when $d > \ell_B$ Halperin (m, m, n) states [77] may emerge if interlayer hopping is suppressed (infinite multilayered Halperin (m, m, n) states have been studied in [78, 79]).

However, it is much less clear, even theoretically, what the situation might be when $d \sim \ell_B$.

This intermediate regime is our interest in this article. For this regime, in [80] Levin and Fisher proposed a theoretical candidate multilayered state for the case of Laughlin states in each layer, which we generalize to Jain states in each layer in this article. Let us first see this regime in terms of various energy scales. Each layer as a cyclotron scale $\omega = eB/m$ and an intralayer Coulomb scale $\epsilon_1 = e^2/\ell_B$. In addition, we have the interlayer Coulomb scale $\epsilon_2 = e^2/d$ and interlayer hopping scale t_{int}. Firstly, we assume ω is much larger than other scales. Our regime is concerned with the situation when the interlayer Coulomb scale is comparable to intralayer Coulomb scale.

The theoretical method is based on the general paradigm of parton description [81, 82] of fractional quantum Hall states. In this description, an electron is imagined to be made up of constituent partons, which are glued together through a gauge field, which arises physically from the redundancy in the labeling of partons, or equivalently, as a Lagrange multiplier for the constraints demanding that the individual parton currents be equal to each other to be able to coherently form an electron. The deconfinement phase of the resultant parton-gauge theory corresponds to the fractionalized physics of the fractional quantum Hall states [1, 82, 83]. In [80], the authors leveraged the parton description of the Laughlin state $\nu = 1/3$ to propose candidate states for the multilayered situation in the intermediate energy scale regime described above. We note that the details of Parton construction in [80] are quite different from those in more conventional Parton constructions as in [81, 82]. We may therefore at times refer to the method of [80] (which we also follow) as a variant of Parton description which is similar is spirit to the more conventional one. We note that proposed partonic states may compete with a nearby Halperin states (of same filling factor) in a real experiment.

We consider specifically two representative Jain states $\nu = 2/5$ and $\nu = 2/3$, as these are experimentally the most prominent ones among the Jain hierarchy. Generalization to other Jain states is straightforward. We begin with the parton description of the Jain's states in the following.

3.2 Parton description of the Jain states

We shall consider the closely related cases of $\nu = 2/5$ and $\nu = 2/3$ FQH states. Among the states in Jain hierarchy, these two are the most prominent ones in experiments. Generalization to general states in the Jain states is straightforward.

In the language of composite fermions (CF) [16, 18], both the states corresponds to integral quantum Hall effect of the CFs at their effective filling of 2. However, the difference arises, within this framework, in the fact that for the $\nu = 2/3$ state, the CFs see a negative (with respect to a fixed conventional direction) effective magnetic field.

We first review the notion of parton description of these states in general terms. In this description, one imagines the electron to be made of constituent *partons*. That is, the electron operator is written as $c = f_1 f_2 f_3$. For the $\nu = 2/5^{th}$ state, f_1, f_2 carry electrical charge of $2e/5$ and f_3 carries $e/5$. For the $\nu = 2/3^{rd}$ state, f_1, f_2 carry electrical charge of $2e/3$ and f_3

carries $-e/3$. As we will see, the parsons are coupled to an emergent gauge degree of freedom which results from the condition that the patrons constitute a composite object (the electron). A fractionalized emergent phase of matter in which the partons are themselves the basic degrees of freedom and not the composite object (electrons) corresponds to the deconfined phase of the resulting gauge-matter theory arising out of this description. The next step in this description is to make an *ansatz* that the individual partons occupy an integral quantum Hall state themselves. A parton description in this way is thus an effective shortcut to describe or obtain a low energy field theory for a given fractional quantum Hall state. For the $\nu = 2/5^{th}$ state, f_1, f_2 are in $\nu' = 1$ state while f_3 is in $\nu' = 2$ state, and for the $\nu = 2/3^{rd}$ state, f_1, f_2 are in $\nu' = 1$ state while f_3 is in $\nu' = -2$ state.

We note here that the more canonical ways of parton descriptions such as in [82] are based on demanding that the partons coherently form an electron, thus their individual currents be equal to each other, which results in an gauge degree of freedom coming up as essentially a Lagrange multiplier. This is not directly the case with the approach of [80] which we use, where as we will see below, a gauge degree of freedom arises from demanding that fluctuations in the hopping amplitudes of the individual partons be such that there is no fluctuation in the hopping of the composite object (the electron). As such, we see that the exact origin of the gauge degree of freedom in the two approaches is somewhat different and this is one crucial technical difference between the two approaches, and it appears to us that the latter approach is better suited in a lattice setup which is the starting point of [80] as well as ours, particularly when dealing with more than one layer of a quantum Hall system since the gauge degree of freedom in the lattice approach (arising from hopping fluctuations over a mean-field theory, as we will see below) is more amenable to the kind of multilayer scenarios studied in [80] (and in this chapter) than the direct field theoretical approach of [82]. See also a remark on this after eq (3.11).

3.2.1 The lattice setup

3.2.1.1 Single layer case

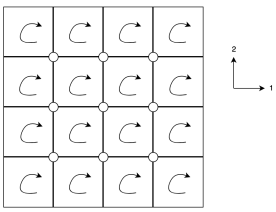

Figure 3.1: The square lattice with electrons (marked with circles) at the sites, and a flux (shown by the circular arrows) per plaquette of $2\pi/M$. The $1-2$ axis correspond to the x_1 and x_2 directions used in the text.

We begin the discussion with a lattice version of the single layer system, in which the Hamiltonian is defined on a square lattice with the (electromagnetic) flux through each plaquette

taken to be $2\pi/M$ (this puts the lattice EM field periodic in a unit cell of size M), and the electron density to be $2/(5M)$ for the $\nu = 2/5$ state and $2/(3M)$ for the $\nu = 2/3$ state, with the limit $M \to \infty$. The lattice Hamiltonian for the electrons (denoted by creation-annihilation operators $c_x^\dagger, c_x$ below) consists of the hopping terms and whichever type of interactions present between them,

$$H = -\sum_{x,i} \left(t c_x^\dagger e^{i\widetilde{A}_{x,i}} c_{x+\hat{x}_i} \right) + \text{interactions} \tag{3.1}$$

