REPRESENTATION FORMULAS FOR CERTAIN QUADRATIC FORMS AND A PROBLEM OF CONSTRUCTING LIFTING MAPS BETWEEN SPACES OF MODULAR FORMS

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. "On the number of representations by certain octonary quadratic forms with coefficients 1, 2, 3, 4 and 6", B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, *Int. J. Number Theory*, **2018**, *14(3)*, 751-812.

2. "On the number of representations of certain quadratic forms and a formula for the Ramanujan Tau function", B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, *Funct. Approx. Comment. Math.*, **2018**, *58(2)*, 233-244.

Conferences

1. "On the representations of a positive integer by certain classes of quadratic forms in eight variables", B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, *Springer Proc. Math. Stat.,* **2017**, *221*,641-664.

2. "Representations of an integer by some quaternary and octonary quadratic forms", B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, *Springer Proc. Math. and Stat.*, **2018**, 251, 383-409.

3. "On the number of representations of certain quadratic forms in 8 variables", B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, AMS Contemporary Mathematics, **2019**, 732, 215-224.

Others

1. Ramanujan–Mordell type formulas associated to certain quadratic forms of discriminant 20^k and 32^k, Anup Kumar Singh and Dongxi Ye.
2. Shimura and Shintani liftings of certain cusp forms of half-integral and integral weight, Manish Kumar Pandey, B. Ramakrishnan and Anup Kumar Singh.

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Dedicated to

My Mother & Brother

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Summary

Three problems in the theory of modular forms are considered in this thesis. The first problem is about obtaining explicit formulas for the number of representations of positive integers by certain types of positive definite integral quadratic forms of even number of variables (≤ 28). The method of obtaining these formulas is to show that the generating functions corresponding to these quadratic forms are modular forms and then uses explicit bases for the spaces of modular forms.

In 1916, S. Ramanujan gave an explicit expression for the generating function for the sum of $2k$ squares in terms of explicitly determined Eisenstein series and a linear combination of certain modular forms (given in terms of Dedekind eta-quotients). Later this was proved by L. J. Mordell, now known as Ramanujan-Mordell theorem. The second part of thesis generalizes this theorem to certain variant of above type of quadratic forms.

Shimura and Shintani correspondence play an important role in the theory of modular forms of degree 1. In 1985, W. Kohnen gave explicit description of the Shintani correspondence from forms of integral weight modular forms to forms of half-integral weight. These correspondences behave nicely on new forms in the respective spaces. In 2017, S. Choi and C. H. Kim constructed explicit Shimura and Shintani maps for the old class (characterized by eigenspaces of Atkin-Lehner involutions) of level $4p$. In the third problem, the above work of Choi-Kim is extended for the level $4N$, N odd and square free.

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CHAPTER 1

Introduction and Preliminaries

In this thesis, we consider three problems, two are related to theta series associated with certain quadratic forms and the third one is about the construction of Shimura and Shintani liftings. The first problem is about finding explicit formulas for the number of representations of these quadratic forms, which extends various works done by many authors ([1, 2, 3, 5, 6, 7, 8, 40, 41]), the second one is about obtaining Ramanujan-Mordell type formulas, which extends the works of Ramanujan [66], Mordell [50], Cooper et. al [22], Ye [82]. The third problem is about the construction of explicit Shimura and Shintani maps between certain subspaces of modular forms of half-integral and integral weights. This work generalizes the result of Choi-Kim [18] for odd square-free levels. These three problems are discussed in three different chapters, and each chapter's introduction gives a detailed account of the history of the problem and the results obtained in the respective chapter.

In this chapter, we give some preliminary facts on modular forms of integral and half-integral weights.

1.1 Notation

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. For a complex number z, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of z respectively. For $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$, we write $a|b$ when b is divisible by a and $a(\mathrm{mod}\ b)$ means that a varies over a complete set of residue classes modulo b. Let $\mathbb{H} = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$ be the complex upper half plane. Unless stated otherwise we denote by $q = e^{2\pi iz}, i = \sqrt{-1}$ for $z \in \mathbb{H}$. For a commutative ring R, we write the set of all $n \times n$ matrices with entries in R by $M_n(R)$. Next we mention few special group of matrices. The general linear group over $\mathbb{Q}$;

$$GL_2^+(\mathbb{Q}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q}) \ : \ ad - bc > 0 \right\}.$$

Special linear group over $\mathbb{Z}$ (full modular group);

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \ : \ ad - bc = 1 \right\}.$$

It is known that $GL_2^+(\mathbb{Q})$ acts on the Poincaré upper half-plane $\mathbb{H}$ by the fractional linear transformation as follows. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ and $z \in \mathbb{H}$, we let

$$\gamma z := \frac{az+b}{cz+d} \in \mathbb{H}. \tag{1.1}$$

Then for any integer k and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, the slash operator on functions $f : \mathbb{H} \to \mathbb{C}$ is defined by

$$(f|_k \gamma)(z) := (\det \gamma)^{k/2} (cz+d)^{-k} f(\gamma z).$$

For a positive integer N, the principal congruence subgroup of level N is denoted by $\Gamma(N)$ and defined as

$$\begin{aligned}\Gamma(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; a \equiv 1(N), b \equiv c \equiv 0(N) \right\}, \\ &= \operatorname{Ker}\left(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\right).\end{aligned}$$

A subgroup Γ of $SL_2(\mathbb{Z})$ is said to be a *congruence subgroup* of $SL_2(\mathbb{Z})$ if it contains a principal congruence sungroup of level N, for some $N \in \mathbb{Z}, N > 0$. The smallest such N is called the *level* of Γ. Besides $\Gamma(N)$, the following two congruence subgroups are equally important. We list them below;

$$\begin{aligned}\Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0(N) \right\}, \\ \Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1(N), c \equiv 0(N) \right\}.\end{aligned}$$

The inclusion relation amongst these congruence subgroups is given by $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$. *Unless stated otherwise, we always let $k \in \mathbb{Z}$ and Γ denotes a congruence subgroup of level N.*

Cusps: Let $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. We then extend the action of $SL_2(\mathbb{Z})$ on $\widehat{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, the extended upper half-plane as follows. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \widehat{\mathbb{H}}$, define

$$\gamma z := \begin{cases} \infty & \text{if } z = -d/c, \\ a/c & \text{if } z = \infty, \\ (az+b)/(cz+d) & \text{otherwise.} \end{cases}$$

A *cusp* of Γ is a Γ-equivalent class of elements in $\mathbb{P}^1(\mathbb{Q})$ under the action of Γ. Note that the group $SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, hence there is only one cusp of $SL_2(\mathbb{Z})$. Since the index of every congruence subgroup Γ of $SL_2(\mathbb{Z})$ is finite, it follows that there are only finitely many cusps of Γ.

Holomorphicity at the cusps: Assume that $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function which satisfies the *modular transformation property* for Γ, namely $f \mid_k \gamma = f$ for all $\gamma \in \Gamma$ (such an f is said to be *weakly holomorphic modular form* of weight k with respect to Γ). Let D' be the open unit disk in $\mathbb{C}$ with the origin removed. Then $z \mapsto q_N := e^{2\pi iz/N}$ defines a map from $\mathbb{H}$ into D'. Since $f \mid_k \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} = f$, it follows that f is periodic with period N and hence there exists a function $F : D' \to \mathbb{C}$ such that $F(q_N) = f(z)$. If for all $q_N \in D'$, we have the Laurent series expansion of the form

$$f(z) = F(q_N) = \sum_{n \geq 0} a(n) q_N^n, \tag{1.2}$$

then f is said to be holomorphic at ∞. Moreover, if $a(0) = 0$, we say that f vanishes at ∞. Eq. (1.2) is called the Fourier expansion of f at ∞ or the q-expansion of f about ∞, and the numbers $a(n) \in \mathbb{C}$ are called the Fourier coefficients of f. Since any cusp $s \in \mathbb{P}^1(\mathbb{Q})$ can be written as $s = \gamma_0 \infty$, for some $\gamma_0 \in SL_2(\mathbb{Z})$ and therefore holomorphy at s is naturally defined in terms of holomorphy at ∞ via the slash operator. More precisely, f is said to be holomorphic (or vanishes) at the cusp s if $f \mid_k \gamma_0$ is holomorphic (or vanishes) at ∞.

1.2 Modular forms

1.2.1 Modular forms of integer weight

Definition 1.2.1 *A holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight k for $\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ with a Dirichlet character χ modulo N, if f satisfies the following conditions:*

1. *For all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ *and* $z \in \mathbb{H}$,

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) \;=:\; f|_k\gamma(z) \;=\; \chi(d)f(z),$$

2. f *is holomorphic at all the cusps of* $\Gamma_0(N)$.

We denote the $\mathbb{C}$-vector space of all such modular forms of weight k, level N with a Dirichlet character χ modulo N by $M_k(N,\chi)$. We denote the subspace of $M_k(N,\chi)$ consisting of all cusp forms by $S_k(N,\chi)$. If the associated Dirichlet character χ (mod N) is trivial, then we denote these spaces simply by $M_k(N)$ and $S_k(N)$ respectively. Further, in the case of full modular group $SL_2(\mathbb{Z})$, we denote the respective spaces by M_k and S_k respectively.

Hecke Operators: Let n be a positive integer. For $(n,N)=1$, the n-th Hecke operator, denoted by T_n, is defined in terms of the Fourier coefficients as follows.

$$f|T_n(z) = \sum_{m=0}^{\infty} \sum_{d|(m,n)} \chi(d)d^{k-1}a(\frac{mn}{d^2})e^{2\pi imz}.$$

The family $\{T_n : (n,N)=1\}$ form a commuting family of Hecke operators on the space $M_k(N,\chi)$. For every positive integer n, we define the $U(n)$ operator on formal sums as follows:

$$U(n) : \sum_{m\geq 0} a(m)e^{2\pi imz} \to \sum_{m\geq 0} a(mn)e^{2\pi imz}. \tag{1.3}$$

For a prime $p|N$, $U(p)$ denotes the p-th Hecke operator on $M_k(N,\chi)$. It is a fact that $\{T_p, p\nmid N; U(p), p|N\}$ generate the Hecke algebra on $M_k(N,\chi)$. Moreover, the operators $T_p, p\nmid N$, $U(p), p|N$ preserve the space of cusp forms. Further, it is a fact that the Hecke operators T_n, $(n,N)=1$ satisfy the following commuting property on $M_k(N,\chi)$:

$$T_mT_n = \sum_{d|gcd(m,n)} \chi(d)d^{k-1}T_{\frac{mn}{d^2}}.$$

Definition 1.2.2 (Petersson inner product) *Let $f, g \in M_k(N,\chi)$ be such that at least one of them is a cusp form. Write $z = x+iy$, then the Petersson inner product of f and g is defined as:*

$$\langle f, g\rangle := i_N^{-1} \int\limits_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}, \tag{1.4}$$

where $\Gamma_0(N)\backslash\mathbb{H}$ is a fundamental domain, $\frac{dxdy}{y^2}$ is an invariant measure under the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ and i_N denotes the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$.

Remark 1.2.1 It is well-known that $S_k(N,\chi)$ is a finite-dimensional Hilbert space with respect to the inner product defined by (1.4). The Hecke operators T_p for all primes $p \nmid N$ are Hermitian with respect to the Petersson scalar product. Further, as they form a commuting family of operators, it follows from linear algebra that the space $S_k(N,\chi)$ has a basis of eigenforms with respect to all T_p, $p \nmid N$. However, by the theory of newforms developed by Atkin and Lehner [10] and W. W. Li [42], there exists a subspace of $S_k(N,\chi)$, denoted by $S_k^{new}(N,\chi)$, called the space of newforms which has an orthogonal basis of eigenforms with respect to all T_p, $p \nmid N$, $U(p)$, W_p, $p|N$. These basis elements, which are eigenforms with respect to all the Hecke operators, are called Hecke eigenforms. In the above, the Atkin-Lehner W-opreators W_p for $p|N$ are defined as follows. Let $p|N$, with $p^\alpha \| N$ (i.e., $p^\alpha | N$, $p^{\alpha+1} \nmid N$), then

$$W_p = \begin{pmatrix} p^\alpha a & b \\ Nc & p^\alpha d \end{pmatrix},$$

where a,b,c,d are integers satisfying the properties, $p^{2\alpha}ad - Nbc = p^\alpha$ and $b \equiv 1 \pmod{N'}$, N' being the conductor of χ. The operators W_p are independent of the choice of the representatives a,b,c,d and satisfy the property that $W_p^2 = \chi(N/p^\alpha)$. In particular, $W_p^2 =$ Identity, if χ is the principal character.

1.2.2 Modular forms of half-integral weight

For complex numbers z and $x \in \mathbb{C} \setminus \{0\}$, we let $z^x = e^{xlogz}$, $logz = log|z| + i\, arg(z)$, $-\pi < arg(z) < \pi$. let ζ be a fourth root of unity. Let G denote the four-sheeted covering of $GL_2^+(\mathbb{Q})$, defined as the set of all ordered pairs $(\alpha, \phi(z))$, where $\phi(z)$ is a holomorphic function on $\mathbb{H}$ such that $\phi^2(z) = \zeta^2 \frac{cz+d}{\sqrt{det\alpha}}$ and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$. Then G is a group with the multiplicative rule: $(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z))$. Let $k \geq 2$ be a natural number. For a function $f : \mathbb{H} \longrightarrow \mathbb{C}$ defined on the upper half-plane $\mathbb{H}$ and an element $(\alpha, \phi(z)) \in G$, define the stroke operator by

$$f|_{k+1/2}(\alpha, \phi(z))(z) = \phi(z)^{-2k-1} f(\alpha z).$$

We omit the subscript $k+1/2$ wherever there is no ambiguity. For the congruence subgroup $\Gamma_0(4)$ and its subgroups, we take the lifting $\Gamma_0(4) \to G$ as the collection $\{(\alpha, j(\alpha, z))\}$, where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, and

$$j(\alpha, z) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(cz+d)^{1/2}.$$

Here $\left(\frac{c}{d}\right)$ denotes the generalized quadratic residue symbol and $\left(\frac{-4}{d}\right)^{-1/2}$ is equal to either 1 or i according as d is 1 or 3 modulo 4.

Definition 1.2.3 *Let M be a natural number. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight $k+1/2$ for $\Gamma_0(4M)$ with character χ(modulo 4M), if*

$$f|_{k+1/2}(\gamma, j(\gamma, z))(z) = \chi(d) f(z), \quad \textit{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4M),$$

and f is holomorphic at all the cusps of $\Gamma_0(4M)$. If further, f vanishes at all the cusps of $\Gamma_0(4M)$, then it is called a cusp form.

The set of modular forms (resp. cusp forms) defined as above becomes a complex vector space denoted by $M_{k+1/2}(4M, \chi)$ (resp. $S_{k+1/2}(4M, \chi)$). If χ is trivial character, then the space is denoted by $M_{k+1/2}(4M)$ (resp. $S_{k+1/2}(4M)$).

The Fourier expansion of a modular form f at the cusp infinity is usually written as

$$f(z) = \sum_{n\geq 0} a_f(n)q^n, \quad q = e^{2\pi iz}.$$

For every positive integer n, the operator $U(n)$ is defined as in (1.3). We define the duplicating operator $B(n)$, which is also defined on formal sums, by

$$B(n) : \sum_{m\geq 0} a(m)q^m \to \sum_{m\geq 0} a(m)q^{mn}.$$

For any $f \in M_{k+1/2}(4M,\chi)$ and a prime $p \nmid 2M$, we define the Hecke operator $T(p^2)$ by

$$f \mid T(p^2) = \sum_{n\geq 0} \left\{ a_f(np^2) + \chi(p)\left(\frac{(-1)^k n}{p}\right) p^{k-1} a_f(n) + \chi(p^2) p^{2k-1} a_f(n/p^2) \right\} q^n,$$

using the recurrence relation and the commutation relations

$$T(p^{2(n+1)}) = T(p^2)T(p^{2n}) - p^{2k-1}T(p^{2(n-1)}) \quad (n \geq 1)$$

and

$$T(n^2m^2) = T(n^2)T(m^2) \quad (n,m) = (mn,2M) = 1,$$

one can extend the definition of $T(n^2)$ for $n \in \mathbb{N}$, $(n,2M) = 1$. The operators $T(n^2)$ for $n \in \mathbb{N}$, $(n,2M) = 1$ and $U(n^2)$ for $n \in \mathbb{N}, n|2M$ are the Hecke operators on $M_{k+1/2}(4M,\chi)$. Further, $B(n)$ maps $M_{k+1/2}(4M,\chi)$ into $M_{k+1/2}(4Mn,\chi\chi_n)$, where χ_n is the quadratic character $\left(\frac{n}{\cdot}\right)$. Finally for $f, g \in M_{k+1/2}(4M,\chi)$ with f or g is a cusp form, the Petersson inner product is defined by

$$\langle f,g\rangle = i_{4M}^{-1} \int_{\mathscr{F}_{4M}} f(z)\overline{g(z)} v^{k-\frac{3}{2}} du dv,$$

where $\mathscr{F}_{4M}$ is a fundamental domain for the action of $\Gamma_0(4M)$ on $\mathbb{H}$, i_{4M} is the index of $\Gamma_0(4M)$ in $SL_2(\mathbb{Z})$ and $z = u + iv$. For more details on the theory of modular forms of half-integral weight, we refer to Koblitz's book [35] and the work of Shimura [73].

1.2.3 Examples of modular forms

Eisenstein series: Let $k \geq 2$ be an even integer. The normalized Eisenstein series $E_k(z)$ of weight k for $SL_2(\mathbb{Z})$ is defined as:

$$E_k(z) := \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (m,n)=1}} \frac{1}{(mz+n)^k}. \tag{1.5}$$

It is known that E_k is a modular form of weight k for $SL_2(\mathbb{Z})$ for $k \geq 4$ and it has the Fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n, \tag{1.6}$$

where $\sigma_r(n) = \sum_{d|n} d^r$, for any positive integer r and B_k's are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{k\geq 0} B_k \frac{x^k}{k!}.$$

Because we use Eisenstein series in our work, we give the Fourier expansion of the first few Eisenstein series:

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, & E_6(z) &= 1 - 504 \sum_{n\geq 1}^{\infty} \sigma_5(n) q^n, \\ E_8(z) &= 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n, & E_{10}(z) &= 1 - 264 \sum_{n\geq 1} \sigma_9(n) q^n, \\ E_{12}(z) &= 1 + \frac{65520}{691} \sum_{n\geq 1} \sigma_{11}(n) q^n, & E_{14}(z) &= 1 - 24 \sum_{n\geq 1} \sigma_{13}(n) q^n. \end{aligned}$$

When $k = 2$, we have $E_2(z) = 1 - 24\sum_{n\geq 1} \sigma(n) q^n$ and it is the fundamental quasimodular form of weight 2 for $SL_2(\mathbb{Z})$.

Generalized Eisenstein series: Suppose that χ and ψ are primitive Dirichlet characters with conductors M and N respectively. For a positive integer k, let

$$E_{k,\chi,\psi}(z) := -\frac{B_{k,\psi}}{2k} \delta_{M,1} + \sum_{n\geq 1} \sigma_{k-1;\chi,\psi}(n)\, q^n, \tag{1.7}$$

where $B_{k,\psi}$ denotes generalized Bernoulli number with respect to the character ψ and

$$\sigma_{k-1;\chi,\psi}(n) := \sum_{d|n} \psi(d)\, \chi(n/d)\, d^{k-1}.$$

Then, $E_{k,\chi,\psi}(z)$ belongs to the space $M_k(\Gamma_0(MN), \chi/\psi)$, provided $\chi(-1)\psi(-1) = (-1)^k$ and $MN \neq 1$. When $\chi = \psi = 1$ (i.e., when $M = N = 1$) and $k \geq 4$, we have $E_{k,\chi,\psi}(z) = -\frac{B_k}{2k} E_k(z)$, where E_k is the normalized Eisenstein series of weight k as defined before. For more details we refer to the book of Miyake [49, Chapter 7] and the book of Stein [77, Section 5.3].

Ramanujan Delta function: The *Ramanujan delta function* is defined as

$$\Delta(z) \ := \ \frac{1}{1728}\left(E_4(z)^3 - E_6(z)^2\right)$$

and it is the unique cusp form of weight 12 for $SL_2(\mathbb{Z})$, with the Fourier expansion

$$\Delta(z) = q \prod_{n\geq 1} (1-q^n)^{24} \ = \ \sum_{n\geq 1} \tau(n) q^n.$$

Here $\tau(n)$ is the *Ramanujan tau function*.

The Dedekind eta function and eta quotients: The Dedekind eta function $\eta(z)$ is defined by

$$\eta(z) = q^{1/24} \prod_{n\geq 1} (1-q^n). \tag{1.8}$$

Note that $\eta^{24}(z) = \Delta(z)$. An eta-quotient is a finite product of integer powers of $\eta(z)$ and we denote it as

$$\prod_{i=1}^{s} \eta^{r_i}(d_i z) := d_1^{r_1} d_2^{r_2} \cdots d_s^{r_s}, \tag{1.9}$$

where d_i's are positive integers and r_i's are non-zero integers.

Poincaré series: Let k, n, N be positive integers. The n-th *Poincaré series* of weight k for

$\Gamma_0(N)$ is defined by

$$P_{k,N;n}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (cz+d)^{-k} e^{2\pi i n \gamma z}, \tag{1.10}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{Z} \right\}$. It is known that $P_{k,N;n} \in S_k(N)$ for $k \geq 2$ and it is characterized by the following property.

Lemma 1.2.4 *Let $f \in S_k(N)$ with Fourier expansion $f(z) = \sum_{m \geq 1} a_f(m) q^m$, $q = e^{2\pi i z}$. Then*

$$\langle f, \, P_{k,N;n} \rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a_f(n).$$

The following familiar result tells about the growth of the Fourier coefficients of a modular form in which the first statement can be easily obtained and the second is due to P. Deligne [24].

Proposition 1.2.5 *Let $a_f(n)$ be the n-th Fourier coefficient of a modular form $f \in M_k(N)$. Then for any $\varepsilon > 0$, we have*

$$a_f(n) \ll_\varepsilon n^{k-1+\varepsilon},$$

and when $f \in S_k^{new}(N)$ is a (normalized) Hecke eigenform, then

$$a_f(n) \ll_\varepsilon n^{\frac{k-1}{2}+\varepsilon}.$$

CHAPTER 2

Representation formulas for certain quadratic forms and some applications

In this chapter, we give a consolidated version of the results published in [61, 62, 63, 64, 65]. In these works, we have obtained explicit formulas for the number of representations of a natural number by certain quadratic forms in $2k$ variables, $k = 2, 4, 7, 9, 11, 12, 14$, with certain integer coefficients. As applications to our formulas, we derive a new formula for the Ramanujan tau function and also deduce certain convolution sum identities of the divisor functions.

2.1 Introduction

One of the classical problems in number theory is to find explicit formulas for the number of representations of a positive integer by positive definite integral quadratic forms. In this connection the classical theta function defined by $\Theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$, $q = e^{2\pi iz}$ and $z \in \mathbb{H}$, plays a key role. After the initial works of Gauss, Jacobi, Lagrange, Glaisher and others, it was S. Ramanujan [66] who first gave a general formula for the number of representations of an integer $n \geq 1$ as a sum of $2k$ squares. This number is denoted by $r_{2k}(n)$ and it is nothing but the n-th Fourier coefficient of $\Theta^{2k}(z)$. There are numerous works on finding formulas for the number of representations corresponding to various types of quadratic forms by many authors and the methods vary from elementary to sophisticated tools. In this chapter (for most part of the results), we use the theory of modular forms to get such formulas. Here we consider different types of quadratic forms in $2k$ variables, where $k = 2,4,7,9,11,12,14$, which are specified below. In this direction, we have obtained such results in five publications and in this chapter we summarize them and give a consolidated presentation of these results. There are two main applications of our results. One of them is a new formula for the Ramanujan tau function (which is the Fourier coefficient of the unique cusp form of weight 12 for the group $SL_2(\mathbb{Z})$) and the other one is finding certain convolution sum identities of the divisor functions as a corollary to some of our results. A few references which motivated us to undertake this investigation are [1, 2, 3, 5, 6, 7, 8, 40, 41] and related references in these works. The quadratic forms considered in our works can be put into three types, which are given below. We present them seperately by giving a brief history of earlier works and about the results that appear in this chapter.

$$\mathbf{1}.\ \mathcal{Q}_k : \sum_{i=1}^{r} a_i x_i^2 + \sum_{i=r+1}^{r+t} b_i(x_{2i-1-r}^2 + x_{2i-1-r}x_{2i-r} + x_{2i-r}^2),\ r+2t = 2k,\ k = 2,4, \\ 7,9,11,12,14 \text{ and } a_i \in \mathcal{A} \subseteq \left\{1,2,3,4,6,8\right\}, b_i \in \mathcal{B} \subseteq \left\{1,2,4,8\right\}. \tag{2.1}$$

When $t = 0$, there are many works in finding formulas for the number of representations of a natural number by sums of $2k$ squares. In this chapter we consider the case $r = 8$. The same case with different set of coefficients were considered by many authors [5, 6, 7, 8, 40, 41]. When $r = 0$, general formulas were given by Lomadze [47], for $k \leq 17$. The case $t = 6$ with $b_i = 1$ was treated by Xia-Yao [80] and the cases $t = 8,10,12$ with $b_i = 1$ were done by Ramakrishnan-Sahu [59, 60]. B. Köklüce [40] also considered similar quadratic forms in 8 variables with certain integer coefficients. In his book [79], K. S. Williams considered certain cases of all these three types of forms.

In this chapter, using techniques from modular forms, we obtain explicit formulas for the cases $r+2t=2k$, with $k=4,7,9,11,12,14$ with the coefficients $a_i \in \mathscr{A} \subseteq \{1,2,3,4,6,8\}, b_i \in \mathscr{B} \subseteq \{1,2,4,8\}$. As a consequence to the formulas for the case $r=0,\ t=12$, we deduce a new formula for the Ramanujan tau function. This is explained in detail in the applications section §2.3.1.

$$\textbf{2}.\ \mathscr{Q}_{a,\ell}\ :\ x_1^2+x_1x_2+ax_2^2+\ell(x_3^2+x_3x_4+ax_4^2),\ (a,\ell)\in\Big\{(1,5),(2,2),(2,3),(2,4),(3,2),(3,3),(4,2),(5,1),(5,2)\Big\}. \tag{2.2}$$

The starting point of this problem is the work of Ramanujan [66], who gave a formula for the number of representations of a positive integer by the quadratic form $\mathscr{Q}_{1,1} = x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2$. There have been many generalizations of this result. In the table below we give the list of results obtained in this direction (including the results obtained in our work [63] and presented as Theorem 2.2.2, in this chapter).

[Theorem 2.2.2] (a,ℓ)	Past works $(a,1)$	Author(s) (earlier works)	References
(1,5)	(1,1)	Huard et al., Lomadze	[30, 47]
(2,2), (2,3), (2,4)	(2,1)	Ramanujan, Berndt, Chan-Ong, Williams	[66, 12] [17, 78]
(3,2), (3,3)	(3,1)	Chan-Cooper	[16]
(4,2)	(4,1)	Cooper-Ye	[23]
(5,1), (5,2)	–	–	–
–	(6,1)	Chan - Cooper	[16]
–	(7,1)	Dongxi Ye	[81]

$$\textbf{3}.\ \mathscr{Q}_{(2,1),j}:\ \mathscr{Q}_{2,1}\oplus j\mathscr{Q}_{2,1};\ 1\le j\le 4 \tag{2.3}$$

Formulas for the quadratic forms $\mathscr{Q}_{2,1}$ were earlier obtained in [12, 17, 66, 78]. In this chapter we use the theory of modular forms to get formulas for the number of representations of a positive integer by quadratic forms $\mathcal{Q}_{(2,1),j}; 1 \leq j \leq 4$. While using this method, as an application, we derive certain convolution sums of the divisor functions (details appear in §2.3.2).

2.2 Statement of theorems

In the formulas presented in the theorems of this section, we denote by $\tau_{k,N;j}(n)$ and $\tau_{k,N,\chi;j}(n)$, the n-th Fourier coefficient of j-th newform in the space $S_k^{new}(N)$ and $S_k^{new}(N,\chi)$ respectively. If there is only one newform in the the respective spaces, we omit the subscript j and write them as $\tau_{k,N}(n)$ and $\tau_{k,N,\chi}(n)$ respectively. Explicit descriptions of these newforms are given in §2.4.3.

In the following theorem (which corresponds to the quadratic form $\mathcal{Q}_k$, given by (2.1)), we give a consolidated version of results obtained in [61, 62, 63].

Theorem 2.2.1 ([61, 62, 63]) *Let $N_k(\mathscr{A};n)$, $N_k(\mathscr{B};n)$ and $N_k(\mathscr{A}:\mathscr{B};n)$, denote the number of representations of a natural number n by the quadratic form $\mathcal{Q}_k$ with $t=0$, $r=0$ and $r,t \neq 0$ respectively. Then*

$$N_k(\mathscr{A};n) = \sum_{1\leq i\leq \ell(k,\mathscr{A})} \alpha_{k,i}\, A_{k,i}(n), \tag{2.4}$$

$$N_k(\mathscr{B};n) = \sum_{1\leq i\leq \ell(k,\mathscr{B})} \beta_{k,i}\, B_{k,i}(n), \tag{2.5}$$

$$N_k(\mathscr{A}:\mathscr{B};n) = \sum_{1\leq i\leq \ell(k,\mathscr{A},\mathscr{B})} \gamma_{k,i}\, C_{k,i}(n). \tag{2.6}$$

Where $\ell(k,\mathscr{A}) = \dim M_k(N_1,\chi_{N_1})$, with $N_1 = lcm(4a_1,4a_2,\ldots,4a_r)$, $a_i \in \mathscr{A}$, $\ell(k,\mathscr{B}) = \dim M_k(N_2,\chi_{N_2})$, with $N_2 = lcm(3b_1,3b_2,\cdots,3b_t)$, $b_i \in \mathscr{B}$, $\ell(k,\mathscr{A},\mathscr{B}) = \dim M_k(N_3,\chi_{N_3})$, with $N_3 = lcm(4a_1,4a_2,\ldots,4a_r,3b_1,3b_2,\cdots,3b_t)$, $a_i \in \mathscr{A}$, $b_i \in \mathscr{B}$ and $\chi_{N_i} = \left(\frac{N_i}{\cdot}\right)$ is the quadratic Dirichlet character modulo N_i (kronecker symbol). Further, $A_{k,i}(n), B_{k,i}(n)$ and $C_{k,i}(n)$ are the n-th Fourier coefficients of i-th basis element of $M_k(N_1,\chi_{N_1})$, $M_k(N_2,\chi_{N_2})$ and $M_k(N_3,\chi_{N_3})$ respectively. Explicit values of k and *the sets $\mathscr{A}$, $\mathscr{B}$ considered in this theorem are tabled below.*

Formula	2k	$\mathscr{A}$	$\mathscr{B}$	N_i	Ref.
$N_k(\mathscr{A};n)$	8	$\{1,2,3,4,6\}$	–	$N_1=48$	[61]
	8	$\{1,2,4,8\}$	–	$N_1=32$	[63]
$N_k(\mathscr{B};n)$	8	–	$\{1,2,4,8\}$	$N_2=24$	[62]
	14, 18, 22, 24, 28	–	$\{1\}$	$N_2=3$	[64]
$N_k(\mathscr{A}:\mathscr{B};n)$	8	$\{1,2,3\}$	$\{1,2,4,8\}$	$N_3=24$	[62]

In the case $r=0$ and $\mathscr{B}=\{1\}$, we give explicit formulas for $N_k(\{1\};n)$ when k=7,9,11,12,14, below. We also use the notation $s_{2k}(n)$ for $N_k(\{1\};n)$.

$$s_{14}(n) = \frac{81}{7}\sum_{d|n}\left(\frac{n/d}{3}\right)d^6 - \frac{3}{7}\sum_{d|n}\left(\frac{d}{3}\right)d^6 + \frac{216}{7}\tau_{7,3,\chi_{-3}}(n), \tag{2.7}$$

$$\begin{aligned} s_{18}(n) &= \frac{2187}{809}\sum_{d|n}\left(\frac{n/d}{3}\right)d^8 + \frac{27}{809}\sum_{d|n}\left(\frac{d}{3}\right)d^8 + \frac{24\times 27\times 1728}{809}\tau_{9,3,\chi_{-3};1}(n) \\ &\quad + \frac{24\times 1728}{809}\tau_{9,3,\chi_{-3};2}(n), \end{aligned} \tag{2.8}$$

$$\begin{aligned} s_{22}(n) &= \frac{729}{1847}\sum_{d|n}\left(\frac{n/d}{3}\right)d^{10} - \frac{3}{1847}\sum_{d|n}\left(\frac{d}{3}\right)d^{10} + \frac{81\times 748}{9235}\tau_{11,3,\chi_{-3};1}(n) \\ &\quad + \frac{729\times 748}{9235}\tau_{11,3,\chi_{-3};2}(n), \end{aligned} \tag{2.9}$$

$$\begin{aligned} s_{24}(n) &= \frac{6552}{73\times 691}\Big(\sigma_{11}(n)+729\sigma_{11}(n/3)\Big) + \frac{240\times 1186848}{50443}\sum_{\substack{a,b\in\mathbb{N}_0\\ a+b=n}}\sigma_3(a)\tau_{8,3}(b) \\ &\quad + \frac{29824}{691}\tau(n) - \frac{504\times 261344}{50443}\sum_{\substack{a,b\in\mathbb{N}_0\\ a+b=n}}\sigma_5(a)\tau_{6,3}(b), \end{aligned} \tag{2.10}$$

$$\begin{aligned} s_{28}(n) &= \frac{12}{1093}\Big(\sigma_{13}(n)-2187\sigma_{13}(n/3)\Big) + \frac{12448\times 504}{1093}\sum_{\substack{a,b\in\mathbb{N}_0\\ a+b=n}}\sigma_5(a)\tau_{8,3}(b) \\ &\quad + \frac{107264\times 12}{1093}\left(\sum_{\substack{a,b\in\mathbb{N}\\ a+b=n}}\sigma(a)\tau(b) - 3\sum_{\substack{a,b\in\mathbb{N}\\ 3a+b=n}}\sigma(a)\tau(b)\right) + \frac{107264}{1093}\tau(n) \end{aligned}$$

$$- \frac{3016 \times 480}{1093} \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b=n}} \sigma_7(a) \tau_{6,3}(b), \qquad (2.11)$$

where $\tau(n)$ is the Ramanujan tau function and $\left(\frac{a}{p}\right)$ is the Legendre symbol. Along with explicit values of the Fourier coefficient $\tau_{k,N}(n), \tau_{k,N,\chi;j}(n)$, we give explicit bases for the spaces $M_k(N_1, \chi_{N_1})$, $M_k(N_2, \chi_{N_2})$ and $M_k(N_3, \chi_{N_3})$, in §2.4.2. The constants $\alpha_{k,i}, \beta_{k,i}$ and $\gamma_{k,i}$ appearing in the (2.5)-(2.6), are given in the tabular form in the Appendix (Table 5.1 – Table 5.16).

In the following table, we list a few values of the Fourier coefficients corresponding to the cusp forms (except for the Ramanujan tau function $\tau(n)$) appearing in the above formulas.

Coeff. \ n	1	2	3	4	5	6	7	8	9	10
$\tau_{6,3}(n)$	1	-6	9	4	6	-54	-40	168	81	-36
$\tau_{8,3}(n)$	1	6	-27	-92	390	-162	-64	-1320	729	2340
$\tau_{7,3,\chi_{-3}}(n)$	1	0	-27	64	0	0	-286	0	729	0
$\tau_{9,3,\chi_{-3};1}(n)$	0	1	-3	0	-10	45	0	8	-270	0
$\tau_{9,3,\chi_{-3};2}(n)$	1	-15	90	-248	150	837	-1750	-120	1593	5040
$\tau_{11,3,\chi_{-3};1}(n)$	1	240	2133	304	-25440	-12960	17234	318720	-174231	76320
$\tau_{11,3,\chi_{-3};2}(n)$	1	0	-27	304	0	6480	17234	0	-57591	76320

In the next theorem, we consider the quadratic form $\mathcal{Q}_{a,\ell}$, given by (2.2), and present a part of the result obtained in [63].

Theorem 2.2.2 ([63]) *Let $N_{a,\ell}(n)$, denote the number of representations of a natural number n by the quadratic form $\mathcal{Q}_{a,\ell}$, where $(a,\ell) \in \{(1,5), (2,2), (2,3), (2,4), (3,2), (3,3), (4,2), (5,1), (5,2)\}$. Then*

$$N_{1,5}(n) = \frac{3}{2}\sigma(n) + \frac{9}{2}\sigma(n/3) - \frac{15}{2}\sigma(n/5) - \frac{45}{2}\sigma(n/15) + \frac{9}{2}\tau_{2,15}(n),$$
$$N_{2,2}(n) = \frac{4}{3}\sigma(n) + \frac{8}{3}\sigma(n/2) - \frac{28}{3}\sigma(n/7) - \frac{56}{3}\sigma(n/14) + \frac{2}{3}\tau_{2,14}(n),$$

$$\begin{aligned}
N_{2,3}(n) &= \frac{63}{40}\sigma(n)-\frac{9}{2}\sigma(n/3)+\frac{21}{2}\sigma(n/7)-\frac{1323}{40}\sigma(n/21)+\frac{1}{2}\tau_{2,21}(n)\\
N_{2,4}(n) &= \frac{2}{3}\sigma(n)+\frac{2}{3}\sigma(n/2)+\frac{8}{3}\sigma(n/4)-\frac{14}{3}\sigma(n/7)-\frac{14}{3}\sigma(n/14)\\
&\quad -\frac{56}{3}\sigma(n/28)+\frac{4}{3}\tau_{2,14}(n)+\frac{8}{3}\tau_{2,14}(n/2),\\
N_{3,2}(n) &= 2\sigma(n)-4\sigma(n/2)+22\sigma(n/11)-44\sigma(n/22),\\
N_{3,3}(n) &= \frac{3}{5}\sigma(n)+\frac{9}{5}\sigma(n/3)-\frac{33}{5}\sigma(n/11)-\frac{99}{5}\sigma(n/33)+\frac{16}{15}\tau_{2,11}(n)\\
&\quad +\frac{16}{5}\tau_{2,11}(n/3)+\frac{1}{3}\tau_{2,33}(n),\\
N_{4,2}(n) &= \frac{1}{2}\sigma(n)+\sigma(n/2)+\frac{3}{2}\sigma(n/3)-\frac{5}{2}\sigma(n/5)+3\sigma(n/6)-5\sigma(n/10)\\
&\quad -\frac{15}{2}\sigma(n/15)-15\sigma(n/30)+\frac{1}{2}\tau_{2,15}(n)+\tau_{2,15}(n/2)+\tau_{2,30}(n),\\
N_{5,1}(n) &= \frac{4}{3}\sigma(n)-\frac{76}{3}\sigma(n/19)+\frac{8}{3}\tau_{2,19}(n),\\
N_{5,2}(n) &= \frac{6}{5}\sigma(n)-\frac{12}{5}\sigma(n/2)+\frac{114}{5}\sigma(n/19)-\frac{228}{5}\sigma(n/38)+\frac{4}{5}\tau_{2,38;2}(n).
\end{aligned}$$

The explicit values of the Fourier coefficient $\tau_{k,N}(n), \tau_{k,N,\chi;j}(n)$ are given in §2.4.4.