where x are site indices, i labels the two directions in the square lattice and the electromagnetic lattice field $\widetilde{A}_{x,i}$ is defined on the links between x and $x+\hat{x}_i$ with the convention that $\widetilde{A}_{x,i}$ defines the field on the link starting at site/vertex x and directed towards i^{th} direction/link. Since the flux per plaquette is $2\pi/M$, we have that $\Delta_1 \widetilde{A}_{x,2} - \Delta_2 \widetilde{A}_{x,1} = 2\pi/M$, where the lattice derivative in the $\hat{x}_i$ directions is defined as $\Delta_i f_x = f_{x+\hat{x}_i} - f_x$. In the Hamiltonian above, we now substitute for $c = f_1 f_2 f_3$, which makes the hopping terms look like $t f_{3,x}^\dagger f_{2,x}^\dagger f_{1,x}^\dagger e^{i\widetilde{A}_{x,i}} f_{1,x+\hat{x}_i} f_{2,x+\hat{x}_i} f_{3,x+\hat{x}_i}$. At this stage the general procedure in interacting systems is to consider saddle-point/mean-field approximations which turn these three-body terms into mean-field one-body terms. Various forms of interactions may stabilize any of these saddle-point mean-field Hamiltonians, and precisely which saddle-point is stabilized depends on the details of the interactions.

We now assume that the interactions are such that a particular type of saddle-point mean-field Hamiltonian is stabilized which leads to a parton description (in lattice form) of FQH states of our interest. This is essentially an *ansatz* in any parton description of an FQH state, and different such *ansatz* lead to different FQH states. We consider the mean-field Hamiltonian of the following form,

$$H_{mf} = -\sum_{x,i} \sum_{m=1,2} t_{m,x,i} f_{m,x}^\dagger e^{i\bar{A}_{x,i}^{(1)}} f_{m,x+\hat{x}_i} - \sum_{x,i,m=3} t_{m,x,i} f_{m,x}^\dagger e^{i\bar{A}_{x,i}^{(2)}} f_{m,x+\hat{x}_i} + \text{h.c.} \tag{3.2}$$

where, the lattice EM flux density seen by the parton f_p ($p = 1, 2, 3$) is q_p times the EM flux density seen by the electron, where q_p is the electromagnetic charge for the p^{th} parton. Thus, for the $\nu = 2/5$ case, the lattice EM flux densities seen by the partons f_1, f_2 are $\Delta_1 \bar{A}_{x,2}^{(1)} - \Delta_2 \bar{A}_{x,1}^{(1)} = 4\pi/(5M)$ and that seen by f_3 is $\Delta_1 \bar{A}_{x,2}^{(2)} - \Delta_2 \bar{A}_{x,1}^{(2)} = 2\pi/(5M)$. Since the density of partons is the same as their parent electron, that is, $2/(5M)$, this puts f_1 and f_2 in a QH state of filling $\nu' = 1$ and f_3 in a QH state of filling $\nu' = 2$. Likewise, for the $\nu = 2/3$ case, we have $\Delta_1 \bar{A}_{x,2}^{(1)} - \Delta_2 \bar{A}_{x,1}^{(1)} = 4\pi/(3M)$ and $\Delta_1 \bar{A}_{x,2}^{(2)} - \Delta_2 \bar{A}_{x,1}^{(2)} = -2\pi/(3M)$, which puts f_1 and f_2 in $\nu' = 1$ and f_3 in $\nu' = -2$. Here, the parton hopping amplitudes $t_{m,x,i}$ described the hopping of the parton type m to the site x from the site $x + \hat{x}_i$. In the mean-field situation (that is, without any fluctuations), the hopping amplitudes are independent of x, i and can be simply written as t_m (that is, constants over the lattice, but different for various partons types).

Next we consider fluctuations over this mean-field setup (we thank M. Levin for a helpful email correspondence about some of what follows below), which are considered through fluctuations in the hopping amplitudes of the form $t_{m,x,i} \to t_{m,x,i}e^{i\theta_{m,x,i}}$ with $\theta_{1,x,i} + \theta_{2,x,i} + \theta_{3,x,i} = 0$. Here, $\theta_{m,x,i}$ denote the phase of the hopping amplitude of the parton type m on the link connecting x and $x + \hat{x}_i$. This constraint among the $\theta_{m,x,i}$ arises because of the essential requirement that the parton hopping fluctuations should not affect the composite object's (that is, the electron's) hopping amplitude. The mean-field Hamiltonian is for the partons dynamics and likewise the fluctuations of the hopping amplitudes is for the partons, but none of this should affect the electron Hamiltonian that we began with because the electron Hamiltonian is not directly being subjected to a mean-field plus fluctuation analysis (which is being done on the partons). Hence the constraint equation for the sum of $\theta_{m,x,i}$ is that the sum should go to zero (modulo 2π), which leaves the electron's hopping amplitude invariant to the fluctuations in the hopping amplitudes of the partons. Since we have three phase variables and one equation for them, we can parametrize them as $\theta_{m,x,i} = q_{mn}\mathcal{A}^{(n)}_{x,i}$, where $\mathcal{A}^{(n)}_{x,i}$ $(n = 1, 2)$, denote new $U(1)$ lattice gauge fields that live on the links connecting x and $x + \hat{x}_i$, and we make the choice $q_{m1} = (1, -1, 0)$ and $q_{m2} = (0, -1, 1)$ (this choice is not unique). We thus have two $U(1)$ gauge fields to which the partons are coupled due to fluctuations. So now we have the total Hamiltonian including the fluctuations as $H = H_t + H_g$, where,

$$\begin{aligned} H_t = \quad & -\sum_{x,i}\sum_{m=1,2} t_{m,x,i} f^\dagger_{m,x} e^{i\bar{A}^{(1)}_{x,i} + iq_{mn}\mathcal{A}^{(n)}_{x,i}} f_{m,x+\hat{x}_i} \\ & -\sum_{x,i,m=3} t_{m,x,i} f^\dagger_{m,x} e^{i\bar{A}^{(2)}_{x,i} + iq_{mn}\mathcal{A}^{(n)}_{x,i}} f_{m,x+\hat{x}_i} + \text{h.c.} \end{aligned} \tag{3.3}$$

and, the lattice gauge field dynamics term consists of (or is assumed to have) the canonical "electric" field and "magnetic" field terms [2, 84],

$$H_g = \sum_{x,i,n} \frac{g}{2}(E^{(n)}_{x,i})^2 - \sum_{x,n} J\cos(\Delta_1\mathcal{A}^{(n)}_{x,2} - \Delta_2\mathcal{A}^{(n)}_{x,1}) \tag{3.4}$$

The weak fluctuation regime of our interest is $g <<< J, t_{m,x,i}$.

3.2.1.2 Multilayer case

We can similarly give a lattice setup for the multilayered cases (assuming a general N number of layers). Here, in addition to the hopping and gauge dynamics terms for each layer (that is, intralayer terms), we will have interlayer hopping and gauge dynamics terms. The physical origin of the interlayer gauge dynamics is the same as before, that is - to begin with we have an interlayer electron hopping term (but without any interlayer background EM field) which under mean-field decomposition in terms of partons yields interlayer parton hopping terms. Accounting for fluctuations of the interlayer hopping amplitudes gives rise to two $U(1)$ gauge fields which

live in the direction between the layers (that is the perpendicular direction for each layer). The coupling "charges" denoted previously by q_{mn} can be chosen to be the same as before, and the partons are minimally coupled to these interlayer gauge degrees of freedom with their coupling charges being q_{mn}.

Explicitly, in the mean-field limit we have (we add another label for the layer z for the partons),

$$\begin{aligned} H_{mf} = \quad & -\sum_{x,i,z}\sum_{m=1,2} t_{m,x,i} f^\dagger_{m,x,z} e^{i\bar{A}^{(1)}_{x,i}} f_{m,x+\hat{x}_i,z} - \sum_{x,i,z,m=3} t_{m,x,i} f^\dagger_{m,x,z} e^{i\bar{A}^{(2)}_{x,i}} f_{m,x+\hat{x}_i,z} \\ & -\sum_{x,z}\sum_{m=1,2} t_{m,x,3} f^\dagger_{m,x,z} f_{m,x,z+1} - \sum_{x,i,m=3} t_{m,x,3} f^\dagger_{m,x,z} f_{m,x,z+1} + \text{h.c.} \end{aligned} \tag{3.5}$$

where $t_{m,x,3}$ are the hopping amplitudes of the partons at sites labelled x (in their respective layers) between the layers z and $z+1$, that is, these are the interlayer hopping amplitudes. Now after considering the gauge fluctuations (which give rise to two additional $U(1)$ gauge fields living in the space between the layers), we have the total Hamiltonian,

$$H = \sum_{z=1}^{N} (H_{t,z} + H_{g,z}) + \sum_{z=1}^{N-1} (H_{t,z,z+1} + H_{g,z,z+1}) \tag{3.6}$$

where $H_{t,z}$ and $H_{g,z}$ describe intralayer hopping and gauge dynamics terms (the intralayer gauge field also is labelled with the layer index z),

$$\begin{aligned} H_{t,z} = \quad & -\sum_{x,i}\sum_{m=1,2} t_{m,x,i} f^\dagger_{m,x,z} e^{i\bar{A}^{(1)}_{x,i} + iq_{mn}\mathcal{A}^{(n)}_{x,i,z}} f_{m,x+\hat{x}_i,z} \\ & -\sum_{x,i,m=3} t_{m,x,i} f^\dagger_{m,x,z} e^{i\bar{A}^{(2)}_{x,i} + iq_{mn}\mathcal{A}^{(n)}_{x,i,z}} f_{m,x+\hat{x}_i,z} + \text{h.c.} \\ H_{g,z} = \quad & \sum_{x,i,n} \frac{g}{2} (E^{(n)}_{x,i,z})^2 - \sum_{x,n} J\cos(\Delta_1 \mathcal{A}^{(n)}_{x,2,z} - \Delta_2 \mathcal{A}^{(n)}_{x,1,z}) \end{aligned} \tag{3.7}$$

and $H_{t,z,z+1}$ and $H_{g,z,z+1}$ describe interlayer hopping and gauge dynamics terms,

$$\begin{aligned} H_{t,z,z+1} = \quad & -\sum_{x,z}\sum_{m=1,2} t_{m,x,3} f^\dagger_{m,x,z} e^{iq_{mn}\mathcal{A}^{(n)}_{x,z,3}} f_{m,x,z+1} \\ & -\sum_{x,i,m=3} t_{m,x,3} f^\dagger_{m,x,z} e^{iq_{mn}\mathcal{A}^{(n)}_{x,z,3}} f_{m,x,z+1} + \text{h.c.} \\ H_{g,z,z+1} = \quad & \sum_{x,n} \frac{g'}{2} (E^{(n)}_{x,z,3})^2 - \sum_{x,i,n} J'\cos(\Delta_i \mathcal{A}^{(n)}_{x,z,3} - \mathcal{A}^{(n)}_{x,i,z} + \mathcal{A}^{(n)}_{x,i,z+1}) \end{aligned} \tag{3.8}$$

where $\mathcal{A}^{(n)}_{x,z,3}$ denotes the gauge fields corresponding to the fluctuations of the hopping amplitudes

for partons at sites labelled x (in their respective layers) between the layers z and $z+1$. As before, our weak fluctuation regime corresponds to the coupling constants of the electric field terms being much smaller than the hopping amplitudes and magnetic field couplings.

3.2.2 Single layer continuum theory

From here we now go to the continuum limit for the single layer case (bilayers and multilayers will be considered in the next sections). For the purpose of describing this as a continuum field theory, let us introduce for f_1, f_2 their respective parton gauge fields $\alpha^{(1)}$, $\alpha^{(2)}$ and write their currents as $j_\mu^{(m)} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda^{(m)}$ for the partons f_1, f_2, i.e., $m = 1, 2$ here. For the third parton f_3, since it occupies a $\nu' = \pm 2$ quantum Hall state (respectively for $\nu = 2/5, 2/3$), we have to introduce two parton gauge fields $\alpha^{(3a),(3b)}$ with which to express its current as $j_\mu^{(3)} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda^{(3a)} + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda^{(3b)}$. The hopping Hamiltonian above is described as an effective theory by the total Lagrangian, for the case of $\nu = 2/5$ state, by the standard Chern-Simons terms for the parton fields along with the minimal coupling of their currents to the two $U(1)$ gauge fields,

$$\begin{aligned}\mathcal{L}_{2/5} = \quad & \frac{1}{4\pi}\sum_{m=1,2}\epsilon_{\mu\nu\lambda}\alpha_\mu^{(m)}\partial_\nu\alpha_\lambda^{(m)} + \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu^{(3a)}\partial_\nu\alpha_\lambda^{(3a)} \\ & + \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu^{(3b)}\partial_\nu\alpha_\lambda^{(3b)} + \sum_{m=1,2}^{n=1,2}\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}q_{mn}\mathcal{A}_\mu^{(n)}\partial_\nu\alpha_\lambda^{(m)} \\ & + \frac{1}{2\pi}q_{3n}\epsilon_{\mu\nu\lambda}\mathcal{A}_\mu^{(n)}\partial_\nu(\alpha_\lambda^{(3a)} + \alpha_\lambda^{(3b)}),\end{aligned} \tag{3.9}$$