Next, we present the result obtained in [65] corresponding to the quadratic forms $\mathcal{Q}_{(2,1),j} := \mathcal{Q}_{2,1}\oplus j\mathcal{Q}_{2,1} = (x_1^2+x_1x_2+2x_2^2+x_3^2+x_3x_4+2x_4^2)+j(x_5^2+x_5x_6+2x_6^2+x_7^2+x_7x_8+2x_8^2)$ for $1\le j\le 4$, as given by (2.3).

Theorem 2.2.3 ([65]) *The number of representations of $n\in\mathbb{N}$ by the quadratic form $\mathcal{Q}_{(2,1),j}$ $(1\le j\le 4)$, denoted by $N_j(n)$ and is given by*

$$N_1(n) = \frac{24}{5}\sigma_3(n)+\frac{1176}{5}\sigma_3(n/7)+\frac{16}{5}\tau_{4,7}(n), \tag{2.12}$$

$$\begin{aligned}
N_2(n) &= \frac{24}{25}\sigma_3(n)+\frac{96}{25}\sigma_3(n/2)+\frac{1176}{25}\sigma_3(n/7)+\frac{4704}{25}\sigma_3(n/14)+\frac{48}{25}\tau_{4,7}(n)\\
&\quad +\frac{192}{25}\tau_{4,7}(n/2)+\frac{28}{25}\tau_{4,14;1}(n),
\end{aligned}\tag{2.13}$$

$$\begin{aligned}
N_3(n) &= \frac{12}{25}\sigma_3(n)+\frac{108}{25}\sigma_3(n/3)+\frac{588}{25}\sigma_3(n/7)+\frac{5292}{25}\sigma_3(n/21)+\frac{32}{25}\tau_{4,7}(n)\\
&\quad +\frac{288}{25}\tau_{4,7}(n/3)+\frac{56}{25}\tau_{4,21;2}(n),
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
N_4(n) &= \frac{6}{25}\sigma_3(n)+\frac{18}{25}\sigma_3(n/2)+\frac{96}{25}\sigma_3(n/4)+\frac{294}{25}\sigma_3(n/7)+\frac{882}{25}\sigma_3(n/14)\\
&\quad +\frac{4704}{25}\sigma_3(n/28)-\frac{356}{175}c_1(n)-\frac{316}{25}c_2(n)+2c_3(n)+\frac{1014}{175}c_4(n)
\end{aligned}$$

$$+\ \frac{1328}{25}c_5(n)+\frac{336}{5}c_6(n)-\frac{664}{25}c_7(n)+\frac{10432}{25}c_8(n)+\frac{10432}{25}c_9(n). \qquad (2.15)$$

The explicit values of the Fourier coefficient $\tau_{k,N}(n), \tau_{k,N;j}(n)$ are given in §2.4.5. Also $c_j(n); 1 \le j \le 9$, are the n-th Fourier coefficients of eta-quotients $f_j(z); 1 \le j \le 9$, which are defined at the end of §2.4.3.

In the following table, we list some values of the Fourier coefficients corresponding to the cusp forms appearing in formulas given in previous Theorem.

2.3 Some applications to our formulas

2.3.1 A formula for the Ramanujan tau function $\tau(n)$.

For a positive integer k, let F_k denote the quadratic form in $2k$ variables defined by

$$F_k(x_1,x_2,\ldots,x_{2k}) = \sum_{j=1}^{k} x_{2j-1}^2 + x_{2j-1}x_{2j} + x_{2j}^2,$$

and the number of representations of a positive integer n by the quadratic form F_k is denoted by $s_{2k}(n)$. Numerically it is defined by

$$s_{2k}(n) = \#\Big\{(x_1,x_2,\cdots,x_{2k}) \in \mathbb{Z}^{2k} : F_k(x_1,x_2,\cdots,x_{2k}) = n\Big\}, \qquad (2.16)$$

We remark that F_k is nothing but k-copies of the quadratic form $\mathcal{Q}_{1,1} = x_1^2 + x_1x_2 + x_2^2$. It is known due to Liouville (ref. [79]) that

$$s_4(n) \ = \ 12\sigma(n) - 36\sigma(n/3).$$

Therefore one can use the duplicating techenique to get formulas for $s_{4k}(n), k \ge 1$ and this method gives an expression for $s_{4k}(n)$ in terms of the convolution sums of $\sigma(n)$, namely $\sum_{\substack{al+bm=n \\ l,m\in\mathbb{N}}} \sigma(l)\sigma(m)$. So in principle one can get formulas for $s_k(n), 4|k$, using the above method and by evaluating convolution sums. We also remark that the evaluation of convolution sum involves two different methods, one using certain combinatorial identities and the other method use the theory of quasimodular forms. For $1 \leq k \leq 6$, formulas for $s_{4k}(n)$ are known due to J. Liouville [44], J. G. Huard et. al. [30], O. X. M. Yao and E. X. W. Xia [80], B. Ramakrishnan and B. Sahu [59]. Most of these works make use of the convolution sums of the divisor functions to evaluate the formulas for $s_{4k}(n)$.

In this section we use our formula for $s_{24}(n)$ as given in Theorem 2.2.1 and the corresponding formulas given by G. A. Lomadze [47, p.12], to get a new expression for the Ramanujan tau function $\tau(n)$.

In [47, p.12], Lomadze gave formulas for $s_{2k}(n)$, $2 \leq k \leq 17$. For $k = 12$, Lomadze's formulas is given by

$$\begin{aligned} s_{24}(n) &= \frac{1}{73\times 691}\Big(6552\left(\sigma_{11}(n)+729\sigma_{11}(n/3)\right)+\frac{291096}{35}L_{12;8}(n) \\ &+ 864L_{12;6}(n)+360L_{12;4}(n)\Big), \end{aligned} \tag{2.17}$$

where $L_{12;4}(n), L_{12;6}(n)$ and $L_{12;8}(n)$ are given by

$$L_{12;4}(n) = \sum_{\substack{x_1\in\mathbb{Z} \\ F_4(x_1,\ldots,x_8)=n}} 1215x_1^8 - 2268nx_1^6 + 1260n^2x_1^4 - 210n^3x_1^2 + 5n^4, \tag{2.18}$$

$$L_{12;8}(n) = \sum_{\substack{x_1\in\mathbb{Z} \\ F_8(x_1,\ldots,x_{16})=n}} 135x_1^4 - 54nx_1^2 + 2n^2, \tag{2.19}$$

$$L_{12;6}(n) = \sum_{\substack{x_1\in\mathbb{Z} \\ F_6(x_1,\ldots,x_{12})=n}} 162x_1^6 - 162nx_1^4 + 36n^2x_1^2 - n^3. \tag{2.20}$$

Comparing (2.17) with our formulas (2.10) (Theorem 2.2.1), we get the following relation (after canceling a factor 50443 in the denominators),

$$29824\times 73\tau(n) \ + \ 1186848\tau_{8,3}(n)+261344\tau_{6,3}(n)+1186848\times 240\sum_{\substack{a,b\in\mathbb{N} \\ a+b=n}}\sigma_3(a)\tau_{8,3}(b)$$

$$-261344\times 504\sum_{\substack{a,b\in\mathbb{N}\\ a+b=n}}\sigma_5(a)\tau_{6,3}(b)=\frac{291096}{35}L_{12;8}(n)+864L_{12;6}(n)+360L_{12;4}(n). \tag{2.21}$$

In [59] and [80], the Fourier coefficients $\tau_{8,3}(n)$ and $\tau_{6,3}(n)$ have been expressed as sums of type that appears in Lomadze's formulas. We give brief proof here (for details we refer to [59, 80]). For $k=6$ and 8, the formula in [47, p.12], are given as follows.

$$s_{12}(n) = \frac{252}{13}(\sigma_5(n)-27\sigma_5(n/3))+\frac{18}{13}L_{6;2}(n) \tag{2.22}$$

$$s_{16}(n) = \frac{240}{41}(\sigma_7(n)+81\sigma_7(n/3))+\frac{16}{41}L_{8;4}(n), \tag{2.23}$$

$$\text{where,}\quad L_{6;2}(n) = \sum_{\substack{x_1\in\mathbb{Z}\\ F_2(x_1,\ldots,x_4)=n}} 9x_1^2-9nx_1^2+n^2 \tag{2.24}$$

$$\text{and}\quad L_{8;4}(n) = \sum_{\substack{x_1\in\mathbb{Z}\\ F_4(x_1,\ldots,x_8)=n}} 45x_1^2-30nx_1^2+2n^2. \tag{2.25}$$

The following are the formulas for $s_{12}(n)$ and $s_{16}(n)$, obtained by Yao – Xia [80] and Ramakrishnan–Sahu [59] respectively:

$$s_{12}(n) = \frac{252}{13}(\sigma_5(n)-27\sigma_5(n/3))+\frac{216}{13}\tau_{6;3}(n), \tag{2.26}$$

$$s_{16}(n) = \frac{240}{41}(\sigma_7(n)+81\sigma_7(n/3))+\frac{1728}{41}\tau_{8;3}(n). \tag{2.27}$$

Comparision of these formulas (2.26) and (2.27) with (2.22) and (2.27), gives

$$\tau_{6,3}(n) = \frac{1}{12}L_{6;2}(n), \tag{2.28}$$

$$\tau_{8,3}(n) = \frac{1}{108}L_{8;4}(n), \tag{2.29}$$

where $L_{6;2}(n)$ and $L_{8;4}(n)$ are defined by (2.24) & (2.25).

On substituting (2.28) and (2.29) in (2.21), we get the following formula for the Ramanujan tau function.

Corollary 2.3.1 *For an integer $n \geq 1$, we have*

$$\tau(n) = \frac{1}{73\times 3728}\Big[\frac{36387}{35}L_{12;8}(n)+108L_{12;6}(n)+\frac{1}{3}\mathscr{L}_4(n)-\frac{32668}{12}L_{6;2}(n)$$

$$-\ 329680\sum_{\substack{a,b\in\mathbb{N}\\ a+b=n}}\sigma_3(a)L_{8;4}(b)+1372056\sum_{\substack{a,b\in\mathbb{N}\\ a+b=n}}\sigma_5(a)L_{6;2}(b)\Big],$$

where the sums $L_{12;8}(n), L_{12;6}, L_{8;4}(n)$ and $L_{6;2}(n)$ are given by the equations (2.19), (2.20), (2.25) and (2.24) respectively. Note that both the sums $L_{8;4}(n)$ and $L_{12;4}(n)$ involve the sum depending on the quadratic form $F_4(x_1,x_2,x_3,x_4)$. So we combine them and denote it by $\mathscr{L}_4(n)$, which is given as

$$\mathscr{L}_4(n) = \sum_{\substack{x_1\in\mathbb{Z}\\ F_4(x_1,\ldots,x_8)=n}}\Big(164025x_1^8-306180nx_1^6+45(3780n^2-4121)x_1^4$$

$$-30(945n^2-4121)nx_1^2+675n^4-8242n^2\Big).$$

2.3.2 Evaluation of certain convolution sums

For positive integers a,b,n, define the convolution sum $W_{a,b}(n)$ by

$$W_{a,b}(n) := \sum_{\substack{l,m\in\mathbb{N}\\ al+bm=n}}\sigma(l)\sigma(m), \tag{2.30}$$

where $\sigma(n)$ is the divisor function. We note that $W_{a,1}(n) = W_{1,a}(n)$, which is denoted by $W_a(n)$. In the previous section, we observed that some of the formulas for $s_{4k}(n)$ can be derived using certain convolution sums of the divisor function $\sigma(n)$. In this section, we demonstrate that a similar method can be adopted to get formulas for the number of representations of a positive integer n by quadratic form of type **3**, namely $\mathscr{Q}_{2,1}\oplus j\mathscr{Q}_{2,1}, 1\leq j\leq 4$. We demonstrate this method here and then compare these two sets of formulas (one by this method and other one given by Theorem 2.2.3). We derive the following convolution sums, presented as corollary to Theorem 2.2.3.

Corollary 2.3.2 *For a natural number n, we have*

$$\begin{aligned}
W_7(n) &= \frac{1}{120}\sigma_3(n)+\frac{49}{120}\sigma_3(n/7)+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma(n)+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma(n/7)\\
&\quad -\frac{1}{70}\tau_{4,7}(n), \qquad (2.31)\\
W_{2,7}(n) &= \frac{1}{600}\sigma_3(n)+\frac{1}{150}\sigma_3(n/2)+\frac{49}{600}\sigma_3(n/7)+\frac{49}{150}\sigma_3(n/14)-\frac{3}{350}\tau_{4,7}(n)\\
&\quad +\left(\frac{1}{24}-\frac{n}{28}\right)\sigma(n/2)+\left(\frac{1}{24}-\frac{n}{8}\right)\sigma(n/7)-\frac{6}{175}\tau_{4,7}(n/2)\\
&\quad +\frac{1}{600}\tau_{4,14;1}(n)+\frac{1}{200}\tau_{4,14;2}(n), \qquad (2.32)\\
W_{3,7}(n) &= \frac{1}{1200}\sigma_3(n)+\frac{3}{400}\sigma_3(n/3)+\frac{49}{1200}\sigma_3(n/7)+\frac{147}{400}\sigma_3(n/21)\\
&\quad -\frac{1}{175}\tau_{4,7}(n)+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma(n/3)+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma(n/7)\\
&\quad -\frac{9}{175}\tau_{4,7}(n/3)-\frac{1}{100}\tau_{4,21;2}(n)+\frac{5}{672}(\tau_{4,21;3}(n)+\tau_{4,21;4}(n))\\
&\quad +\frac{11}{12768}\sqrt{57}(\tau_{4,21;3}(n)-\tau_{4,21;4}(n)), \qquad (2.33)\\
W_{4,7}(n) &= \frac{1}{2400}\sigma_3(n)+\frac{1}{800}\sigma_3(n/2)+\frac{1}{150}\sigma_3(n/4)+\frac{49}{2400}\sigma_3(n/7)\\
&\quad +\frac{49}{800}\sigma_3(n/14)+\frac{49}{150}\sigma_3(n/28)+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma(n/4)\\
&\quad +\left(\frac{1}{24}-\frac{n}{16}\right)\sigma(n/7)+\frac{697}{470400}c_1(n)+\frac{139}{22400}c_2(n)\\
&\quad -\frac{9}{896}c_3(n)-\frac{893}{470400}c_4(n)+\frac{43}{1400}c_5(n)-\frac{7}{40}c_6(n)\\
&\quad +\frac{241}{1400}c_7(n)-\frac{881}{1050}c_8(n)-\frac{178}{525}c_9(n). \qquad (2.34)
\end{aligned}$$

The explicit values of the Fourier coefficient $\tau_{k,N}(n), \tau_{k,N;j}(n)$ are given in §2.4.5. Also $c_j(n); 1 \le j \le 9$, are the n-th Fourier coefficients of eta-quotients $f_j(z); 1 \le j \le 9$, as defined at the end of §2.4.3.

For simplicity, we denote the quadratic form $\mathcal{Q}_{2,1} = x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2$ by $\mathcal{Q}$ and let $R_{\mathcal{Q}}(n)$ be the number of representations of a positive integer n by the quadratic form $\mathcal{Q}$. In [78], K.S. Williams obtained a formula for $R_{\mathcal{Q}}(n)$ by using an elementary method, which is given by

$$R_{\mathcal{Q}}(n) = 4\sigma(n) - 28\sigma(n/7). \qquad (2.35)$$

Let $N_j(n)$ be the number of representations of a positive integer n by the quadratic form $\mathscr{Q}_{2,1} \oplus j\mathscr{Q}_{2,1}, 1 \leq j \leq 4$. Then a formula for $N_j(n)$ for $1 \leq j \leq 4$, can be given as

$$\begin{aligned} N_j(n) &= \sum_{\substack{a,b\in\mathbb{N}_0 \\ a+jb=n}} \Big(\sum_{Q(x_1,\cdots,x_4)=a} 1\Big)\Big(\sum_{Q(x_5,\cdots,x_8)=b} 1\Big) \\ &= R_Q(n) + R_Q(n/j) + \sum_{\substack{a,b\in\mathbb{N} \\ a+jb=n}} R_Q(a)R_Q(b) \\ &= 4\sigma(n) - 28\sigma(n/7) + 4\sigma(n/j) - 28\sigma(n/7j) + 16W_j(n) \\ &\quad -4\times 28W_{7j}(n) - 4\times 28W_{j,7}(n) + 28^2W_j(n/7). \end{aligned} \tag{2.36}$$

We need the following convolution sums which are obtained earlier by E. Royer [69] and A. Alaca, S. Alaca & Ntienjem [2]. For a natural number n, we have

$$W_1(n) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{n}{2}\right)\sigma(n), \tag{2.37}$$

$$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(n/2) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/2), \tag{2.38}$$

$$W_3(n) = \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3(n/3) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/3), \tag{2.39}$$

$$\begin{aligned} W_4(n) &= \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4) + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma(n) \\ &\quad + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/4), \end{aligned} \tag{2.40}$$

$$\begin{aligned} W_{14}(n) &= \frac{1}{600}\sigma_3(n) + \frac{1}{150}\sigma_3(n/2) + \frac{49}{600}\sigma_3(n/7) + \frac{49}{150}\sigma_3(n/14) \\ &\quad + \left(\frac{1}{24} - \frac{n}{56}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/14) - \frac{3}{350}\tau_{4,7}(n) \\ &\quad - \frac{6}{175}\tau_{4,7}(n/2) - \frac{1}{84}\tau_{4,14;1}(n) - \frac{1}{200}\tau_{4,14;2}(n), \end{aligned} \tag{2.41}$$

$$\begin{aligned} W_{28}(n) &= \frac{1}{2400}\sigma_3(n) + \frac{1}{800}\sigma_3(n/2) + \frac{1}{150}\sigma_3(n/4) + \frac{49}{2400}\sigma_3(n/7) + \frac{49}{800}\sigma_3(n/14) \\ &\quad + \frac{49}{150}\sigma_3(n/28) + \left(\frac{1}{24} - \frac{n}{112}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/28) \\ &\quad + \frac{1121}{67200}c_1(n) + \frac{2389}{22400}c_2(n) - \frac{1}{128}c_3(n) - \frac{3349}{67200}c_4(n) - \frac{101}{200}c_5(n) \\ &\quad - \frac{17}{40}c_6(n) + \frac{13}{200}c_7(n) - \frac{433}{150}c_8(n) - \frac{254}{75}c_9(n). \end{aligned} \tag{2.42}$$

Before we proceed to prove Corollary 2.3.2, we need the convolution sum $W_{21}(n)$ also, which we present in the following Lemma.

Lemma 2.3.1 *For a natural number n, we have*

$$\begin{aligned}W_{21}(n) \;=\; & \frac{1}{24}\sigma(n)+\frac{1}{24}\sigma(n/21)-\frac{1}{4}n\sigma(n/21)-\frac{1}{84}n\sigma(n)+\frac{1}{1200}\sigma_3(n)\\ & +\frac{3}{400}\sigma_3(n/3)+\frac{49}{1200}\sigma_3(n/7)+\frac{147}{400}\sigma_3(n/21)-\frac{1}{175}\tau_{4,7}(n)\\ & -\frac{9}{175}\tau_{4,7}(n/3)-\frac{1}{100}\tau_{4,21;2}(n)-\frac{5}{672}\Big(\tau_{4,21;3}(n)+\tau_{4,21;4}(n)\Big)\\ & -\frac{11}{12768}\sqrt{57}\Big(\tau_{4,21;3}(n)-\tau_{4,21;4}(n)\Big). \qquad (2.43)\end{aligned}$$

The explicit values of the Fourier coefficient $\tau_{k,N}(n), \tau_{k,N;j}(n)$ are given in §2.4.5.

Remark 2.3.1 *The dimension of $S_4^{new}(21)$ is 4 and out of which two newforms have rational Fourier coefficients and the other two newforms do not have rational Fourier coefficients. In the latter case, the eigenvalues of the Hecke operators $T(p)$ for $p \neq 3,7$ satisfy the quadratic polynomial $x^2+3x-12$, and therefore the eigenvalues (and hence the Fourier coefficients) of these two newforms belong to the number field $\mathbb{Q}(\sqrt{57})$ (57 is the discriminant of the quadratic polynomial). More precisely, the Fourier coefficients $\tau_{4,21;3}(n)$ and $\tau_{4,21;4}(n)$ belong to the number field $\mathbb{Q}(\sqrt{57})$. However, the Fourier coefficients $\tau_{4,21;3}(n)+\tau_{4,21;4}(n)$ and $\sqrt{57}\Big(\tau_{4,21;3}(n)-\tau_{4,21;4}(n)\Big)$, appearing in above formula, are indeed rational numbers.*

Proof. The vector space $M_2(21)$ is 4 dimensional with a basis

$$\{\Phi_{1,3}(z), \Phi_{1,7}(z), \Phi_{1,21}(z), \Delta_{2,21}(z)\},$$

where the function $\Phi_{a,b}(z)$ (for natural numbers a,b with $a|b$, $a \neq b$) is defined as:

$$\Phi_{a,b}(z) = \frac{1}{b-a}\Big(bE_2(bz)-aE_2(az)\Big). \qquad (2.44)$$

Since $E_2(z)E_2(21z)$ is a quasimodular form of weight 4, level 21 and depth 2, therefore we can use the structure theorem for the space of quasimodular forms and the above basis for the space $M_2(21)$, to express $E_2(z)E_2(21z)$ as follows,

$$\begin{aligned} E_2(z)E_2(21z) &= \frac{1}{500}E_4(z)+\frac{9}{500}E_4(3z)+\frac{49}{500}E_4(7z)+\frac{441}{500}E_4(21z)-\frac{576}{175}\Delta_{4,7}(z) \\ &- \frac{5184}{175}\Delta_{4,7}(3z)-\frac{144}{25}\Delta_{4,21;2}(z)+\frac{40}{7}D\Phi_{1,21}(z)+\frac{4}{7}DE_2(z) \\ &+ \left(\frac{66}{133}\sqrt{57}-\frac{30}{7}\right)\Delta_{4,21;4}(z)-\left(\frac{66}{133}\sqrt{57}+\frac{30}{7}\right)\Delta_{4,21;3}(z). \end{aligned}$$

Writing the n-th Fourier coefficients on both the sides of above equation, we have

$$\begin{aligned} 576W_{21}(n) &= 24\sigma(n)+24\sigma(n/21)+\frac{12}{25}\sigma_3(n)+\frac{108}{25}\sigma_3(n/3)+\frac{588}{25}\sigma_3(n/7) \\ &+ \frac{5292}{25}\sigma_3(n/21)-\frac{576}{175}\tau_{4,7}(n)-\frac{5184}{175}\tau_{4,7}(n/3)-\frac{144}{25}\tau_{4,21;2}(n) \\ &- \frac{48}{7}n\sigma(n)-144n\sigma(n/21)-\left(\frac{66}{133}\sqrt{57}+\frac{30}{7}\right)\tau_{4,21;3}(n) \\ &+ \left(\frac{66}{133}\sqrt{57}\frac{30}{7}\right)\tau_{4,21;4}(n). \end{aligned}$$

This equation gives the required formula for $W_{21}(n)$ as given by (2.43).

Proof of Corollary 2.3.2

We are now ready to prove the Corollary. Comparing the formulas for $N_j(n)$ obtained in Theorem 2.2.3 and in (2.36), for each $j=1,2,3,4$, we get an expression for $W_{j,7}(n)$, which involves the convolution sums $W_j(n)$ and $W_{7j}(n)$ for $j=1,2,3,4$. Now making use of the convolution sums $W_j(n), 1\le j\le 4$, $W_{14}(n), W_{21}(n)$ and $W_{28}(n)$ (given by (2.37)-(2.43)), we get the required formulas given in Corollary 2.3.2.

To illustrate the above method, we give details for the case $j=3$ (other cases follows in a similar way). From Theorem 2.2.3, we have

$$\begin{aligned} N_3(n) &= \frac{12}{25}\sigma_3(n)+\frac{108}{25}\sigma_3(n/3)+\frac{588}{25}\sigma_3(n/7)+\frac{5292}{25}\sigma_3(n/21)+\frac{32}{25}\tau_{4,7}(n) \\ &+ \frac{288}{25}\tau_{4,7}(n/3)+\frac{56}{25}\tau_{4,21;2}(n), \end{aligned} \quad (2.45)$$

and from the equation (2.36), we have

$$\begin{aligned} N_3(n) &= 4\sigma(n)+4\sigma(n/3)-28\sigma(n/7)-28\sigma(n/21)+16W_3(n)-112W_{21}(n) \\ &- 4\times 28W_{3,7}(n)+28^2W_3(n/7). \end{aligned} \quad (2.46)$$

On comparing these equations (2.45) , (2.46) and substituting the values of the convolution sums $W_3(n)$ and $W_{21}(n)$ from (2.39) and (2.43), we get

$$\begin{aligned} W_{3,7}(n) &= \frac{1}{1200}\sigma_3(n)+\frac{3}{400}\sigma_3(n/3)+\frac{49}{1200}\sigma_3(n/7)+\frac{147}{400}\sigma_3(n/21)-\frac{1}{175}\tau_{4,7}(n) \\ &+ \left(\frac{1}{24}-\frac{n}{28}\right)\sigma(n/3)+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma(n/7)-\frac{9}{175}\tau_{4,7}(n/3)-\frac{1}{100}\tau_{4,21;2}(n) \\ &+ \frac{5}{672}\Big(\tau_{4,21;3}(n)+\tau_{4,21;4}(n)\Big)+\frac{11}{12768}\sqrt{57}\Big(\tau_{4,21;3}(n)-\tau_{4,21;4}(n)\Big). \end{aligned}$$

This completes the proof.

2.3.3 Fourier coefficients of certain newforms

Recall that, for a positive integers k and n, $s_{2k}(n)$ denotes the number of representations of n by the quadratic form F_k as defined in (2.16). In [59, 80], the authors obtained formulas for the Fourier coefficients of certain newforms (as expressed in (2.28) and (2.29)). Using our Theorem 2.2.1 (equations (2.47), (2.47) and (2.47)) and comparing the respective formulas for $k = 7,9,11$ as given in [47, p.12], we get expressions for $\tau_{7,3,\chi_{-3}}(n)$, $27\tau_{9,3,\chi_{-3};1}(n)+\tau_{9,3,\chi_{-3};2}(n)$ and $\tau_{11,3,\chi_{-3};1}(n)+9\tau_{11,3,\chi_{-3};2}(n)$. More precisely, from the equations (2.47), (2.47) and (2.47) of Theorem 2.2.1, we have

$$s_{14}(n) = \frac{81}{7}\sum_{d|n}\left(\frac{n/d}{3}\right)d^6-\frac{3}{7}\sum_{d|n}\left(\frac{d}{3}\right)d^6+\frac{216}{7}\tau_{7,3,\chi_{-3}}(n),$$

$$\begin{aligned} s_{18}(n) &= \frac{2187}{809}\sum_{d|n}\left(\frac{n/d}{3}\right)d^8+\frac{27}{809}\sum_{d|n}\left(\frac{d}{3}\right)d^8+\frac{24\times 27\times 1728}{809}\tau_{9,3,\chi_{-3};1}(n) \\ &+ \frac{24\times 1728}{809}\tau_{9,3,\chi_{-3};2}(n), \end{aligned}$$

$$\begin{aligned} s_{22}(n) &= \frac{729}{1847}\sum_{d|n}\left(\frac{n/d}{3}\right)d^{10}-\frac{3}{1847}\sum_{d|n}\left(\frac{d}{3}\right)d^{10}+\frac{81\times 748}{9235}\tau_{11,3,\chi_{-3};1}(n) \\ &+\frac{729\times 748}{9235}\tau_{11,3,\chi_{-3};2}(n), \end{aligned}$$

These formulas given by Lomadze in [47, p.12], are as follows:

$$s_{14}(n) = \frac{81}{7}\sum_{d|n}\left(\frac{n/d}{3}\right)d^{6}-\frac{3}{7}\sum_{d|n}\left(\frac{d}{3}\right)d^{6}+\frac{36}{35}\sum_{F_3(x_1,\cdots,x_6)=n}15x_1^4-12nx_1^2+n^2,$$

$$s_{18}(n) = \frac{2187}{809}\sum_{d|n}\left(\frac{n/d}{3}\right)d^{8}+\frac{27}{809}\sum_{d|n}\left(\frac{d}{3}\right)d^{8}+\frac{1728}{5663}\sum_{F_5(x_1,\cdots,x_{10})=n}63x_1^4-36nx_1^2+2n^2,$$

$$s_{22}(n) = \frac{729}{1847}\sum_{d|n}\left(\frac{n/d}{3}\right)d^{10}-\frac{3}{1847}\sum_{d|n}\left(\frac{d}{3}\right)d^{10}+\frac{748}{1847}\sum_{F_7(x_1,\cdots,x_{14})=n}54x_1^4-24nx_1^2+n^2.$$

By comapring these two sets of formulas, we get the following corollary.

Corollary 2.3.3 *The Fourier coefficients $\tau_{7,3,\chi_{-3}}(n)$, $27\tau_{9,3,\chi_{-3};1}(n)+\tau_{9,3,\chi_{-3};2}(n)$ and $\tau_{11,3,\chi_{-3};1}(n)+9\tau_{11,3,\chi_{-3};2}(n)$ of cusp forms of weight $k=7,9,11$, level 3 with character χ_{-3}, are given respectively by*

$$\begin{aligned} \tau_{7,3,\chi_{-3}}(n) &= \frac{1}{30}\sum_{F_3(x_1,\cdots,x_6)=n}15x_1^4-12nx_1^2+n^2, \\ 27\tau_{9,3,\chi_{-3};1}(n)+\tau_{9,3,\chi_{-3};2}(n) &= \frac{1}{168}\sum_{F_5(x_1,\cdots,x_{10})=n}63x_1^4-36nx_1^2+2n^2, \\ \tau_{11,3,\chi_{-3};1}(n)+9\tau_{11,3,\chi_{-3};2}(n) &= \frac{5}{81}\sum_{F_7(x_1,\cdots,x_{14})=n}54x_1^4-24nx_1^2+n^2. \end{aligned}$$

2.4 Proofs of Theorems 2.2.1, 2.2.2 and 2.2.3

We show that the generating functions for the quadratic forms corresponding to these theorems are modular forms in the space $M_k(N,\chi)$. The weight k is determined by the number of variables in the respective quadratic form and N,χ are determined by the set of coefficients appearing in the linear combinations of these quadratic forms. This is the main ingredient in the proofs. First, we present some preliminary facts and results.

2.4.1 Preliminary facts and results

The quadratic forms considered in this chapter are of the following two types:

(i) Sum of squares.

(ii) Linear combination of quadratic forms of the type $x^2+xy+ay^2, a\in\mathbb{N}, a\leq 5$.

The generating functions corresponding to these quadratic forms are given respectively by

$$\begin{aligned}\Theta(z) &= \sum_{n\in\mathbb{Z}} q^{n^2},\\ \mathcal{G}_a(z) &= \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+an^2}, \quad q=e^{2\pi iz}.\end{aligned}$$

It is known that $\Theta(z)$ is a modular form of weight $1/2$ on $\Gamma_0(4)$ [35, Chap. 4] and the generating function $\mathcal{G}_a(z)\in M_1\left(4a-1, \left(\frac{\cdot}{4a-1}\right)\right)$ [71, Theorem 4].

Before proceeding to the proofs, we present below some basic facts about modular forms.

Lemma 2.4.1 *If $f_i\in M_{k_i}(N_i,\chi_i), k\in\mathbb{N}, i=1,2$ then $f_1f_2\in M_{k_1+k_2}(N,\chi_1\chi_2)$, where N = lcm $[N_1,N_2]$.*

Proof. This is a basic fact on modular forms. For details, we refer to [35, Chap. 3].

Lemma 2.4.2 *Let λ be an integer or half-integer. If $f\in M_\lambda(N,\chi)$ and $d\in\mathbb{N}$, then*

$$f(dz)\in\begin{cases} M_\lambda(Nd,\chi) & \text{if} \quad \lambda\in\mathbb{Z},\\ M_\lambda(Nd,\chi\chi_d) & \text{if} \quad \lambda\in\frac{1}{2}\mathbb{Z},\end{cases}$$

where $\chi_d=\left(\frac{d}{\cdot}\right)$ is the Kronecker symbol.

Proof. We refer to [35, Chap. 3, Proposition 17], for the case of integer weight modular forms and [73, Prop. 1.3] for the case of half-integer weight modular forms.

Lemma 2.4.3 *For positive integers r and d, we have*

$$\Theta^r(dz) \in \begin{cases} M_{r/2}(4d,\chi_d) & \text{if } r \text{ is odd,} \\ M_{r/2}(4d,\chi_{-4}) & \text{if } r \equiv 2 \pmod 4, \\ M_{r/2}(4d) & \text{if } r \equiv 0 \pmod 4, \end{cases}$$

and for positive integers d_1,d_2,r_1,r_2 with $2 \nmid r_1r_2$, we have

$$\Theta^{r_1}(d_1z)\Theta^{r_2}(d_2z) \in \begin{cases} M_{\frac{r_1+r_2}{2}}\left(4[d_1,d_2],\chi_{-d_1d_2}\right) & r_1+r_2 \equiv 2 \pmod 4, \\ M_{\frac{r_1+r_2}{2}}\left(4[d_1,d_2],\chi_{d_1d_2}\right) & r_1+r_2 \equiv 0 \pmod 4. \end{cases}$$

Proof. By Lemma 2.4.2, if $f \in M_k(N,\chi)$, then $f(dz)$ belongs to $M_k(dN,\chi')$, where $\chi' = \chi$ if k is an integer and $\chi' = \chi\chi_d$, if k is a half-integer. Next, if $f_i \in M_{k_i}(4N_i,\chi_i)$, $i = 1,2$ are two modular forms of weight k_i (where k_1 and k_2 are integers or half-integers). Then, it follows that the product f_1f_2 is a modular form in $M_{k_1+k_2}(4[N_1,N_2],\psi)$, where ψ is a character modulo $4[N_1,N_2]$. If both the weights k_1 and k_2 are integers, then the resulting form is of weight k_1+k_2 (which is an integer) and so $\psi(-1) = (-1)^{k_1+k_2} = \chi_1(-1)\chi_2(-1)$, implying $\psi = \chi_1\chi_2$. If k_1+k_2 is half-integer (say) with k_1 integer and k_2 half-integer, then $\chi_1(-1) = (-1)^{k_1}$ and χ_2 is an even character modulo $4N_2$. Since the resulting form is of half-integral weight, we must have ψ an even character. Therefore, $\psi = \chi_1\chi_2$, if k_1 is even and $\psi = \chi_1\chi_2\chi_{-4}$, if k_1 is odd. If both k_1 and k_2 are half-integers, then the resulting form is of integer weight k_1+k_2. If k_1+k_2 is an odd integer then $\chi_1\chi_2$ is an even character and the character of the space should be $\chi_1\chi_2\chi_{-4}$ instead of $\chi_1\chi_2$. For details we refer to [35, Chap. 4, Proposition 3] and [73, Proposition 1.3].

Lemma 2.4.4 *The vector space $M_k(\Gamma_1(N))$ is decomposed into the space of modular forms of Nebentypus as follows.*

$$M_k(\Gamma_1(N)) = \oplus_\chi M_k(N,\chi),$$

where the direct sum varies over all Dirichlet characters modulo N if the weight k is a *positive integer and varies over all even Dirichlet characters modulo N,* $4|N$*, if the weight k is half-integer. Further, if k is an integer, one has* $M_k(N,\chi) = \{0\}$*, if* $\chi(-1) \neq (-1)^k$*. We also have the following decomposition of the space into subspaces of Eisenstein series and cusp forms:*

$$M_k(N,\chi) = \mathscr{E}_k(N,\chi) \oplus S_k(N,\chi),$$

where $\mathscr{E}_k(N,\chi)$ *is the space generated by the Eisenstein series of weight k on* $\Gamma_0(N)$ *with character* χ *and* $S_k(N,\chi)$ *is the subspace of* $M_k(N,\chi)$ *consisting of cusp forms.*

Proof. For a proof, we refer to [35, Chap. 3].

Lemma 2.4.5 *By the Atkin-Lehner theory of newforms, the space* $S_k(N,\chi)$ *can be decomposed into the space of newforms and oldforms:*

$$S_k(N,\chi) = S_k^{new}(N,\chi) \oplus S_k^{old}(N,\chi),$$

where the above is an orthogonal direct sum (with respect to the Petersson scalar product) and

$$S_k^{old}(N,\chi) = \bigoplus_{\substack{r|N, r<N \\ rd|N}} S_k^{new}(r,\chi)|B(d).$$

In the above, $S_k^{new}(N,\chi)$ *is the space of newforms and* $S_k^{old}(N,\chi)$ *is the space of oldforms and the operator* $B(d)$ *is given by* $f(z) \mapsto f(dz)$*.*

Proof. For a proof, we refer to the works of Atkin-Lehner and W.W. Li [10, 42].

For more details on the theory of modular forms of integral and half-integral weights, we refer to [10, 35, 42, 49, 73].

2.4.2 Proof of Theorem 2.2.1

First we use Lemmas 2.4.1 and 2.4.2 to show that the generating functions are modular forms. Since $\Theta(z) \in M_{1/2}(4)$, by Lemma 2.4.2, we see that

$$\prod_{i=1}^{r} \Theta(a_i z) \in M_{r/2}(N_1, \chi_1),$$

where $a_1, a_2, \cdots, a_r$ belongs to $\mathscr{A} \subseteq \{1,2,3,4,6,8\}$ and $N_1 = lcm[4a_1, 4a_2, \ldots, 4a_r]$. Also $\chi_1 = \left(\frac{N_1}{\cdot}\right)$ is the quadratic Dirichlet character modulo N_1.

Next, we consider forms of type $x^2 + xy + y^2$. We have already observed in §2.4.1 that the generating function for the quadratic form $x^2 + xy + y^2$, is given by $\mathscr{G}_1(z) \in M_1\left(3, \left(\frac{\cdot}{3}\right)\right)$. So by Lemma 2.4.2,

$$\mathscr{G}_1(b_i z) \in M_1\left(3b_i, \left(\frac{\cdot}{3}\right)\right).$$

Now by Lemma 2.4.1, we have

$$\prod_{i=1}^{t} \mathscr{G}_1(b_i z) \in M_t(N_2, \chi_2),$$

where $b_1, b_2, \cdots, b_t$ belongs to $\mathscr{B} \subseteq \{1,2,4,8\}$ and $N_2 = lcm[3b_1, 3b_2, \cdots, 3b_t]$. Also $\chi_2 = \left(\frac{\cdot}{3}\right)^t$ is the quadratic Dirichlet character modulo N_2.