and likewise for the $\nu = 2/3$ state,

$$\begin{aligned}\mathcal{L}_{2/3} = \quad & \frac{1}{4\pi}\sum_{m=1,2}\epsilon_{\mu\nu\lambda}\alpha_\mu^{(m)}\partial_\nu\alpha_\lambda^{(m)} - \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu^{(3a)}\partial_\nu\alpha_\lambda^{(3a)} \\ & - \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu^{(3b)}\partial_\nu\alpha_\lambda^{(3b)} + \sum_{m=1,2}^{n=1,2}\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}q_{mn}\mathcal{A}_\mu^{(n)}\partial_\nu\alpha_\lambda^{(m)} \\ & - \frac{1}{2\pi}q_{3n}\epsilon_{\mu\nu\lambda}\mathcal{A}_\mu^{(n)}\partial_\nu(\alpha_\lambda^{(3a)} + \alpha_\lambda^{(3b)}).\end{aligned} \tag{3.10}$$

As for the continuum terms corresponding to H_g, we see that the electric field term is quadratic and expanding the cosine term (the magnetic field term) to leading order in the variables $\mathcal{A}_{x,i}^{(n)}$ also gives a quadratic term, so we have Maxwell terms for these gauge fields in the continuum (as expected, since the lattice terms were also Maxwell). Since integrating out the gapped partons would produce Chern-Simons terms for the dynamics of the gauge fields $\mathcal{A}_\mu^{(n)}$, with respect to which the Maxwell terms are irrelevant in the low-energy effective theory, we can

safely ignore/drop them at the outset. We will later see that in the multilayer situations, this will generally not be the case for the interlayer gauge field that we will introduce later. For brevity we have also not explicitly written the terms corresponding to the minimal coupling of the parton currents to the external electromagnetic field.

Let us verify that our Lagrangians above do indeed describe the intended quantum Hall states of $\nu = 2/5, 2/3$. To do so, we now integrate the gauge fields $\mathcal{A}_\mu^{(n)}$, which produces the constraints that the parton currents are fixed to be equal to each other. That is, $j_\mu^{(1)} = j_\mu^{(2)} = j_\mu^{(3)}$. A general solution to this can be taken as $\alpha_\mu^{(1)} = \alpha_\mu^{(2)} = \alpha_\mu^{(3a)} + \alpha_\mu^{(3b)} = \alpha_\mu$. Denoting $\alpha_\mu^{(3a)} = \beta_\mu$, we thus have two independent fields α_μ and β_μ. Substituting these in the Lagrangians above, we have,

$$\begin{aligned}
\mathcal{L}_{2/5} = \quad & \frac{2}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu\partial_\nu\alpha_\lambda + \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\beta_\mu\partial_\nu\beta_\lambda \\
& + \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}(\alpha-\beta)_\mu\partial_\nu(\alpha-\beta)_\lambda + \frac{1}{2\pi}eA_{EM,\mu}\partial_\nu\alpha_\lambda \\
= \quad & \frac{3}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu\partial_\nu\alpha_\lambda + \frac{2}{4\pi}\epsilon_{\mu\nu\lambda}\beta_\mu\partial_\nu\beta_\lambda - \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu\partial_\nu\beta_\lambda \\
& - \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\beta_\mu\partial_\nu\alpha_\lambda + \frac{1}{2\pi}eA_{EM,\mu}\partial_\nu\alpha_\lambda
\end{aligned} \tag{3.11}$$

We note that we could have obtained the above Lagrangian directly in the continuum by arguing that because the partons have to be confined to each other to form an electron, their individual currents must equal each other (which is what we obtained above after integrating out the fields $\mathcal{A}_\mu^{(n)}$, which would have yielded the above Lagrangian directly. However, an approach that directly starts with the field theoretic constraint of the parton currents being equal to each other results in a $SU(3)$ gauge field that couples to the partons and keeps them "glued" to each other to form an electron [85]. In our language presented above following [80], this would have meant that assuming that the hopping amplitudes for all the partons are equal to each other, which is a different saddle point choice than the one we (and [80]) have made. It appears that choosing the saddle point that has the parton hopping amplitudes different from each other is a more general scenario which in particular seems to be more suitable for considering interlayer hopping in the bilayer and multilayer scenarios because it enables to treat the interlayer hopping in this seemingly more general scenario where the partons may hop between the layers with different amplitudes. With this remark, we now continue with this approach.

Using the K-matrix notation, we introduce a two-component vector $\Lambda = (\alpha, \beta)^T$, and $s = (1, 0)$, and write the above Lagrangian as,

$$\mathcal{L}_{2/5} = \frac{1}{4\pi}\mathcal{K}_{2/5}\epsilon_{\mu\nu\lambda}\Lambda_\mu\partial_\nu\Lambda_\lambda + \frac{1}{2\pi}esA_{EM,\mu}\partial_\nu\Lambda_\lambda \tag{3.12}$$

where,

$$\mathcal{K}_{2/5} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \tag{3.13}$$

This matrix is related to the one in [82] by a similarity transformation, which means that the theory described by our K-matrix is in the same topological class as that in [82]. In addition, the ground state degeneracy on a torus of a quantum Hall state whose effective theory is described through such a K-matrix is given by $|det(\mathcal{K})|$. We give a simple derivation of this in the Appendix I at the end of this chapter. For the above K-matrix then the ground state degeneracy on torus is 5, as it should be for the FQH state at $\nu = 2/5$.

Likewise, for the $\nu = 2/3$ state,

$$\begin{aligned} \mathcal{L}_{2/3} = \quad & \frac{2}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu\partial_\nu\alpha_\lambda - \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\beta_\mu\partial_\nu\beta_\lambda \\ & -\frac{1}{4\pi}\epsilon_{\mu\nu\lambda}(\alpha-\beta)_\mu\partial_\nu(\alpha-\beta)_\lambda + \frac{1}{2\pi}eA_{EM,\mu}\partial_\nu\alpha_\lambda \\ = \quad & \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu\partial_\nu\alpha_\lambda - \frac{2}{4\pi}\epsilon_{\mu\nu\lambda}\beta_\mu\partial_\nu\beta_\lambda - \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu\partial_\nu\beta_\lambda \\ & -\frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\beta_\mu\partial_\nu\alpha_\lambda + \frac{1}{2\pi}eA_{EM,\mu}\partial_\nu\alpha_\lambda \\ = \quad & \frac{1}{4\pi}\mathcal{K}_{2/3}\epsilon_{\mu\nu\lambda}\Lambda_\mu\partial_\nu\Lambda_\lambda + \frac{1}{2\pi}esA_{EM,\mu}\partial_\nu\Lambda_\lambda \end{aligned}$$

where,

$$\mathcal{K}_{2/3} = \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix} \tag{3.14}$$

As before, this matrix is also equivalent by a similarity transformation to the one in [82]. The ground state degeneracy on torus of our theory is 3, as is should be for the FQH state at $\nu = 2/3$.