Finally, in the case of the mixed type, we have the following. Using the above modular properties of $\prod_{i=1}^{r} \Theta(a_i z)$, $\prod_{i=1}^{t} \mathscr{G}_1(b_i z)$ and Lemma 2.4.1, we see that the generating function for $\mathscr{Q}_k$ with $r + 2t = 2k, r, t \neq 0$, is given by

$$\prod_{i=1}^{r} \prod_{j=1}^{t} \Theta(a_i z) \mathscr{G}_1(b_j z) \in M_k(N_3, \chi_3),$$

where $a_1, a_2, \cdots, a_r$ belongs to $\mathscr{A} \subseteq \{1,2,3,4,6,8\}$ and $b_1, b_2, \cdots, b_t$ belongs to $\mathscr{B} \subseteq \{1,2,4,8\}$. Also $N_3 = lcm\ [4a_1, \ldots, 4a_r, 3b_1, \cdots, 3b_t]$ and $\chi_3 = \left(\frac{N_1}{\cdot}\right)\left(\frac{\cdot}{3}\right)^t$ is the quadratic Dirichlet character modulo N_3.

Case (i): The case $t = 0$ and $r = 8$.

This is the first case of sums of squares with coefficients $a_1, a_2, \cdots, a_8$ belongs to $\mathscr{A} \subseteq \{1,2,3,4,6,8\}$. The generating function is $\prod_{i=1}^{8} \Theta(a_i z) \in M_4(N_1, \chi_1)$, as observed above. Where $(N_1, \chi_1) \in \Big\{(16,\chi_8), (24,\chi_0), (24,\chi_8), (24,\chi_{12}), (24,\chi_{24}), (32,\chi_0), (32,\chi_8), (48,\chi_0), (48,\chi_8), (48,\chi_{12}), (48,\chi_{24})\Big\}$. The list of modular form bases corresponding to these values of (N_1, χ_1) are given in the following table 1. Now, we express the generating function corresponding to each case as a linear combination of basis elements in $M_4(N, \chi)$. By comparing the n-th Fourier coefficients both the sides we get the required formula for $N_4(\mathscr{A}; n)$. The constants $\alpha_{4,i}$ appearing in the linear combinations (2.5), are determined using the known bounds (for example, Sturm bound) for the Fourier coefficients of modular forms and we used SAGE [70] for carrying out these computations. We present the coefficients $\alpha_{4,i}$ (appearing in (2.5)), in the appendix at the end of the thesis (Table 5.1 – Table 5.11).

Table 1 : Basis for $M_4(N,\chi)$

Space	$\mathscr{E}_k(N,\chi)$		$S_k(N,\chi)$	
$M_k(N,\chi)$	dim.	Basis	dim.	Basis
$M_4(16,\chi_8)$	4	$\{E_{4,\mathbf{1},\chi_8}(tz), E_{4,\chi_8,\mathbf{1}}(tz); t \mid 2\}$	4	$\{f_{4,8,\chi_8;1}(tz), f_{4,8,\chi_8;2}(tz); t \mid 2\}$
$M_4(24,\chi_0)$	8	$\{E_4(tz); t\|24\}$	8	$\{f_{4,6}(t_1z); t_1\|4, f_{4,8}(t_2z); t_2\|3, f_{4,12}(t_3z); t_3\|2, f_{4,24}\otimes\chi_4(z)\}$
$M_4(24,\chi_8)$	4	$\{E_{4,\mathbf{1},\chi_8}(tz),\ E_{4,\chi_8,\mathbf{1}}(tz); t\|3\}$	10	$\{f_{4,8,\chi_8;1}(t_1z), f_{4,8,\chi_8;2}(t_1z); t_1\|3, f_{4,24,\chi_8;j}(z); 1\le j\le 6\}$
$M_4(24,\chi_{12})$	8	$\{E_{4,\mathbf{1},\chi_{12}}(tz), E_{4,\chi_{12},\mathbf{1}}(tz), E_{4,\chi_{-4},\chi_{-3}}(tz),$ $E_{4,\chi_{-3},\chi_{-4}}(tz); t\|2\}$	8	$\{f_{4,12,\chi_{12};j}(t_1z); t_1\|2, 1\le j\le 4\}$
$M_4(24,\chi_{24})$	4	$\{E_{4,\mathbf{1},\chi_{24}}(z), E_{4,\chi_{24},\mathbf{1}}(z),$ $E_{4,\chi_{-8},\chi_{-3}}(z), E_{4,\chi_{-3},\chi_{-8}}(z)\}$	10	$\{f_{4,24,\chi_{24};j}(z); 1\le j\le 10\}$
$M_4(32,\chi_0)$	8	$\{E_4(t_1z); t_1\|32, E_{4,\chi_{-4},\chi_{-4}}(t_2z); t_2\|2\}$	8	$\{f_{4,8}(t_1z); t_1\|4, f_{4,16}(t_2z); t_2\|2, f_{4,32;j}(z); 1\le j\le 3\}$
$M_4(32,\chi_8)$	8	$\{E_{4,\mathbf{1},\chi_8}(tz), E_{4,\chi_8,\mathbf{1}}(tz); t\|4,$ $E_{4,\chi_{-4},\chi_{-8}}(z), E_{4,\chi_{-8},\chi_{-4}}(z)\}$	8	$\{f_{4,8,\chi_8;j}(t_1z); t_1\|4, f_{4,32,\chi_8;j}(z); 1\le j\le 2\}$
$M_4(48,\chi_0)$	12	$\{E_{4,\chi_{-4},\chi_{-4}}(z), E_{4,\chi_{-4},\chi_{-4}}(3z),$ $E_4(tz); t\|48\}$	18	$\{f_{4,6}(t_1z); t_1\|8, f_{4,8}(t_2z); t_2\|6, f_{4,12}(t_3z); t_3\|4, f_{4,16}(t_4z); t_4\|3,$ $f_{4,24}\otimes\chi_4(t_5z); t_5\|2, f_{4,6}\otimes\chi_{-4}(z), f_{4,12}\otimes\chi_{-4}(z), f_{4,24}\otimes\chi_{-4}(z)\}$
$M_4(48,\chi_8)$	8	$\{E_{4,\mathbf{1},\chi_8}(tz),\ E_{4,\chi_8,\mathbf{1}}(tz); t\|6\}$	20	$\{f_{4,8,\chi_8;1}(t_1z), f_{4,8,\chi_8;2}(t_1z); t_1\|6, f_{4,24,\chi_8;j}(t_2z); t_2\|2, 1\le j\le 6\}$
$M_4(48,\chi_{12})$	12	$\{E_{4,\mathbf{1},\chi_{12}}(tz), E_{4,\chi_{12},\mathbf{1}}(tz),$ $E_{4,\chi_{-4},\chi_{-3}}(tz), E_{4,\chi_{-3},\chi_{-4}}(tz); t\|4\}$	18	$\{f_{4,12,\chi_{12};j_1}(t_1z); t_1\|4, 1\le j_1\le 4, f_{4,48,\chi_{12};j_2}(z); 1\le j_2\le 6\}$
$M_4(48,\chi_{24})$	8	$\{E_{4,\mathbf{1},\chi_{24}}(tz), E_{4,\chi_{24},\mathbf{1}}(tz),$ $E_{4,\chi_{-8},\chi_{-3}}(tz), E_{4,\chi_{-3},\chi_{-8}}(tz); t\|2\}$	20	$\{f_{4,24,\chi_{24};j}(tz); t\|2, 1\le j\le 10\}$

2.4.3 Description of some of the basis elements appearing in Table 1

The normalized Eisenstein series of weight k for $SL_2(\mathbb{Z})$ is denoted as $E_k(z)$ and the generalized eisenstein series of weight k for characters χ and ψ is denoted by $E_{k,\chi,\psi}(z)$. Both these Eisenstein series are defined in §1.2.3.

The unique newform in the space $S_k(N,\chi)$, is denoted by $\Delta_{k,N,\chi}(z)$ and we write it's Fourier expansion as follows:

$$\Delta_{k,N,\chi}(z) = \sum_{n\geq 1} \tau_{k,N,\chi}(n)q^n. \tag{2.47}$$

In case if there are more newforms in the space $S_k(N,\chi)$, we put an indexing in the suffix and denote the j-th cusp form in $S_k^{new}(N,\chi)$, by $\Delta_{k,N,\chi;j}(z)$. We denote $\tau_{k,N,\chi;j}(n)$ to be the n-th Fourier coefficient of $\Delta_{k,N,\chi;j}(z)$. When character χ is trivial, we drop the symbol χ from the notation and denote it just by $\Delta_{k,n;j}(z)$ and the respective Fourier coefficient by $\tau_{k,N;j}(n)$. In particular $\tau_{k,N}(n)$ denotes the n-th Fourier coefficient of unique cusp form of weight k and level N.

In some case, instead of newforms of weight k, level N and quadratic charater χ modulo N, we give general cusp form in the space $S_k(N,\chi)$ and we denote it by $f_{k,N,\chi}(z)$ with the corresponding fourier coefficients $a_{k,N,\chi}(n)$. If there are more than one such cusp forms, we put an indexing in the notation and we write it as $f_{k,N,\chi;j}(z)$ (in order to distinguish the forms). The corresponding n-th Fourier coefficients is denoted by $a_{k,N,\chi;j}(n)$. Also, $f_{k,N;j}(z)$ denote (j-th) cusp form of weight k, level N with a trivial character χ. Moreover $f_{k,N}(z)$ denotes a cusp form of weight k, level N with trivial character χ.

Following the notation $\prod_{i=1}^{s} \eta^{r_i}(d_i z) := d_1^{r_1} d_2^{r_2} \cdots d_s^{r_s}$ for the eta quotients, we define all the cusp forms appearing in above tables. First we define $f_{4,N}(z), N = 6, 8, 12, 24$ as follows,

$$f_{4,6}(z) = 1^2 2^2 3^2 6^2, \qquad f_{4,12}(z) = 1^{-1}2^2 3^3 4^3 6^2 12^{-1} - 1^3 2^2 3^{-1} 4^{-1} 6^2 12^3,$$

$$f_{4,8}(z) = 2^4 4^4, \qquad f_{4,16}(z) = 2^{-4}4^{16}8^{-4}, \quad f_{4,24}(z) = 1^{-4}2^{11}3^{-4}4^{-3}6^{11}12^{-3}.$$

Now in the following list, we give expressions for remaining cusp forms $f_{k,N,\chi;j}(z)$.

$f_{4,8,\chi_8;1}(z) = 1^{-2}2^{11}4^{-3}8^2$, $\quad f_{4,8,\chi_8;2}(z) = 1^2 2^{-3}4^{11}8^{-2}$,

$f_{4,12,\chi_{12};1}(z) = 2^{-1}3^4 4^2 6^5 12^{-2}$, $\quad f_{4,12,\chi_{12};2}(z) = 3^4 4^3 6^{-2}12^3$,

$f_{4,12,\chi_{12};3}(z) = 2^2 3^4 4^{-1}6^{-4}12^7$, $\quad f_{4,12,\chi_{12};4}(z) = 1^4 4^{-1}6^{-2}12^7$,

$f_{4,24,\chi_8;1}(z) = 1^2 2^1 3^{-4}4^1 6^{10}8^2 12^{-4}$, $\quad f_{4,24,\chi_8;2}(z) = 1^1 2^3 3^{-1}4^1 6^4 8^{-1}24^1$,

$f_{4,24,\chi_8;3}(z) = 1^{-1}2^4 3^1 6^3 8^1 12^1 24^{-1}$, $\quad f_{4,24,\chi_8;4}(z) = 1^{-2}2^4 4^2 6^1 8^2 12^1$,

$f_{4,24,\chi_8;5}(z) = 2^1 3^{-2}4^1 6^4 12^2 24^2$, $\quad f_{4,24,\chi_8;6}(z) = 1^{-6}2^{14}6^1 8^{-2}12^1$

$f_{4,24,\chi_{24};1}(z) = 3^{-2}6^7 8^3 12^3 24^{-3}$, $\quad f_{4,24,\chi_{24};2}(z) = 3^2 4^7 6^{-3}8^{-2}12^4$,

$f_{4,24,\chi_{24};3}(z) = 3^2 4^{-3}6^1 8^6 12^2$, $\quad f_{4,24,\chi_{24};4}(z) = 3^2 6^{-3}8^3 12^5 24^1$,

$f_{4,24,\chi_{24};5}(z) = 3^2 4^2 6^{-3}8^{-1}12^3 24^5$, $\quad f_{4,24,\chi_{24};6}(z) = 3^2 4^1 6^1 8^{-2}12^{-2}24^8$,

$f_{4,24,\chi_{24};7}(z) = 3^2 4^1 6^1 8^{-2}12^{-2}24^8$, $\quad f_{4,24,\chi_{24};8}(z) = 1^1 3^{-1}6^1 8^{-2}12^1 24^8$,

$f_{4,24,\chi_{24};9}(z) = 2^2 3^6 4^1 6^{-3}8^2$, $\quad f_{4,24,\chi_{24};10}(z) = 3^2 4^3 6^5 12^{-4}24^2$.

In order to define some modular forms in the space $S_4(48,\chi_{12})$, below we define three eta-quotients as:

$$\begin{aligned}
1^{-4}2^7 4^5 6^{-3}8^{-3}12^9 24^{-3} &:= \sum_{n\geq 1} a_{4,48,\chi_{12};1}(n)q^n,\\
2^{-3}3^4 4^9 6^7 8^{-3}12^5 24^{-3} &:= \sum_{n\geq 1} a_{4,48,\chi_{12};2}(n)q^n,\\
1^{-2}2^2 3^2 4^2 8^1 12^2 24^1 &:= \sum_{n\geq 1} a_{4,48,\chi_{12};3}(n)q^n.
\end{aligned}$$

Using the Fourier coefficients of these three eta quotients, we define modular forms $f_{4,48,\chi_{12};j}(z); 1\leq j\leq 6$ as follows,

$$\begin{aligned}
f_{4,48,\chi_{12};1}(z) &= \sum_{\substack{n\geq 1\\ n\equiv 1 \pmod 4}} a_{4,48,\chi_{12};1}(n)q^n, & f_{4,48,\chi_{12};2}(z) &= \sum_{\substack{n\geq 1\\ n\equiv 3 \pmod 4}} a_{4,48,\chi_{12};1}(n)q^n,\\
f_{4,48,\chi_{12};3}(z) &= \sum_{\substack{n\geq 1\\ n\equiv 1 \pmod 4}} a_{4,48,\chi_{12};2}(n)q^n, & f_{4,48,\chi_{12};4}(z) &= \sum_{\substack{n\geq 1\\ n\equiv 3 \pmod 4}} a_{4,48,\chi_{12};2}(n)q^n,\\
f_{4,48,\chi_{12};5}(z) &= \sum_{\substack{n\geq 1\\ n\equiv 1 \pmod 4}} a_{4,48,\chi_{12};3}(n)q^n, & f_{4,48,\chi_{12};6}(z) &= \sum_{\substack{n\geq 1\\ n\equiv 3 \pmod 4}} a_{4,48,\chi_{12};3}(n)q^n,
\end{aligned}$$

We have also used the notation $f\otimes\chi(z)$, for a function f with Fourier expansion $f(z)=\sum_{n\geq1}a(n)q^n$ and a Dirichlet character χ. We define the twisted function $f\otimes\chi(z)$ as $f\otimes\chi(z)=\sum_{n\geq1}\chi(n)a(n)q^n$.

Following the notation for the eta-quotient mentioned above, we define

$$\begin{array}{lll} f_1(z) = 1^5 2^{-1} 7^5 14^{-1}, & f_2(z) = 1^2 2^2 7^2 14^2, & f_3(z) = 1^6 2^{-2} 7^{-2} 14^2, \\ f_4(z) = 1^{-2} 2^6 7^6 14^{-2}, & f_5(z) = 4^2 14^4 28^2, & f_6(z) = 2^6 4^{-2} 14^{-2} 28^6, \\ f_7(z) = 2^4 4^{-2} 28^6, & f_8(z) = 1^1 2^1 7^1 14^{-3} 28^8, & f_9(z) = 2^1 4^1 14^{-3} 28^9. \end{array}$$

For $1\leq j\leq 9$, we denote $c_j(n)$ to be the n-th Fourier coefficient of the cusp form $f_j(z)$ defined above.

Remark 2.4.1 In most of the cases, we have obtained a basis of $S_k(N,\chi)$ in terms of eta-quotients. One can also construct bases which are not necessarily eta-quotients. We remark that one of the problems in the theory of modular forms is to find a basis for the space of modular forms of weight k, level N which are holomorphic eta-quotients. In this direction, we refer to a work of J. Rouse and J.J. Webb [68], in which they give an intrinsic characterization of level N eta-quotients. They also show that in the case when the modular form is having integer Fourier coefficients and non-vanishing in the upper half-plane, the modular form is an integer multiple of an eta-quotient.

Sample formulas ($t=0$)

Here we give some sample formula for $N_4(\mathscr{A};n)$, for some choice of the set $\mathscr{A}$. i.e. we shall give some sample formulas for the following tuples: $\{(1^2,2^3,4^3)$, $(1^5,3^2,4^1)$, $(1^4,2^1,3^2,4^1)$, $(1^1,2^1,4^1,8^5)$, $(1^1,2^1,3^1,4^2,6^3)\}$.

The tuple $(1^2,2^3,4^3)$ is corresponding to the quadratic form $x_1^2+x_2^2+2(x_3^2+x_4^2+x_5^2)+4(x_6^2+x_7^2+x_8^2)$ and the associated theta series is $\Theta^2(z)\Theta^3(2z)\Theta^3(4z)$, which belongs to $M_4(16,\chi_8)$. Using the basis given in Table 1, we get

$$\Theta^2(z)\Theta^3(2z)\Theta^3(4z) = \frac{8}{11}E_{4;\chi_8,\mathbf{1}}(z)+\frac{2}{11}E_{4;\mathbf{1},\chi_8}(2z)+\frac{2}{11}f_{4,8,\chi_8;1}(z)$$

$$+ \quad \frac{34}{11}f_{4,8,\chi_8;2}(n) + \frac{92}{11}f_{4,8,\chi_8;1}(2z) + \frac{16}{11}f_{4,8,\chi_8;2}(2z).$$

(The constants $[\frac{8}{11}, 0, 0, \frac{2}{11}, \frac{2}{11}, \frac{34}{11}, \frac{92}{11}, \frac{16}{11}]$ appear in the table given in appendix (row 6 in Table 5.1, page 93). We have used SAGE to get these constants.)

Now comparing the n-th Fourier coefficient on both the sides of the above identity, we obtain

$$N_4(1^2,2^3,4^3;n) = \frac{8}{11}\sigma_{3;\chi_8,\mathbf{1}}(n) + \frac{2}{11}\sigma_{3;\mathbf{1},\chi_8}(n/2) + \frac{2}{11}a_{4,8,\chi_8;1}(n) + \frac{34}{11}a_{4,8,\chi_8;2}(n)$$
$$+ \frac{92}{11}a_{4,8,\chi_8;1}(n/2) + \frac{16}{11}a_{4,8,\chi_8;2}(n/2).$$

The other sample formulas given below follows in similar way.

$$N_4(1^5,3^2,4^1;n) = \frac{14}{5}\sigma_3(n) - \frac{54}{5}\sigma_3(n/3) - \frac{238}{5}\sigma_3(n/4) + \frac{252}{5}\sigma_3(n/8) + \frac{918}{5}\sigma_3(n/12)$$
$$- \frac{448}{5}\sigma_3(n/16) - \frac{972}{5}\sigma_3(n/24) + \frac{1728}{5}\sigma_3(n/48) + \frac{7}{20}\sigma_{3;\chi_{-4},\chi_{-4}}(n) - \frac{4}{5}a_{4,6}(n)$$
$$+ \frac{27}{20}\sigma_{3;\chi_{-4},\chi_{-4}}(n/3) + \frac{16}{5}a_{4,6}(n/2) - \frac{176}{5}a_{4,6}(n/4) - \frac{1408}{5}a_{4,6}(n/8) + a_{4,8}(n/2)$$
$$+ 27a_{4,8}(n/6) + 4a_{4,12}(n) + 20a_{4,12}(n/4) + \frac{1}{4}a_{4,16}(n) - \frac{27}{4}a_{4,16}(n/3)$$
$$+ 9a_{4,24}(n/2)\left(\frac{4}{n/2}\right) + \frac{22}{5}a_{4,6}(n)\left(\frac{-4}{n}\right) + \frac{5}{4}a_{4,12}(n)\left(\frac{-4}{n}\right) - \frac{9}{4}a_{4,24}(n)\left(\frac{-4}{n}\right),$$

$$N_4(1^4,2^1,3^2,4^1;n) = -\frac{26}{451}\sigma_{3;\mathbf{1},\chi_8}(n/2) + \frac{108}{451}\sigma_{3;\mathbf{1},\chi_8}(n/6) + \frac{832}{451}\sigma_{3;\chi_8,\mathbf{1}}(n)$$
$$+\frac{3456}{451}\sigma_{3;\chi_8,\mathbf{1}}(n/3) - \frac{12448}{451}a_{4,8,\chi_8;1}(n) + \frac{38416}{451}a_{4,8,\chi_8;1}(n/2) + \frac{1296}{451}a_{4,8,\chi_8;1}(n/3)$$
$$+\frac{98496}{451}a_{4,8,\chi_8;1}(n/6) + \frac{14016}{451}a_{4,8,\chi_8;2}(n) - \frac{28032}{451}a_{4,8,\chi_8;2}(n/2) - \frac{1728}{451}a_{4,8,\chi_8;2}(n/3)$$
$$-\frac{3456}{451}a_{4,8,\chi_8;2}(n/6) + \frac{668}{41}a_{4,24,\chi_8;1}(n) - \frac{2368}{41}a_{4,24,\chi_8;1}(n/2) + \frac{3240}{41}a_{4,24,\chi_8;2}(n)$$
$$-\frac{11208}{41}a_{4,24,\chi_8;2}(n/2) - \frac{216}{41}a_{4,24,\chi_8;3}(n) + \frac{2856}{41}a_{4,24,\chi_8;3}(n/2) - \frac{6240}{41}a_{4,24,\chi_8;4}(n)$$
$$+\frac{20672}{41}a_{4,24,\chi_8;4}(n/2) - \frac{11232}{41}a_{4,24,\chi_8;5}(n) + \frac{46656}{41}a_{4,24,\chi_8;5}(n/2) + \frac{932}{41}a_{4,24,\chi_8;6}(n)$$
$$-\frac{3568}{41}a_{4,24,\chi_8;6}(n/2),$$

$$N_4(1^1,2^1,4^1,8^5;n) = \frac{1}{32}\sigma_3(n) - \frac{1}{32}\sigma_3(n/2) - 16\sigma_3(n/16) - 256\sigma_3(n/32) + \frac{11}{32}a_{4,8}(n)$$

$$+\frac{3}{4}a_{4,8}(n/2)+2a_{4,8}(n/4)+\frac{5}{8}a_{4,16}(n)+\frac{11}{8}a_{4,32,1}(n)+\frac{1}{4}a_{4,32,2}(n)-\frac{3}{8}a_{4,32,3}(n),$$

$$N_4(1^1,2^1,3^1,4^2,6^3;n)=\frac{1}{8}\sigma_3(n)-\frac{1}{8}\sigma_3(n/2)-\frac{9}{8}\sigma_3(n/3)+\frac{9}{8}\sigma_3(n/6)+2\sigma_3(n/8)$$
$$-32\sigma_3(n/16)-18\sigma_3(n/24)+288\sigma_3(n/48)-4a_{4,6}(n/4)-32a_{4,6}(n/8)-\frac{1}{8}a_{4,8}(n)$$
$$-\frac{5}{2}a_{4,8}(n/2)+\frac{9}{8}a_{4,8}(n/3)-\frac{27}{2}a_{4,8}(n/6)+\frac{3}{8}a_{4,12}(n)+2a_{4,12}(n/2)+6a_{4,12}(n/4)$$
$$-\frac{1}{4}a_{4,16}(n)-\frac{9}{4}a_{4,16}(n/3)+\frac{5}{8}a_{4,24}(n)\left(\frac{4}{n}\right)+\frac{3}{2}a_{4,24}(n/2)\left(\frac{4}{n/2}\right)+a_{4,6}(n)\left(\frac{-4}{n}\right)$$
$$+\frac{1}{2}a_{4,12}(n)\left(\frac{-4}{n}\right)-\frac{1}{4}a_{4,24}(n)\left(\frac{-4}{n}\right).$$

Case (ii): The case $r=0$ and $t=4$.

This corresponds to the second type of generating function $\prod_{i=1}^{4}\mathcal{G}_1(b_iz)\in M_4(24,\chi_0)$, for the coefficients b_1,b_2,b_3 and b_4 belongs to $\{1,2,4,8\}$. Explicit basis for $M_4(24,\chi_0)$ are already provided in above Table 1. We use these bases and proceed as in the first case to get the corresponding formulas for $N_4(\mathcal{B};n)$. The constants $\beta_{4,i}$ (appearing in (2.6)), are given in the appendix at the end of the thesis (Table 5.12).

Sample formulas ($r=0$)

Here we give some sample formula for $N_4(\mathcal{B};n)$, for some choice of the set $\mathcal{B}$. i.e. we shall give some sample formulas for the tuples $(1^3,2)$, $(1^3,4)$ and $(1,2,4,8)$.

The tuple $(1^3,2)$ is corresponding to the quadratic form $x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+x_5^2+x_5x_6+x_6^2+2(x_7^2+x_7x_8+x_8^2)$ and the associated theta series is $\mathcal{G}_1^3(z)\mathcal{G}_1(2z)$ which belongs to $M_4(6)$. Using the basis for the space $M_4(24,\chi_0)$ given in Table 1, we get

$$\mathcal{G}_1^3(z)\mathcal{G}_1(2z) = \frac{3}{40}E_4(z)-\frac{1}{5}E_4(2z)-\frac{27}{40}E_4(3z)+\frac{9}{5}E_4(6z). \qquad (2.48)$$

(The constants $[\frac{3}{40},-\frac{1}{5},-\frac{27}{40},0,\frac{9}{5},0,0,0,0,0,0,0,0,0,0,0]$ appear in the table given in appendix (row 1 in Table 5.12, page 123). We have used SAGE to get these constants.)

Now comparing the n-th Fourier coefficient on both the sides of (2.49), we obtain

$$N_4(1^3,2;n) = 18\sigma_3(n) - 48\sigma_3(n/2) - 162\sigma_3(n/3) + 432\sigma_3(n/6).$$

The other sample formulas given below follows in similar way.

$$N_4(1^3,4;n) = \frac{36}{5}\sigma_3(n) - \frac{108}{5}\sigma_3(n/2) + \frac{324}{5}\sigma_3(n/3) + \frac{192}{5}\sigma_3(n/4) - \frac{972}{5}\sigma_3(n/6) + \frac{1728}{5}\sigma_3(n/12) + \frac{54}{5}a_{4,6}(n) + \frac{432}{5}a_{4,6}(n/2),$$

$$N_4(1,2,4,8;n) = \frac{3}{5}\sigma_3(n) - 3\sigma_3(n/2) + \frac{27}{5}\sigma_3(n/3) - 12\sigma_3(n/4) - 27\sigma_3(n/6) + \frac{192}{5}\sigma_3(n/8) - 108\sigma_3(n/12) + \frac{1728}{5}\sigma_3(n/24) + \frac{9}{10}a_{4,6}(n) + \frac{27}{5}a_{4,6}(n/2) + \frac{72}{5}a_{4,6}(n/4) + \frac{9}{2}a_{4,8}(n) + \frac{81}{2}a_{4,8}(n/3)$$

Case (iii): The case $r+2t=8$ and $r,t \neq 0$. As prooved earlier, the generating function in this case is given by

$$\prod_{i=1}^{r}\prod_{j=1}^{t}\Theta(a_iz)\mathscr{F}(b_jz) \in M_4(N_3,\chi_3),$$

where $a_1,a_2,\cdots,a_r \in \{1,2,3\}$, $b_1,b_2,\cdots,b_t \in \{1,2,4,8\}$ and $(N_3,\chi_3) \in \Big\{(24,\chi_0), (24,\chi_8),(24,\chi_{12}),(24,\chi_{24})\Big\}$. Using the bases presented in Table 1, we express these generating functions as linear combination of basis elements of $M_4(24,\chi)$ and compare the n-th Fourier coefficients to get required formula for $N_4(\mathscr{A}:\mathscr{B};n)$. The coefficients $\gamma_{4,i}$ (appearing in (2.6)), are given in the appendix at the end of the thesis (Table 5.13 – Table 5.16).

Sample formulas ($r,t \neq 0$)

Here we give a few explicit formulas for $N_4(\mathscr{A}:\mathscr{B};n)$, where $\mathscr{A} \subseteq \{1,2,3\}$ and $\mathscr{B} \subseteq \{1,2,4,8\}$. We give five explicit formulas corresponding to the tuples $(1^4:1^2)$, $(1^3,2^1:1^2)$, $(1^3,3^1:1^2)$, $(1^3,3:1,2)$ and $(1^2,2^1,3^1:1,2)$.

The tuple $(1^4:1^2)$ is corresponding to the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2$ and the associated theta series is $\Theta^4(z)\mathscr{G}_1^2(z)$ which belongs to $M_4(12)$. Using the basis for the space $M_4(24,\chi_0)$ given in Table 1, we get

$$\begin{aligned}\Theta^4(z)\mathscr{G}_1^2(z) &= \frac{7}{75}E_4(z)-\frac{7}{100}E_4(2z)-\frac{9}{25}E_4(3z)-\frac{28}{75}E_4(4z)+\frac{27}{100}E_4(6z)\\ &+\frac{36}{25}E_4(12z)-\frac{72}{5}f_{4,6}(n)-\frac{288}{5}f_{4,6}(n/2)+12f_{4,12}(n). \qquad (2.49)\end{aligned}$$

(The constants $[\frac{7}{75},-\frac{7}{100},-\frac{9}{25},-\frac{28}{75},\frac{27}{100},0,\frac{36}{25},0,-\frac{72}{5},-\frac{288}{5},0,0,0,12,0,0]$ appear in the table given in appendix (row 1 in Table 5.13, page 124). We have used SAGE to get these constants.)

Now comparing the n-th Fourier coefficient on both the sides of (2.49), we obtain

$$\begin{aligned}N_4(1^4:1^2;n) &= \frac{112}{5}\sigma_3(n)-\frac{84}{5}\sigma_3(n/2)-\frac{432}{5}\sigma_3(n/3)-\frac{448}{5}\sigma_3(n/4)\\ &+\frac{324}{5}\sigma_3(n/6)+\frac{1728}{5}\sigma_3(n/12)-\frac{72}{5}a_{4,6}(n)-\frac{288}{5}a_{4,6}(n/2)+12a_{4,12}(n).\end{aligned}$$

The other sample formulas given below follows in similar way.

$$\begin{aligned}N_4(1^3,2^1:\ 1^2;n) &= -\frac{26}{451}\sigma_{3;\mathbf{1},\chi_8}(n)+\frac{108}{451}\sigma_{3;\mathbf{1},\chi_8}(n/3)+\frac{6656}{451}\sigma_{3;\chi_8,\mathbf{1}}(n)\\ &+\frac{27648}{451}\sigma_{3;\chi_8,\mathbf{1}}(n/3)+\frac{168}{451}a_{4,8,\chi_8;1}(n)+\frac{11448}{451}a_{4,8,\chi_8;1}(n/3)-\frac{2496}{451}a_{4,8,\chi_8;2}(n)\\ &-\frac{17280}{451}a_{4,8,\chi_8;2}(n/3)+\frac{24}{41}a_{4,24,\chi_8;1}(n)+\frac{936}{41}a_{4,24,\chi_8;2}(n)+\frac{144}{41}a_{4,24,\chi_8;3}(n)\\ &-\frac{384}{41}a_{4,24,\chi_8;4}(n)+\frac{4032}{41}a_{4,24,\chi_8;5}(n)-\frac{48}{41}a_{4,24,\chi_8;6}(n),\end{aligned}$$

$$\begin{aligned}N_4(1^3,3^1:\ 1^2;n) &= \frac{1}{23}\sigma_{3;\mathbf{1},\chi_{12}}(n)+\frac{288}{23}\sigma_{3;\chi_{12},\mathbf{1}}(n)+\frac{32}{23}\sigma_{3;\chi_{-4},\chi_{-3}}(n)\\ &+\frac{9}{23}\sigma_{3;\chi_{-3},\chi_{-4}}(n)+\frac{84}{23}a_{4,12,\chi_{12};1}(n)+\frac{720}{23}a_{4,12,\chi_{12};2}(n)+\frac{336}{23}a_{4,12,\chi_{12};3}(n)\\ &+\frac{864}{23}a_{4,12,\chi_{12};4}(n),\end{aligned}$$

$$\begin{aligned}N_4(1^3,3:\ 1,2;n) &= \frac{1}{23}\sigma_{3;\mathbf{1},\chi_{12}}(n)+\frac{144}{23}\sigma_{3;\chi_{12},\mathbf{1}}(n)-\frac{16}{23}\sigma_{3;\chi_{-4},\chi_{-3}}(n)\\ &-\frac{9}{23}\sigma_{3;\chi_{-3},\chi_{-4}}(n)+\frac{156}{23}a_{4,12,\chi_{12};1}(n)-\frac{48}{23}a_{4,12,\chi_{12};2}(n)-\frac{168}{23}a_{4,12,\chi_{12};3}(n)\end{aligned}$$

$$- \frac{456}{23} a_{4,12,\chi_{12};4}(n),$$

$$\begin{aligned} N_4(1^2,2^1,3^1 : 1,2;n) &= \frac{1}{261}\sigma_{3;\mathbf{1},\chi_{24}}(n) + \frac{128}{29}\sigma_{3;\chi_{24},\mathbf{1}}(n) + \frac{128}{261}\sigma_{3;\chi_{-8},\chi_{-3}}(n) \\ &+ \frac{1}{29}\sigma_{3;\chi_{-3},\chi_{-8}}(n) + \frac{208}{87}a_{4,24,\chi_{24};1}(n) - \frac{32}{29}a_{4,24,\chi_{24};2}(n) - \frac{284}{87}a_{4,24,\chi_{24};3}(n) \\ &- \frac{368}{29}a_{4,24,\chi_{24};4}(n) + \frac{1048}{29}a_{4,24,\chi_{24};5}(n) - \frac{6224}{87}a_{4,24,\chi_{24};6}(n) - \frac{7100}{87}a_{4,24,\chi_{24};7}(n) \\ &+ \frac{21248}{87}a_{4,24,\chi_{24};8}(n) + \frac{8}{3}a_{4,24,\chi_{24};9}(n) + \frac{500}{87}a_{4,24,\chi_{24};10}(n). \end{aligned}$$

The case $r=0$, $\mathscr{B}=\{1\}$ and $t=k=7,9,11,12,14$: a proof of formulas for $N_k(\{1\};n) := s_{2k}(n)$, for $k=7,9,11,12,14$.

As mentioned in the statement, we use $s_{2k}(n)$ in place of $N_k(\{1\};n)$. Recalling from §2.3.1, for a positive integer k, let F_k denote the quadratic form in $2k$ variables defined by

$$F_k(x_1,x_2,\ldots,x_{2k}) = \sum_{j=1}^{k} x_{2j-1}^2 + x_{2j-1}x_{2j} + x_{2j}^2,$$

and the number of representations of a positive integer n by the quadratic form F_k is denoted by $s_{2k}(n)$. Numerically it is defined by

$$s_{2k}(n) = \#\Big\{(x_1,x_2,\cdots,x_{2k}) \in \mathbb{Z}^{2k} : F_k(x_1,x_2,\cdots,x_{2k}) = n\Big\}.$$

We remark that the quadratic form F_k is nothing but k-copies of the quadratic form $x_1^2 + x_1x_2 + x_2^2$. Therefore, the generating function corresponding to F_k is given by

$$\mathscr{G}_1^k(z) = \sum_{n\geq 0} s_{2k}(n)q^n. \tag{2.50}$$

Using the facts from §2.4.1, it is clear that, in order to get the required formulas for $s_{2k}(n)$, $k=7,9,11,12,14$, we need to find explicit bases for the vector spaces $M_k(3,\chi_{-3})$ for $k=7,9,11$ and $M_k(3)$ for $k=12,14$, which are presented in the following table.

Table : bases for the spaces $M_k(3,\chi_{-3})$ for $k = 7, 9, 11$, $M_{12}(3)$ and $M_{14}(3)$.

Space	$\mathcal{E}_k(N,\chi)$		$S_k(N,\chi)$	
$M_k(N,\chi)$	dim.	Basis	dim.	Basis
$M_7(3,\chi_{-3})$	2	$\{E_{7,\mathbf{1},\chi_{-3}}(z), E_{7,\chi_{-3},\mathbf{1}}(z)\}$	1	$\{\Delta_{7,3,\chi_{-3}}(z)\}$
$M_9(3,\chi_{-3})$	2	$\{E_{9,\mathbf{1},\chi_{-3}}(z), E_{9,\chi_{-3},\mathbf{1}}(z)\}$	2	$\{\Delta_{9,3,\chi_{-3};j}(z); 1 \le j \le 2\}$
$M_{11}(3,\chi_{-3})$	2	$\{E_{11,\mathbf{1},\chi_{-3}}(z), E_{11,\chi_{-3},\mathbf{1}}(z)\}$	2	$\{\Delta_{11,3,\chi_{-3};j}(z); 1 \le j \le 2\}$
$M_{12}(3)$	2	$\{E_{12}(z), E_{12}(3z)\}$	3	$\{\Delta(z), E_4(z)\Delta_{8,3}(z),$ $E_6(z)\Delta_{6,3}(z)\}$
$M_{14}(3)$	2	$\{E_{14}(z), E_{14}(3z)\}$	3	$\{E_8(z)\Delta_{6,3}(z), E_6(z)\Delta_{8,3}(z),$ $\frac{1}{2}(3E_2(3z) - E_2(z))\Delta(z)\}$

In the above table, the Eisenstein series $E_k(z)$, $E_{k,\chi,\psi}(z)$ are defined in §1.2.3. $\Delta(z)$ is the cusp form of weight 12 for $SL_2(\mathbb{Z})$ and the remaining cusp forms used in the above table, are described below.

$$\Delta_{7,3,\chi_{-3}}(z) = \big(E_4(z) - E_4(3z)\big)\eta^9(z)\eta^{-3}(3z),$$

$$\Delta_{9,3,\chi_{-3};1}(z) = \eta^3(z)\eta^{15}(3z), \qquad \Delta_{9,3,\chi_{-3};2}(z) = \eta^{15}(z)\eta^3(3z),$$

$$\Delta_{11,3,\chi_{-3};1}(z) = E_4(z)\Delta_{7,3,\chi_{-3}}(z), \qquad \Delta_{11,3,\chi_{-3};2}(z) := E_4(3z)\Delta_{7,3,\chi_{-3}}(z).$$

Here, the cusp forms $\Delta_{9,3,\chi_{-3};1}(z)$ and $\Delta_{9,3,\chi_{-3};2}(z)$ forms a basis for the space $S_9(3,\chi_{-3})$. As explained in the proof of Theorem 2.2.1, we express each of the generating function as a linear combination of the corresponding basis elements. Below we give explicit linear combination (we have used SAGE [70] to carry out these computations).

$$\begin{aligned}
\mathcal{G}_1^7(z) &= \frac{81}{7}E_{7,\mathbf{1},\chi_{-3}}(z) - \frac{3}{7}E_{7,\chi_{-3},\mathbf{1}}(z) + \frac{216}{7}\Delta_{7,3,\chi_{-3}}(z),\\
\mathcal{G}_1^9(z) &= \frac{2187}{809}E_{9,\mathbf{1},\chi_{-3}}(z) + \frac{27}{809}E_{9,\chi_{-3},\mathbf{1}}(z) + \frac{1119744}{809}\Delta_{9,3,\chi_{-3};1}(z)\\
&\quad + \frac{41472}{809}\Delta_{9,3,\chi_{-3};2}(z),\\
\mathcal{G}_1^{11}(z) &= \frac{729}{1847}E_{11,\mathbf{1},\chi_{-3}}(z) - \frac{3}{1847}E_{11,\chi_{-3},\mathbf{1}}(z) + \frac{60588}{9235}\Delta_{11,3,\chi_{-3};1}(z)\\
&\quad + \frac{545292}{9235}\Delta_{11,3,\chi_{-3};2}(z),
\end{aligned}$$

$$\begin{aligned}\mathscr{G}_1^{12}(z) &= \frac{1}{730}E_{12}(z)+\frac{729}{730}E_{12}(3z)+\frac{29824}{691}\Delta(z)+\frac{1186848}{50443}E_4(z)\Delta_{8,3}(z)\\ &\quad+\frac{261344}{50443}E_6(z)\Delta_{6,3}(z),\\ \mathscr{G}_1^{14}(z) &= -\frac{1}{2186}E_{14}(z)+\frac{2187}{2186}E_{14}(3z)-\frac{3016}{1093}E_8(z)\Delta_{6,3}(z)\\ &\quad-\frac{12448}{1093}E_6(z)\Delta_{8,3}(z)+\frac{53632}{1093}\big(3E_2(3z)-E_2(z)\big)\Delta(z).\end{aligned}$$

Comparison of the n-th Fourier coefficients on both the sides of the above equations gives the formulas for $s_{2k}(n), k=7,9,11,12,14$ respectively, as given in Theorem 2.2.1. For example, for the first identity, comparison of n-th Fourier coefficients gives the formula for $s_{14}(n)$, as follows:

$$\begin{aligned}s_{14}(n) &= \frac{81}{7}\sigma_{6,\mathbf{1},\chi_{-3}}(n)-\frac{3}{7}\sigma_{6,\chi_{-3},\mathbf{1}}(n)+\frac{216}{7}\tau_{7,3,\chi_{-3}}(n)\\ &= \frac{81}{7}\sum_{d|n}\left(\frac{d}{3}\right)d^6-\frac{3}{7}\sum_{d|n}\left(\frac{n/d}{3}\right)d^6+\frac{216}{7}\tau_{7,3,\chi_{-3}}(n)\\ &= \frac{3}{7}\sum_{d|n}\left(27\left(\frac{d}{3}\right)-\left(\frac{n/d}{3}\right)\right)d^6+\frac{216}{7}\tau_{7,3,\chi_{-3}}(n).\end{aligned} \tag{2.51}$$

The other formulas for $s_{2k}(n); k=9,11,12,14$, are obtained in a similar fashion.

2.4.4 Proof of Theorem 2.2.2

Let $\mathscr{G}_{a,\ell}(z)$ denote the theta series associated to the quadratic form $\mathscr{Q}_{a,\ell}$. Then

$$\mathscr{G}_{a,\ell}(z)=\mathscr{G}_a(z)\mathscr{G}_a(\ell z), \tag{2.52}$$

where $\mathscr{G}_a(z)$ is the theta function associated to the quadratic form $x^2+xy+ay^2$, which is defined in §2.4.1.

$$\mathscr{G}_a(z)=\sum_{x,y\in\mathbb{Z}}q^{x^2+xy+ay^2}. \tag{2.53}$$

We already observed in §2.4.1 that $\mathcal{G}_a(z)$ is a modular form of weight 1 on $\Gamma_0(4a-1)$ with character $\left(\frac{\cdot}{4a-1}\right)$.

$$\begin{aligned} \text{By Lamma 2.4.2,} \quad & \mathcal{G}_a(\ell z) && \in M_1\left(\ell(4a-1), \left(\frac{\cdot}{4a-1}\right)\right), \\ \text{and by Lamma 2.4.1,} \quad & \mathcal{G}_{a,\ell}(z) && = \mathcal{G}_a(z)\mathcal{G}_a(\ell z) \in M_2\Big(\ell(4a-1)\Big). \end{aligned}$$

Therefore the theta series $\mathcal{G}_{a,\ell}(z)$ is a modular form of weight 2 on $\Gamma_0\Big(\ell(4a-1)\Big)$ with trivial character. As described in the proof of Theorem 2.2.1, we construct explicit basis for the spaces $M_2\Big(\ell(4a-1)\Big)$ for the values of (a,ℓ) as given in Theorem 2.2.2. We express each of these theta series as a linear combination of basis elements of the respective space of modular forms . Using SAGE [70] we compute the explicit constants. Explicit bases for all the cases appearing here, are given in the following table.

Bases for the spaces $M_2(N)$, for $N = 14, 15, 19, 21, 22, 28, 30, 33$ and 38.

Space	$\mathcal{E}_k(N)$		$S_k(N)$	
$M_k(N)$	dim.	Basis	dim.	Basis
$M_2(14)$	3	$\{\Phi_{1,2}(z), \Phi_{1,7}(z), \Phi_{1,14}(z)\}$	1	$\{\Delta_{2,14}(z)\}$
$M_2(15)$	3	$\{\Phi_{1,3}(z), \Phi_{1,5}(z), \Phi_{1,15}(z)\}$	1	$\{\Delta_{2,15}(z)\}$
$M_2(19)$	1	$\{\Phi_{1,19}(z)\}$	1	$\{\Delta_{2,19}(z)\}$
$M_2(21)$	3	$\{\Phi_{1,3}(z), \Phi_{1,7}(z), \Phi_{1,21}(z)\}$	1	$\{\Delta_{2,21}(z)\}$
$M_2(22)$	3	$\{\Phi_{1,2}(z), \Phi_{1,11}(z), \Phi_{1,22}(z)\}$	2	$\{\Delta_{2,11}(z), \Delta_{2,11}(2z)\}$
$M_2(28)$	5	$\{\Phi_{1,d}(z); 1 < d \mid 28\}$	2	$\{\Delta_{2,14}(z), \Delta_{2,14}(2z)\}$
$M_2(30)$	7	$\{\Phi_{1,d}(z); 1 < d \mid 30\}$	3	$\{\Delta_{2,15}(z), \Delta_{2,15}(2z), \Delta_{2,30}(z)\}$
$M_2(33)$	3	$\{\Phi_{1,3}(z), \Phi_{1,11}(z), \Phi_{1,33}(z)\}$	3	$\{\Delta_{2,11}(z), \Delta_{2,11}(3z), \Delta_{2,33}(z)\}$
$M_2(38)$	3	$\{\Phi_{1,2}(z), \Phi_{1,19}(z), \Phi_{1,38}(z)\}$	3	$\{\Delta_{2,19}(z), \Delta_{2,19}(2z), \Delta_{2,38;1}(z), \Delta_{2,38;2}(z)\}$

Here the function $\Phi_{a,b}(z)$ is defined by

$$\Phi_{a,b}(z) = \frac{1}{b-a}\Big(bE_2(bz) - aE_2(az)\Big), \tag{2.54}$$

where $E_2(z)$ is the Eisenstein series of weight 2. The newforms which appear in above table are described below.

$$\begin{aligned}
\Delta_{2,14}(z) &= \eta(z)\eta(2z)\eta(7z)\eta(14z),\\
\Delta_{2,15}(z) &= \eta(z)\eta(3z)\eta(5z)\eta(15z),\\
\Delta_{2,30}(z) &= \eta(3z)\eta(5z)\eta(6z)\eta(10z)-\eta(z)\eta(2z)\eta(15z)\eta(30z),\\
\Delta_{2,21}(z) &= \frac{\eta(7z)}{2\eta^2(z)\eta(3z)\eta(9z)\eta(21z)}\Big(3\eta^2(z)\eta^2(7z)\eta^4(9z)-\eta^5(3z)\eta(7z)\eta(9z)\eta(21z)\\
&+ 3\eta^4(z)\eta^2(9z)\eta^2(63z)+7\eta(z)\eta^2(3z)\eta(9z)\eta^4(21z)+3\eta^3(z)\eta(7z)\eta^3(9z)\eta(63z)\\
&- 3\eta(z)\eta^5(3z)\eta(21z)\eta(63z)\Big),\\
\Delta_{2,33}(z) &= q+q^2-q^3-q^4-2q^5-q^6+4q^7-3q^8+q^9+O(q^{10})\\
\Delta_{2,38;1}(z) &= q-q^2+q^3+q^4-q^6-q^7-q^8-2q^9+O(q^{10})\\
\Delta_{2,38;2}(z) &= q+q^2-q^3+q^4-4q^5-q^6+3q^7+q^8-2q^9+O(q^{10}),\\
\Delta_{2,19}(z) &= q\left\{\Psi(4z)\Theta(38z)-q^2\Psi(z)\Psi(19z)+q^9\Theta(2z)\Psi(76z)\right\}^2,
\end{aligned}$$

where the Ramanujan theta functions $\Theta(z)$ and $\Psi(z)$ are defined as:

$$\Theta(z):=\frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)} \quad , \quad \Psi(z):=q^{-1/8}\frac{\eta^2(2z)}{\eta(z)}.$$

Note that we could not find explicit expressions for the newforms $\Delta_{2,33}(z), \Delta_{2,38;1}(z)$ and $\Delta_{2,38;2}(z)$. Using these bases given in above table, one can express $\mathscr{G}_{a,\ell}(z)$ (for each (a,ℓ) as given in Theorem 2.2.2) as follows.

$$\mathscr{G}_{1,5}(z) = -\frac{1}{8}\Phi_{1,3}(z)+\frac{1}{4}\Phi_{1,5}(z)+\frac{7}{8}\Phi_{1,15}(z)+\frac{9}{2}\Delta_{2,15}(z), \tag{2.55}$$

$$\mathscr{G}_{2,2}(z) = -\frac{1}{18}\Phi_{1,2}(z)+\frac{1}{3}\Phi_{1,7}(z)+\frac{13}{18}\Phi_{1,14}(z)+\frac{2}{3}\Delta_{2,14}(z) \tag{2.56}$$

$$\mathscr{G}_{2,3}(z) = \frac{1}{8}\Phi_{1,3}(z)-\frac{3}{8}\Phi_{1,7}(z)+\frac{21}{16}\Phi_{1,21}(z)+\frac{1}{2}\Delta_{2,21}(z) \tag{2.57}$$

$$\begin{aligned}\mathscr{G}_{2,4}(z) &= -\frac{1}{72}\Phi_{1,2}(z)-\frac{1}{12}\Phi_{1,4}(z)+\frac{1}{6}\Phi_{1,7}(z)+\frac{13}{72}\Phi_{1,14}(z)+\frac{3}{4}\Phi_{1,28}(z)\\
&+\frac{4}{3}\Delta_{2,14}(z)+\frac{8}{3}\Delta_{2,14}(2z)\end{aligned} \tag{2.58}$$

$$\mathscr{G}_{3,2}(z) = \frac{1}{12}\Phi_{1,2}(z)-\frac{5}{6}\Phi_{1,11}(z)+\frac{7}{4}\Phi_{1,22}(z) \tag{2.59}$$

$$\begin{aligned}
\mathscr{G}_{3,3}(z) &= -\frac{1}{20}\Phi_{1,3}(z)+\frac{1}{4}\Phi_{1,11}(z)+\frac{4}{5}\Phi_{1,33}(z)+\frac{16}{15}\Delta_{2,11}(z)+\frac{16}{5}\Delta_{2,11}(3z)\\
&\quad+\frac{1}{3}\Delta_{2,33}(z) &(2.60)\\
\mathscr{G}_{4,2}(z) &= -\frac{1}{48}\Phi_{1,2}(z)-\frac{1}{24}\Phi_{1,3}(z)+\frac{1}{12}\Phi_{1,5}(z)-\frac{5}{48}\Phi_{1,6}(z)+\frac{3}{16}\Phi_{1,10}(z)\\
&\quad+\frac{7}{24}\Phi_{1,15}(z)+\frac{29}{48}\Phi_{1,30}(z)+\frac{1}{2}\Delta_{2,15}(z)+\Delta_{2,15}(2z)+\Delta_{2,30}(z) &(2.61)\\
\mathscr{G}_{5,1}(z) &= \Phi_{1,19}(z)+\frac{8}{3}\Delta_{2,19}(z) &(2.62)\\
\mathscr{G}_{5,2}(z) &= \frac{1}{20}\Phi_{1,2}(z)-\frac{9}{10}\Phi_{1,19}(z)+\frac{37}{20}\Phi_{1,5}(z)+\frac{4}{5}\Delta_{2,38;2}(z). &(2.63)
\end{aligned}$$

Using the definition of function $\Phi_{a,b}(z)$ defined by (2.54), we get the following identities,

$$\begin{aligned}
\mathscr{G}_{1,5}(z) &= -\frac{1}{16}E_2(z)-\frac{3}{16}E_2(3z)+\frac{5}{16}E_2(5z)+\frac{15}{16}E_2(15z)+\frac{9}{2}\Delta_{2,15}(z), &(2.64)\\
\mathscr{G}_{2,2}(z) &= -\frac{1}{18}E_2(z)-\frac{1}{9}E_2(2z)+\frac{7}{18}E_2(7z)+\frac{7}{9}E_2(14z)+\frac{2}{3}\Delta_{2,14}(z), &(2.65)\\
\mathscr{G}_{2,3}(z) &= -\frac{21}{320}E_2(z)+\frac{3}{16}E_2(3z)-\frac{7}{16}E_2(7z)+\frac{441}{320}E_2(21z)+\frac{1}{2}\Delta_{2,21}(z), &(2.66)\\
\mathscr{G}_{2,4}(z) &= -\frac{1}{36}E_2(z)-\frac{1}{36}E_2(2z)-\frac{1}{9}E_2(4z)+\frac{7}{36}E_2(7z)+\frac{7}{36}E_2(14z)+\frac{7}{9}E_2(28z)\\
&\quad+\frac{4}{3}\Delta_{2,14}(z)+\frac{8}{3}\Delta_{2,14}(2z), &(2.67)\\
\mathscr{G}_{3,2}(z) &= -\frac{1}{12}E_2(z)+\frac{1}{6}E_2(2z)-\frac{11}{12}E_2(11z)+\frac{11}{6}E_2(22z), &(2.68)\\
\mathscr{G}_{3,3}(z) &= -\frac{1}{40}E_2(z)-\frac{3}{40}E_2(3z)+\frac{11}{40}E_2(11z)+\frac{33}{40}E_2(33z)+\frac{16}{15}\Delta_{2,11}(z)\\
&\quad+\frac{16}{5}\Delta_{2,11}(3z)+\frac{1}{3}\Delta_{2,33}(z), &(2.69)\\
\mathscr{G}_{4,2}(z) &= -\frac{1}{48}E_2(z)-\frac{1}{24}E_2(2z)-\frac{1}{16}E_2(3z)+\frac{5}{48}E_2(5z)-\frac{1}{8}E_2(6z)\\
&\quad+\frac{5}{24}E_2(10z)+\frac{5}{16}E_2(15z)+\frac{5}{8}E_2(30z)+\frac{1}{2}\Delta_{2,15}(z)+\Delta_{2,15}(2z)\\
&\quad+\Delta_{2,30}(z), &(2.70)\\
\mathscr{G}_{5,1}(z) &= -\frac{1}{18}E_2(z)+\frac{19}{18}E_2(19z)+\frac{8}{3}\Delta_{2,19}(z), &(2.71)\\
\mathscr{G}_{5,2}(z) &= -\frac{1}{20}E_2(z)+\frac{1}{10}E_2(2z)-\frac{19}{20}E_2(19z)+\frac{19}{10}E_2(38z)+\frac{4}{5}\Delta_{2,38;2}(z). &(2.72)
\end{aligned}$$

Comparing of n-th Fourier coefiicients on both sides of above equations (2.64) - (2.72), gives all the required formulas, as given in Theorem 2.2.2.

2.4.5 Proof of Theorem 2.2.3

As described in the proof of Theorem 2.2.2, $\mathcal{G}_{a,\ell}(z) \in M_2(\ell(4a-1))$. For $a=2, \ell=1$, we see that $\mathcal{G}_{2,1}(z) \in M_2(7)$. Now the generating function for the quadratic forms $\mathcal{Q}_{(2,1),j}(z) = \mathcal{Q}_{2,1} \oplus j\mathcal{Q}_{2,1};\ 1 \le j \le 4$ is nothing but $\mathcal{G}_{(2,1),j}(z) := \mathcal{G}_{2,1}(z)\mathcal{G}_{2,1}(jz); 1 \le j \le 4$ and by Lemma 2.4.1, it belongs to the space $M_4(7j), 1 \le j \le 4$. We write its Fourier expansion as,

$$\mathcal{G}_{(2,1),j}(z) = \sum_{n=0}^{\infty} N_j(n) q^n.$$

Therefore, in order to get formulas for $N_j(n)$, it is sufficient to obtain explicit bases for the spaces of modular forms $M_4(7j); 1 \le j \le 4$. The required formulas will follow by expressing each theta series as a linear combination of basis elements and comparing the n-th Fourier coefficients. In the following table, we list all the basis elements, which will be used in obtaining the required formulas of Theorem 2.2.3.

Bases for the spaces $M_4(7j), 1 \le j \le 4$

Space	Dimension	Basis
$M_4(7)$	3	$\{E_4(z), E_4(7z), \Delta_{4,7}(z)\}$
$M_4(14)$	8	$\{E_4(az); a\vert 14, \Delta_{4,7}(bz); b\vert 2, \Delta_{4,14;1}(z), \Delta_{4,14;2}(z)\}$
$M_4(21)$	10	$\{E_4(az); a\vert 21, \Delta_{4,7}(bz); b\vert 2, \Delta_{4,21;j}(z); 1 \le j \le 4\}$
$M_4(28)$	15	$\{E_4(az), a\vert 28; f_i(z), 1 \le i \le 9\}$

Here the Eisenstein series $E_k(z)$ are defined in §1.2.3, and the cusp forms $\Delta_{4,7}(z)$, $\Delta_{4,14;1}(z)$, $\Delta_{4,14;2}(z)$, $\Delta_{4,21;j}(z); 1 \le j \le 4\}$ are defined below:

$$\begin{aligned}
\Delta_{4,7}(z) &= \frac{(\eta^3(z)\eta^3(7z) + 4\eta^3(2z)\eta^3(14z))\eta^2(z)\eta^2(7z)}{\eta(2z)\eta(14z)},\\
\Delta_{4,14;1}(z) &= -\frac{9}{4}\Delta_{4,7}(z) - 9\Delta_{4,7}(2z) + \frac{\Delta_{2,14}(z)}{4}\left(24\Delta_{2,14}(z) + 14E_2(14z) - E_2(z)\right),\\
\Delta_{4,14;2}(z) &= \Delta_{4,7}(z) - 4\Delta_{4,7}(2z) - 5\Delta_{2,14}^2(z),\\
\Delta_{4,21;1}(z) &= \Delta_{2,21}(z)\left(-\frac{1}{24}E_2(z) - \frac{1}{8}E_2(3z) + \frac{7}{24}E_2(7z) + \frac{7}{8}E_2(21z)\right),\\
\Delta_{4,21;2}(z) &= \Delta_{2,21}(z)\left(\frac{1}{12}E_2(z) - \frac{1}{4}E_2(3z) - \frac{7}{12}E_2(7z) + \frac{7}{4}E_2(21z)\right).
\end{aligned}$$

Further the newform $\Delta_{2,14}(z)$ appearing in above expressions, is defined by $\Delta_{2,14}(z) = \eta(z)\eta(2z)\eta(7z)\eta(14z)$ and the functions $f_i(z)$ appearing in above table, are defined at the end of §2.4.3.

The dimension of the space $S_4^{new}(21)$ is 4 and out of which two newforms have rational Fourier coefficients and the other two newforms do not have rational Fourier coefficients. In the latter case, the eigenvalues of the Hecke operators $T(p)$ for $p \neq 3,7$ satisfy the quadratic polynomial $x^2+3x-12$, and therefore the eigenvalues (and hence the Fourier coefficients) of these two newforms belong to the number field $\mathbb{Q}(\sqrt{57})$ (57 is the discriminant of the quadratic polynomial). More precisely, the Fourier coefficients $\tau_{4,21;3}(n)$ and $\tau_{4,21;4}(n)$ belong to the number field $\mathbb{Q}(\sqrt{57})$. However, we note that both $\tau_{4,21;3}(n)+\tau_{4,21;4}(n)$ and $\sqrt{57}(\tau_{4,21;3}(n)-\tau_{4,21;4}(n))$ are rational numbers. The Fourier coefficients of newforms $\Delta_{4,21;3}(z)$ and $\Delta_{4,21;4}(z)$ lie in the real quadratic field $\mathbb{Q}(\sqrt{57})$. Using the L-function database [46], we give their first few Fourier coefficients, which we use in our computations.

$$\begin{aligned}
\Delta_{4,21;3}(z) &= q+\frac{-3+\sqrt{57}}{2}q^2+3q^3+\frac{17-3\sqrt{57}}{2}q^4+(3-\sqrt{57})q^5+\frac{-9+\sqrt{57}}{2}q^6\\
&+ 7q^7+\frac{-87+5\sqrt{57}}{2}q^8+9q^9-3(11+\sqrt{57})q^{10}+(-3+5\sqrt{57})q^{11}+\cdots,\\
\Delta_{4,21;4}(z) &= q+\frac{-3-\sqrt{57}}{2}q^2+3q^3+\frac{17+3\sqrt{57}}{2}q^4+(3+\sqrt{57})q^5-\frac{9+\sqrt{57}}{2}q^6+7q^7\\
&- \frac{87+5\sqrt{57}}{2}q^8+9q^9-3(11-\sqrt{57})q^{10}-(3+5\sqrt{57})q^{11}+\cdots.
\end{aligned}$$

Using the bases given in above table, we get following expressions for generating functions,

$$\mathscr{G}_{(2,1),1}(z) = \frac{1}{50}E_4(z)+\frac{49}{50}E_4(7z)+\frac{16}{5}\Delta_{4,7}(z), \tag{2.73}$$

$$\begin{aligned}
\mathscr{G}_{(2,1),2}(z) &= \frac{1}{250}E_4(z)+\frac{2}{125}E_4(2z)+\frac{49}{250}E_4(7z)+\frac{98}{125}E_4(14z)\\
&\quad+\frac{48}{25}\Delta_{4,7}(z)+\frac{192}{25}\Delta_{4,7}(2z)+\frac{28}{25}\Delta_{4,14;1}(z),
\end{aligned} \tag{2.74}$$

$$\begin{aligned}
\mathscr{G}_{(2,1),3}(z) &= \frac{1}{500}E_4(z)+\frac{9}{500}E_4(3z)+\frac{49}{500}E_4(7z)+\frac{441}{500}E_4(21z)\\
&\quad+\frac{32}{25}\Delta_{4,7}(z)+\frac{288}{25}\Delta_{4,7}(3z)+\frac{56}{25}\Delta_{4,21;2}(z),
\end{aligned} \tag{2.75}$$

$$\mathscr{G}_{(2,1),4}(z) = \frac{1}{1000}E_4(z)+\frac{3}{1000}E_4(2z)+\frac{2}{125}E_4(4z)+\frac{49}{1000}E_4(7z)$$

$$+\frac{147}{1000}E_4(14z)+\frac{98}{125}E_4(28z)-\frac{356}{175}f_1(z)-\frac{316}{25}f_2(z)+2f_3(z)$$
$$+\frac{1014}{175}f_4(z)+\frac{1328}{25}f_5(z)+\frac{336}{5}f_6(z)-\frac{664}{25}f_7(z)$$
$$+\frac{10432}{25}f_8(z)+\frac{10432}{25}f_9(z). \tag{2.76}$$

The formulas in Theorem 2.2.3 follow from comparing the n-th Fourier coefficients from both the sides of the above expressions.

CHAPTER 3

Ramanujan–Mordell type formulas associated to quadratic forms

In this chapter, we establish Ramanujan–Mordell type formulas associated to the quadratic forms $x_1^2+x_2^2+\cdots+x_{k-1}^2+x_k^2+m(x_{k+1}^2+x_{k+2}^2+\cdots+x_{2k-1}^2+x_{2k}^2)$ with $m\in\{5,8\}$. This is an extension of earlier works by Ramanujan [66] ($m=1$), Cooper et. al [22] ($m=3,7,11,23$) and Dongxi Ye [82] ($m=2,4$). This chapter is an expanded version of results obtained in [76].

3.1 Introduction and the main theorem

In the long history of number theory, one of the classical problems is to determine exact formulas for the number of representations of a positive integer n as a sum of $2k$ squares, that is, the number of integral solutions of

$$x_1^2+x_2^2+\cdots+x_{2k-1}^2+x_{2k}^2=n,$$

denoted usually by $r_{2k}(n)$. However, in this chapter we denote it by $\mathscr{R}_k(n)$. Such a classical but interesting problem has been explored and studied by many mathematicians since it was introduced. As a result, formulas for $\mathscr{R}_k(n)$ for various cases have been found. For example, for the sums of $2,4,6$ and 8 squares, (reformulated) formulas for $\mathscr{R}_k(n)$ are originally due to Jacobi [32]:

$$\mathscr{R}_1(n) = 4\sum_{d|n}\left(\frac{-4}{d}\right),$$

$$\mathscr{R}_2(n) = 8\sum_{d|n} d - 32\sum_{d|\frac{n}{4}} d,$$

$$\mathscr{R}_3(n) = -4\sum_{d|n}\left(\frac{-4}{d}\right)d^2 + 16\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2,$$

$$\mathscr{R}_4(n) = 16\sum_{d|n} d^3 - 32\sum_{d|\frac{n}{2}} d^3 + 256\sum_{d|\frac{n}{4}} d^3,$$

where, here and throughout this chapter, $\left(\frac{D}{\cdot}\right)$ denotes the quadratic character for the discriminant D. The result for $k=5$, i.e., sum of 10 squares, was due (without proof) in part to Eisenstein [26], and fully described (without proof) by Liouville [45]. The results for $1 \le k \le 9$ were all proved by Glaisher [27]. Now if one sets

$$\Theta(\tau) := \sum_{m=-\infty}^{\infty} q^{m^2}, \quad q = e^{2\pi i\tau},\ \tau \in \mathbb{H}, \tag{3.1}$$

then one has the well known relation

$$\Theta^{2k}(\tau) = \sum_{m=0}^{\infty} \mathscr{R}_k(m) q^m,$$

where the theta function on the left hand side is a modular form of weight k with character $\left(\frac{-4}{\cdot}\right)^k$ for $\Gamma_0(4)$ (see Chapter 2 for more datails). By such a nice relation and thanks to the theory of classical modular forms, we know that the modular form $\Theta^{2k}(\tau)$ has the following interesting decomposition

$$\Theta^{2k}(\tau) = E_k^*(\tau) + C_k(\tau), \tag{3.2}$$

where $E_k^*(\tau)$ is an Eisenstein series whose n-th Fourier coefficient is some divisor function in n, and $C_k(\tau)$ is a cusp form whose n-th Fourier coefficient is of order substantially lower than that of $E_k^*(\tau)$. Then as a consequence, formulas for $\mathscr{R}_k(n)$ follows from equating the Fourier coefficients on both sides of (3.2). Such a modular form theoretic intrinsic of $\Theta^{2k}(\tau)$ indirectly allows Ramanujan [66], [67, Eqs. (145)–(147)] to "completely" solve (without proof) the problem in 1916. Now we state Ramanujan's fascinating results and for that we need the Dedekind eta function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, defined in Chapter 1. Here and throughout the remainder of this work, we write η_m for $\eta(m\tau)$ for any positive integer m.

Theorem 3.1.1 (Ramanujan [66]) *Suppose k is a positive integer. Let $\Theta(\tau)$ be the classical theta function defined by (3.1). Then*

$$\Theta^{2k}(\tau) = F_k(\tau) + \Theta^{2k}(\tau) \sum_{1\le j\le \frac{(k-1)}{4}} c_{k,j}x^j \tag{3.3}$$

where $c_{k,j}$ are numerical rational constants that depend on j and k,

$$x = x(\tau) := \frac{\eta_1^{24}\eta_4^{24}}{\eta_2^{48}},$$

and $F_k(\tau)$ is an Eisenstein series defined by

$$\begin{aligned}
F_1(\tau) &:= 1 + 4\sum_{j=1}^{\infty}\frac{q^j}{1+q^{2j}},\\
F_{2k}(\tau) &:= 1 - \frac{4k(-1)^k}{(2^{2k}-1)\mathscr{B}_{2k}}\sum_{j=1}^{\infty}\frac{j^{2k-1}q^j}{1-(-1)^{k+j}q^j},\ (k\ge 1),\\
F_{2k+1}(\tau) &:= 1 + \frac{4(-1)^k}{\mathscr{E}_{2k}}\sum_{j=1}^{\infty}\left(\frac{(2j)^{2k}q^j}{1+q^{2j}} - \frac{(-1)^{k+j}(2j-1)^{2k}q^{2j-1}}{1-q^{2j-1}}\right),\ (k\ge 1).
\end{aligned}$$

Here $\mathscr{B}_k$ and $\mathscr{E}_k$ are the Bernoulli numbers and Euler numbers, respectively, defined by

$$\frac{u}{e^u-1} = \sum_{k=0}^{\infty}\frac{\mathscr{B}_k}{k!}u^k \quad \text{and} \quad \frac{1}{\cosh u} = \sum_{k=0}^{\infty}\frac{\mathscr{E}_k}{k!}u^k.$$

Formula (3.3) of Theorem 3.1.1 was first proved by Mordell [50], utilizing the theory of modular forms, and thus it is called the Ramanujan–Mordell Theorem (also called as the Ramanujan – Mordell formula). Inspired by the beauty of this formula, it is natural to consider extensions of this kind of results for other types of $2k$-ary quadratic forms. Recently S. Cooper, B. Kane and D. Ye [22], extended the Ramanujan–Mordell formula to the $2k$-ary quadratic forms $x_1^2+\cdots+x_k^2+m\left(x_{k+1}^2+\cdots+x_{2k}^2\right)$, for $m\in\{3,7,11,23\}$ (for which the associated modular form is $\Theta^k(\tau)\Theta^k(m\tau)$) and analogously obtained a beautiful unified *polynomial representation* (or called Ramanujan–Mordell type representation) for the cusp form component $C_k(\tau)$. In [82], Ye further, extended this result to the cases $m\in\{2,4\}$. In this chapter, we treat the uncharted cases, $m=5$ and $m=8$, whose associated quadratic forms have discriminants 20^k and 32^k, respectively, and obtain the corresponding Ramanujan–Mordell type formulas. We conclude this section by stating the main results of this chapter and presenting some explicit examples.

The generalized Eisenstein series $E_{k,\chi,\psi}(\tau)$ corresponding to the characters χ and ψ has been defined in Chapter 1 (§1.2.3). However, in this chapter we use a different normalization and define it as

$$G_{k,\chi,\psi}(\tau):=\frac{2k}{B_{k,\psi}}E_{k,\chi,\psi}(\tau).$$

Then its Fourier expansion is given by

$$G_{k,\chi,\psi}(\tau):=\frac{2k}{B_{k,\psi}}E_{k,\chi,\psi}(\tau)=\delta_{M,1}-\frac{2k}{B_{k,\psi}}\sum_{n=1}^{\infty}\sigma_{k-1,\chi,\psi}(n)q^n. \tag{3.4}$$

Now we state the main theorem of this chapter.

Theorem 3.1.2 (Theorem 1.2, [76]) *For $m\in\{5,8\}$, let $x_m=x_m(\tau)$ be defined by*

$$x_m=\begin{cases}\dfrac{\eta_1^3\eta_4\eta_5\eta_{20}^3}{\eta_2^4\eta_{10}^4} & \text{if } m=5,\\[2ex] \dfrac{\eta_1^2\eta_4\eta_8\eta_{32}^2}{\eta_2^3\eta_{16}^3} & \text{if } m=8,\end{cases} \tag{3.5}$$

where $\eta_m = \eta(m\tau)$. *Let* $F_{k,m}(\tau)$ *be defined by*

$$F_{k,m}(\tau) = \begin{cases} \frac{(-1)^\ell E_{2\ell}(\tau)-(-1)^\ell 2E_{2\ell}(2\tau)+(-4)^\ell E_{2\ell}(4\tau)+5^\ell E_{2\ell}(5\tau)-5^\ell 2E_{2\ell}(10\tau)+20^\ell E_{2\ell}(20\tau)}{-(-1)^\ell+(-4)^\ell-5^\ell+20^\ell} & \\ & \text{if } k=2\ell \text{ and } m=5, \\ G_{2\ell+1,\mathbf{1},\chi_{-20}}(\tau)+(-20)^\ell G_{2\ell+1,\chi_{-20},\mathbf{1}}(\tau) & \text{if } k=2\ell+1\geq 3 \text{ and } m=5, \\ G_{1,\mathbf{1},\chi_{-20}}(\tau) & \text{if } k=1 \text{ and } m=5, \\ \frac{(-1)^\ell E_{2\ell}(\tau)-(-1)^\ell E_{2\ell}(2\tau)-8^\ell E_{2\ell}(16\tau)+32^\ell E_{2\ell}(32\tau)}{-8^\ell+32^\ell} & \text{if } k=2\ell \text{ and } m=8, \\ G_{2\ell+1,\mathbf{1},\chi_{-8}}(4\tau)+(-1)^\ell 2^{\ell-1} G_{2\ell+1,\chi_{-8},\mathbf{1}}(\tau) & \\ & \text{if } k=2\ell+1\geq 3 \text{ and } m=8, \\ \frac{1}{3}G_{1,\mathbf{1},\chi_{-8}}(\tau)+\frac{2}{3}G_{1,\mathbf{1},\chi_{-8}}(4\tau) & \text{if } k=1 \text{ and } m=8. \end{cases}$$

Furthermore, let $l(k,m)$ *be defined by*

$$l(k,m) = \begin{cases} 3\ell-2 & \text{if } k=2\ell \text{ and } m=5, \\ 3\ell+1 & \text{if } k=2\ell+1 \text{ and } m=5, \\ 2k-1 & \text{if } k\geq 2 \text{ and } m=8, \\ 2 & \text{if } k=1 \text{ and } m=8. \end{cases}$$

Then there exist rational numbers $c_{k,m,j}$ *depending on k, m and j such that*

$$\Theta^k(\tau)\Theta^k(m\tau) = F_{k,m}(\tau) + \Theta^k(\tau)\Theta^k(m\tau)\sum_{j=1}^{l(k,m)} c_{k,m,j}x_m^j(\tau). \tag{3.6}$$

Examples: For $k=1$ or 2, the general formula (3.6) gives the following explicit identities.

$$\Theta(\tau)\Theta(5\tau) = 1+\sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n}+\frac{\eta_1\eta_2\eta_{10}\eta_{20}}{\eta_4\eta_5}, \tag{3.7}$$

$$\Theta(\tau)\Theta(8\tau) = 1+\frac{2}{3}\sum_{n=1}^{\infty}\left(\left(\frac{-8}{n}\right)\frac{q^n}{1-q^n}+2\left(\frac{-8}{n}\right)\frac{q^{4n}}{1-q^{4n}}\right)+\frac{4}{3}\frac{\eta_2^2\eta_{16}^2}{\eta_4\eta_8} - \frac{2}{3}\frac{\eta_1^2\eta_{32}^2}{\eta_2\eta_{16}}, \tag{3.8}$$

$$\Theta^2(\tau)\Theta^2(5\tau) = 1+\sum_{n=1}^{\infty}\left(2\frac{nq^n}{1-q^n}-4\frac{nq^{2n}}{1-q^{2n}}+8\frac{nq^{4n}}{1-q^{4n}}-10\frac{nq^{5n}}{1-q^{5n}}\right.$$
$$\left.+20\frac{nq^{10n}}{1-q^{10n}}-40\frac{nq^{20n}}{1-q^{20n}}\right)+2\frac{\eta_2^6\eta_{10}^6}{\eta_1\eta_4^3\eta_5^3\eta_{20}}, \quad (3.9)$$

$$\Theta^2(\tau)\Theta^2(8\tau) = 1+\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n}-\frac{nq^{2n}}{1-q^{2n}}+8\frac{nq^{16n}}{1-q^{16n}}-32\frac{nq^{32n}}{1-q^{32n}}\right)$$
$$+3\frac{\eta_2^7\eta_{16}^7}{\eta_1^2\eta_4^3\eta_8^3\eta_{32}^2}-4\frac{\eta_2^4\eta_{16}^4}{\eta_4^2\eta_8^2}+2\frac{\eta_1^2\eta_2\eta_{16}\eta_{32}^2}{\eta_4\eta_8}. \quad (3.10)$$

Deduction of identity (3.7) from Theorem 3.1.2

This identity (3.7) corresponds to the case $k=1$ and $m=5$. In this case $l(k,m)=1$. Now from (3.6), we have

$$\Theta(\tau)\Theta(5\tau) = F_{1,5}(\tau)+\Theta(\tau)\Theta(5\tau)c_{1,5,1}x_5(\tau). \quad (3.11)$$

Following the proof of Theorem 3.1.2, we find that $c_{1,5,1}=1$. Using this along with the expressions for $F_{1,5}(\tau)$, $x_5(\tau)$ from Theorem 3.1.2 and using the eta representation for $\Theta(\tau)$, given by $\Theta(\tau)=\frac{\eta_2^5}{\eta_1^2\eta_4^2}$, above equation (3.11) becomes:

$$\begin{aligned}\Theta(\tau)\Theta(5\tau) &= G_{1,\mathbf{1},\chi_{-20}}(\tau)+\frac{\eta_2^5}{\eta_1^2\eta_4^2}\times\frac{\eta_{10}^5}{\eta_5^2\eta_{20}^2}\times\frac{\eta_1^3\eta_4\eta_5\eta_{20}^3}{\eta_2^4\eta_{10}^4}\\ &= G_{1,\mathbf{1},\chi_{-20}}(\tau)+\frac{\eta_1\eta_2\eta_{10}\eta_{20}}{\eta_4\eta_5}.\end{aligned}$$

Now the identity (3.7) follows from above equation. The other identities (3.8), (3.9) and (3.9)follow in a similar fashion.

Note: The identity (3.7) was first given by Berkovich and Yesilyurt in [11], making use of Ramanujan's summation formula. To the best of our knowledge, identities (3.8)–(3.10) have not appeared in any literature, though equivalent forms of (3.9) and (3.10) were given in [3] and [1], respectively.

In the next section, we give preliminary results needed for the proof of Theorem 3.1.2 and the proof is given in the last section.

3.2 Some preliminary results

In this section, we present some preliminary results that will be used in the proof of Theorem 3.1.2. The group generated by $\Gamma_0(N)$ and the Fricke involution $H_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is denoted by $\Gamma_0(N)^+$. Further, for $p^\alpha \| N$, $\Gamma_0(N)^{+p^\alpha}$ denotes the group generated by $\Gamma_0(N)$ and the Atkin-Lehner operator W_p corresponding to the prime p. The Atkin-Lehner operator W_p corresponding to the prime p, is defined in Remark 1.2.1 (given in Chapter 1). We shall need the cusps of the group $\Gamma_0(4m)^+$ for $m = 5, 8$, which are obtained among the inequivalent cusps of the group $\Gamma_0(4m)$, which are also inequivalent under the Fricke involution H_{4m}. For more details about the cusps of $\Gamma_0(4m)$, we refer to [25, Chapter 3, pp. 98].