Thus we have described the effective theories of $\nu = 2/5, 2/3$ starting from a parton construction. Now in the following sections, we shall use similar procedure to describe bilayers and quasi-3D limit of these states.

3.3 Bilayer continuum theory

3.3.1 the $\nu = 2/5$ case

The case of bilayer corresponds to $N = 2$ in our previous discussion of multilayer lattice setup. The main difference here from the single layer case comes from the terms corresponding to the interlayer hopping (and the consequent interlayer gauge field) of the partons. As before, the

intralayer gauge field's Maxwell terms can be ignored at the outset in the low-energy limit as the more dominant Chern-Simons terms for these gauge fields will be induced by the partons. However, this is not the case with the interlayer gauge field, and thus its Maxwell terms have to be considered in the continuum theory, and this is ultimately responsible for the "coupling" between the layers in the continuum picture.

Following the conventions of [80], we enlarge, notationally, our $\mathcal{A}_{\mu}^{(n)}$ fields from having two spatial components to now having three spatial components (the first two spatial components of this enlarged notation come from the intralayer gauge fields, while the third spatial component is just the interlayer gauge field). We label the parton fields with the layer indices, $\alpha_{l,\mu}^{(m)}$, as well as the gauge fields, $\mathcal{A}_{l,\mu}^{(n)}$. The total effective action is $\mathcal{L}_{2/5}^{(blr)} = \sum_{l=1,2} \mathcal{L}_{2/5}^{l} + \mathcal{L}_{\perp}$, where $i = 1, 2$ denotes the planar spatial indices, and,

$$\begin{aligned}\mathcal{L}_{2/5}^{l} = \quad & \frac{1}{4\pi}\sum_{m=1,2}\epsilon_{\mu\nu\lambda}\alpha_{l,\mu}^{(m)}\partial_{\nu}\alpha_{l,\lambda}^{(m)} + \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_{l,\mu}^{(3a)}\partial_{\nu}\alpha_{l,\lambda}^{(3a)} \\ & + \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_{l,\mu}^{(3b)}\partial_{\nu}\alpha_{l,\lambda}^{(3b)} + \sum_{m=1,2}^{n=1,2}\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}q_{mn}\mathcal{A}_{l,\mu}^{(n)}\partial_{\nu}\alpha_{l,\lambda}^{(m)} \\ & + \frac{1}{2\pi}q_{3n}\epsilon_{\mu\nu\lambda}\mathcal{A}_{l,\mu}^{(n)}\partial_{\nu}(\alpha_{l,\lambda}^{(3a)} + \alpha_{l,\lambda}^{(3b)}),\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\mathcal{L}_{\perp} = \quad & \sum_{n=1,2}\eta_1(\partial_0\mathcal{A}_{1,3}^{(n)} - \mathcal{A}_{1,0}^{(n)} + \mathcal{A}_{2,0}^{(n)})^2 \\ & - \sum_{n=1,2}^{i=1,2}\eta_2(\partial_i\mathcal{A}_{1,3}^{(n)} - \mathcal{A}_{1,i}^{(n)} + \mathcal{A}_{2,i}^{(n)})^2.\end{aligned} \tag{3.16}$$

where $\eta_1 = 1/g'$ and $\eta_2 = J'/2$ in terms of the coupling constants defined on the lattice previously, and the layer spacing has been put to unity.

Let us again explain our notation here, which has been adopted from [80] (see for instance eq.(25) in that article). The $\mathcal{L}_{\perp}$ term is a generic Maxwell-like term for the dynamics of the interlayer gauge field (layer spacing has been put to unity here). The corresponding $U(1)$ gauge fields, which we may call $a_l^{(n)}$ (l is the layer index, and $n = 1, 2$), exists only in the third (or z-) direction (i.e., $a_{l,3}^{(n)}$ is the only non-vanishing component of this field), while the intralayer gauge fields have spatial components in only the planar 1-2 (or x-y) directions (i.e., $\mathcal{A}_{l,(1,2)}^{(n)}$). In writing the above equation, we have thus simply enlarged the notational definition (which we so far used) of $\mathcal{A}_{l,(1,2)}^{(n)}$ to contain as its third spatial component the field $a_{l,3}^{(n)}$, to write it in its enlarged form as $\mathcal{A}_{l,(1,2,3)}^{(n)}$. The $(1, 2)$ components of this enlarged field now are the intralayer gauge fields while the (3) component is simply the interlayer $a_{l,3}^{(n)}$. Note also that the first term in the above equation

is just the electric field term in the interlayer direction and the second term is the magnetic field term in the interlayer direction.

We now choose the gauge $\mathcal{A}^{(n)}_{1,3} = 0$, go to the basis $\mathcal{A}^{(n)}_{\pm,\mu} = \mathcal{A}^{(n)}_{1,\mu} \pm \mathcal{A}^{(n)}_{2,\mu}$, the total Lagrangian becomes,

$$\begin{aligned}
\mathcal{L}^{(blr)}_{2/5} = \quad & \frac{1}{4\pi}\sum_{m=1,2}^{l=1,2}\epsilon_{\mu\nu\lambda}\alpha^{(m)}_{l,\mu}\partial_\nu\alpha^{(m)}_{l,\lambda} + \frac{1}{4\pi}\sum_{l=1,2}\epsilon_{\mu\nu\lambda}\alpha^{(3a)}_{l,\mu}\partial_\nu\alpha^{(3a)}_{l,\lambda} \\
+ \quad & \frac{1}{4\pi}\sum_{l=1,2}\epsilon_{\mu\nu\lambda}\alpha^{(3b)}_{l,\mu}\partial_\nu\alpha^{(3b)}_{l,\lambda} \\
+ \quad & \sum_n \eta_1(\mathcal{A}^{(n)}_{-,0})^2 - \sum_{n,i}\eta_2(\mathcal{A}^{(n)}_{-,i})^2 \\
+ \quad & \frac{1}{4\pi}\sum_{m=1,2}^{n=1,2}\sum_{\pm}\epsilon_{\mu\nu\lambda}q_{mn}(\mathcal{A}^{(n)}_{\pm,\mu}\partial_\nu(\alpha^{(m)}_{1,\lambda} \pm \alpha^{(m)}_{2,\lambda})) \\
+ \quad & \frac{1}{4\pi}\sum_{\pm}^{n=1,2}\epsilon_{\mu\nu\lambda}q_{3n}(\mathcal{A}^{(n)}_{\pm,\mu}\partial_\nu(\alpha^{(3a)}_{1,\lambda} \pm \alpha^{(3a)}_{2,\lambda} \pm \alpha^{(3b)}_{1,\lambda} \pm \alpha^{(3b)}_{2,\lambda})).
\end{aligned}$$

To obtain an effective theory in terms of the parton fields, we now integrate out the $\mathcal{A}^{(n)}_{\pm,\mu}$ fields. Clearly, integrating out the $\mathcal{A}^{(n)}_{-,\mu}$ fields generate Maxwellian terms for the parton fields which are irrelevant compared to the Chern-Simons terms and thus may be ignored hereafter. Integrating out the $\mathcal{A}^{(n)}_{+,\mu}$ fields produce the constraints that the parton currents (for each parton type) over both the layers are equal to each other, that is (suppressing the vector indices for notational clarity for now), $\alpha^{(1)}_1 + \alpha^{(1)}_2 = \alpha^{(2)}_1 + \alpha^{(2)}_2 = \alpha^{(3a)}_1 + \alpha^{(3a)}_2 + \alpha^{(3b)}_1 + \alpha^{(3b)}_2 = a$. We have eight field variables and two equations, thus there are six independent field variables. Parametrizing the various variables as $\alpha^{(1)}_1 = a - b_1$, $\alpha^{(1)}_2 = b_1$, $\alpha^{(2)}_1 = a - b_2$, $\alpha^{(2)}_2 = b_2$, $\alpha^{(3a)}_1 = b_3$, $\alpha^{(3a)}_2 = b_4$, $\alpha^{(3b)}_1 = b_5$, $\alpha^{(3b)}_2 = a - b_3 - b_4 - b_5$, substituting these in the Lagrangian above and introducing the vector $\tilde{\alpha} = (a, b_1, b_2, b_3, b_4, b_5)^T$, we have the effective Lagrangian of the bilayer as,

$$\mathcal{L}_{bA} = \frac{1}{4\pi}\mathcal{K}_{bA}\epsilon_{\mu\nu\lambda}\tilde{\alpha}_\mu\partial_\nu\tilde{\alpha}_\lambda + \frac{1}{2\pi}esA_{EM,\mu}\partial_\nu\tilde{\alpha}_\lambda \tag{3.17}$$

where the charge vector is $s = (1, 0, 0, 0, 0, 0)$ and

$$\mathcal{K}_{bA} = \begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 1 & 1 \\ -1 & 0 & 0 & 1 & 2 & 1 \\ -1 & 0 & 0 & 1 & 1 & 2 \end{pmatrix} \tag{3.18}$$

Let us see some observable properties of this effective theory. The ground state degeneracy

of this bilayered system on a torus is $|det(\mathcal{K}_{bA})| = 20$. This differentiates the partonic bilayer from a system of two decoupled $\nu = 2/5$ layers whose toric ground state degeneracy (by which we mean ground state degeneracy on torus) would be 25, as from a FQH state of total filling factor $\nu = 2/5 + 2/5 = 4/5$, whose toric ground state degeneracy would be 5.

A K-matrix theory also allows us to readily calculate the statistics of parton excitations described by the theory. To do this, let us first label the partons with their corresponding vectors $k_1 = (0, 1, 0, 0, 0, 0)^T$, $k_2 = (0, 0, 1, 0, 0, 0)^T$ and $k_3 = (1, 0, 0, 0, 0, 0)^T$. Then, the self-exchange statistics of a parton labeled by k_i is given by $\theta_i = \pi k_i^T K^{-1} k_i$ and the mutual exchange statistics between partons labeled by k_i and k_j is given by $\theta_{ij} = 2\pi k_i^T K^{-1} k_j$ [26].

Thus, for the K-matrix of the partonic bilayer given above, $\mathcal{K}_{bA}$, we have $\theta_1 = \theta_2 = 7\pi/10$ and $\theta_3 = 4\pi/5$. This is distinct from the self-exchange statistical angle of $2\pi/5$ for the $2e/5$ charged excitations and $3\pi/5$ for the $e/5$ charged excitations of the $\nu = 2/5$ FQH state. The mutual braiding statistics of the partons in our bilayer is given as $\theta_{12} = \theta_{23} = \theta_{13} = 4\pi/5$.

3.3.2 the $\nu = 2/3$ case

Similar considerations follow as in the case of the above subsection, except with the changes in the Lagrangians corresponding to the individual layers as described in the previous section. For the sake of not cluttering the article, we do not repeat writing the steps of the calculation, and simply present below the final expression for the effective Lagrangian of the bilayer after simplification in terms of the K-matrix,

$$\mathcal{L}_{bB} = \frac{1}{4\pi}\mathcal{K}_{bB}\epsilon_{\mu\nu\lambda}\tilde{\alpha}_\mu\partial_\nu\tilde{\alpha}_\lambda + \frac{1}{2\pi}esA_{EM,\mu}\partial_\nu\tilde{\alpha}_\lambda \tag{3.19}$$

where the charge vector is $s = (1, 0, 0, 0, 0, 0)$ and

$$\mathcal{K}_{bB} = \begin{pmatrix} 2 & -1 & -1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & -1 & -1 \\ 1 & 0 & 0 & -1 & -2 & -1 \\ 1 & 0 & 0 & -1 & -1 & -2 \end{pmatrix} \tag{3.20}$$

The ground state degeneracy of this bilayer system on a torus is $|det(\mathcal{K}_{bB})| = 28$. This differentiates the partonic bilayer from a system of two decoupled $\nu = 2/3$ layers whose toric ground state degeneracy would be 9, as from a FQH state of total filling factor $\nu = 2/3 + 2/3 = 1 + 1/3$, whose toric ground state degeneracy would be 3. The parton self-exchange statistical angle for this case are $\theta_1 = \theta_2 = 9\pi/14$ and $\theta_3 = 4\pi/7$. The mutual exchange statistical angles are $\theta_{12} = \theta_{23} = \theta_{13} = 4\pi/7$.