The following lemma gives the transformation formulas for the concerned Eisenstein series under the Fricke involution.

Lemma 3.2.1 *Let $G_{k,\chi,\psi}(\tau)$ be defined as in (3.4). Then for $k = 1$,*

$$G_{k,\mathbf{1},\chi_{-20}}\left(-\frac{1}{20\tau}\right) = \frac{(20\tau)^k}{i\sqrt{20}} G_{k,\mathbf{1},\chi_{-20}}(\tau),$$
$$G_{k,\mathbf{1},\chi_{-8}}\left(-\frac{1}{8\tau}\right) = \frac{(8\tau)^k}{i\sqrt{8}} G_{k,\mathbf{1},\chi_{-8}}(\tau),$$

and for odd $k \geq 3$,

$$G_{k,\mathbf{1},\chi_{-20}}\left(-\frac{1}{20\tau}\right) = \frac{(20\tau)^k}{i\sqrt{20}} G_{k,\chi_{-20},\mathbf{1}}(\tau),$$
$$G_{k,\chi_{-20},\mathbf{1}}\left(-\frac{1}{20\tau}\right) = \frac{\sqrt{20}\tau^k}{i} G_{k,\mathbf{1},\chi_{-20}}(\tau),$$
$$G_{k,\mathbf{1},\chi_{-8}}\left(-\frac{1}{8\tau}\right) = \frac{(8\tau)^k}{i\sqrt{8}} G_{k,\chi_{-8},\mathbf{1}}(\tau),$$
$$G_{k,\chi_{-8},\mathbf{1}}\left(-\frac{1}{8\tau}\right) = \frac{\sqrt{8}\tau^k}{i} G_{k,\mathbf{1},\chi_{-8}}(\tau).$$

Proof. It is well known that the Fricke involution permutes the Eisenstein series. Moreover, as k is odd, the eigenvalues of the Fricke involution H_N are $\pm i$. Therefore, it follows that

$$\begin{aligned} G_{k,\mathbf{1},\chi_{-20}}\left(-\frac{1}{20\tau}\right) &= \frac{(20\tau)^k}{i\sqrt{20}} G_{k,\chi_{-20},\mathbf{1}}(\tau), \\ G_{k,\mathbf{1},\chi_{-8}}\left(-\frac{1}{8\tau}\right) &= \frac{(8\tau)^k}{i\sqrt{8}} G_{k,\chi_{-8},\mathbf{1}}(\tau). \end{aligned}$$

Changing τ to $\frac{-1}{20\tau}$ and $\frac{-1}{8\tau}$ respectively, in above equation, we get the other identities. For $k=1$, The corresponding space is one dimensional and so they follow easily. For more details we refer to [49, Chapter 7].

Lemma 3.2.2 *Under the transformation $\tau \to \tau + \frac{1}{2}$, the following identities hold. When k is even,*

$$E_k\left(\tau+\frac{1}{2}\right) = -E_k(\tau) + (2^k+2)E_k(2\tau) - 2^k E_k(4\tau), \tag{3.12}$$

and when k is odd, we have

$$G_{k,\mathbf{1},\chi_{-20}}\left(\tau+\frac{1}{2}\right) = -G_{k,\mathbf{1},\chi_{-20}}(\tau) + 2G_{k,\mathbf{1},\chi_{-20}}(2\tau), \tag{3.13}$$

$$G_{k,\chi_{-20},\mathbf{1}}\left(\tau+\frac{1}{2}\right) = -G_{k,\chi_{-20},\mathbf{1}}(\tau) + 2^k G_{k,\chi_{-20},\mathbf{1}}(2\tau), \tag{3.14}$$

$$G_{k,\mathbf{1},\chi_{-8}}\left(\tau+\frac{1}{2}\right) = -G_{k,\mathbf{1},\chi_{-8}}(\tau) + 2G_{k,\mathbf{1},\chi_{-8}}(2\tau), \tag{3.15}$$

$$G_{k,\chi_{-8},\mathbf{1}}\left(\tau+\frac{1}{2}\right) = -G_{k,\chi_{-8},\mathbf{1}}(\tau) + 2^k G_{k,\chi_{-8},\mathbf{1}}(2\tau). \tag{3.16}$$

Proof. First of all, from the definitions of $E_k(\tau)$ and $G_{k,\chi,\psi}(\tau)$, we have

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{3.17}$$

$$G_{2k+1,\mathbf{1},\chi_{-20}}(\tau) = 1 - \frac{4k+2}{B_{2k+1,\chi_{-20}}} \sum_{n=1}^{\infty} \sigma_{2k,\mathbf{1},\chi_{-20}}(n) q^n, \tag{3.18}$$

$$G_{2k+1,\chi_{-20},\mathbf{1}}(\tau) = -\frac{4k+2}{B_{2k+1,\mathbf{1}}} \sum_{n=1}^{\infty} \sigma_{2k,\chi_{-20},\mathbf{1}}(n) q^n, \tag{3.19}$$

To prove (3.12), we observe that $\sigma_k(2n) = (1+2^k)\sigma_k(n) - 2^k\sigma_k(n/2)$, where $\sigma_k(n/2)$ is defined to be zero if $n/2$ is not a positive integer. Then

$$\begin{aligned}
& E_{2k}\left(\tau+\frac{1}{2}\right)+E_{2k}(\tau) \\
= {} & 2\left(1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(2n)q^{2n}\right) \\
= {} & 2\left(1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\left((1+2^{2k-1})\sigma_{2k-1}(n)-2^{2k-1}\sigma_{2k-1}(n/2)\right)q^{2n}\right) \\
= {} & 2\left((1+2^{2k-1})\left(1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{2n}\right)-2^{2k-1}\left(1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n/2)q^{2n}\right)\right) \\
= {} & (2+2^{2k})E_{2k}(2\tau)-2^{2k}E_{2k}(4\tau).
\end{aligned}$$

For the proof of (3.13), we observe that

$$\begin{aligned}
& G_{2k+1,\mathbf{1},\chi_{-20}}(\tau+\frac{1}{2})+G_{2k+1,\mathbf{1},\chi_{-20}}(\tau) \\
& \qquad = 2\left(1-\frac{4k+2}{B_{2k+1,\chi_{-20}}}\sum_{n=1}^{\infty}(1+(-1)^n)\sigma_{2k,\mathbf{1},\chi_{-20}}(n)q^n\right)
\end{aligned} \tag{3.20}$$

and

$$2G_{2k+1,\mathbf{1},\chi_{-20}}(2\tau)=2\left(1-\frac{4k+2}{B_{2k+1,\chi_{-20}}}\sum_{n=1}^{\infty}\sigma_{2k,\mathbf{1},\chi_{-20}}(n/2)q^n\right). \tag{3.21}$$

It is now easy to check that the n-th Fourier coefficients of both the equations (3.20) and (3.21) match. This proves the identity given by (3.13).

The proofs of other cases (3.14)-(3.16), are similar. We also refer to similar proofs given in [22].

Along with all the above mentioned results, the order of vanishing of a modular form at the cusps plays a crucial role in the proof. Throughout this chapter, we write $\mathrm{ord}_s(f)$ for the order of vanishing of a complex function f at the point s. The preceding two lemmas are now used to compute the order of vanishing of $F_{k,m}(\tau)$ at the cusp $\frac{1}{2}$ and yield the following results.

Lemma 3.2.3 *Let $F_{k,m} = F_{k,m}(\tau)$ be defined as in Theorem 3.1.2. Then*

$$\mathrm{ord}_{1/2}(F_{k,m}) = \begin{cases} 2 & \text{if } k \text{ even and } m = 5, \\ \frac{1}{2} & \text{if } k \text{ odd and } m = 5, \\ 1 & \text{if } k \geq 2 \text{ and } m = 8, \\ 0 & \text{if } k = 1 \text{ and } m = 8. \end{cases}$$

Proof. Since $F_{k,m}(\tau)$ is a linear combination of the Eisenstein series or the generalized Eisenstein series, it follows that it is a modular form of weight k with character $\left(\frac{-4m}{\cdot}\right)^k$ for $\Gamma_0(4m)$. Also, we note that $\pi : X(\Gamma_0(4m)) \twoheadrightarrow X(\Gamma_0(4m)^+)$ is a degree 2 projective morphism, and that the cusp $\frac{1}{2}$ of the latter modular curve ramifies as two cusps in the former modular curve, each of ramification index 1. Thus, the local coordinate of the cusp $\frac{1}{2}$ remains the same, which is of width m, and one can proceed with the standard way to compute the order of vanishing of $F_{k,m}$ as a modular form for $\Gamma_0(4m)$. We give the proof to the case $k \geq 3$ odd and $m = 5$, and the other cases follow in a similar way. When $k = 2\ell + 1 \geq 3$, we have

$$\begin{aligned}
&(2\tau+1)^{-k} F_k\left(\frac{\tau}{2\tau+1}\right) \\
&= (2\tau+1)^{-k}\left(G_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{\tau}{2\tau+1}\right) + (-20)^{\ell} G_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{\tau}{2\tau+1}\right)\right) \\
&= (2\tau+1)^{-k}\left(G_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{-1}{4\tau+2}+\frac{1}{2}\right) + (-20)^{\ell} G_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{-1}{4\tau+2}+\frac{1}{2}\right)\right) \\
&= (2\tau+1)^{-k}\Bigg(-G_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{-1}{4\tau+2}\right) + 2G_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{-1}{2\tau+1}\right) \\
&\qquad - (-20)^{\ell} G_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{-1}{4\tau+2}\right) + 2^{2\ell+1}(-20)^{\ell} G_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{-1}{2\tau+1}\right)\Bigg)
\end{aligned}$$

by Lemma 3.2.2

$$\begin{aligned}
&= (2\tau+1)^{-k}\Bigg(-\frac{20^{2\ell+1}}{i\sqrt{20}}\left(\frac{2\tau+1}{10}\right)^{2\ell+1} G_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{2\tau+1}{10}\right) \\
&\qquad + 2\frac{20^{2\ell+1}}{i\sqrt{20}}\left(\frac{2\tau+1}{20}\right)^{2\ell+1} G_{2\ell+1,\chi_{-20},\mathbf{1}}\left(\frac{2\tau+1}{20}\right)
\end{aligned}$$

$$-(-20)^{\ell}\frac{\sqrt{20}}{i}(4\tau+2)^{2\ell+1}G_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{4\tau+2}{20}\right)$$
$$+2^{2\ell+1}(-20)^{\ell}\frac{\sqrt{20}}{i}(2\tau+1)^{2\ell+1}G_{2\ell+1,\mathbf{1},\chi_{-20}}\left(\frac{2\tau+1}{20}\right)\Bigg)$$

by Lemma 3.2.1

$$=Cq^{\frac{1}{10}}+O(q^{\frac{1}{5}})$$

for some nonzero constant C. Hence, the order of vanishing of $F_{k,m}$ for odd $k \geq 3$ and $m=5$ at the cusp $\frac{1}{2}$ is $1/2$.

The next two lemmas are useful for computing the order of vanishing of $\Theta(\tau)\Theta(m\tau)$ and $x_m(\tau)$ at cusps.

Lemma 3.2.4 *If $f(\tau)=\prod_{d|N}\eta_d^{r_d}$ for some positive integer N with $k=\frac{1}{2}\sum_{d|N}r_d\in\mathbb{Z}$, with the additional properties that*

$$\sum_{d|N} dr_d \equiv 0 \pmod{24}$$

and

$$\sum_{d|N}\frac{N}{d}r_d \equiv 0 \pmod{24},$$

then $f(\tau)$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\chi(d)(c\tau+d)^k f(\tau)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(N)$, where the character χ is defined by Jacobi symbol $\chi(d)=\left(\frac{(-1)^k s}{d}\right)$, $s=\prod_{d|N}d^{r_d}$.

Proof. This is a well known theorem on the modular properties of an eta-quotient. For more details, we refer to [28, 52, 53].

In the following lemma we present the results obtained by Ligozat [43], Biagioli [14] and Martin [48] on the order of vanishing of an eta-quotient at the cusps.

Lemma 3.2.5 *Let a, c and N be positive integers with $c|N$ and $\gcd(a,c)=1$. If $f(\tau)=\prod_{d|N}\eta_d^{r_d}$ satisfies the conditions of Lemma 3.2.4 for N, then the order of vanishing $\mathrm{ord}_{a/c}(f)$ of $f(\tau)$ at the cusp a/c is*

$$\frac{N}{24}\sum_{d|N}\frac{\gcd(c,d)^2 r_d}{\gcd(c,N/c)cd}.$$

As described in the Chapter 2, the classical theta series is given by an eta-quotient : $\Theta(\tau)=\frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)}$ and in our new notation, it is given by

$$\Theta(\tau)=\frac{\eta_2^5}{\eta_1^2\eta_4^2}, \tag{3.22}$$

and $x_m(\tau)$ is defined in Theorem 3.1.2 is also an eta-quotient. Here, we recall that $\Gamma_0(N)^+$ means the group generated by $\Gamma_0(N)$ and the Fricke involution $H_N=\begin{pmatrix}0 & -1\\ N & 0\end{pmatrix}$. Also, for $p^\alpha||N$, $\Gamma_0(N)^{+p^\alpha}$ denotes the group generated by $\Gamma_0(N)$ and the Atkin-Lehner operator W_p corresponding to the prime p.

Now by using Lemma 3.2.4 and Lemma 3.2.5, we have the following Lemma giving the order of vanishing of $\Theta(\tau)\Theta(m\tau)$ and $x_m(\tau)$ at the cusps 0, $\frac{1}{2}$ and $\frac{1}{4}$, of the congruence subgroup $\Gamma_0(4m)^+$ for $m\in\{5,8\}$. It is also to be noted that the theta series $\Theta(\tau)\Theta(m\tau)$ is automorphic with respect to $\Gamma_0(4m)^+$.

Lemma 3.2.6 *Let $\Theta(\tau)$ and $x_m=x_m(\tau)$ be defined as in (3.1) and (3.5), respectively. Then*

$$\mathrm{ord}_s(\Theta(\tau)\Theta(m\tau))=\begin{cases}0 & \text{if } s=0 \text{ or } \frac{1}{4}, \text{ and } m=5 \text{ or } 8,\\ \frac{3}{2} & \text{if } s=\frac{1}{2} \text{ and } m=5,\\ 2 & \text{if } s=\frac{1}{2} \text{ and } m=8,\end{cases}$$

and for $m \in \{5,8\}$,

$$\mathrm{ord}_s(x_m) = \begin{cases} 1 & \text{if } s = 0, \\ -1 & \text{if } s = \frac{1}{2}, \\ 0 & \text{if } s = \frac{1}{4}. \end{cases}$$

3.3 Proof of Theorem 3.1.2

Proof of Theorem 3.1.2. Let $F_{k,m}(\tau)$ and $l(k,m)$ be defined as in theorem. Since $F_{k,m}(\tau)$ and $\Theta^k(\tau)\Theta^k(m\tau)$ are modular forms of weight k, the ratio $\frac{F_{k,m}(\tau)}{\Theta^k(\tau)\Theta^k(m\tau)}$ is a modular function for $\Gamma_0(4m)$ (Since both these functions are modular forms of same weight.). Further, using the action of H_{4m} on the Eisenstein series as obtained in Lemma 3.2.1 and using the fact that $\Theta(-\frac{1}{4\tau}) = \sqrt{\frac{2\tau}{i}}\Theta(\tau)$, it follows that the ratio $\frac{F_{k,m}(\tau)}{\Theta^k(\tau)\Theta^k(m\tau)}$ is an eigenfunction with respect to H_{4m}. So, it is a modular function on $\Gamma_0(4m)^+$. Using Lemma 3.2.3 and Lemma 3.2.6, we find that the only pole of the function $\frac{F_{k,m}(\tau)}{\Theta^k(\tau)\Theta^k(m\tau)}$ is at the cusp $\frac{1}{2}$, of order $l(k,m)$. By Lemma 3.2.6, the modular function $x_m(\tau)$ has a simple pole at $\frac{1}{2}$ and has simple zero at the cusp 0. This implies that the pole at the cusp $\frac{1}{2}$ of order $l(k,m)$ of the function $\frac{F_{k,m}(\tau)}{\Theta^k(\tau)\Theta^k(m\tau)}$ cancels with the zero of $x_m^{l(k,m)}(\tau)$ at $\frac{1}{2}$ (of order $l(k,m)$). Moreover it is an eigenfunction under H_{4m}. So, it is enough to make this function holomorphic at the cusp 0. To do this, inductively we find the rational numbers $b_{k,m,j}$ such that

$$\frac{F_{k,m}(\tau)}{\Theta^k(\tau)\Theta^k(m\tau)x_m^{l(k,m)}(\tau)} - \sum_{j=1}^{l(k,m)} b_{k,m,j}x_m^{-j}(\tau)$$

is holomorphic at the cusp 0 and thus it is holomorphic everywhere on the compact modular curve $X(\Gamma_0(4m)^+)$. Therefore, it must be a constant (say C). Since $x_m^{-1}(\tau)$ has a zero at the cusp $\frac{1}{2}$, while $\frac{F_{k,m}(\tau)}{\Theta^k(\tau)\Theta^k(m\tau)x_m^{l(k,m)}(\tau)}$ is holomorphic and nonvanishing at the cusp $\frac{1}{2}$, we see that the constant C is nonzero. Multiplying both sides by $x_m^{l(k,m)}$ and rearranging the terms on both sides, we get

$$b_{k,m,l(k,m)}\Theta^k(\tau)\Theta^k(m\tau) = F_{k,m}(\tau) + \Theta^k(\tau)\Theta^k(m\tau)\sum_{j=1}^{l(k,m)} c_{k,m,j}x_m^j(\tau)$$

by setting $c_{k,m,j} = b_{k,m,l(k,m)-j}$ and $c_{k,m,l(k,m)} = -C$. Taking $\tau \to i\infty$, using the definition of $F_{k,m}(\tau)$, we obtain $b_{k,m,l(k,m)} = 1$. This completes the proof. $\square$

CHAPTER 4

Construction of Shimura and Shintani maps for certain subspaces of cusp forms

In 1985, W. Kohnen [38] constructed explicit Shintani lifts from the space of cusp forms of weight $2k$ on $\Gamma_0(N)$, which are mapped into the plus space $S^+_{k+1/2}(4N)$, where N is an odd integer. This lifting is adjoint to the (modified) Shimura map defined by him with respect to the Petersson scalar product. Using this construction along with the theory of newforms on $S^+_{k+1/2}(\Gamma_0(4N))$ (where N is an odd square-free natural number) developed in [37], he derived explicit Waldspurger theorem for the newforms belonging to the space $S^+_{k+1/2}(4N))$. Further, in this work, Kohnen considered the ± 1 eigen subspaces corresponding to the Atkin-Lehner involution and showed that the intersection of this $\pm$ spaces with the corresponding newform spaces are isomorphic under the Shimura correspondence. A natural question is whether a similar result holds good in the case of intersecting this subspace with the oldforms. In this direction, S. Choi and C.H. Kim [18] considered the case where N is an odd prime p and constructed similar Shimura and Shintani maps between subspaces of forms of half-integral and integral weights. The subspaces considered by Choi and Kim are nothing but $+1$ eigenspace under the Fricke involution. In this chapter, we present our work [55] in which we generalize the results of Choi and Kim to the case where N is an odd square-free natural number.

4.1 Introduction

Let $\Gamma_0(N)$ denote the congruence subgroup of the full modular group $SL_2(\mathbb{Z})$. The work of G. Shimura [73] in 1973 gave the foundation of the theory of modular forms of half-integral weight and also a correspondence between the spaces of modular forms of half-integral weight and integral weight. Later in 1975, T. Shintani [75] used theta kernel to construct a modular form of half-integral weight from a given modular form of integral weight. These are referred to in the literature as the Shimura and Shintani correspondences. In [36, 37], W. Kohnen constructed a canonical subspace of $S_{k+1/2}(4N)$ (the $\mathbb{C}$-vector space of all cusp forms of weight $k+1/2$ on $\Gamma_0(4N)$), when N is an odd positive integer. This subspace is called the plus space, denoted by $S^+_{k+1/2}(4N)$ is defined as follows.

$$S^+_{k+1/2}(4N) = \left\{ f \in S_{k+1/2}(4N) : a_f(n) \neq 0 \implies (-1)^k n \equiv 0,1 \pmod 4 \right\},$$

where $f(\tau) = \sum_{n\geq 1} a_f(n) e^{2\pi i n\tau}$. Following Kohnen, we also denote the plus space $S^+_{k+1/2}(4N)$ simply by $S_{k+1/2}(N)$. Since we shall be mainly discussing results in the Kohnen plus space, throughout this chapter, N denotes an odd positive integer. Whenever necessary we shall be further assuming that N is square-free. In [37], Kohnen defined a modified Shimura lift on $S_{k+1/2}(N)$, which is mapped into space $S_{2k}(N)$, the vector space of cusp forms of weight $2k$ on $\Gamma_0(N)$. In that work he also developed an analogous theory of newforms on $S_{k+1/2}(N)$, when N is odd and square-free and showed that the subspaces spanned by newforms in the respective spaces $S_{k+1/2}(N)$ and $S_{2k}(N)$ are isomorphic under a linear combination of the modified Shimura lifts.

Here we define the modified Shimura map constructed by Kohnen. Let D be a fundamental discriminant such that $(-1)^k D > 0$. Then the D-th Shimura-Kohnen map on

$S_{k+1/2}(N)$ is defined as follows.

$$\mathscr{S}_D^+(f)(\tau) = \sum_{n\geq 1}\left(\sum_{\substack{d|n \\ (d,N)=1}}\left(\frac{D}{d}\right)d^{k-1}a_f(|D|n^2/d^2)\right)e^{2\pi i n\tau}. \tag{4.1}$$

Then, he showed that $\mathscr{S}_D^+ : S_{k+1/2}(N) \to S_{2k}(N)$. In another work [38], Kohnen defined the D-th Shintani lifting, denoted by $\mathscr{S}_D^{+*}$, which is adjoint to the D-th Shimura map by constructing a kernel function. This has the following mapping property:

$$\mathscr{S}_D^{+*} : S_{2k}(N) \to S_{k+1/2}(N).$$

We shall use this kernel function later in the chapter.

The W-operators (introduced by Atkin and Lehner in [10] in the case of integral weight modular forms) play a crucial role in the study of the theory of newforms in the space of modular forms (of integral and half-integral weights). In [37], Kohnen introduced analogous W-operators in the space $S_{k+1/2}(N)$, which we define below. Let p be a prime dividing N (since N is odd and square-free, p is an odd prime and $p^2 \nmid N$). Then the Atkin-Lehner W-operator in the space $S_{k+1/2}(N)$ is defined by

$$w_p := p^{-k/2+1/4}U(p)W(p), \tag{4.2}$$

where $U(p)$ is the usual U-operator as defined by (1.3), and $W(p)$ is the operator defined in a similar manner as the Atkin-Lehner W-operator in the case of integral weight. Explicitly, it is defined as follows:

$$W(p) = \left(\begin{pmatrix} p & a \\ 4N & pb \end{pmatrix}, \left(\frac{-4}{p}\right)^{-k-1/2} p^{-k/2-1/4}(4Nz+pb)^{k+1/2}\right),$$

where a and b are integers with $p^2b - 4Na = p$. It is a fact that $W(p)$ maps $S_{k+1/2}(4N)$ to $S_{k+1/2}(4N, \left(\frac{4p}{\cdot}\right))$ and $\left(\frac{-4}{p}\right)^{-k/2-1/4} W(p)$ acts as an unitary involution on the sum of these spaces. Since this W-operator doesn't preserve the space $S_{k+1/2}(N)$, Kohnen defined the above w_p operator, which is considered as the analogue of the Atkin-Lehner operator in the case of half-integral weight. He showed that the operator w_p is a hermitian involution and further characterised its ± 1 eigenspaces in terms of certain properties of the Fourier coefficients of the forms. In fact, for a prime $p|N$, the ± 1 subspace corresponding to the W-operator w_p is defined as

$$S^{\pm,p}_{k+1/2}(N) := \left\{ f \in S_{k+1/2}(N) \middle| f|w_p = \pm f \right\}. \tag{4.3}$$

In [37], Kohnen showed that

$$S^{\pm,p}_{k+1/2}(N) = \left\{ f \in S_{k+1/2}(N) \middle| a_f(n) = 0, \text{ if } \left(\frac{(-1)^k n}{p}\right) = \mp 1 \right\} \tag{4.4}$$

and proved that $S^{new}_{k+1/2}(N)$ and $S^{new}_{2k}(N)$ (respective subspaces generated by newforms) are isomorphic under a linear combination of the modified Shimura lifts. Further, he proved that the intersection of the newforms space with the above $\pm, p$ subspace is also isomorphic via the (modified) Shimura correspondence to the subspace $S^{new}_{2k}(N) \cap S^{\pm,p}_{2k}(N)$, where

$$S^{\pm,p}_{2k}(N) = \left\{ F \in S_{2k}(N) \middle| F|W_p = \pm F \right\}.$$

Here W_p is the Atkin-Lehner operator in the space $S_{2k}(N)$ for primes $p|N$ (see Remark 1.2.1). A natural question is whether such a correspondence via the Shimura maps can be given for the intersection of the W-operator eigenspace with the old class.

In 2017, S. Choi and C. H. Kim [18] considered this problem when N is an odd prime and showed that one can use the kernel function constructed by Kohnen [38] to define a new kernel function for the required mappings for the old class.

In this Chapter, we generalize the work of Choi and Kim to the case of odd and square-free level N. More precisely, we construct Shimura and Shintani maps corresponding to the subspaces of $S_{k+1/2}(N)$ and $S_{2k}(N)$, consisting of cusp forms which are eigenforms (with $+1$ eigenvalue) for the operator $\prod_{p|N} w_p$ and $\prod_{p|N} W_p$ respectively. We first decompose this $+1$ eigen subspace into a direct sum of component subspaces and we shall be deriving our mapping correspond to these component subspaces. We note that there is a difference between the result announced in the synopsis and the actual statement of the theorem in this chapter. The reason is that while proving the theorem, we realized that one should consider another subspace (see the definition in (4.8)) in order to get the required mapping property for our purpose. The main result is not affected, rather we introduce intermediate steps to achieve our main result.

After presenting the necessary preliminary details we describe our main results in §4.2 and in §4.3, we give a proof of our results.

4.2 Preliminaries and statement of results

Throughout this chapter $N > 1$ denotes an odd square-free natural number and we assume that N is a product of v prime divisors, i.e., $\nu(N) = v$.

As mentioned in the introduction, we use the notation $S_{k+1/2}(N)$ for the Kohnen plus space $S^{+}_{k+1/2}(\Gamma_0(4N))$. Let w_p be the Atkin-Lehner W-operator defined by (4.2) (due to Kohnen). The ± 1 eigen subspace (in $S_{k+1/2}(N)$) corresponding to each of the primes p dividing N is denoted as $S^{\pm,p}_{k+1/2}(N)$ (equation (4.3)). We now define the following subspaces of $S_{k+1/2}(N)$ and $S_{2k}(N)$, which are ± 1 eigen subspaces with respect to the product of the W-operators:

$$S^{\pm,N}_{k+1/2}(N) := \Big\{ f \in S_{k+1/2}(N) \mid f|\prod_{p|N} w_p = \pm f \Big\}, \tag{4.5}$$

$$S^{\pm,N}_{2k}(N) := \Big\{ F \in S_{2k}(N) \mid F|\prod_{p|N} W_p = \pm F \Big\}. \tag{4.6}$$

It is to be noted that in the case of integral weight, the Fricke involution H_N (which is defined before Lemma 3.2.6) is equivalent to the product of all the W-operators. i.e., $H_N = \prod_{p|N} W_p$ on $S_{2k}(N)$.

The spaces $S_{k+1/2}(N)$ and $S_{2k}(N)$ are decomposed as orthogonal direct sums (with respect to the Petersson scalar product) of these $\pm$ subspaces as follows:

$$\begin{aligned} S_{k+1/2}(N) &= S^{+,N}_{k+1/2}(N) \oplus S^{-,N}_{k+1/2}(N), \\ S_{2k}(N) &= S^{+,N}_{2k}(N) \oplus S^{-,N}_{2k}(N). \end{aligned} \tag{4.7}$$

In this chapter we shall be constructing the required correspondences between the spaces $S^{+,N}_{k+1/2}(N)$ and $S^{+,N}_{2k}(N)$. In order to get our construction of the maps, we need to further decompose these spaces as follows. As per our assumption, N is a product of v primes and so we consider the $(\pm, p)$ subspaces corresponding to each of the v primes and consider their intersection. Totally there are 2^v subspaces consisting of forms in $S_{k+1/2}(N)$ which are eigenfunctions w.r.t all the w_p operators, $p|N$ with ± 1 eigenvalues. Let us assume that $f \in S_{k+1/2}(N)$ is an eigenfunction with respect to w_p with $+1$ eigenvalue for r number of primes (say $p_1, p_2, \ldots, p_r$) and with -1 eigenvalue for s number of primes (say $q_1, q_2, \ldots, q_s$), with $r+s=v$. The notation to keep the primes as p_i and q_j are only local and we do not assume these in the general situation. i.e., when we have a partition $r+s=v$, r,s are non-negative integers, we assume that the r primes are $p_1, p_2, \ldots, p_r$ and the s primes are $q_1, \ldots, q_s$. So, when we vary the numbers r,s in the partition of v, the number of primes in each group will also vary corresponding to the partition. Then it is clear that such an f belongs to $S^{+,N}_{k+1/2}(N)$ (resp. $S^{-,N}_{k+1/2}(N)$) only when s is even (resp. s is odd). Moreover, the number of such subspaces is equal to $\binom{v}{s}$. Let us denote the subspace as defined above for a given partition (r,s) of v as $S^{(r,s)}_{k+1/2}(N)$. To be precise, let r,s be non-negative integers such that $r+s=v$. Then we define

$$S^{(r,s)}_{k+1/2}(N) = \Big\{ f \in S_{k+1/2}(N) \big| f|w_p = \pm f, p|N, s = \#\{p : f|w_p = -f\} \Big\}. \tag{4.8}$$

It follows that when s is a non-negative integer, then

$$S^{(r,s)}_{k+1/2}(N) \subset \begin{cases} S^{+,N}_{k+1/2}(N) & \text{if } s \text{ is even,} \\ S^{-,N}_{k+1/2}(N) & \text{if } s \text{ is odd.} \end{cases}$$

We denote by $S^{(r,s)_e}_{k+1/2}(N)$ (resp. $S^{(r,s)_o}_{k+1/2}(N)$), the subspace $S^{(r,s)}_{k+1/2}(N)$ with s even (resp. s odd). From the construction of these spaces it is clear that the spaces $S^{\pm,N}_{k+1/2}(N)$ can be decomposed as follows:

$$\begin{aligned} S^{+,N}_{k+1/2}(N) &= \bigoplus_{\substack{s=0,2|s \\ r+s=v}}^{v} \oplus_\sigma S^{(r,s)_e}_{k+1/2}(N), \\ S^{-,N}_{k+1/2}(N) &= \bigoplus_{\substack{s=1,s \text{ odd} \\ r+s=v}}^{v} \oplus_{\sigma'} S^{(r,s)_o}_{k+1/2}(N). \end{aligned} \tag{4.9}$$

In the second direct sum, σ runs over all the $\binom{v}{s}$ choices of s with $2|s$ and σ' runs over similar choices when s is odd. Correspondingly we also decompose the space $S_{2k}(N)$ in a similar fashion as follows:

$$\begin{aligned} S^{+,N}_{2k}(N) &= \bigoplus_{\substack{s=0,2|s \\ r+s=v}}^{v} \oplus_\sigma S^{(r,s)_e}_{2k}(N), \\ S^{-,N}_{2k}(N) &= \bigoplus_{\substack{s=1,s \text{ odd} \\ r+s=v}}^{v} \oplus_{\sigma'} S^{(r,s)_o}_{2k}(N), \end{aligned} \tag{4.10}$$

where $S^{(r,s)_e}_{2k}(N)$ (resp. $S^{(r,s)_o}_{2k}(N)$) is the subspace of $S_{2k}(N)$ consisting of forms F with the property the $F|W_p = \pm F$, $p|N$ and r number of primes p with $+1$ eigenvalue and s number of primes with -1 eigenvalue such the s is even (resp. odd) and $r+s=v$. In the above direct sums, σ and σ' have the same property as in (4.9).

Using the above decompositions, it is clear that in order to get the required correspondences between $S^{+,N}_{k+1/2}(N)$ and $S^{+,N}_{2k}(N)$, it is enough to construct the mappings between the spaces $S^{(r,s)}_{k+1/2}(N)$ and $S^{(r,s)}_{2k}(N)$ when s is even. For this reason, from now onwards we shall be assuming the following:

Assumption: Let r, s be non-negative integers such that $r+s=\nu$ with $2|s$.

Due to the above assumption that $2|s$, we shall remove the subscript 'e' and write the subspace as $S_{k+1/2}^{(r,s)}(N)$, to simplify the notation.

Kernel function constructed by Kohnen:

We first describe the kernel function introduced by Kohnen, which will be used for our construction. Let $\mathscr{D}$ denote the set of all discriminants, i.e.,

$$\mathscr{D} = \left\{ d \in \mathbb{Z} \mid d \equiv 0,1 \pmod 4 \right\}.$$

For $d \in \mathscr{D}$, let us denote by $\mathscr{Q}_{d,N}$, the set of all integral binary quadratic forms $Q(x,y) = ax^2+bxy+cy^2$ with $N|a$ and $b^2-4ac=d$. Since N divides a, it is to be noted that $\mathscr{Q}_{d,N}$ is an empty set unless $d \equiv \square\ (4N)$. The meaning of this symbol is that d is a square modulo $4N$. We make use of this symbol from now onwards. For an integer $k \geq 2$ and $D, D' \in \mathscr{D}$ with $DD' > 0$, define a function denoted by $F_{k,N}(z; D, D')$ as follows.

$$F_{k,N}(z;D,D') = \sum_{Q \in \mathscr{Q}_{DD',N}} \frac{\omega_D(Q)}{Q(z,1)^k}, \tag{4.11}$$

$z \in \mathscr{H}$, the upper half-plane. In the above definition, $\omega_D(Q)$ denotes the generalised genus character whose value is equal to $\left(\frac{D}{r}\right)$, if $(a,b,c,D)=1$ and Q represents r with $(r,D)=1$ and takes the value zero otherwise. The series converges absolutely and uniformly on compact sets and it defines a cusp form of weight $2k$ on $\Gamma_0(N)$. Moreover, it is non-zero only when $DD' \equiv \square\ (4N)$.

For $D \in \mathscr{D}$ with $(-1)^k D > 0$, let $P_{k,N;|D|}(\tau)$ denote the $|D|$-th Poincaré series in the space $S_{k+1/2}(N)$, characterised by

$$\langle f, P_{k,N;|D|} \rangle = c(|D|) a_f(|D|), \tag{4.12}$$

where $f \in S_{k+1/2}(N)$ and

$$c(|D|) = i_{4N}^{-1} \frac{\Gamma(k-1/2)}{(4\pi|D|)^{k-1/2}}, \tag{4.13}$$

where i_N is the index of the subgroup $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$. The kernel function constructed by Kohnen is defined as follows. For a fundamental discriminant $D \in \mathscr{D}$ with $(-1)^k D > 0$ and for $z, \tau \in \mathscr{H}$, let

$$\Omega_{k,N}(z,\tau;D) = \frac{i_N}{c_{k,D}} \sum_{\substack{m\geq 1\\ (-1)^k m\equiv \square\ (4)}} m^{k-\frac{1}{2}} \Big(\sum_{t|N} \mu(t) \left(\frac{D}{t}\right) t^{k-1} F_{k,\frac{N}{t}}(tz;D,(-1)^k m)\Big) e^{2\pi i m\tau}, \quad (4.14)$$

where

$$c_{k,D} = (-1)^{[\frac{k}{2}]} |D|^{-k+\frac{1}{2}} \pi \binom{2k-2}{k-1} 2^{-3k+2}. \quad (4.15)$$

Since $F_{k,N}(z;D,(-1)^k m)$ belongs to $S_{2k}(N)$, it follows that $\Omega_{k,N}(z,\tau;D)$ belongs to $S_{2k}(N)$ with respect to the z variable (for a fixed $\tau \in \mathscr{H}$). In Theorem 1 of [38], Kohnen showed that the omega function defined above can be expressed in terms of the Poincaré series in $S_{k+1/2}(N)$. We give below the theorem proved by Kohnen.

Theorem A. ([38, Theorem 1])
The function $\Omega_{k,N}(z,\tau;D)$ defined by (4.14) has the Fourier development

$$\Omega_{k,N}(z,\tau;D) = i_N c_{k,D}^{-1} \delta_k \sum_{n\geq 1} n^{k-1} \Big(\sum_{\substack{d|n\\ (d,N)=1}} \left(\frac{D}{d}\right) (n/d)^k P_{k,N;n^2|D|/d^2}(\tau) \Big) e^{2\pi i n z} \quad (4.16)$$

with respect to z, where

$$\delta_k = \frac{(-1)^{[\frac{k}{2}]} 3 (2\pi)^k}{(k-1)!}. \quad (4.17)$$

In particular, for a fixed $z \in \mathscr{H}$, $\Omega_{k,N}(z,\tau;D)$ is a cusp form in $S_{k+1/2}(N)$.

In the following, we make use of the above kernel function defined by Kohnen to construct the required kernel function for our liftings. For $D \in \mathscr{D}$ with $(-1)^k D > 0$, set

$$\varphi_{k,N}(z,\tau) := \sum_{\substack{m\geq 1\\ (-1)^k m\equiv \square\ (4)}} m^{k-\frac{1}{2}} F_{k,N}(z;D,(-1)^k m) e^{2\pi i m\tau}. \quad (4.18)$$

As remarked earlier, the function $F_{k,N}(z;D,(-1)^k m)$ is non-zero only when $|D|m$ is a square modulo $4N$. Therefore, we have

$$\varphi_{k,N}(z,\tau) = \sum_{\substack{m\geq 1\\ |D|m\equiv\square\ (4N)}} m^{k-\frac{1}{2}} F_{k,N}(z;D,(-1)^k m)e^{2\pi i m\tau}. \tag{4.19}$$

Our first result is to show (using Theorem A) that the function $\varphi_{k,N}(z,\tau)$ belongs to the Kohnen plus space with respect to the τ variable.

Proposition 4.2.1 *For a fundamental discriminant $D \in \mathscr{D}$ with $(-1)^k D > 0$, the function $\varphi_{k,N}(z,\cdot)$ belongs to the space $S_{k+1/2}(N)$ for a fixed $z \in \mathscr{H}$.*

Next, we use the following definition to choose the fundamental discriminants for the purpose of Shimura and Shintani liftings.