3.4 quasi-3D multilayers

Let us review our procedure so far. We described the bilayers as follows - we started from a parton description of an individual layer (described by a corresponding gauge theory with Chern-Simons terms) which captures the essential low-energy physics of the layer, in addition we had an interlayer gauge degree of freedom with Maxwell-like dynamics, and finally we integrated out the various gauge fields to obtain effective K-matrix theory of the bilayered system in terms of the parton fields. This procedure can be straightforwardly generalized to constructing a quasi-three dimensional multilayer formed from stacking a very large number of layers, which we describe below.

The total Lagrangian is readily written, $\mathcal{L}^{(3d)} = \sum_l \mathcal{L}^l_{2/5} + \mathcal{L}_\perp$, where $\mathcal{L}^l_{2/5}$ is as in eq(3.9), l labels the layers, and

$$\begin{aligned}\mathcal{L}_\perp = \quad & \sum_{n=1,2}^{l} \eta_1(\partial_0 \mathcal{A}^{(n)}_{l,3} - \mathcal{A}^{(n)}_{l,0} + \mathcal{A}^{(n)}_{l+1,0})^2 \\ & - \sum_{n=1,2}^{i=1,2} \sum_l \eta_2(\partial_i \mathcal{A}^{(n)}_{l,3} - \mathcal{A}^{(n)}_{l,i} + \mathcal{A}^{(n)}_{l+1,i})^2. \end{aligned} \tag{3.21}$$

As before, intralayer Maxwell terms have been ignored for our purposes of being in the low-energy limit. As before, we may integrate out the intralayer and interlayer gauge fields $\mathcal{A}^{(n)}_{l,\mu}$ and generate an effective theory in terms of a large number N of parton fields, and then take the limit $N \to \infty$ to access the quasi-three dimensional limit. However, this requires dealing with a very large K-matrix of dimension of the order of $N \times N$. An alternative is, for the case of many layers, to instead integrate out the parton fields and generate an effective theory in terms of the gauge fields $\mathcal{A}^{(n)}_{l,\mu}$. This, as we shall see, generates an effective theory with much smaller K-matrix, which makes the theory easier to analyse for calculating, for eg., its dispersion.

Integrating out the gapped parton fields yields the effective Lagrangian as (all summations over $n = 1, 2$ and $i = 1, 2$ below),

$$\begin{aligned}\mathcal{L}^{(3d)} = \quad & \frac{1}{4\pi} \sum_l \epsilon_{\mu\nu\lambda} \mathcal{K} \mathcal{A}^T_{l,\mu} \partial_\nu \mathcal{A}_{l,\lambda} \\ & + \sum_{n,l} \eta_1(\partial_0 \mathcal{A}^{(n)}_{l,3} - \mathcal{A}^{(n)}_{l,0} + \mathcal{A}^{(n)}_{l+1,0})^2 \\ & - \sum_{n,i,l} \eta_2(\partial_i \mathcal{A}^{(n)}_{l,3} - \mathcal{A}^{(n)}_{l,i} + \mathcal{A}^{(n)}_{l+1,i})^2, \end{aligned} \tag{3.22}$$

where we defined for the intralayer components in a given layer $\mathcal{A}_{l,\mu} = (\mathcal{A}^{(1)}_{l,\mu}, \mathcal{A}^{(2)}_{l,\mu})^T$ and,

$$\mathcal{K} = \begin{pmatrix} -2 & -1 \\ -1 & -3 \end{pmatrix} \tag{3.23}$$

It is readily seen that for the purpose of deriving the effective theory in terms of $\mathcal{A}_\mu$ fields, the system formed from many layers of $\nu = 2/3$ states has the exact same form of the effective theory except with the resultant $\mathcal{K}$-matrix for $\mathcal{A}_\mu$ is simply the negative of the above equation, which reveals that as far as universal properties are concerned, both the stacked systems approach the same quasi-3D system.

From eq (3.14), we can readily find the dispersion of the gauge field. The Fourier transformed Lagrangian (note that Fourier transformation does away with the layer indices), in the basis $\mathcal{A}_{\pm,\mu} = \frac{1}{\sqrt{2}}(\mathcal{A}_\mu^{(1)} \pm \mathcal{A}_\mu^{(2)})$ with temporal gauge $\mathcal{A}_{\pm,0} = 0$, can be written as $\mathcal{L}^{(3d)}(\vec{p}) = \sum_{r=\pm} \mathcal{A}_r(-\vec{p})^\dagger L_r(\vec{p}) \mathcal{A}_r(\vec{p})$, where

$$L_r = \begin{bmatrix} -\eta_2 p_z^2 & \frac{ik_r p_o}{4\pi} & \eta_2 p_1 p_3 \\ -\frac{ik_r p_o}{4\pi} & -\eta_2 p_z^2 & \eta_2 p_2 p_3 \\ \eta_2 p_1 p_3 & \eta_2 p_2 p_3 & \eta_1 p_0^2 - \eta_2(p_1^2 + p_2^2) \end{bmatrix} \tag{3.24}$$

where $k_r = -1.382, -3.618$ are eigenvalues of $\mathcal{K}$. From L_r, we find a gapless mode with dispersion,

$$\epsilon_r^2 = \frac{\eta_2}{\eta_1}(p_1^2 + p_2^2) + \frac{16\pi^2\eta_2^2}{k_r^2} p_3^4 \tag{3.25}$$

Thus, for the realistic case of large but finite N number of layers, $p_3 \sim \pi/N$, and the lowest mode has the dispersion,

$$\epsilon_r = \frac{\eta_2}{\eta_1}(p_1^2 + p_2^2) + \frac{16\pi^6\eta_2^2}{k_r^2 N^4}. \tag{3.26}$$

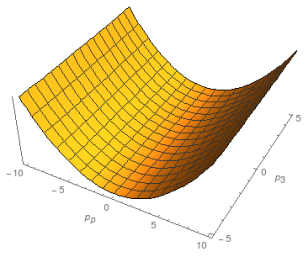

Figure 3.2: dispersion plot of the gauge mode at specific chosen values of the parameters in the coefficients

Similar to the Laughlin case, the gauge mode disperses anisotropically with similar form of the dispersion relation as in the Laughlin case of [80]. The quasi-3D system has gapless (or nearly gapless, in a realistic situation) gauge boson excitation, as well as gapped partons f_i,

which are minimally coupled to it and therefore have long-range interactions, and moreover, their interactions are naturally anisotropic. In addition, the partons due to being minimally coupled to the gauge field component in the $z-$direction are thus free to move around in the 3D bulk and not confined to the 2D layers of their original FQH states. However, the parton excitations are only fractionally charged by construction but can not have fractional statistics in this quasi-3D system since point-particle-like excitations can not have fractional statistics in 3D.