Definition 1. *For a fixed partition $r+s=v$ with r,s non-negative integers, and for an integer a, we say that $a \equiv \square_{r,s} \pmod{4N}$, if $a \equiv \square(4)$ and the number of primes $p|N$ for which a is a quadratic residue mod p is equal to r and the number of primes $p|N$ for which a is a quadratic non-residue mod p is equal to s. In other words, $a \in \mathscr{D}$ is such that $r = \#\{p|N : (\frac{a}{p}) = 1\}$, $s = \#\{p|N : (\frac{a}{p}) = -1\}$ and $r+s=v$.*

Remark 4.2.1 Using the characterisation (4.4), obtained by Kohnen, we see that the space $S^{(r,s)}_{k+1/2}(N)$ can be defined in terms of the Fourier coefficients as follows.

$$S^{(r,s)}_{k+1/2}(N) = \Big\{ f \in S_{k+1/2}(N) \Big| a_f(n) = 0 \text{ unless } (-1)^k n \equiv \square_{r,s} \pmod{4N} \Big\}. \tag{4.20}$$

For a fixed pair (r,s) as chosen above with $2|s$ and $r+s=v$, let D be a fundamental discriminant with $(-1)^k D > 0$ satisfying the condition $D \equiv \square_{r,s} \pmod{4N}$. The genus character has the following property for the action of the Atkin-Lehner operators W_t, $t|N$, where $W_t = \prod_{p|t} W_p$ (see [38, p.243]):

$$\omega_D(Q\circ W_t) = \left(\frac{D}{t}\right)\omega_D(Q). \tag{4.21}$$

Further, $Q_{DD',N}$ is invariant under the action of the W-operators. Therefore, it follows that for $D \equiv \square_{r,s} \pmod{4N}$, the function $F_{k,N}(z;D,(-1)^k m)$ belongs to the space $S_{2k}^{(r,s)}(N)$. Therefore, the function $\varphi_{k,N}(z,\tau)$ defined by (4.18) (which is equivalent to (4.19)), is indeed a function (with respect to the z variable) belonging to the space $S_{2k}^{(r,s)}(N)$. Hence, for a fundamental discriminant D with $(-1)^k D > 0$ and $D \equiv \square_{r,s} \pmod{4N}$, we consider the function $\varphi_{k,N}(z,\tau)$, defined by (4.19) which belongs to the space $S_{2k}^{(r,s)}(N)$ (due to our choice of D) as our kernel function w.r.t the z variable.

By Proposition 4.2.1, $\varphi_{k,N}(z,\tau)$ belongs to $S_{k+1/2}(N)$, and the space is decomposed as follows:

$$S_{k+1/2}(N) = S_{k+1/2}^{(r,s)}(N) \oplus S_{k+1/2}^{(r,s)\perp}(N), \tag{4.22}$$

where the above is an orthogonal direct sum w.r.t. the Petersson scalar product. So, for a fixed $z \in \mathscr{H}$, as a function of τ, we project the function $\varphi_{k,N}(z,\tau)$ onto the subspace $S_{k+1/2}^{(r,s)}(N)$ and write it as $\varphi_{k,N}^{(r,s)}(z,\tau)$. So, we have

$$\varphi_{k,N}(z,\tau) = \varphi_{k,N}^{(r,s)}(z,\tau) + \varphi_{k,N}^{(r,s)\perp}(z,\tau). \tag{4.23}$$

To make it uniform, we denote our kernel function as $\varphi_{k,N}^{(r,s)}(z,\tau)$ and as a function of z it is nothing but (4.19). In fact, as a function of z, $\varphi_{k,N}^{(r,s)\perp}(z,\tau) = 0$. Using this kernel function, we define the required Shimura and Shintani maps. Let $D \in \mathscr{D}$ be a fundamental discriminant with $(-1)^k D > 0$ and assume that $D \equiv \square_{r,s} \pmod{4N}$. Then the $|D|$-th Shimura map $\mathscr{S}_{D,(r,s)}$ on $S_{k+1/2}^{(r,s)}(N)$ and the $|D|$-th Shintani map on $S_{2k}^{(r,s)}(N)$ are defined as follows.

$$f|\mathscr{S}_{D,(r,s)}(z) := \frac{i_N}{c_{k,D}^*}\langle f, \varphi_{k,N}^{r,s}(-\bar{z},\cdot)\rangle_\tau, \tag{4.24}$$

$f \in S_{k+1/2}^{(r,s)}(N)$, and

$$F|\mathscr{S}_{D,(r,s)}^*(\tau) := \frac{i_N}{c_{k,D}^*}\langle F, \varphi_{k,N}^{r,s}(\cdot,-\bar{\tau})\rangle_z, \tag{4.25}$$

$F \in S_{2k}^{(r,s)}(N)$, where the constant $c_{k,D}^*$ is defined by

$$c_{k,D}^* = (-1)^{[k/2]} 2^k c_{k,D}, \tag{4.26}$$

with $c_{k,D}$ as in (4.15). In the above $\langle\cdot,\cdot\rangle_\tau$ and $\langle\cdot,\cdot\rangle_z$ denote the inner products with respect to τ and z respectively. The following mapping property follows from the fact that the function $\varphi_{k,N}^{(r,s)}(z,\tau)$ belongs to $S_{k+1/2}^{(r,s)}(N)$ as a function of τ and belongs to $S_{2k}^{(r,s)}(N)$ as a function of z:

$$\begin{aligned} \mathscr{S}_{D,(r,s)} &: S_{k+1/2}^{(r,s)}(N) \to S_{2k}^{(r,s)}(N), \\ \mathscr{S}^*_{D,(r,s)} &: S_{2k}^{(r,s)}(N) \to S_{k+1/2}^{(r,s)}(N). \end{aligned} \tag{4.27}$$

In the following theorem we give the properties of the Shimura and Shintani maps as defined above.

Theorem 4.2.2 *Let r and s be non-negative integers such that $r+s=v$ and $2|s$. Assume that $D\in\mathscr{D}$ is a fundamental discriminant with $(-1)^kD>0$ and $D\equiv\square_{r,s} \pmod{4N}$. Then the Shimura and Shintani maps defined by (4.24), (4.25) satisfy the following properties.*

(a) We have the following explicit description of the Shimura and Shintani maps in terms of the Fourier coefficients.

$$\begin{aligned} f|\mathscr{S}_{D,(r,s)} = \delta_k \Bigg[&\sum_{\substack{n\geq 1\\(n,N)=1}} n^{k-1}\Big(\sum_{\substack{d|n,\\(d,N)=1}} \left(\frac{D}{d}\right)(n/d)^k a_f(n^2|D|/d^2)c(n^2|D|/d^2)\Big)e^{2\pi inz} \\ &+\sum_{\substack{t|N\\t\neq 1}} \lambda_{k,t,N,D}\sum_{n\geq 1} n^{k-1}\Big(\sum_{\substack{d|n\\(d,N/t)=1}} \left(\frac{D}{d}\right)(n/d)^k\langle f, P^{r,s}_{k,\frac{N}{t};n^2|D|/d^2}(\tau)\rangle\Big)e^{2\pi intz}\Bigg], \end{aligned} \tag{4.28}$$

where δ_k is the constant defined by (4.17) and $\lambda_{k,t,N,D}$ is a constant depending only on k,t,N,D and $c(\alpha)$ is defined by (4.13). (For the definition of $P^{r,s}_{k,N;n}(\tau)$, we refer to §3.3.*)*

(b) The Shimura and Shintani liftings defined above commute with the action of Hecke

operators. Indeed, if $(\ell, N) = 1$, then for $f \in S^{(r,s)}_{k+1/2}(N)$ and $F \in S^{(r,s)}_{2k}(N)$, we have

$$\begin{aligned} f|\mathscr{S}_{D,(r,s)}|T(\ell) &= f|T(\ell^2)|\mathscr{S}_{D,(r,s)}, \\ F|\mathscr{S}^*_{D,(r,s)}|T(\ell^2) &= F|T(\ell)|\mathscr{S}^*_{D,(r,s)}. \end{aligned} \tag{4.29}$$

The next theorem is concerned about the nature of the intersection of the space $S^{(r,s)}_{k+1/2}(N)$ with the old class. We show that exactly half of the oldforms (in terms of the dimension of the space) belong to the space $S^{+,N}_{k+1/2}(N)$. We prove this fact by producing explicit generators for the old class with respect to the component spaces $S^{(r,s)}_{k+1/2}(N)$.

Theorem 4.2.3 *Let $S^{+,N;old}_{k+1/2}(N) = S^{old}_{k+1/2}(N) \cap S^{+,N}_{k+1/2}(N)$, where the space $S^{old}_{k+1/2}(N)$ is the orthogonal complement of the space of newforms $S^{new}_{k+1/2}(N)$ in $S_{k+1/2}(N)$. Then the dimension of the space $S^{+,N;old}_{k+1/2}(N)$ is given by*

$$\dim S^{+,N;old}_{k+1/2}(N) = \frac{1}{2} \dim S^{old}_{k+1/2}(N). \tag{4.30}$$

4.3 Proofs of theorems

4.3.1 Proof of Proposition 4.2.1

Using the kernel function $\Omega_{k,N}(z,\tau;D)$ defined in (4.14), it follows that the function $\varphi_{k,N}(z,\tau)$ given by (4.19) can be expressed as follows.

$$\begin{aligned} \varphi_{k,N}(z,\tau) &= i_N^{-1} c_{k,D}\, \Omega_{k,N}(z,\tau;D) \\ &\quad - \sum_{\substack{t|N \\ t\neq 1}} \mu(t) \left(\frac{D}{t}\right) t^{k-1} \sum_{\substack{m\geq 1 \\ |D|m\equiv\square \pmod{4N}}} m^{k-\frac{1}{2}} F_{k,N/t}(tz;D,(-1)^k m) e^{2\pi i m\tau} \\ &= i_N^{-1} c_{k,D} \Omega_{k,N}(z,\tau;D) - \sum_{\substack{t|N \\ t\neq 1}} \mu(t) \left(\frac{D}{t}\right) t^{k-1} \varphi_{k,N/t}(tz,\tau). \end{aligned} \tag{4.31}$$

Now we use induction on the number of prime factors of N. If $N = p$ is a prime, then the second part in the last line of the above equation has only one term $\left(\frac{D}{p}\right) p^{k-1} \varphi_{k,1}(pz,\tau)$.

But $\varphi_{k,1}(z,\tau)$ is nothing but (upto a constant) $\Omega_{k,1}(z,\tau;D)$ which is an element of $S_{k+1/2}(1)$ and $\Omega_{k,p}(z,\tau;D)\in S_{k+1/2}(p)$. Therefore, $\varphi_{k,p}(z,\tau)\in S_{k+1/2}(p)$. Now, we can show inductively (induction on the number of prime factors of N) that $\varphi_{k,t}(z,\tau)\in S_{k+1/2}(t)$ for each $t|N$, $t<N$, from which we conclude that $\varphi_{k,N}(z,\tau)$ belongs to $S_{k+1/2}(N)$.

4.3.2 Proof of Theorem 4.2.2(a)

To get the properties on the Fourier coefficients of the images under these mappings, we need the following properties of the Poincaré series in $S_{k+1/2}(N)$. For a natural number n with $(-1)^k n\equiv\square(4)$, we have the n-th Poincaré series $P_{k,N;n}(\tau)$ in $S_{k+1/2}(N)$. As per the assumptions of the theorem, let us take the pair (r,s) where r and s are non-negative integers such that $r+s=\nu$ and $2|s$. We consider the projection of $P_{k,N;n}(\tau)$ onto the space $S^{(r,s)}_{k+1/2}(N)$ by $P^{r,s}_{k,N;n}(\tau)$. Further, one also has the n-th Poincaré series in $S^{(r,s)}_{k+1/2}(N)$, denoted by $P^{r,s}_n(\tau)$ (with the property $\langle f,P^{r,s}_n\rangle=c(n)a_f(n)$, for $f\in S^{(r,s)}_{k+1/2}(N)$). The following proposition gives some properties of these Poincaré series.

Proposition 4.3.1 *Let $\alpha\in\mathbb{N},(-1)^k\alpha\equiv\square(4)$. Then for fixed non-negative integers r,s with $r+s=\nu$, $2|s$, we have the following properties.*

(i) If $(-1)^k\alpha\equiv\square_{r,s}\pmod{4N}$, then $P^{r,s}_{k,N;\alpha}=P^{r,s}_\alpha$. As $(\alpha,N)=1$, we also have $P_{k,N;\alpha}=P^{r,s}_{k,N;\alpha}$.

(ii) If $(-1)^k\alpha\not\equiv\square_{r,s}\pmod{4N}$, then $P^{r,s}_{k,N;\alpha}=0$.

Proof of Proposition 4.3.1. We write the Poincaré series $P_{k,N;\alpha}(\tau)$ as

$$P_{k,N;\alpha}(\tau)=P^{r,s}_{k,N;\alpha}(\tau)+P^{r,s\perp}_{k,N;\alpha}(\tau),$$

where $P^{r,s\perp}_{k,N;\alpha}(\tau)$ belongs to the orthogonal complement (with respect to the Petersson scalar product) of $S^{(r,s)}_{k+1/2}(N)$ in $S_{k+1/2}(N)$. Write the Fourier expansion of $P^{r,s}_{k,N;\alpha}(\tau)$ as

$$P^{r,s}_{k,N;\alpha}(\tau)=\sum_{\substack{\ell\geq 1,(-1)^k\ell\equiv\square(4),\\ (-1)^k\ell\equiv\square_{r,s}\pmod{4N}}} a_{P^{r,s}_{k,N;\alpha}}(\ell)e^{2\pi i\ell\tau}.$$

Then for each $\ell \in \mathbb{N}$ with $(-1)^k \ell \equiv \square(4)$, $(-1)^k \ell \equiv \square_{r,s} \pmod{4N}$, we have

$$a_{P_\ell^{r,s}}(\alpha)c(\alpha) = \langle P_\ell^{r,s}, P_{k,N;\alpha}\rangle = \langle P_\ell^{r,s}, P_{k,N;\alpha}^{r,s}\rangle = \overline{\langle P_{k,N;\alpha}^{r,s}, P_\ell^{r,s}\rangle} = c(\ell)\overline{a_{P_{k,N;\alpha}^{r,s}}(\ell)}, \qquad (4.32)$$

where $c(\alpha)$ is defined by (4.13). This implies that

$$a_{P_{k,N;\alpha}^{r,s}}(\ell) = \frac{c(\alpha)}{c(\ell)}\overline{a_{P_\ell^{r,s}}(\alpha)} = 0, \text{ unless } (-1)^k\alpha \equiv \square_{r,s} \pmod{4N},$$

since $P_\ell^{r,s} \in S_{k+1/2}^{(r,s)}(N)$. This implies that if $(-1)^k\alpha \not\equiv \square_{r,s} \pmod{4N}$, then $P_{k,N;\alpha}^{r,s} = 0$, which proves (ii). To prove (i), consider

$$a_{P_\ell^{r,s}}(\alpha)c(\alpha) = \langle P_\ell^{r,s}, P_\alpha^{r,s}\rangle = \overline{\langle P_\alpha^{r,s}, P_\ell^{r,s}\rangle} = c(\ell)\overline{a_{P_\alpha^{r,s}}(\ell)}. \qquad (4.33)$$

Combining (4.32) and (4.33), we get $P_{k,N;\alpha}^{r,s} = P_\alpha^{r,s}$, which proves the first part of (i). Since $(-1)^k\alpha \equiv \square_{r,s} \pmod{4N}$ it follows that $(\alpha, N) = 1$. For α, β with $(-1)^k\alpha \equiv \square_{r,s} \pmod{4N}$ and $(-1)^k\beta \equiv \square_{r,s} \pmod{4N}$, we have

$$\begin{aligned}
\langle P_{k,N;\alpha}^{r,s\perp}, P_{k,N;\beta}^{r,s\perp}\rangle &= \langle P_{k,N;\alpha} - P_\alpha^{r,s}, P_{k,N;\beta} - P_\beta^{r,s}\rangle \\
&= c(\beta)a_{P_{k,N;\alpha}}(\beta) - \langle P_{k,N;\alpha}, P_\beta^{r,s}\rangle - \langle P_\alpha^{r,s}, P_{k,N;\beta}\rangle + \langle P_\alpha^{r,s}, P_\beta^{r,s}\rangle \\
&= c(\beta)a_{P_{k,N;\alpha}}(\beta) - c(\alpha)\overline{a_{P_\beta^{r,s}}(\alpha)} - c(\beta)a_{P_\alpha^{r,s}}(\beta) + c(\beta)a_{P_\alpha^{r,s}}(\beta) \\
&= c(\beta)\left(a_{P_{k,N;\alpha}}(\beta) - a_{P_\alpha^{r,s}}(\beta)\right) \qquad \text{(using (4.33))} \\
&= c(\beta)a_{P_{k,N;\alpha}^{r,s\perp}}(\beta).
\end{aligned}$$

This implies that

$$\begin{aligned}
\langle P_{k,N;\alpha}, P_{k,N;\beta}^{r,s\perp}\rangle &= \langle P_\alpha^{r,s} + P_{k,N;\alpha}^{r,s\perp}, P_{k,N;\beta}^{r,s\perp}\rangle \\
&= c(\beta)\, a_{P_{k,N;\alpha}^{r,s\perp}}(\beta) \\
&= c(\alpha)\, \overline{a_{P_{k,N;\beta}^{r,s\perp}}(\alpha)} \\
&= 0 \qquad \text{since } (-1)^k\alpha \equiv \square_{r,s} \pmod{4N} \text{ by our assumption.}
\end{aligned}$$

This shows that $a_{P^{r,s\perp}_{k,N;\alpha}}(\beta) = 0$ for all β with $(-1)^k\beta \equiv \square_{r,s} \pmod{4N}$, i.e., whenever $(-1)^k\alpha \equiv \square_{r,s} \pmod{4N}$ and $(\alpha,N)=1$, the Poincaré series $P_{k,N;\alpha}$ behaves like a Poincaré series in the (r,s) space. This implies that $P_{k,N;\alpha} = P^{r,s}_{\alpha}$. This completes the proof of Proposition 4.3.1.

Now we return to the proof of (a). Using inductive arguments, it follows from (4.31) that the function $\varphi_{k,N}(z,\tau)$ can be expressed as follows.

$$\varphi_{k,N}(z,\tau) = i_N^{-1}\, c_{k,D}\, \Omega_{k,N}(z,\tau;D) + \sum_{\substack{t|N\\ t\neq 1}} \lambda_{k,t,N,D}\, i_{N/t}^{-1}\, c_{k,D}\, \Omega_{k,N/t}(tz,\tau;D), \tag{4.34}$$

where $\lambda_{k,t,N,D}$ is a constant depending only on t,k,N,D, which can be given explicitly. Now using Theorem A, we can express the omega functions in terms of the half-integral weight Poincaré series. Explicitly, we have the following expression for $\varphi_{k,N}(z,\tau)$:

$$\begin{aligned}\varphi_{k,N}(z,\tau) = \delta_k \Bigg(&\sum_{n\geq 1} n^{k-1}\Big(\sum_{\substack{d|n,\\ (d,N)=1}} \left(\frac{D}{d}\right)(n/d)^k P_{k,N;n^2|D|/d^2}(\tau)\Big) e^{2\pi i n z} \\ &+ \sum_{\substack{t|N\\ t\neq 1}} \lambda_{k,t,N,D} \sum_{n\geq 1} n^{k-1}\Big(\sum_{\substack{d|n,\\ (d,N/t)=1}} \left(\frac{D}{d}\right)(n/d)^k P_{k,N/t;n^2|D|/d^2}(\tau)\Big) e^{2\pi i n t z}\Bigg).\end{aligned} \tag{4.35}$$

As a function of the τ variable, we project the above function onto the space $S^{(r,s)}_{k+1/2}(N)$ and obtain the following expression.

$$\begin{aligned}\varphi^{r,s}_{k,N}(z,\tau) = \delta_k \Bigg(&\sum_{n\geq 1} n^{k-1}\Big(\sum_{\substack{d|n,\\ (d,N)=1}} \left(\frac{D}{d}\right)(n/d)^k P^{r,s}_{k,N;n^2|D|/d^2}(\tau)\Big) e^{2\pi i n z} \\ &+ \sum_{\substack{t|N\\ t\neq 1}} \lambda_{k,t,N,D} \sum_{n\geq 1} n^{k-1}\Big(\sum_{\substack{d|n,\\ (d,N/t)=1}} \left(\frac{D}{d}\right)(n/d)^k P^{r,s}_{k,N/t;n^2|D|/d^2}(\tau)\Big) e^{2\pi i n t z}\Bigg).\end{aligned} \tag{4.36}$$

Since $D \equiv \square_{r,s} \pmod{4N}$, by Proposition 2, the first part of the above expression is non-zero only when $(n,N)=1$. Moreover, in this case the projected Poincaré series coincides with the Poincaré series in the space (Proposition 2 (i)) and hence we get the following required expression for the Fourier expansion of the Shimura map.

$$f|\mathscr{S}_{D,(r,s)} = \delta_k \left[\sum_{\substack{n\geq 1\\(n,N)=1}} n^{k-1} \Big(\sum_{\substack{d|n,\\(d,N)=1}} \left(\frac{D}{d}\right) (n/d)^k a_f(n^2|D|/d^2) c(n^2|D|/d^2) \Big) e^{2\pi inz} \right.$$
$$\left. + \sum_{\substack{t|N\\t\neq 1}} \lambda_{k,t,N,D} \sum_{n\geq 1} n^{k-1} \Big(\sum_{\substack{d|n\\(d,N/t)=1}} \left(\frac{D}{d}\right) (n/d)^k \langle f, P^{r,s}_{k,\frac{N}{t};n^2|D|/d^2}(\tau)\rangle \Big) e^{2\pi intz} \right], \tag{4.37}$$

where δ_k, $\lambda_{k,t,N,D}$ and $c(\alpha)$ are the constants as defined before. This completes the proof of Theorem 4.2.2 (a).

4.3.3 Proof of Theorem 4.2.2(b)

In this section, we prove the required commutative properties with respect to the Hecke operators. For a positive integer $\ell \geq 1$ with $(\ell, N) = 1$, the Hecke operators $T(\ell^2)$ and $T(\ell)$ on the spaces $S_{k+1/2}(N)$ and $S_{2k}(N)$ are hermitian with respect to the Petersson scalar product. Further, the following are orthogonal direct sums (in their respective spaces):

$$S_{k+1/2}(N) = S^{(r,s)}_{k+1/2}(N) \oplus S^{(r,s)\perp}_{k+1/2}(N)$$
$$S_{2k}(N) = S^{(r,s)}_{2k}(N) \oplus S^{(r,s)\perp}_{2k}(N)$$

Therefore, the Hecke operators $T(\ell^2)$ and $T(\ell)$ preserve the respective subspaces. Thus, we have the following lemma.

Lemma 4.3.2 *Let $(\ell, N) = 1$. Then one has*

$$\big(\varphi_{k,N}|T(\ell^2)\big)^{r,s} = \varphi^{r,s}_{k,N}|T(\ell^2),$$

and

$$\big(\varphi_{k,N}|T(\ell)\big)^{r,s} = \varphi^{r,s}_{k,N}|T(\ell).$$

In the following we shall be using the inner products in both the spaces of half-integral and integral weights. To distinguish this, we use the notation $\langle\cdot,\cdot\rangle_\tau$ for the inner product in $S_{k+1/2}(N)$ and the notation $\langle\cdot,\cdot\rangle_z$ for the inner product in $S_{2k}(N)$. Also, we denote the D-th Shimura map on $S_{k+1/2}(N)$ defined by Kohnen [37] as $\mathscr{S}^+_{D,N}$. For a positive integer $t|N$, let $f \in S_{k+1/2}(t)$. Then

$$\begin{aligned}
\langle f, \Omega_{k,t}(-\bar{z},\tau;D)|T(\ell^2)\rangle_\tau &= \langle f|T(\ell^2), \Omega_{k,t}(-\bar{z},\tau;D)\rangle_\tau \\
&= f|T(\ell^2)\mathscr{S}^+_{D,t} \\
&= f|\mathscr{S}^+_{D,t}T(\ell) \\
&= \langle f, \Omega_{k,t}(-\bar{z},\tau;D)\rangle_\tau|T(\ell) \\
&= \langle f, \Omega_{k,t}(-\bar{z},\tau;D)|T(\ell)\rangle_\tau.
\end{aligned}$$

In the above we used the fact that the Shimura map $\mathscr{S}^+_{D,t}$ commutes with Hecke operators. Therefore, the above computation shows that for each $t|N$,

$$\Omega_{k,t}(-\bar{z},\tau;D)|T(\ell^2) = \Omega_{k,t}(-\bar{z},\tau;D)|T(\ell). \tag{4.38}$$

In particular,

$$\begin{aligned}
\Omega_{k,N/t}(-t\bar{z},\tau;D)|T(\ell^2) &= \Omega_{k,N/t}(-\bar{z},\tau;D)|T(\ell^2)|B(t) \\
&= \Omega_{k,N/t}(-\bar{z},\tau;D)|T(\ell)|B(t) \qquad \text{(by (4.38))} \\
&= \Omega_{k,N/t}(-t\bar{z},\tau;D)|T(\ell) \qquad \text{(since } (\ell,N)=1).
\end{aligned}$$

Now, proceeding as done in [18, p. 312], it is easy to see that the following commuting rule holds.

$$f|\mathscr{S}_{D,(r,s)}|T(\ell) = f|T(\ell^2)|\mathscr{S}_{D,(r,s)}, \tag{4.39}$$

where $f \in S^{(r,s)}_{k+1/2}(N)$. In a similar way one can prove that for $F \in S_{2k}(t)$, $t|N$, one has

$$\langle F, \Omega_{k,t}(\cdot,-\bar{\tau};D)|T(\ell)\rangle_z = \langle F, \Omega_{k,t}(\cdot,-\bar{\tau};D)|T(\ell^2)\rangle_z, \tag{4.40}$$

from which it follows that

$$F|\mathscr{S}^*_{D,(r,s)}|T(\ell^2) = F|T(\ell)|\mathscr{S}^*_{D,(r,s)}, \tag{4.41}$$

4.3.4 Proof of Theorem 4.2.3

The case $N = p$ was already considered by Choi and Kim in [18]. So, we assume that N has at least two prime factors. We first consider the case $N = p_1p_2$, $p_1 \neq p_2$ and then discuss about the general case of N as a product of v prime factors. When $N = p_1p_2$, the oldforms in $S_{k+1/2}(p_1p_2)$ consists of 3 eigenclasses, namely, $S_{k+1/2}(1)$, $S^{new}_{k+1/2}(p_i)$, $i = 1,2$. By the theory of newforms developed by Kohnen, it is enough to determine forms belonging to each eigenclass in the projected space $S^{+,N}_{k+1/2}(N)$. In this case the pairs (r,s) are given by $(2,0),(0,2)$ and $(1,1)$, of which we need to consider the first two cases where s is even. Note that the pair $(1,1)$ corresponds to the two subspaces $(+,-),(-,+)$, the ± 1 subspaces corresponding to the primes p_1, p_2. First we assume that $(r,s) = (2,0)$. i.e., the subspace consisting of forms which are eigenfunctions w.r.t w_{p_i}, $i = 1,2$ with eigenvalue $+1$. The old class generated by a Hecke eigenform in $S_{k+1/2}(1)$ in $S_{k+1/2}(p_1p_2)$ is of the form

$$g = f_1 + f_2|w_{p_1} + f_3|w_{p_2} + f_4|w_{p_1p_2}, f_i \in S_{k+1/2}(1), 1 \leq i \leq 4,$$

where $w_{p_1p_2} = w_{p_1}w_{p_2}$. We want the function g to satisfy the property that $g|w_{p_i} = g$, $i = 1,2$. Then one can check easily that in this case g takes the form $g = f + f|w_{p_1} + f|w_{p_2} + f|w_{p_1p_2}$. Similarly, for the case $(r,s) = (0,2)$, g takes the form $g = f - f|w_{p_1} - f|w_{p_2} + f|w_{p_1p_2}$. Therefore, the contribution from $S_{k+1/2}(1)$ to the intersection $S^{+,N}_{k+1/2}(N) \cap S^{old}_{k+1/2}(N)$ is given by

$$\langle f + f|w_{p_1} + f|w_{p_2} + f|w_{p_1}w_{p_2}, f - f|w_{p_1} - f|w_{p_2} + f|w_{p_1}w_{p_2} \ : \ f \in S_{k+1/2}(1)\rangle.$$

This shows that the (dimension) contribution of the space $S_{k+1/2}(1)$ in the subspace $S^{+,N}_{k+1/2}(N)$ is exactly $2\dim S_{k+1/2}(1)$. Next, we consider the contributions from $S^{new}_{k+1/2}(p_i)$, $i = 1,2$.

An element in the old class will be of the form $g = f_1 + f_2|w_{N/p_i}$, $f_1, f_2 \in S^{new}_{k+1/2}(p_i)$. We need the properties that $g|w_{p_i} = g$, $i = 1,2$. This will imply that $f_1 = f_2 = f$ (say) and moreover, as f_i's are newforms, we must have $f|w_{p_i} = f$, in other words, $g = f + f|w_{N/p_i}$. The above is for the case $(2,0)$. Similar arguments imply that the contribution from $S^{new}_{k+1/2}(p_i)$ (for the case $(0,2)$) is given as $g = f - f|w_{N/p_i}$, $f \in S^{(-,p_i),new}_{k+1/2}(N)$. Therefore, the contribution from $S^{new}_{k+1/2}(p_i)$ in $S^{+,N}_{k+1/2}(N)$ is given by

$$\langle f \pm f|w_{N/p_i}, f \in S^{(\pm,p_i),new}_{k+1/2}(N)\rangle.$$

In terms of dimension contribution of the space $S^{new}_{k+1/2}(p_i)$ in the subspace $S^{+,N}_{k+1/2}(N)$ is exactly $\dim S^{new}_{k+1/2}(p_i)$. Thus, we have

$$\dim S^{+,N}_{k+1/2}(N) \cap S^{old}_{k+1/2}(N) = 2\dim S_{k+1/2}(1) + \sum_{i=1}^{2} \dim S^{new}_{k+1/2}(p_i) = \frac{1}{2}\dim S^{old}_{k+1/2}(N).$$

Now we consider the general case. Let $t|N$ be a positive divisor of N with $t < N$. We determine the old class generated by the space $S^{new}_{k+1/2}(t)$ in $S^{+,N}_{k+1/2}(N)$. From the newform theory, it is known that this old class has dimension equal to $d(N/t)\dim S^{new}_{k+1/2}(t)$, where $d(n)$ is the number of divisors of n. Note that as N is square-free, $d(N/t) = 2^{\alpha}$, where α is the number of prime factors of N/t. We require that an old form g in this class to be an eigenform with respect to the W-operators w_p, where $p|N/t$. Note that as the base functions are newforms in $S^{new}_{k+1/2}(t)$, they are already eigenforms with respect to w_p, for $p|t$. For each prime $p|N/t$, if we require the condition that g is an eigenfunction w.r.t w_p, then the number of components in the old class reduces by a factor of 2. We shall be repeating this process for each prime $p|N/t$ and so we will be repeating α times. So, finally in order that g is an eigenfunction under each W-operator for $p|N/t$, the number of components becomes $2^{\alpha}/2^{\alpha} = 1$. That is, only one Hecke eigenform (newform) in $S^{new}_{k+1/2}(t)$ will be generating the required old class. Since we have to consider the (r,s) case, we shall get into the precise contribution of a newform in the required old class. If p_i, $1 \le i \le r$ is one of the primes dividing the level t, then the newform $f \in S^{new}_{k+1/2}(t)$ is an eigenfunction w.r.t w_{p_i}, for $p_i|t$

with eigenvalue $+1$. Similarly, if q_j divides t, then $f|w_{q_j} = -f$. For the rest of the primes p_i and q_j not dividing t, we need to assume the respective eigenvalue (+1 or -1) and finally we will end up with only one linear combination generated by a newform in $S^{new}_{k+1/2}(t)$, which is described below.

$$g = \pm^{r,s} \sum_{l|N/t} f|w_l, \tag{4.42}$$

where $w_l = \prod_{p|l} w_p$ and the sign in the linear combination is -1, if the number of primes q_j, $1 \leq j \leq s$ that divide N/t is odd. Now, we compute the dimension of the space $S^{+,N;old}_{k+1/2}(N)$.

Case (i): Let $f \in S^{+,t;new}_{k+1/2}(t)$. This implies that $\#\{q_j : 1 \leq j \leq s, q_j|N/t\}$ is even. As the number of prime divisors of N/t is α, the total contribution (in $S^{+,N;old}_{k+1/2}(N)$) from $S^{+,t;new}_{k+1/2}(t)$ is given by

$$\sum_{s=0, s \text{ even}}^{\alpha} \binom{\alpha}{s} \dim S^{+,t;new}_{k+1/2}(t). \tag{4.43}$$

Case (ii): Let $f \in S^{-,t;new}_{k+1/2}(t)$. In this case, $\#\{q_j : 1 \leq j \leq s, q_j|N/t\}$ is odd. The total contribution of $S^{+,t;new}_{k+1/2}(t)$ in $S^{+,N;old}_{k+1/2}(N)$ is then given by

$$\sum_{s=0, s \text{ odd}}^{\alpha} \binom{\alpha}{s} \dim S^{-,t;new}_{k+1/2}(t). \tag{4.44}$$

Since $\sum_{s=0, s \text{ even}}^{\alpha} \binom{\alpha}{s} = \sum_{s=0, s \text{ odd}}^{\alpha} \binom{\alpha}{s} = 2^{\alpha-1}$, combining these two cases, we see that the total contribution from $S^{new}_{k+1/2}(t)$ in the old class $S^{+,N;old}_{k+1/2}(N)$ is

$$2^{\alpha-1} \left(\dim S^{+,t;new}_{k+1/2}(t) + \dim S^{-,t;new}_{k+1/2}(t) \right) = 2^{\alpha-1} \dim S^{new}_{k+1/2}(t).$$

Therefore we have,

$$\dim S^{+,N;old}_{k+1/2}(N) = \sum_{t|N, t<N} 2^{\alpha-1} \dim S^{new}_{k+1/2}(t) = \frac{1}{2} \dim S^{old}_{k+1/2}(N). \tag{4.45}$$

Note that explicit description of the space $S^{+,N;old}_{k+1/2}(N)$ follows from (4.42). This completes the proof of Theorem 4.2.3. □

CHAPTER 5

Appendix

Table 5.1, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,4\}$. The tuple ijk represents quadratic form corresponding to tuple $(1^i, 2^j, 4^k)$, $i+j+k=8$ The corresponding generating function belongs to the space $M_4(16,\chi_8)$, with dimension 8.

ijk	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$
251	0	$\frac{16}{11}$	$\frac{2}{11}$	0	$\frac{4}{11}$	$\frac{24}{11}$	$\frac{48}{11}$	$\frac{16}{11}$
152	0	$\frac{8}{11}$	$\frac{2}{11}$	0	$\frac{2}{11}$	$\frac{12}{11}$	$\frac{48}{11}$	$\frac{16}{11}$
431	0	$\frac{32}{11}$	$\frac{2}{11}$	0	$\frac{8}{11}$	$\frac{48}{11}$	$\frac{136}{11}$	$\frac{16}{11}$
134	0	$\frac{4}{11}$	$\frac{2}{11}$	0	$\frac{1}{11}$	$\frac{17}{11}$	$\frac{48}{11}$	$\frac{16}{11}$
332	0	$\frac{16}{11}$	$\frac{2}{11}$	0	$\frac{4}{11}$	$\frac{46}{11}$	$\frac{136}{11}$	$\frac{16}{11}$
233	0	$\frac{8}{11}$	$\frac{2}{11}$	0	$\frac{2}{11}$	$\frac{34}{11}$	$\frac{92}{11}$	$\frac{16}{11}$
611	0	$\frac{64}{11}$	$\frac{2}{11}$	0	$\frac{-28}{11}$	$\frac{96}{11}$	$\frac{224}{11}$	$\frac{192}{11}$
126	0	$\frac{2}{11}$	$\frac{2}{11}$	0	$\frac{6}{11}$	$\frac{14}{11}$	$\frac{48}{11}$	$\frac{-28}{11}$
512	0	$\frac{32}{11}$	$\frac{2}{11}$	0	$\frac{-14}{11}$	$\frac{92}{11}$	$\frac{224}{11}$	$\frac{192}{11}$
215	0	$\frac{4}{11}$	$\frac{2}{11}$	0	$\frac{12}{11}$	$\frac{28}{11}$	$\frac{92}{11}$	$\frac{-28}{11}$
413	0	$\frac{16}{11}$	$\frac{2}{11}$	0	$\frac{4}{11}$	$\frac{68}{11}$	$\frac{180}{11}$	$\frac{104}{11}$
314	0	$\frac{8}{11}$	$\frac{2}{11}$	0	$\frac{13}{11}$	$\frac{45}{11}$	$\frac{136}{11}$	$\frac{16}{11}$

Table 5.2, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,3,4\}$. $ijkl$ represents quadratic form corresponding to tuple $(1^i,2^j,3^k,4^l)$, $i+j+k+l=8$, whose generating functions belong to the space $M_4(48,\chi_0)$, with dimension 30.