Let us also comment on boundary transport of the quasi-3D system. The boundary transport is now a sheet-like surface transport, with components in the $xy-$direction, as well as in the $z-$direction, the former being quantized due to arising from the QH nature of the individual layers in the bulk, while the latter is unquantized and non-topological. In the large but finite N layers, there are order $N \times N$ chiral modes of quantized transport in the $xy-$planes due to the bulk K-matrix (in the effective theory of the partons) being order $N \times N$. In the 3d limit, however, due to the gapless gauge mode in the bulk, the quantized xy-transport may continue to be gapless and chiral due to the underlying QH nature of the layers in the bulk, however, the gapless gauge mode decoheres the z-transport, which is non-quantized to begin with and is basically an artifact of the interlayer hopping events. The gapless mode in the bulk may also affect the *a priori* chiral $xy-$transport due to the lack of energetic distinction between these gapless edge modes and the bulk gapless mode. This eventual lack of distinction may serve as an experimental signature of the existence of a bulk gapless mode in such quasi-3D multilayered systems, in particular, by closely observing such vanishing energetic distinction between the gapless chiral edge modes and gapless bulk gauge mode as a function of the number of layers.

3.5 Summary

In this chapter, we have extended the multilayered Laughlin partonic theory of [80] to the case of bilayered and multilayered quasi-3D stacks of Jain series FQH states using their parton descriptions, suggesting that these may be useful candidate states for the situation when the interlayer coupling (via interlayer hopping) is comparable to the intralayer interactions. For the bilayered cases, we presented a K-matrix theory from which we showed how the bilayers represent experimentally different QH states from the cases when the layers are either decoupled or have bonded together to form an additive QH liquid. For the quasi-3D stacks, we constructed its effective theory in terms of the gauge fields which have a gapless anisotropically dispersing mode in the low-energy, and discussed its plausible consequences on the observable edge/surface transport. The most important future direction would be to construct ways to analyse the surface sheet transport, which, as we mentioned earlier, has not been successful even for the simpler case of $\nu = 1/3$ FQH states in each layer.

3.6 Appendix - Toric ground state degeneracy

In this appendix, we will present a brief derivation of the ground state degeneracy on a torus of a quantum Hall fluid in the K-matrix description, following [26, 83].

Let us first show that the toric degeneracy of a Chern-Simons theory at level k is $|k|$. This corresponds to a 1×1 $\mathcal{K}$-matrix, i.e., $\mathcal{K} = k$, and would describe the effective theory of a fractional quantum Hall liquid at filling $\nu = 1/k$. That is, we have

$$\mathcal{L} = \frac{k}{4\pi}\epsilon_{\mu\nu\lambda}\alpha_\mu\partial_\nu\alpha_\lambda. \tag{3.27}$$

Consider a torus of dimensions (L_1, L_2). We may define gauge field configurations (essentially a gauge choice) on the torus as

$$\alpha_0(x, y, t) = 0, \alpha_1(x, y, t) = \frac{2\pi X}{L_1}, \alpha_2(x, y, t) = \frac{2\pi Y}{L_2},$$

where X and Y are periodic coordinates on the two big circles of the torus.

Substituting this configuration into the Chern-Simons Lagrangian above gives us,

$$\mathcal{L} = \pi k(Y\dot{X} - X\dot{Y}). \tag{3.28}$$

On the torus, there exist two large gauge transformations, $g_1 = \exp(2\pi ix/L_1)$ and $g_2 = \exp(2\pi iy/L_2)$, under which $\alpha_\mu \to \alpha_\mu - ig_{1,2}^{-1}\partial_\mu g_{1,2}$. This is readily seen to transform $(X, Y) \to (X + 1, Y)$ and $(X, Y) \to (X, Y + 1)$ for g_1, g_2 respectively. Thus $(X, Y) \sim (X, Y + 1) \sim (X + 1, Y)$ is an equivalence relation.

From the substituted Lagrangian above, we have the "momentum" conjugate to Y,

$$p = \frac{\delta\mathcal{L}}{\delta\dot{Y}} = -2\pi kX, \tag{3.29}$$

so that we have the noncommutativity relation,

$$[X, Y] = \frac{-i}{2\pi k}, \tag{3.30}$$

and the Hamiltonian vanishes, a defining feature of a "pure" topological field theory,

$$\mathcal{H} = p\dot{Y} - \mathcal{L} = 0. \tag{3.31}$$

Thus it appears *a priori* that any arbitrary function qualifies as the ground-state wavefunction of such a vanishing Hamiltonian. However, legitimate wavefunctions need to satisfy the aforementioned equivalence relation. This condition filters out a set of finite number of ground state wavefunctions, and the number of elements of this set is the ground state degeneracy of the level-k Chern-Simons theory on the torus.

The condition $\psi(Y) = \psi(Y + 1)$ implies we can write, for integer n,

$$\psi(Y) = \sum_{n=-\infty}^{\infty} c_n \exp(2i\pi nY). \tag{3.32}$$

Since X is essentially a conjugate momentum variable to Y, we have to Fourier transform the above wavefunction to impose the condition $\psi(X) = \psi(X+1)$, with $p = i\partial/\partial Y$,

$$\tilde{\psi}(p) = \sum c_n \delta(p - 2\pi n). \tag{3.33}$$

Since, as noted above, $p = -2\pi kX$, we may equivalently write,

$$\phi(X) = \sum c_n \delta(kX + n). \tag{3.34}$$

$X \sim X + 1$ thus implies that $c_n = c_{n+k}$, and thereby that there are k independent c_n's. This implies that the number of degenerate ground state wavefunctions on the torus is k.

Same conclusion is reached if we had instead chosen X as our canonical position variable and Y as its corresponding conjugate momentum.

We recall that the above derivation was done for a 1×1 $\mathcal{K}$-matrix. Now consider a $m \times m$ $\mathcal{K}$-matrix which we assume to be invertible so that there is no eigenvalue $= 0$, and which in general may have non-zero off-diagonal entries. There always exists a diagonalizing transformation to a new basis (new gauge fields α') which brings the $\mathcal{K}$-matrix to its diagonal form with its eigenvalues being the diagonal entries, and thus the determinant (ground state degeneracy on torus) is simply the product of the eigenvalues. Since eigenvalues, and thus determinant, of a matrix are independent of basis or similarity transformations, it follows that the ground state degeneracy on torus of a general invertible $m \times m$ $\mathcal{K}$-matrix Chern-Simons theory is simply $|det(\mathcal{K})|$.

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