$ijkl$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$
0062	$\frac{1}{1200}$	$\frac{-1}{1200}$	$\frac{-7}{400}$	0	$\frac{7}{400}$	$\frac{1}{300}$	0	$\frac{-4}{75}$	$\frac{-7}{100}$	$\frac{28}{25}$	$\frac{1}{20}$	$\frac{21}{20}$	$\frac{2}{15}$	$\frac{-8}{3}$	$\frac{-112}{3}$
0044	$\frac{1}{2400}$	$\frac{1}{800}$	$\frac{3}{800}$	$\frac{-17}{600}$	$\frac{9}{800}$	$\frac{1}{50}$	$\frac{-51}{200}$	$\frac{8}{75}$	$\frac{9}{50}$	$\frac{24}{25}$	0	0	$\frac{2}{5}$	$\frac{28}{5}$	$\frac{112}{5}$
0026	$\frac{7}{19200}$	$\frac{-7}{19200}$	$\frac{-9}{6400}$	0	$\frac{9}{6400}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{-7}{80}$	$\frac{-27}{80}$	$\frac{-19}{10}$	-7	-8
0422	$\frac{7}{4800}$	$\frac{-7}{4800}$	$\frac{-9}{1600}$	0	$\frac{9}{1600}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	0	0	$\frac{-8}{5}$	-4	0
0242	$\frac{1}{1200}$	$\frac{-1}{1200}$	$\frac{3}{400}$	0	$\frac{-3}{400}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	0	0	$\frac{4}{5}$	8	16
0224	$\frac{7}{9600}$	$\frac{-7}{9600}$	$\frac{-9}{3200}$	0	$\frac{9}{3200}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	0	0	$\frac{-9}{5}$	-6	-8
5021	$\frac{7}{600}$	0	$\frac{-9}{200}$	$\frac{-119}{600}$	0	$\frac{21}{100}$	$\frac{153}{200}$	$\frac{-28}{75}$	$\frac{-81}{100}$	$\frac{36}{25}$	$\frac{7}{20}$	$\frac{27}{20}$	$\frac{-4}{5}$	$\frac{16}{5}$	$\frac{-176}{5}$
4022	$\frac{7}{1200}$	$\frac{-7}{1200}$	$\frac{-9}{400}$	0	$\frac{9}{400}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{7}{20}$	$\frac{27}{20}$	$\frac{-2}{5}$	0	-48
3041	$\frac{1}{300}$	$\frac{-1}{150}$	$\frac{3}{100}$	$\frac{17}{300}$	$\frac{-3}{50}$	$\frac{-3}{50}$	$\frac{51}{100}$	$\frac{8}{75}$	$\frac{-27}{50}$	$\frac{24}{25}$	$\frac{-1}{10}$	$\frac{9}{10}$	$\frac{16}{5}$	$\frac{64}{5}$	$\frac{96}{5}$
3023	$\frac{7}{2400}$	$\frac{-7}{800}$	$\frac{-9}{800}$	$\frac{119}{1200}$	$\frac{27}{800}$	$\frac{-7}{100}$	$\frac{-153}{400}$	$\frac{-28}{75}$	$\frac{27}{100}$	$\frac{36}{25}$	$\frac{7}{40}$	$\frac{27}{40}$	$\frac{-1}{5}$	$\frac{-8}{5}$	$\frac{-212}{5}$
2042	$\frac{1}{600}$	$\frac{-1}{600}$	$\frac{3}{200}$	0	$\frac{-3}{200}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{-1}{10}$	$\frac{9}{10}$	$\frac{8}{5}$	8	32
2024	$\frac{7}{4800}$	$\frac{-7}{960}$	$\frac{-9}{1600}$	$\frac{119}{1200}$	$\frac{9}{320}$	$\frac{-7}{100}$	$\frac{-153}{400}$	$\frac{-28}{75}$	$\frac{27}{100}$	$\frac{36}{25}$	0	0	$\frac{-3}{5}$	$\frac{-18}{5}$	$\frac{-152}{5}$
1061	$\frac{1}{600}$	0	$\frac{-7}{200}$	$\frac{-17}{600}$	0	$\frac{3}{100}$	$\frac{119}{200}$	$\frac{-4}{75}$	$\frac{-63}{100}$	$\frac{28}{25}$	$\frac{1}{20}$	$\frac{21}{20}$	$\frac{4}{15}$	$\frac{-16}{15}$	$\frac{-304}{15}$
1043	$\frac{1}{1200}$	$\frac{1}{1200}$	$\frac{3}{400}$	$\frac{-17}{600}$	$\frac{3}{400}$	$\frac{1}{50}$	$\frac{-51}{200}$	$\frac{8}{75}$	$\frac{9}{50}$	$\frac{24}{25}$	$\frac{-1}{20}$	$\frac{9}{20}$	$\frac{4}{5}$	$\frac{28}{5}$	$\frac{152}{5}$
1025	$\frac{7}{9600}$	$\frac{-7}{1920}$	$\frac{-9}{3200}$	$\frac{119}{2400}$	$\frac{9}{640}$	$\frac{-7}{300}$	$\frac{-153}{800}$	$\frac{-28}{75}$	$\frac{9}{100}$	$\frac{36}{25}$	$\frac{-7}{80}$	$\frac{-27}{80}$	$\frac{-13}{10}$	$\frac{-29}{5}$	$\frac{-96}{5}$
3221	$\frac{7}{1200}$	$\frac{-7}{1200}$	$\frac{-9}{400}$	0	$\frac{9}{400}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	0	0	$\frac{-2}{5}$	0	-24
2222	$\frac{7}{2400}$	$\frac{-7}{2400}$	$\frac{-9}{800}$	0	$\frac{9}{800}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	0	0	$\frac{-6}{5}$	-4	-24
1421	$\frac{7}{2400}$	$\frac{-7}{2400}$	$\frac{-9}{800}$	0	$\frac{9}{800}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	0	0	$\frac{-6}{5}$	-4	-16
1241	$\frac{1}{600}$	$\frac{-1}{600}$	$\frac{3}{200}$	0	$\frac{-3}{200}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	0	0	$\frac{8}{5}$	8	16
1223	$\frac{7}{4800}$	$\frac{-7}{4800}$	$\frac{-9}{1600}$	0	$\frac{9}{1600}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	0	0	$\frac{-8}{5}$	-6	-20
$ijkl$	$\alpha_{4,16}$	$\alpha_{4,17}$	$\alpha_{4,18}$	$\alpha_{4,19}$	$\alpha_{4,20}$	$\alpha_{4,21}$	$\alpha_{4,22}$	$\alpha_{4,23}$	$\alpha_{4,24}$	$\alpha_{4,25}$	$\alpha_{4,26}$	$\alpha_{4,27}$	$\alpha_{4,28}$	$\alpha_{4,29}$	$\alpha_{4,30}$
0062	$\frac{-2432}{15}$	$\frac{-3}{2}$	-9	$\frac{-9}{2}$	-27	$\frac{2}{3}$	$\frac{10}{3}$	$\frac{68}{3}$	$\frac{-9}{4}$	$\frac{27}{4}$	$\frac{1}{2}$	7	$\frac{38}{15}$	$\frac{17}{12}$	$\frac{-7}{4}$

0044	$\frac{128}{5}$	$\frac{3}{2}$	6	$\frac{27}{2}$	54	$\frac{-1}{2}$	-6	-8	1	9	$\frac{-3}{2}$	-6	-2	0	1
0026	$\frac{32}{5}$	$\frac{-9}{16}$	$\frac{-3}{2}$	$\frac{-243}{16}$	$\frac{-81}{2}$	$\frac{17}{16}$	$\frac{5}{2}$	8	$\frac{9}{16}$	$\frac{-243}{16}$	$\frac{21}{16}$	$\frac{3}{2}$	$\frac{19}{10}$	$\frac{-17}{16}$	$\frac{-21}{16}$
0422	$\frac{-128}{5}$	$\frac{-3}{4}$	-1	$\frac{-81}{4}$	-27	$\frac{5}{4}$	4	4	$\frac{1}{2}$	$\frac{-27}{2}$	$\frac{3}{4}$	3	2	-1	$\frac{-3}{2}$
0242	$\frac{128}{5}$	1	6	9	54	-1	-4	-8	1	9	-1	-6	-2	0	1
0224	$\frac{-128}{5}$	$\frac{-5}{8}$	-1	$\frac{-135}{8}$	-27	$\frac{9}{8}$	3	4	$\frac{1}{2}$	$\frac{-27}{2}$	$\frac{9}{8}$	3	2	-1	$\frac{-3}{2}$
5021	$\frac{-1408}{5}$	0	1	0	27	4	0	20	$\frac{1}{4}$	$\frac{-27}{4}$	0	9	$\frac{22}{5}$	$\frac{5}{4}$	$\frac{-9}{4}$
4022	$\frac{-1408}{5}$	$\frac{-1}{2}$	1	$\frac{-27}{2}$	27	2	2	20	$\frac{1}{4}$	$\frac{-27}{4}$	$\frac{3}{2}$	9	$\frac{22}{5}$	$\frac{5}{4}$	$\frac{-9}{4}$
3041	$\frac{768}{5}$	0	10	0	18	0	0	−8	$\frac{5}{2}$	$\frac{-9}{2}$	0	−10	$\frac{-12}{5}$	$\frac{-1}{2}$	$\frac{5}{2}$
3023	$\frac{-928}{5}$	$\frac{-3}{4}$	0	$\frac{-81}{4}$	0	1	3	20	$\frac{3}{8}$	$\frac{-81}{8}$	$\frac{9}{4}$	6	$\frac{37}{10}$	$\frac{5}{8}$	$\frac{-15}{8}$
2042	$\frac{768}{5}$	1	10	9	18	0	-4	-8	$\frac{5}{2}$	$\frac{-9}{2}$	-1	-10	$\frac{-12}{5}$	$\frac{-1}{2}$	$\frac{5}{2}$
2024	$\frac{-448}{5}$	$\frac{-3}{4}$	-1	$\frac{-81}{4}$	-27	$\frac{3}{4}$	3	20	$\frac{1}{2}$	$\frac{-27}{2}$	$\frac{9}{4}$	3	3	0	$\frac{-3}{2}$
1061	$\frac{-2432}{15}$	0	−9	0	−27	$\frac{4}{3}$	0	$\frac{68}{3}$	$\frac{-9}{4}$	$\frac{27}{4}$	0	7	$\frac{38}{15}$	$\frac{17}{12}$	$\frac{-7}{4}$
1043	$\frac{448}{5}$	$\frac{3}{2}$	8	$\frac{27}{2}$	36	0	−6	−8	$\frac{7}{4}$	$\frac{9}{4}$	$\frac{-3}{2}$	−8	$\frac{-11}{5}$	$\frac{-1}{4}$	$\frac{7}{4}$
1025	$\frac{-128}{5}$	$\frac{-5}{8}$	$\frac{-3}{2}$	$\frac{-135}{8}$	$\frac{-81}{2}$	$\frac{7}{8}$	$\frac{5}{2}$	16	$\frac{9}{16}$	$\frac{-243}{16}$	$\frac{15}{8}$	$\frac{3}{2}$	$\frac{12}{5}$	$\frac{-9}{16}$	$\frac{-21}{16}$
3221	$\frac{-448}{5}$	$\frac{-1}{2}$	-1	$\frac{-27}{2}$	-27	2	2	20	$\frac{1}{2}$	$\frac{-27}{2}$	$\frac{3}{2}$	3	3	0	$\frac{-3}{2}$
2222	$\frac{-448}{5}$	$\frac{-1}{2}$	-1	$\frac{-27}{2}$	-27	$\frac{3}{2}$	2	20	$\frac{1}{2}$	$\frac{-27}{2}$	$\frac{3}{2}$	3	3	0	$\frac{-3}{2}$
1421	$\frac{-128}{5}$	$\frac{-1}{2}$	−1	$\frac{-27}{2}$	−27	$\frac{3}{2}$	2	4	$\frac{1}{2}$	$\frac{-27}{2}$	$\frac{3}{2}$	3	2	−1	$\frac{-3}{2}$
1241	$\frac{128}{5}$	1	6	9	54	0	−4	−8	1	9	−1	−6	−2	0	1
1223	$\frac{-288}{5}$	$\frac{-1}{2}$	−1	$\frac{-27}{2}$	−27	$\frac{5}{4}$	2	12	$\frac{1}{2}$	$\frac{-27}{2}$	$\frac{3}{2}$	3	$\frac{5}{2}$	$\frac{-1}{2}$	$\frac{-3}{2}$

Table 5.3, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,3,4\}$. $ijkl$ represents quadratic form corresponding to tuple $(1^i,2^j,3^k,4^l)$, $i+j+k+l=8$, whose generating function belongs to the space $M_4(48,\chi_8)$, with dimension 28.

$ijkl$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$
0521	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{208}{451}$	0	$\frac{864}{451}$	0	$\frac{-3112}{451}$	$\frac{16768}{451}$	$\frac{324}{451}$	$\frac{98496}{451}$	$\frac{3504}{451}$	$\frac{832}{451}$
0341	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{160}{451}$	0	$\frac{-1152}{451}$	0	$\frac{14744}{1353}$	$\frac{-32392}{451}$	$\frac{-432}{451}$	$\frac{-64224}{451}$	$\frac{-7296}{451}$	$\frac{-320}{451}$
0323	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{104}{451}$	0	$\frac{432}{451}$	0	$\frac{-4713}{451}$	$\frac{16768}{451}$	$\frac{162}{451}$	$\frac{98496}{451}$	$\frac{5360}{451}$	$\frac{832}{451}$
0161	0	$\frac{-2}{451}$	0	$\frac{84}{451}$	$\frac{64}{451}$	0	$\frac{2688}{451}$	0	$\frac{-96320}{4059}$	$\frac{158704}{4059}$	$\frac{1008}{451}$	$\frac{33312}{451}$	$\frac{14400}{451}$	$\frac{-28800}{451}$
0143	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{80}{451}$	0	$\frac{-576}{451}$	0	$\frac{7268}{451}$	$\frac{628}{451}$	$\frac{-216}{451}$	$\frac{-64224}{451}$	$\frac{-10864}{451}$	$\frac{-14752}{451}$
0125	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{52}{451}$	0	$\frac{216}{451}$	0	$\frac{-3935}{451}$	$\frac{11356}{451}$	$\frac{81}{451}$	$\frac{98496}{451}$	$\frac{4484}{451}$	$\frac{8048}{451}$
4121	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{832}{451}$	0	$\frac{3456}{451}$	0	$\frac{-12448}{451}$	$\frac{38416}{451}$	$\frac{1296}{451}$	$\frac{98496}{451}$	$\frac{14016}{451}$	$\frac{-28032}{451}$
3122	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{416}{451}$	0	$\frac{1728}{451}$	0	$\frac{-12538}{451}$	$\frac{38416}{451}$	$\frac{648}{451}$	$\frac{98496}{451}$	$\frac{14224}{451)}$	$\frac{-28032}{451}$
2321	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{416}{451}$	0	$\frac{1728}{451}$	0	$\frac{-6224}{451}$	$\frac{16768}{451}$	$\frac{648}{451}$	$\frac{98496}{451}$	$\frac{7008}{451}$	$\frac{832}{451}$
2141	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{320}{451}$	0	$\frac{-2304}{451}$	0	$\frac{24076}{1353}$	$\frac{36160}{1353}$	$\frac{-864}{451}$	$\frac{-64224}{451}$	$\frac{-14592}{451}$	$\frac{-29184}{451}$
2123	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{208}{451}$	0	$\frac{864}{451}$	0	$\frac{-9426}{451}$	$\frac{27592}{451}$	$\frac{324}{451}$	$\frac{98496}{451}$	$\frac{10720}{451}$	$\frac{-13600}{451}$
1322	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{208}{451}$	0	$\frac{864}{451}$	0	$\frac{-6269}{451}$	$\frac{16768}{451}$	$\frac{324}{451}$	$\frac{98496}{451}$	$\frac{7112}{451}$	$\frac{832}{451}$
1142	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{160}{451}$	0	$\frac{-1152}{451}$	0	$\frac{26470}{1353}$	$\frac{36160}{1353}$	$\frac{-432}{451}$	$\frac{-64224}{451}$	$\frac{-14512}{451}$	$\frac{-29184}{451}$
1124	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{104}{451}$	0	$\frac{432}{451}$	0	$\frac{-12583}{902}$	$\frac{16768}{451}$	$\frac{162}{451}$	$\frac{98496}{451}$	$\frac{7164}{451}$	$\frac{832}{451}$

$ijkl$	$\alpha_{4,15}$	$\alpha_{4,16}$	$\alpha_{4,17}$	$\alpha_{4,18}$	$\alpha_{4,19}$	$\alpha_{4,20}$	$\alpha_{4,21}$	$\alpha_{4,22}$	$\alpha_{4,23}$	$\alpha_{4,24}$	$\alpha_{4,25}$	$\alpha_{4,26}$	$\alpha_{4,27}$	$\alpha_{4,28}$
0521	$\frac{-432}{451}$	$\frac{-3456}{451}$	$\frac{208}{41}$	$\frac{-1712}{41}$	$\frac{810}{41}$	$\frac{-7272}{41}$	$\frac{-54}{41}$	$\frac{2856}{41}$	$\frac{-904}{41}$	$\frac{12800}{41}$	$\frac{-2808}{41}$	$\frac{23040}{41}$	$\frac{110}{41}$	$\frac{-2256}{41}$
0341	$\frac{576}{451}$	$\frac{-2304}{451}$	$\frac{-832}{123}$	$\frac{2480}{123}$	$\frac{-1080}{41}$	$\frac{4992}{41}$	$\frac{72}{41}$	$\frac{-1376}{41}$	$\frac{4928}{123}$	$\frac{-32128}{123}$	$\frac{3744}{41}$	$\frac{-16128}{41}$	$\frac{-256}{41}$	$\frac{1776}{41}$
0323	$\frac{-216}{451}$	$\frac{-3456}{451}$	$\frac{309}{41}$	$\frac{-1548}{41}$	$\frac{1266}{41}$	$\frac{-6780}{41}$	$\frac{-150}{41}$	$\frac{2364}{41}$	$\frac{1764}{41}$	$\frac{11488}{41}$	$\frac{-4356}{41}$	$\frac{23040}{41}$	$\frac{260}{41}$	$\frac{-2092}{41}$
0161	$\frac{-1344}{451}$	$\frac{-2688}{451}$	$\frac{4348}{369}$	$\frac{-10016}{369}$	$\frac{7232}{123}$	$\frac{-14344}{123}$	$\frac{-832}{123}$	$\frac{4040}{123}$	$\frac{-46304}{369}$	$\frac{92224}{369}$	$\frac{-8736}{41}$	$\frac{25792}{41}$	$\frac{2284}{123}$	$\frac{-5264}{123}$
0143	$\frac{288}{451}$	$\frac{-2304}{451}$	$\frac{-412}{41}$	$\frac{280}{41}$	$\frac{-1688}{41}$	$\frac{2040}{41}$	$\frac{200}{41}$	$\frac{-1048}{41}$	$\frac{3008}{41}$	$\frac{-3712}{41}$	$\frac{5808}{41}$	$\frac{-8256}{41}$	$\frac{-456}{41}$	$\frac{792}{41}$
0125	$\frac{-108}{451}$	$\frac{-3456}{451}$	$\frac{298}{41}$	$\frac{-1220}{41}$	$\frac{2127}{82}$	$\frac{-5304}{41}$	$\frac{-273}{82}$	$\frac{1872}{41}$	$\frac{-1210}{41}$	$\frac{8208}{41}$	$\frac{-3654}{41}$	$\frac{17136}{41}$	$\frac{383}{82}$	$\frac{-1600}{41}$
4121	$\frac{-1728}{451}$	$\frac{-3456}{451}$	$\frac{668}{41}$	$\frac{-2368}{41}$	$\frac{3240}{41}$	$\frac{-11208}{41}$	$\frac{-216}{41}$	$\frac{2856}{41}$	$\frac{-6240}{41}$	$\frac{20672}{41}$	$\frac{-11232}{41}$	$\frac{46656}{41}$	$\frac{932}{41}$	$\frac{-3568}{41}$

3122	$\frac{-864}{451}$	$\frac{-3456}{451}$	$\frac{744}{41}$	$\frac{-2368}{41}$	$\frac{3342}{41}$	$\frac{-11208}{41}$	$\frac{-354}{41}$	$\frac{2856}{41}$	$\frac{-6400}{41}$	$\frac{20672}{41}$	$\frac{-11520}{41}$	$\frac{46656}{41}$	$\frac{958}{41}$	$\frac{-3568}{41}$
2321	$\frac{-864}{451}$	$\frac{-3456}{451}$	$\frac{416}{41}$	$\frac{-1384}{41}$	$\frac{1620}{41}$	$\frac{-6288}{41}$	$\frac{-108}{41}$	$\frac{1872}{41}$	$\frac{-2464}{41}$	$\frac{10176}{41}$	$\frac{-5616}{41}$	$\frac{23040}{41}$	$\frac{384}{41}$	$\frac{-1928}{41}$
2141	$\frac{1152}{451}$	$\frac{-2304}{451}$	$\frac{-1664}{123}$	$\frac{800}{123}$	$\frac{-1832}{41}$	$\frac{-912}{41}$	$\frac{472}{41}$	$\frac{-720}{41}$	$\frac{9856}{123}$	$\frac{9856}{123}$	$\frac{7488}{41}$	$\frac{-384}{41}$	$\frac{-512}{41}$	$\frac{-192}{41}$
2123	$\frac{-432}{451}$	$\frac{-3456}{451}$	$\frac{618}{41}$	$\frac{-1876}{41}$	$\frac{2532}{41}$	$\frac{-8748}{41}$	$\frac{-300}{41}$	$\frac{2364}{41}$	$\frac{-4512}{41}$	$\frac{15424}{41}$	$\frac{-8712}{41}$	$\frac{34848}{41}$	$\frac{684}{41}$	$\frac{-2748}{41}$
1322	$\frac{-432}{451}$	$\frac{-3456}{451}$	$\frac{413}{41}$	$\frac{-1384}{41}$	$\frac{1671}{41}$	$\frac{-6288}{41}$	$\frac{-177}{41}$	$\frac{1872}{41}$	$\frac{-2544}{41}$	$\frac{10176}{41}$	$\frac{-5760}{41}$	$\frac{23040}{41}$	$\frac{397}{41}$	$\frac{-1928}{41}$
1142	$\frac{576}{451}$	$\frac{-2304}{451}$	$\frac{-1652}{123}$	$\frac{-800}{123}$	$\frac{-2064}{41}$	$\frac{-912}{41}$	$\frac{400}{41}$	$\frac{-720}{41}$	$\frac{11488}{123}$	$\frac{9856}{123}$	$\frac{7680}{41}$	$\frac{-384}{41}$	$\frac{-584}{41}$	$\frac{-192}{41}$
1124	$\frac{-216}{451}$	$\frac{-3456}{451}$	$\frac{905}{82}$	$\frac{-1384}{41}$	$\frac{3393}{82}$	$\frac{-6288}{41}$	$\frac{-423}{82}$	$\frac{1872}{41}$	$\frac{-2584}{41}$	$\frac{10176}{41)}$	$\frac{-5832}{41}$	$\frac{23040}{41}$	$\frac{807}{82}$	$\frac{-1928}{41}$

Table 5.4, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,3,4\}$. $ijkl$ represents quadratic form corresponding to tuple $(1^i,2^j,3^k,4^l)$, $i+j+k+l=8$, whose generating function belongs to the space $M_4(48,\chi_{12})$, with dimension 30.

$ijkl$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$
0071	$\frac{1}{46}$	$\frac{4}{23}$	$\frac{4}{23}$	$\frac{1}{46}$	$\frac{-1}{46}$	$\frac{-4}{23}$	$\frac{4}{23}$	$\frac{1}{46}$	$\frac{1}{23}$	$\frac{64}{23}$	$\frac{64}{23}$	$\frac{1}{23}$	$\frac{14}{23}$	0	$\frac{28}{23}$
0053	$\frac{-1}{92}$	$\frac{3}{23}$	$\frac{-1}{23}$	$\frac{3}{92}$	$\frac{1}{92}$	$\frac{6}{23}$	$\frac{2}{23}$	$\frac{3}{92}$	$\frac{1}{23}$	$\frac{-96}{23}$	$\frac{32}{23}$	$\frac{-3}{23}$	$\frac{-5}{46}$	$\frac{38}{23}$	$\frac{191}{23}$
0035	$\frac{-1}{184}$	$\frac{9}{92}$	$\frac{1}{92}$	$\frac{-9}{184}$	$\frac{1}{184}$	$\frac{9}{23}$	$\frac{-1}{23}$	$\frac{-9}{184}$	$\frac{1}{23}$	$\frac{-144}{23}$	$\frac{-16}{23}$	$\frac{9}{23}$	$\frac{-5}{92}$	$\frac{98}{23}$	$\frac{-677}{92}$
0017	$\frac{1}{368}$	$\frac{27}{368}$	$\frac{-1}{368}$	$\frac{-27}{368}$	$\frac{-1}{368}$	$\frac{-27}{46}$	$\frac{-1}{46}$	$\frac{-27}{368}$	$\frac{1}{23}$	$\frac{216}{23}$	$\frac{-8}{23}$	$\frac{-27}{23}$	0	$\frac{-273}{46}$	$\frac{819}{184}$
0611	0	$\frac{27}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{10}{23}$	$\frac{-112}{23}$	$\frac{94}{23}$
0431	0	$\frac{9}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{13}{23}$	$\frac{120}{23}$	$\frac{-149}{23}$
0413	0	$\frac{27}{92}$	$\frac{-1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{-13}{46}$	$\frac{-148}{23}$	$\frac{209}{46}$
0251	0	$\frac{6}{23}$	$\frac{-2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{3}{23}$	$\frac{19}{23}$	$\frac{28}{23}$	$\frac{94}{23}$
0233	0	$\frac{9}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{-5}{23}$	$\frac{106}{23}$	$\frac{-241}{46}$
0215	0	$\frac{27}{184}$	$\frac{-1}{184}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{-13}{92}$	$\frac{-143}{23}$	$\frac{104}{23}$
6011	$\frac{1}{46}$	$\frac{108}{23}$	$\frac{-4}{23}$	$\frac{-27}{46}$	$\frac{-1}{46}$	$\frac{108}{23}$	$\frac{4}{23}$	$\frac{-27}{46}$	$\frac{1}{23}$	$\frac{-1728}{23}$	$\frac{64}{23}$	$\frac{-27}{23}$	$\frac{70}{23}$	$\frac{56}{23}$	$\frac{28}{23}$
5012	0	$\frac{54}{23}$	$\frac{2}{23}$	0	0	$\frac{108}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	$\frac{-1728}{23}$	$\frac{64}{23}$	$\frac{-27}{23}$	$\frac{63}{23}$	$\frac{12}{23}$	$\frac{54}{23}$
4031	$\frac{1}{46}$	$\frac{36}{23}$	$\frac{4}{23}$	$\frac{9}{46}$	$\frac{-1}{46}$	$\frac{-36}{23}$	$\frac{4}{23}$	$\frac{9}{46}$	$\frac{1}{23}$	$\frac{576}{23}$	$\frac{64}{23}$	$\frac{9}{23}$	$\frac{70}{23}$	$\frac{208}{23}$	$\frac{-68}{23}$

4013	$\frac{-1}{92}$	$\frac{27}{23}$	$\frac{-1}{23}$	$\frac{27}{92}$	$\frac{1}{92}$	$\frac{54}{23}$	$\frac{2}{23}$	$\frac{27}{92}$	$\frac{1}{23}$	$\frac{-864}{23}$	$\frac{32}{23}$	$\frac{-27}{23}$	$\frac{119}{46}$	$\frac{-10}{23}$	$\frac{67}{23}$
3032	0	$\frac{18}{23}$	$\frac{2}{23}$	0	0	$\frac{-36}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	$\frac{576}{23}$	$\frac{64}{23}$	$\frac{9}{23}$	$\frac{49}{23}$	$\frac{148}{23}$	$\frac{-206}{23}$
3014	$\frac{-1}{92}$	$\frac{27}{46}$	$\frac{-1}{46}$	$\frac{27}{92}$	$\frac{1}{92}$	0	0	$\frac{27}{92}$	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{99}{46}$	$\frac{-36}{23}$	$\frac{153}{46}$
2051	$\frac{1}{46}$	$\frac{12}{23}$	$\frac{-4}{23}$	$\frac{-3}{46}$	$\frac{-1}{46}$	$\frac{12}{23}$	$\frac{4}{23}$	$\frac{-3}{46}$	$\frac{1}{23}$	$\frac{-192}{23}$	$\frac{64}{23}$	$\frac{-3}{23}$	$\frac{62}{23}$	$\frac{8}{23}$	$\frac{-100}{23}$
2033	$\frac{-1}{92}$	$\frac{9}{23}$	$\frac{1}{23}$	$\frac{-9}{92}$	$\frac{1}{92}$	$\frac{-18}{23}$	$\frac{2}{23}$	$\frac{-9}{92}$	$\frac{1}{23}$	$\frac{288}{23}$	$\frac{32}{23}$	$\frac{9}{23}$	$\frac{77}{46}$	$\frac{118}{23}$	$\frac{-275}{23}$
2015	$\frac{-1}{184}$	$\frac{27}{92}$	$\frac{-1}{92}$	$\frac{27}{184}$	$\frac{1}{184}$	$\frac{-27}{23}$	$\frac{-1}{23}$	$\frac{27}{184}$	$\frac{1}{23}$	$\frac{432}{23}$	$\frac{-16}{23}$	$\frac{-27}{23}$	$\frac{145}{92}$	$\frac{-64}{23}$	$\frac{337}{92}$
1052	0	$\frac{6}{23}$	$\frac{-2}{23}$	0	0	$\frac{12}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	$\frac{-192}{23}$	$\frac{64}{23}$	$\frac{-3}{23}$	$\frac{19}{23}$	$\frac{28}{23}$	$\frac{94}{23}$
1034	$\frac{-1}{92}$	$\frac{9}{46}$	$\frac{1}{46}$	$\frac{-9}{92}$	$\frac{1}{92}$	0	0	$\frac{-9}{92}$	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{41}{46}$	$\frac{104}{23}$	$\frac{-493}{46}$
1016	0	$\frac{27}{184}$	$\frac{-1}{184}$	0	0	$\frac{-27}{23}$	$\frac{-1}{23}$	0	$\frac{1}{23}$	$\frac{432}{23}$	$\frac{-16}{23}$	$\frac{-27}{23}$	$\frac{79}{92}$	$\frac{-97}{23}$	$\frac{185}{46}$
4211	0	$\frac{54}{23}$	$\frac{-2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{63}{23}$	$\frac{12}{23}$	$\frac{54}{23}$
3212	0	$\frac{27}{23}$	$\frac{-1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{43}{23}$	$\frac{-40}{23}$	$\frac{73}{23}$
2411	0	$\frac{27}{23}$	$\frac{-1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{43}{23}$	$\frac{-40}{23}$	$\frac{73}{23}$
2231	0	$\frac{18}{23}$	$\frac{2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{49}{23}$	$\frac{148}{23}$	$\frac{-206}{23}$
2213	0	$\frac{27}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{33}{23}$	$\frac{-66}{23}$	$\frac{165}{46}$
1412	0	$\frac{27}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{10}{23}$	$\frac{-112}{23}$	$\frac{94}{23}$
1232	0	$\frac{9}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{13}{23}$	$\frac{120}{23}$	$\frac{-149}{23}$
1214	0	$\frac{27}{92}$	$\frac{-1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0	0	$\frac{-27}{23}$	$\frac{33}{46}$	$\frac{-102}{23}$	$\frac{93}{23}$
ijkl	$\alpha_{4,16}$	$\alpha_{4,17}$	$\alpha_{4,18}$	$\alpha_{4,19}$	$\alpha_{4,20}$	$\alpha_{4,21}$	$\alpha_{4,22}$	$\alpha_{4,23}$	$\alpha_{4,24}$	$\alpha_{4,25}$	$\alpha_{4,26}$	$\alpha_{4,27}$	$\alpha_{4,28}$	$\alpha_{4,29}$	$\alpha_{4,30}$
0071	$\frac{28}{23}$	0	$\frac{364}{23}$	$\frac{1092}{23}$	0	$\frac{-728}{23}$	0	$\frac{5824}{23}$	$\frac{5824}{23}$	-5	$\frac{7}{3}$	4	$\frac{28}{3}$	$\frac{80}{3}$	$\frac{-112}{3}$
0053	$\frac{-43}{23}$	$\frac{-78}{23}$	$\frac{-342}{23}$	$\frac{186}{23}$	$\frac{1224}{23}$	$\frac{204}{23}$	$\frac{1296}{23}$	$\frac{-5760}{23}$	$\frac{-4368}{23}$	$\frac{15}{4}$	$\frac{1}{4}$	$\frac{-15}{4}$	$\frac{-17}{4}$	-20	12
0035	$\frac{595}{92}$	$\frac{-122}{23}$	$\frac{350}{23}$	$\frac{-776}{23}$	$\frac{-1136}{23}$	$\frac{380}{23}$	$\frac{-2464}{23}$	$\frac{3992}{23}$	$\frac{1720}{23}$	$\frac{-5}{8}$	$\frac{-21}{8}$	$\frac{5}{8}$	$\frac{9}{8}$	-2	18
0017	$\frac{-1365}{184}$	$\frac{273}{46}$	$\frac{-1365}{46}$	$\frac{1911}{46}$	$\frac{273}{23}$	$\frac{140}{23}$	$\frac{112}{23}$	$\frac{56}{23}$	$\frac{-280}{23}$	0	$\frac{29}{8}$	0	$\frac{15}{8}$	0	-30
0611	$\frac{-170}{23}$	12	-60	84	24	$\frac{1360}{23}$	$\frac{3008}{23}$	$\frac{-896}{23}$	$\frac{-320}{23}$	0	4	-1	3	-16	-48
0431	$\frac{147}{23}$	0	12	-36	-48	$\frac{312}{23}$	$\frac{-2752}{23}$	$\frac{5472}{23}$	$\frac{1312}{23}$	$\frac{-3}{2}$	$\frac{-7}{2}$	$\frac{1}{2}$	$\frac{-3}{2}$	8	24
0413	$\frac{-331}{46}$	12	-48	48	12	$\frac{716}{23}$	$\frac{1168}{23}$	$\frac{-160}{23}$	$\frac{-688}{23}$	$\frac{3}{4}$	$\frac{15}{4}$	$\frac{-3}{4}$	$\frac{9}{4}$	-12	-36

0251	$\frac{-70}{23}$	0	0	0	40	$\frac{-320}{23}$	$\frac{2112}{23}$	$\frac{-4256}{23}$	$\frac{-2912}{23}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{-7}{2}$	$\frac{-1}{2}$	-8	-8
0233	$\frac{331}{46}$	-2	14	-26	-44	$\frac{220}{23}$	$\frac{-2384}{23}$	$\frac{5104}{23}$	$\frac{2048}{23}$	-1	$\frac{-5}{2}$	1	$\frac{3}{2}$	4	12
0215	$\frac{-169}{23}$	9	-39	45	12	$\frac{348}{23}$	$\frac{432}{23}$	$\frac{24}{23}$	$\frac{-504}{23}$	$\frac{3}{8}$	$\frac{29}{8}$	$\frac{-3}{8}$	$\frac{15}{8}$	-6	-30
6011	$\frac{-140}{23}$	$\frac{316}{23}$	$\frac{-316}{23}$	$\frac{-1580}{23}$	$\frac{-632}{23}$	$\frac{632}{23}$	$\frac{-2528}{23}$	$\frac{2528}{23}$	$\frac{-5056}{23}$	9	3	-4	0	-64	0
5012	$\frac{-174}{23}$	$\frac{348}{23}$	$\frac{-924}{23}$	$\frac{-12}{23}$	$\frac{-120}{23}$	$\frac{632}{23}$	$\frac{-2528}{23}$	$\frac{2528}{23}$	$\frac{-5056}{23}$	9	3	-4	0	-64	0
4031	$\frac{140}{23}$	$\frac{120}{23}$	$\frac{348}{23}$	$\frac{804}{23}$	$\frac{-240}{23}$	$\frac{-696}{23}$	$\frac{-960}{23}$	$\frac{8448}{23}$	$\frac{5568}{23}$	-1	1	4	12	16	-48
4013	$\frac{-191}{23}$	$\frac{282}{23}$	$\frac{-1134}{23}$	$\frac{1146}{23}$	$\frac{288}{23}$	$\frac{444}{23}$	$\frac{-1968}{23}$	$\frac{1920}{23}$	$\frac{-3792}{23}$	$\frac{27}{4}$	$\frac{13}{4}$	$\frac{-11}{4}$	$\frac{3}{4}$	-44	-12
3032	$\frac{110}{23}$	$\frac{32}{23}$	$\frac{780}{23}$	$\frac{-76}{23}$	$\frac{-912}{23}$	$\frac{-696}{23}$	$\frac{-960}{23}$	$\frac{8448}{23}$	$\frac{5568}{23}$	-1	1	4	12	16	48
3014	$\frac{-387}{46}$	$\frac{200}{23}$	$\frac{-1040}{23}$	$\frac{1520}{23}$	$\frac{440}{23}$	$\frac{256}{23}$	$\frac{-1408}{23}$	$\frac{1312}{23}$	$\frac{-2528}{23}$	$\frac{9}{2}$	$\frac{7}{2}$	$\frac{-3}{2}$	$\frac{3}{2}$	-24	-24
2051	$\frac{-124}{23}$	$\frac{-60}{23}$	$\frac{-364}{23}$	$\frac{-972}{23}$	$\frac{120}{23}$	$\frac{728}{23}$	$\frac{480}{23}$	$\frac{-7264}{23}$	$\frac{-5824}{23}$	5	-1	-4	-8	-32	32
2033	$\frac{95}{23}$	$\frac{-74}{23}$	$\frac{790}{23}$	$\frac{-662}{23}$	$\frac{-1252}{23}$	$\frac{-284}{23}$	$\frac{-1488}{23}$	$\frac{6592}{23}$	$\frac{4176}{23}$	$\frac{-3}{4}$	$\frac{-1}{4}$	$\frac{11}{4}$	$\frac{33}{4}$	8	-24
2015	$\frac{-755}{92}$	$\frac{128}{23}$	$\frac{-818}{23}$	$\frac{1430}{23}$	$\frac{434}{23}$	$\frac{116}{23}$	$\frac{-944}{23}$	$\frac{824}{23}$	$\frac{-1528}{23}$	$\frac{21}{8}$	$\frac{29}{8}$	$\frac{-5}{8}$	$\frac{15}{8}$	-10	-30
1052	$\frac{-70}{23}$	$\frac{-108}{23}$	$\frac{-524}{23}$	$\frac{-300}{23}$	$\frac{824}{23}$	$\frac{728}{23}$	$\frac{480}{23}$	$\frac{-7264}{23}$	$\frac{-5824}{23}$	5	-1	-4	-8	-32	32
1034	$\frac{227}{46}$	$\frac{-136}{23}$	$\frac{584}{23}$	$\frac{-808}{23}$	$\frac{-1256}{23}$	$\frac{128}{23}$	$\frac{-2016}{23}$	$\frac{4736}{23}$	$\frac{2784}{23}$	$\frac{-1}{2}$	$\frac{-3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	0	0
1016	$\frac{-361}{46}$	$\frac{97}{23}$	$\frac{-643}{23}$	$\frac{1153}{23}$	$\frac{352}{23}$	$\frac{24}{23}$	$\frac{-576}{23}$	$\frac{456}{23}$	$\frac{-792}{23}$	$\frac{9}{8}$	$\frac{29}{8}$	$\frac{-1}{8}$	$\frac{15}{8}$	-2	-30
4211	$\frac{-174}{23}$	8	-32	32	8	$\frac{256}{23}$	$\frac{-1408}{23}$	$\frac{1312}{23}$	$\frac{-2528}{23}$	$\frac{9}{2}$	$\frac{7}{2}$	$\frac{-3}{2}$	$\frac{3}{2}$	-24	-24
3212	$\frac{-179}{23}$	8	-32	32	8	$\frac{256}{23}$	$\frac{-1408}{23}$	$\frac{1312}{23}$	$\frac{-2528}{23}$	$\frac{9}{2}$	$\frac{7}{2}$	$\frac{-3}{2}$	$\frac{3}{2}$	-24	24
2411	$\frac{-179}{23}$	4	-28	52	16	$\frac{72}{23}$	$\frac{-672}{23}$	$\frac{576}{23}$	$\frac{-1056}{23}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	-8	-24
2231	$\frac{110}{23}$	-4	16	-16	-40	$\frac{128}{23}$	$\frac{-2016}{23}$	$\frac{4736}{23}$	$\frac{2784}{23}$	$\frac{-1}{2}$	$\frac{-3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	0	0
2213	$\frac{-363}{46}$	6	-30	42	12	$\frac{164}{23}$	$\frac{-1040}{23}$	$\frac{944}{23}$	$\frac{-1792}{23}$	3	$\frac{7}{2}$	-1	$\frac{3}{2}$	-16	-24
1412	$\frac{-170}{23}$	8	-32	32	8	$\frac{72}{23}$	$\frac{-672}{23}$	$\frac{576}{23}$	$\frac{-1056}{23}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	-8	-24
1232	$\frac{147}{23}$	-4	16	-16	-40	$\frac{128}{23}$	$\frac{-2016}{23}$	$\frac{4736}{23}$	$\frac{2784}{23}$	$\frac{-1}{2}$	$\frac{-3}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	0	0
1214	$\frac{-177}{23}$	6	-30	42	12	$\frac{72}{23}$	$\frac{-672}{23}$	$\frac{576}{23}$	$\frac{-1056}{23}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	-8	-24

Table 5.5, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,3,4\}$. $ijkl$ represents quadratic form corresponding to tuple $(1^i,2^j,3^k,4^l)$, $i+j+k+l=8$, whose generating function belongs to $M_4(48,\chi_{24})$, with dimension 20.

$ijkl$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$
0512	0	$\frac{1}{261}$	$\frac{4}{261}$	0	$\frac{12}{29}$	0	0	$\frac{-3}{29}$	$\frac{3234}{261}$	$\frac{16192}{261}$
0332	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{1}{29}$	$\frac{-3788}{261}$	$\frac{-17648}{261}$
0314	0	$\frac{1}{261}$	$\frac{2}{261}$	0	$\frac{6}{29}$	0	0	$\frac{-3}{29}$	$\frac{3800}{261}$	$\frac{16192}{261}$
0152	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{16}{87}$	0	0	$\frac{-1}{87}$	$\frac{1960}{261}$	$\frac{28432}{261}$
0134	0	$\frac{1}{261}$	$\frac{-4}{261}$	0	$\frac{4}{29}$	0	0	$\frac{1}{29}$	$\frac{-2416}{261}$	$\frac{-17648}{261}$
0116	0	$\frac{1}{261}$	$\frac{1}{261}$	0	$\frac{3}{29}$	0	0	$\frac{-3}{29}$	$\frac{3988}{261}$	$\frac{13060}{261}$
5111	0	$\frac{1}{261}$	$\frac{32}{261}$	0	$\frac{96}{29}$	0	0	$\frac{-3}{29}$	$\frac{9644}{261}$	$\frac{45424}{261}$
4112	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{48}{29}$	0	0	$\frac{-3}{29}$	$\frac{7432}{261}$	$\frac{45424}{261}$
3311	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{48}{29}$	0	0	$\frac{-3}{29}$	$\frac{5344}{261}$	$\frac{24544}{261}$
3131	0	$\frac{1}{261}$	$\frac{-32}{261}$	0	$\frac{32}{29}$	0	0	$\frac{1}{29}$	$\frac{-1580}{261}$	$\frac{-38528}{261}$
3113	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{24}{29}$	0	0	$\frac{-3}{29}$	$\frac{5282}{261}$	$\frac{34984}{261}$
2312	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{24}{29}$	0	0	$\frac{-3}{29}$	$\frac{5804}{261}$	$\frac{24544}{261}$
2132	0	$\frac{1}{261}$	$\frac{-16}{261}$	0	$\frac{16}{29}$	0	0	$\frac{1}{29}$	$\frac{-1312}{261}$	$\frac{-38528}{261}$
2114	0	$\frac{1}{261}$	$\frac{4}{261}$	0	$\frac{12}{29}$	0	0	$\frac{-3}{29}$	$\frac{4468}{261}$	$\frac{24544}{261}$
1511	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{24}{29}$	0	0	$\frac{-3}{29}$	$\frac{3716}{261}$	$\frac{16192}{261}$
1331	0	$\frac{1}{261}$	$\frac{-16}{261}$	0	$\frac{16}{29}$	0	0	$\frac{1}{29}$	$\frac{-1312}{261}$	$\frac{-17648}{261}$
1313	0	$\frac{1}{261}$	$\frac{4}{261}$	0	$\frac{12}{29}$	0	0	$\frac{-3}{29}$	$\frac{4990}{261}$	$\frac{20368}{261}$
1151	0	$\frac{1}{261}$	$\frac{32}{261}$	0	$\frac{32}{87}$	0	0	$\frac{-1}{87}$	$\frac{2876}{261}$	$\frac{28432}{261}$
1133	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{1}{29}$	$\frac{-1178}{261}$	$\frac{-28088}{261}$
1115	0	$\frac{1}{261}$	$\frac{2}{261}$	0	$\frac{6}{29}$	0	0	$\frac{-3}{29}$	$\frac{4322}{261}$	$\frac{17236}{261}$
$ijkl$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$	$\alpha_{4,16}$	$\alpha_{4,17}$	$\alpha_{4,18}$	$\alpha_{4,19}$	$\alpha_{4,20}$
0512	$\frac{1258}{87}$	$\frac{5224}{87}$	$\frac{-9470}{261}$	$\frac{-38912}{261}$	$\frac{-8264}{87}$	$\frac{-45728}{87}$	$\frac{-4844}{87}$	$\frac{-1184}{87}$	$\frac{57688}{261}$	$\frac{223456}{261}$
0332	$\frac{-1100}{87}$	$\frac{-6752}{87}$	$\frac{7078}{261}$	$\frac{1208}{261}$	$\frac{10696}{87}$	$\frac{1808}{3}$	$\frac{2524}{87}$	$\frac{6880}{87}$	$\frac{-45608}{261}$	$\frac{-295952}{261}$
0314	$\frac{1151}{87}$	$\frac{5224}{87}$	$\frac{-8650}{261}$	$\frac{-38912}{261}$	$\frac{-10048}{87}$	$\frac{-45728}{87}$	$\frac{-3988}{87}$	$\frac{-1184}{87}$	$\frac{54944}{261}$	$\frac{223456}{261}$
0152	$\frac{1114}{87}$	$\frac{9304}{87}$	$\frac{-2750}{261}$	$\frac{-45296}{261}$	$\frac{-2008}{29}$	$\frac{-83296}{87}$	$\frac{-428}{87}$	$\frac{-34784}{87}$	$\frac{24280}{261}$	$\frac{470176}{261}$
0134	$\frac{-811}{87}$	$\frac{-6752}{87}$	$\frac{5366}{261}$	$\frac{1208}{9}$	$\frac{7088}{87}$	$\frac{1808}{3}$	$\frac{740}{87}$	$\frac{6880}{87}$	$\frac{-28024}{261}$	$\frac{-295952}{261}$
0116	$\frac{1228}{87}$	$\frac{5224}{87}$	$\frac{-8240}{261}$	$\frac{-34736}{261}$	$\frac{-11288}{87}$	$\frac{-39464}{87}$	$\frac{-3560}{87}$	$\frac{-2992}{87}$	$\frac{56704}{261}$	$\frac{200488}{261}$
5111	$\frac{2234}{87}$	$\frac{13576}{87}$	$\frac{-12598}{261}$	$\frac{-80672}{261}$	$\frac{-23656}{87}$	$\frac{-123680}{87}$	$\frac{-8476}{87}$	$\frac{-42944}{87}$	$\frac{91928}{261}$	$\frac{624352}{261}$
4112	$\frac{1378}{87}$	$\frac{13576}{87}$	$\frac{-10214}{261}$	$\frac{-80672}{261}$	$\frac{-17048}{87}$	$\frac{-123680}{87}$	$\frac{-5804}{87}$	$\frac{-42944}{87}$	$\frac{57448}{261}$	$\frac{624352}{261}$
3311	$\frac{1378}{87}$	$\frac{9400}{87}$	$\frac{-8126}{261}$	$\frac{-51440}{261}$	$\frac{-13568}{87}$	$\frac{-72176}{87}$	$\frac{-3716}{87}$	$\frac{-13712}{87}$	$\frac{51184}{261}$	$\frac{377968}{261}$
3131	$\frac{-746}{87}$	$\frac{-10928}{87}$	$\frac{4822}{261}$	$\frac{2216}{9}$	$\frac{5896}{87}$	$\frac{3584}{3}$	$\frac{700}{87}$	$\frac{36112}{87}$	$\frac{-30008}{261}$	$\frac{-542336}{261}$
3113	$\frac{950}{87}$	$\frac{11488}{87}$	$\frac{-7978}{261}$	$\frac{-66056}{261}$	$\frac{-12004}{87}$	$\frac{-97928}{87}$	$\frac{-3424}{87}$	$\frac{-28328}{87}$	$\frac{37076}{261}$	$\frac{501160}{261}$
2312	$\frac{1472}{87}$	$\frac{9400}{87}$	$\frac{-9022}{261}$	$\frac{-51440}{261}$	$\frac{-15136}{87}$	$\frac{-72176}{87}$	$\frac{-4468}{87}$	$\frac{-13712}{87}$	$\frac{61088}{261}$	$\frac{377968}{261}$
2132	$\frac{-634}{87}$	$\frac{-10928}{87}$	$\frac{4238}{261}$	$\frac{2216}{9}$	$\frac{4688}{87}$	$\frac{3584}{3}$	$\frac{-172}{87}$	$\frac{36112}{87}$	$\frac{-20224}{261}$	$\frac{-542336}{261}$
2114	$\frac{997}{87}$	$\frac{9400}{87}$	$\frac{-7382}{261}$	$\frac{-51440}{261}$	$\frac{-11048}{87}$	$\frac{-72176}{87}$	$\frac{-2756}{87}$	$\frac{-13712}{87}$	$\frac{38896}{261}$	$\frac{377968}{261}$

1511	$\frac{950}{87}$	$\frac{5224}{87}$	$\frac{-6934}{261}$	$\frac{-38912}{261}$	$\frac{-10264}{87}$	$\frac{-45728}{87}$	$\frac{-2380}{87}$	$\frac{-1184}{87}$	$\frac{42296}{261}$	$\frac{223456}{261}$
1331	$\frac{-634}{87}$	$\frac{-6752}{87}$	$\frac{4238}{261}$	$\frac{1208}{9}$	$\frac{4688}{87}$	$\frac{1808}{3}$	$\frac{-172}{87}$	$\frac{6880}{87}$	$\frac{-20224}{261}$	$\frac{-295952}{261}$
1313	$\frac{1258}{87}$	$\frac{7312}{87}$	$\frac{-8426}{261}$	$\frac{-45176}{261}$	$\frac{-13484}{87}$	$\frac{-58952}{87}$	$\frac{-3800}{87}$	$\frac{-7448}{87}$	$\frac{56644}{261}$	$\frac{300712}{261}$
1151	$\frac{1010}{87}$	$\frac{9304}{87}$	$\frac{-3934}{261}$	$\frac{-45296}{261}$	$\frac{-2392}{29}$	$\frac{-83296}{87}$	$\frac{-1900}{87}$	$\frac{-34784}{87}$	$\frac{25592}{261}$	$\frac{470176}{261}$
1133	$\frac{-578}{87}$	$\frac{-8840}{87}$	$\frac{3946}{261}$	$\frac{1712}{9}$	$\frac{4084}{87}$	$\frac{2696}{3}$	$\frac{-608}{87}$	$\frac{21496}{87}$	$\frac{-15332}{261}$	$\frac{-419144}{261}$
1115	$\frac{1151}{87}$	$\frac{7312}{87}$	$\frac{-7606}{261}$	$\frac{-41000}{261}$	$\frac{-11788}{87}$	$\frac{-52688}{87}$	$\frac{-2944}{87}$	$\frac{-3272}{87}$	$\frac{48680}{261}$	$\frac{277744}{261}$

$ijkl$	$\alpha_{4,21}$	$\alpha_{4,22}$	$\alpha_{4,23}$	$\alpha_{4,24}$	$\alpha_{4,25}$	$\alpha_{4,26}$	$\alpha_{4,27}$	$\alpha_{4,28}$
0512	$\frac{26914}{261}$	$\frac{52480}{261}$	$\frac{14720}{261}$	$\frac{-52480}{261}$	$\frac{-3536}{261}$	$\frac{-14600}{261}$	$\frac{-3562}{261}$	$\frac{-14656}{261}$
0332	$\frac{-21050}{261}$	$\frac{-45080}{261}$	$\frac{-20800}{261}$	$\frac{129536}{261}$	$\frac{3724}{261}$	$\frac{18160}{261}$	$\frac{3704}{261}$	$\frac{18176}{261}$
0314	$\frac{26246}{261}$	$\frac{52480}{261}$	$\frac{15712}{261}$	$\frac{-52480}{261}$	$\frac{-3856}{261}$	$\frac{-14600}{261}$	$\frac{-3869}{261}$	$\frac{-14656}{261}$
0152	$\frac{-3038}{261}$	$\frac{171184}{261}$	$\frac{25088}{261}$	$\frac{-265216}{261}$	$\frac{-2024}{261}$	$\frac{-28952}{261}$	$\frac{-2554}{261}$	$\frac{-28960}{261}$
0134	$\frac{-14962}{261}$	$\frac{-45080}{261}$	$\frac{-27104}{261}$	$\frac{129536}{261}$	$\frac{2384}{261}$	$\frac{18160}{261}$	$\frac{2113}{261}$	$\frac{18176}{261}$
0116	$\frac{25912}{261}$	$\frac{35776}{261}$	$\frac{16208}{261}$	$\frac{-35776}{261}$	$\frac{-4984}{261}$	$\frac{-4016}{261}$	$\frac{-12512}{261}$	$\frac{-3892}{261}$
5111	$\frac{19562}{261}$	$\frac{252928}{261}$	$\frac{17536}{261}$	$\frac{-252928}{261}$	$\frac{-7930}{261}$	$\frac{-39656}{261}$	$\frac{-8138}{261}$	$\frac{-43888}{261}$
4112	$\frac{14218}{261}$	$\frac{252928}{261}$	$\frac{25472}{261}$	$\frac{-252928}{261}$	$\frac{-5792}{261}$	$\frac{-39656}{261}$	$\frac{-6418}{261}$	$\frac{-43888}{261}$
3311	$\frac{14218}{261}$	$\frac{119296}{261}$	$\frac{25472}{261}$	$\frac{-119296}{261}$	$\frac{-4226}{261}$	$\frac{-22952}{261}$	$\frac{-4330}{261}$	$\frac{-25096}{261}$
3131	$\frac{-24170}{261}$	$\frac{-178712}{261}$	$\frac{-16384}{261}$	$\frac{263168}{261}$	$\frac{2890}{261}$	$\frac{39040}{261}$	$\frac{2810}{261}$	$\frac{36968}{261}$
3113	$\frac{11546}{261}$	$\frac{186112}{261}$	$\frac{29440}{261}$	$\frac{-186112}{261}$	$\frac{-3940}{261}$	$\frac{-31304}{261}$	$\frac{-4514}{261}$	$\frac{-34492}{261}$
2312	$\frac{19898}{261}$	$\frac{119296}{261}$	$\frac{21088}{261}$	$\frac{-119296}{261}$	$\frac{-4984}{261}$	$\frac{-22952}{261}$	$\frac{-5036}{261}$	$\frac{-25096}{261}$
2132	$\frac{-16522}{261}$	$\frac{-178712}{261}$	$\frac{-24896}{261}$	$\frac{263168}{261}$	$\frac{2228}{261}$	$\frac{39040}{261}$	$\frac{1666}{261}$	$\frac{36968}{261}$
2114	$\frac{14386}{261}$	$\frac{119296}{261}$	$\frac{27248}{261}$	$\frac{-119296}{261}$	$\frac{-3536}{261}$	$\frac{-22952}{261}$	$\frac{-3823}{261}$	$\frac{-25096}{261}$
1511	$\frac{19898}{261}$	$\frac{52480}{261}$	$\frac{21088}{261}$	$\frac{-52480}{261}$	$\frac{-3418}{261}$	$\frac{-14600}{261}$	$\frac{-3470}{261}$	$\frac{-14656}{261}$
1331	$\frac{-16522}{261}$	$\frac{-45080}{261}$	$\frac{-24896}{261}$	$\frac{129536}{261}$	$\frac{1706}{261}$	$\frac{18160}{261}$	$\frac{1666}{261}$	$\frac{18176}{261}$
1313	$\frac{22738}{261}$	$\frac{85888}{261}$	$\frac{18896}{261}$	$\frac{-85888}{261}$	$\frac{-4580}{261}$	$\frac{-18776}{261}$	$\frac{-4606}{261}$	$\frac{-19876}{261}$
1151	$\frac{3842}{261}$	$\frac{171184}{261}$	$\frac{16768}{261}$	$\frac{-265216}{261}$	$\frac{-2482}{261}$	$\frac{-28952}{261}$	$\frac{-2498}{261}$	$\frac{-28960}{261}$
1133	$\frac{-12698}{261}$	$\frac{-111896}{261}$	$\frac{-29152}{261}$	$\frac{196352}{261}$	$\frac{1636}{261}$	$\frac{28600}{261}$	$\frac{1094}{261}$	$\frac{27572}{261}$
1115	$\frac{19982}{261}$	$\frac{69184}{261}$	$\frac{21976}{261}$	$\frac{-69184}{261}$	$\frac{-3856}{261}$	$\frac{-16688}{261}$	$\frac{-3869}{261}$	$\frac{-17788}{261}$

Table 5.6, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,3,4,6\}$. $ijklm$ represents quadratic form corresponding to tuple $(1^i,2^j,3^k,4^l,6^m)$, $i+j+k+l+m=8$, whose generating functions belong to $M_4(48,\chi_0)$, with dimension 30.

$ijklm$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$
00026	0	$\frac{1}{600}$	0	$\frac{-1}{600}$	$\frac{-7}{200}$	$\frac{1}{300}$	$\frac{7}{200}$	$\frac{-4}{75}$	$\frac{-7}{100}$	$\frac{28}{25}$
00062	0	$\frac{7}{2400}$	0	$\frac{-7}{2400}$	$\frac{-9}{800}$	$\frac{7}{300}$	$\frac{9}{800}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
00044	0	$\frac{1}{600}$	0	$\frac{-1}{600}$	$\frac{3}{200}$	$\frac{-1}{150}$	$\frac{-3}{200}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
00422	$\frac{1}{2400}$	$\frac{-1}{2400}$	$\frac{-7}{800}$	0	$\frac{7}{800}$	$\frac{1}{300}$	0	$\frac{-4}{75}$	$\frac{-7}{100}$	$\frac{28}{25}$
00242	$\frac{1}{4800}$	$\frac{-1}{4800}$	$\frac{3}{1600}$	0	$\frac{-3}{1600}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
00224	$\frac{1}{4800}$	$\frac{-1}{4800}$	$\frac{-7}{1600}$	0	$\frac{7}{1600}$	$\frac{1}{300}$	0	$\frac{-4}{75}$	$\frac{-7}{100}$	$\frac{28}{25}$
02222	$\frac{1}{2400}$	$\frac{-1}{2400}$	$\frac{3}{800}$	0	$\frac{-3}{800}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
04022	0	$\frac{7}{600}$	0	$\frac{-7}{600}$	$\frac{-9}{200}$	$\frac{7}{300}$	$\frac{9}{200}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
02042	0	$\frac{7}{1200}$	0	$\frac{-7}{1200}$	$\frac{-9}{400}$	$\frac{7}{300}$	$\frac{9}{400}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
02024	0	$\frac{1}{300}$	0	$\frac{-1}{300}$	$\frac{3}{100}$	$\frac{-1}{150}$	$\frac{-3}{100}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
01511	$\frac{1}{2400}$	$\frac{-1}{2400}$	$\frac{13}{800}$	0	$\frac{-13}{800}$	$\frac{-1}{600}$	0	$\frac{2}{75}$	$\frac{-13}{200}$	$\frac{26}{25}$
03311	$\frac{1}{960}$	$\frac{-1}{960}$	$\frac{-3}{320}$	0	$\frac{3}{320}$	$\frac{1}{120}$	0	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$
05111	$\frac{13}{9600}$	$\frac{-13}{9600}$	$\frac{9}{3200}$	0	$\frac{-9}{3200}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
01313	$\frac{1}{4800}$	$\frac{-1}{4800}$	$\frac{13}{1600}$	0	$\frac{-13}{1600}$	$\frac{-1}{600}$	0	$\frac{2}{75}$	$\frac{-13}{200}$	$\frac{26}{25}$
03113	$\frac{1}{1920}$	$\frac{-1}{1920}$	$\frac{-3}{640}$	0	$\frac{3}{640}$	$\frac{1}{120}$	0	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$
01133	$\frac{1}{3840}$	$\frac{-1}{3840}$	$\frac{-3}{1280}$	0	$\frac{3}{1280}$	$\frac{1}{120}$	0	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$
01115	$\frac{1}{9600}$	$\frac{-1}{9600}$	$\frac{13}{3200}$	0	$\frac{-13}{3200}$	$\frac{-1}{600}$	0	$\frac{2}{75}$	$\frac{-13}{200}$	$\frac{26}{25}$
01331	$\frac{1}{1920}$	$\frac{-1}{1920}$	$\frac{-3}{640}$	0	$\frac{3}{640}$	$\frac{1}{120}$	0	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$
03131	$\frac{13}{19200}$	$\frac{-13}{19200}$	$\frac{9}{6400}$	0	$\frac{-9}{6400}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
01151	$\frac{13}{38400}$	$\frac{-13}{38400}$	$\frac{9}{12800}$	0	$\frac{-9}{12800}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
11123	$\frac{1}{1920}$	$\frac{-1}{1920}$	$\frac{-3}{640}$	0	$\frac{3}{640}$	$\frac{1}{120}$	0	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$
11141	$\frac{13}{19200}$	$\frac{-13}{19200}$	$\frac{9}{6400}$	0	$\frac{-9}{6400}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
11321	$\frac{1}{960}$	$\frac{-1}{960}$	$\frac{-3}{320}$	0	$\frac{3}{320}$	$\frac{1}{120}$	0	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$
13121	$\frac{13}{9600}$	$\frac{-13}{9600}$	$\frac{9}{3200}$	0	$\frac{-9}{3200}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
10412	$\frac{1}{1200}$	$\frac{-1}{1200}$	$\frac{-7}{400}$	0	$\frac{7}{400}$	$\frac{1}{300}$	0	$\frac{-4}{75}$	$\frac{-7}{100}$	$\frac{28}{25}$
12212	$\frac{1}{1200}$	$\frac{-1}{1200}$	$\frac{3}{400}$	0	$\frac{-3}{400}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
14012	$\frac{7}{4800}$	$\frac{-7}{4800}$	$\frac{-9}{1600}$	0	$\frac{9}{1600}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
10034	$\frac{1}{4800}$	$\frac{-1}{4800}$	$\frac{3}{1600}$	0	$\frac{-3}{1600}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
10232	$\frac{1}{2400}$	$\frac{-1}{2400}$	$\frac{3}{800}$	0	$\frac{-3}{800}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
12032	$\frac{7}{9600}$	$\frac{-7}{9600}$	$\frac{-9}{3200}$	0	$\frac{9}{3200}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
10052	$\frac{7}{19200}$	$\frac{-7}{19200}$	$\frac{-9}{6400}$	0	$\frac{9}{6400}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
10214	$\frac{1}{2400}$	$\frac{-1}{2400}$	$\frac{-7}{800}$	0	$\frac{7}{800}$	$\frac{1}{300}$	0	$\frac{-4}{75}$	$\frac{-7}{100}$	$\frac{28}{25}$
12014	$\frac{1}{2400}$	$\frac{-1}{2400}$	$\frac{3}{800}$	0	$\frac{-3}{800}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
10016	$\frac{1}{4800}$	$\frac{-1}{4800}$	$\frac{-7}{1600}$	0	$\frac{7}{1600}$	$\frac{1}{300}$	0	$\frac{-4}{75}$	$\frac{-7}{100}$	$\frac{28}{25}$
22022	$\frac{7}{4800}$	$\frac{-7}{4800}$	$\frac{-9}{1600}$	0	$\frac{9}{1600}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
21311	$\frac{1}{480}$	$\frac{-1}{480}$	$\frac{-3}{160}$	0	$\frac{3}{160}$	$\frac{1}{120}$	0	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$

23111	$\frac{13}{4800}$	$\frac{-13}{4800}$	$\frac{9}{1600}$	0	$\frac{-9}{1600}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
21113	$\frac{1}{960}$	$\frac{-1}{960}$	$\frac{-3}{320}$	0	$\frac{3}{320}$	$\frac{1}{120}$	0	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$
21131	$\frac{13}{9600}$	$\frac{-13}{9600}$	$\frac{9}{3200}$	0	$\frac{-9}{3200}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
20222	$\frac{1}{1200}$	$\frac{-1}{1200}$	$\frac{3}{400}$	0	$\frac{-3}{400}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
20042	$\frac{7}{9600}$	$\frac{-7}{9600}$	$\frac{-9}{3200}$	0	$\frac{9}{3200}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
20024	$\frac{1}{2400}$	$\frac{-1}{2400}$	$\frac{3}{800}$	0	$\frac{-3}{800}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
31121	$\frac{13}{4800}$	$\frac{-13}{4800}$	$\frac{9}{1600}$	0	$\frac{-9}{1600}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
30212	$\frac{1}{600}$	$\frac{-1}{600}$	$\frac{3}{200}$	0	$\frac{-3}{200}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
32012	$\frac{7}{2400}$	$\frac{-7}{2400}$	$\frac{-9}{800}$	0	$\frac{9}{800}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
30032	$\frac{7}{4800}$	$\frac{-7}{4800}$	$\frac{-9}{1600}$	0	$\frac{9}{1600}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
30014	$\frac{1}{1200}$	$\frac{-1}{1200}$	$\frac{3}{400}$	0	$\frac{-3}{400}$	$\frac{-1}{150}$	0	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$
41111	$\frac{13}{2400}$	$\frac{-13}{2400}$	$\frac{9}{800}$	0	$\frac{-9}{800}$	$\frac{-13}{600}$	0	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$
40022	$\frac{7}{2400}$	$\frac{-7}{2400}$	$\frac{-9}{800}$	0	$\frac{9}{800}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$
50012	$\frac{7}{1200}$	$\frac{-7}{1200}$	$\frac{-9}{400}$	0	$\frac{9}{400}$	$\frac{7}{300}$	0	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$

$ijklm$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$	$\alpha_{4,16}$	$\alpha_{4,17}$	$\alpha_{4,18}$	$\alpha_{4,19}$	$\alpha_{4,20}$
00026	0	0	0	$\frac{4}{15}$	$\frac{-16}{3}$	$\frac{-512}{15}$	0	-3	0	-9
00062	0	0	0	$\frac{-16}{5}$	-8	$\frac{32}{5}$	0	$\frac{-3}{2}$	0	$\frac{-81}{2}$
00044	0	0	0	$\frac{8}{5}$	16	$\frac{128}{5}$	0	2	0	18
00422	0	0	$\frac{-4}{15}$	$\frac{-4}{3}$	-16	$\frac{-384}{5}$	$\frac{-1}{2}$	-5	$\frac{-3}{2}$	-15
00242	0	0	$\frac{1}{5}$	2	16	$\frac{128}{5}$	$\frac{3}{4}$	2	$\frac{27}{4}$	18
00224	0	0	$\frac{-2}{15}$	0	$\frac{-16}{3}$	$\frac{-512}{15}$	$\frac{-1}{4}$	-3	$\frac{-3}{4}$	-9
02222	0	0	$\frac{2}{5}$	4	16	$\frac{128}{5}$	$\frac{1}{2}$	2	$\frac{9}{2}$	8
04022	0	0	0	$\frac{-4}{5}$	0	$\frac{-128}{5}$	0	-1	0	-27
02042	0	0	0	$\frac{-12}{5}$	-8	$\frac{-128}{5}$	0	-1	0	-27
02024	0	0	0	$\frac{16}{5}$	16	$\frac{128}{5}$	0	2	0	18
01511	0	0	$\frac{2}{5}$	$\frac{8}{3}$	16	$\frac{896}{15}$	$\frac{3}{4}$	$\frac{9}{2}$	$\frac{21}{4}$	$\frac{63}{2}$
03311	0	0	$\frac{-1}{2}$	-3	-12	-32	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{-9}{2}$	$\frac{-63}{2}$
05111	0	0	$\frac{4}{5}$	4	16	$\frac{256}{5}$	$\frac{7}{8}$	$\frac{7}{2}$	$\frac{81}{8}$	$\frac{81}{2}$
01313	0	0	$\frac{11}{30}$	$\frac{5}{3}$	$\frac{28}{3}$	$\frac{416}{15}$	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{9}{2}$	$\frac{27}{2}$
03113	0	0	0	0	0	0	$\frac{-1}{8}$	$\frac{1}{2}$	$\frac{9}{8}$	$\frac{-9}{2}$
01133	0	0	$\frac{-1}{4}$	$\frac{-3}{2}$	-6	-16	$\frac{-7}{16}$	-1	$\frac{-9}{16}$	-9
01115	0	0	$\frac{4}{15}$	$\frac{4}{3}$	$\frac{16}{3}$	$\frac{256}{15}$	$\frac{3}{8}$	$\frac{3}{2}$	$\frac{21}{8}$	$\frac{21}{2}$
01331	0	0	$\frac{-3}{4}$	$\frac{-9}{2}$	-18	-48	$\frac{-7}{8}$	$\frac{-5}{2}$	$\frac{-45}{8}$	$\frac{-63}{2}$
03131	0	0	$\frac{13}{20}$	$\frac{7}{2}$	10	$\frac{176}{5}$	$\frac{9}{16}$	3	$\frac{135}{16}$	27
01151	0	0	$\frac{7}{10}$	3	8	$\frac{96}{5}$	$\frac{21}{32}$	$\frac{7}{4}$	$\frac{243}{32}$	$\frac{81}{4}$
11123	0	0	0	0	-4	-32	$\frac{-1}{8}$	$\frac{-5}{2}$	$\frac{9}{8}$	$\frac{-27}{2}$
11141	0	0	$\frac{9}{10}$	3	12	$\frac{96}{5}$	$\frac{17}{16}$	$\frac{5}{2}$	$\frac{135}{16}$	$\frac{27}{2}$
11321	0	0	$\frac{-1}{2}$	-3	-16	-64	$\frac{-1}{2}$	$\frac{-9}{2}$	$\frac{-9}{2}$	$\frac{-63}{2}$

13121	0	0	$\frac{4}{5}$	4	12	$\frac{96}{5}$	$\frac{7}{8}$	$\frac{5}{2}$	$\frac{81}{8}$	$\frac{27}{2}$
10412	0	0	$\frac{2}{15}$	0	$\frac{-16}{3}$	$\frac{-384}{5}$	$\frac{1}{2}$	-5	$\frac{3}{2}$	-15
12212	0	0	$\frac{4}{5}$	4	16	$\frac{128}{5}$	1	2	9	18
14012	0	0	$\frac{2}{5}$	0	0	$\frac{-128}{5}$	$\frac{1}{4}$	-1	$\frac{27}{4}$	-27
10034	0	0	$\frac{1}{5}$	2	16	$\frac{128}{5}$	$\frac{1}{4}$	2	$\frac{9}{4}$	18
10232	0	0	$\frac{2}{5}$	2	20	$\frac{288}{5}$	1	3	9	9
12032	0	0	$\frac{1}{5}$	-2	-8	$\frac{-128}{5}$	$\frac{1}{8}$	-1	$\frac{27}{8}$	-27
10052	0	0	$\frac{3}{5}$	-2	-8	$\frac{32}{5}$	$\frac{-1}{16}$	$\frac{-3}{2}$	$\frac{-27}{16}$	$\frac{-81}{2}$
10214	0	0	$\frac{2}{5}$	$\frac{4}{3}$	0	$\frac{-512}{15}$	$\frac{1}{2}$	-3	$\frac{3}{2}$	-9
12014	0	0	$\frac{2}{5}$	4	16	$\frac{128}{5}$	$\frac{1}{2}$	2	$\frac{9}{2}$	18
10016	0	0	$\frac{8}{15}$	$\frac{4}{3}$	$\frac{-16}{3}$	$\frac{-512}{15}$	$\frac{3}{4}$	-3	$\frac{9}{4}$	-9
22022	0	0	$\frac{2}{5}$	0	0	$\frac{-128}{5}$	$\frac{1}{4}$	-1	$\frac{27}{4}$	-27
21311	0	0	0	0	-8	-64	$\frac{1}{4}$	$\frac{-9}{2}$	$\frac{-9}{4}$	$\frac{-63}{2}$
23111	0	0	$\frac{11}{10}$	5	20	$\frac{96}{5}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{27}{2}$	$\frac{27}{2}$
21113	0	0	$\frac{1}{2}$	3	4	-32	$\frac{1}{2}$	$\frac{-5}{2}$	$\frac{9}{2}$	$\frac{-27}{2}$
21131	0	0	$\frac{21}{20}$	$\frac{7}{2}$	18	$\frac{176}{5}$	$\frac{11}{8}$	$\frac{9}{2}$	$\frac{81}{8}$	$\frac{27}{2}$
20222	0	0	$\frac{4}{5}$	4	24	$\frac{448}{5}$	1	4	9	0
20042	0	0	$\frac{6}{5}$	0	-8	$\frac{-128}{5}$	$\frac{-1}{8}$	-1	$\frac{-27}{8}$	-27
20024	0	0	$\frac{2}{5}$	4	16	$\frac{128}{5}$	$\frac{1}{2}$	2	$\frac{9}{2}$	18
31121	0	0	$\frac{11}{10}$	5	24	$\frac{256}{5}$	$\frac{3}{2}$	$\frac{13}{2}$	$\frac{27}{2}$	$\frac{27}{2}$
30212	0	0	$\frac{8}{5}$	8	24	$\frac{448}{5}$	1	4	9	0
32012	0	0	$\frac{4}{5}$	4	16	$\frac{-128}{5}$	$\frac{1}{2}$	-1	$\frac{27}{2}$	-27
30032	0	0	$\frac{7}{5}$	2	-8	$\frac{-448}{5}$	0	0	0	0
30014	0	0	$\frac{4}{5}$	8	16	$\frac{128}{5}$	1	2	9	18
41111	0	0	$\frac{6}{5}$	8	32	$\frac{256}{5}$	$\frac{7}{4}$	$\frac{13}{2}$	$\frac{81}{4}$	$\frac{27}{2}$
40022	0	0	$\frac{4}{5}$	4	0	$\frac{-768}{5}$	$\frac{1}{2}$	1	$\frac{27}{2}$	27
50012	0	0	$\frac{-2}{5}$	8	32	$\frac{-768}{5}$	$\frac{3}{2}$	1	$\frac{81}{2}$	27

$ijklm$	$\alpha_{4,21}$	$\alpha_{4,22}$	$\alpha_{4,23}$	$\alpha_{4,24}$	$\alpha_{4,25}$	$\alpha_{4,26}$	$\alpha_{4,27}$	$\alpha_{4,28}$	$\alpha_{4,29}$	$\alpha_{4,30}$
00026	0	$\frac{4}{3}$	$\frac{20}{3}$	0	0	0	1	0	0	0
00062	0	$\frac{5}{2}$	8	0	0	0	$\frac{3}{2}$	0	0	0
00044	0	-2	-8	0	0	0	-2	0	0	0
00422	$\frac{1}{6}$	2	12	-1	3	$\frac{1}{2}$	3	$\frac{4}{3}$	$\frac{2}{3}$	-1
00242	$\frac{-1}{4}$	-2	-8	$\frac{1}{2}$	$\frac{9}{2}$	$\frac{-3}{4}$	-2	-1	0	$\frac{1}{2}$
00224	$\frac{1}{12}$	$\frac{4}{3}$	$\frac{20}{3}$	$\frac{-1}{2}$	$\frac{3}{2}$	$\frac{1}{4}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{-1}{2}$
02222	$\frac{-1}{2}$	0	-8	$\frac{1}{2}$	$\frac{9}{2}$	$\frac{-1}{2}$	-2	-1	0	$\frac{1}{2}$
04022	0	4	4	0	0	0	3	0	0	0
02042	0	3	4	0	0	0	3	0	0	0
02024	0	0	-8	0	0	0	-2	0	0	0
01511	0	$\frac{-5}{3}$	$\frac{-50}{3}$	1	3	$\frac{-1}{4}$	$\frac{-7}{2}$	$\frac{-5}{3}$	$\frac{-5}{6}$	$\frac{1}{2}$

03311	$\frac{3}{4}$	3	6	$\frac{-3}{4}$	$\frac{-27}{4}$	1	$\frac{7}{2}$	1	$\frac{-1}{2}$	$\frac{-3}{4}$
05111	$\frac{-5}{8}$	0	10	0	0	$\frac{-3}{8}$	$\frac{3}{2}$	-2	1	0
01313	$\frac{1}{12}$	$\frac{-1}{3}$	$\frac{-26}{3}$	$\frac{1}{4}$	$\frac{9}{4}$	0	$\frac{-3}{2}$	-1	$\frac{-1}{2}$	$\frac{1}{4}$
03113	$\frac{3}{8}$	2	6	$\frac{-1}{2}$	$\frac{-9}{2}$	$\frac{5}{8}$	$\frac{5}{2}$	0	0	$\frac{-1}{2}$
01133	$\frac{3}{16}$	$\frac{3}{2}$	6	$\frac{-3}{8}$	$\frac{-27}{8}$	$\frac{7}{16}$	2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{-3}{8}$
01115	$\frac{5}{24}$	0	$\frac{-10}{3}$	0	0	$\frac{1}{8}$	$\frac{-1}{2}$	$\frac{-2}{3}$	$\frac{-1}{3}$	0
01331	$\frac{5}{8}$	3	10	$\frac{-5}{8}$	$\frac{-45}{8}$	$\frac{7}{8}$	$\frac{7}{2}$	$\frac{3}{2}$	0	$\frac{-7}{8}$
03131	$\frac{-13}{16}$	$\frac{-1}{2}$	2	$\frac{3}{8}$	$\frac{27}{8}$	$\frac{-9}{16}$	0	$\frac{-3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$
01151	$\frac{-25}{32}$	$\frac{-5}{4}$	0	$\frac{1}{4}$	$\frac{27}{4}$	$\frac{-21}{32}$	$\frac{-3}{4}$	$\frac{-5}{4}$	$\frac{5}{8}$	$\frac{3}{8}$
11123	$\frac{3}{8}$	2	6	$\frac{-1}{4}$	$\frac{-9}{4}$	$\frac{5}{8}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{-1}{4}$
11141	$\frac{-9}{16}$	$\frac{-3}{2}$	-6	$\frac{3}{4}$	$\frac{27}{4}$	$\frac{-9}{16}$	$\frac{-3}{2}$	-1	$\frac{1}{2}$	$\frac{3}{4}$
11321	$\frac{3}{4}$	3	14	$\frac{-1}{2}$	$\frac{-9}{2}$	1	$\frac{7}{2}$	2	$\frac{1}{2}$	-1
13121	$\frac{-5}{8}$	0	-6	$\frac{3}{4}$	$\frac{27}{4}$	$\frac{-3}{8}$	$\frac{-3}{2}$	-1	$\frac{1}{2}$	$\frac{3}{4}$
10412	$\frac{2}{3}$	$\frac{2}{3}$	12	-1	3	$\frac{1}{2}$	3	$\frac{4}{3}$	$\frac{2}{3}$	-1
12212	0	0	-8	$\frac{1}{2}$	$\frac{9}{2}$	0	-2	-1	0	$\frac{1}{2}$
14012	$\frac{1}{4}$	4	4	0	0	$\frac{3}{4}$	3	0	0	0
10034	$\frac{1}{4}$	-2	-8	$\frac{1}{2}$	$\frac{-9}{2}$	$\frac{1}{4}$	-2	0	0	$\frac{1}{2}$
10232	0	-2	-8	1	0	$\frac{-1}{2}$	-3	-1	0	1
12032	$\frac{1}{8}$	3	4	0	0	$\frac{3}{8}$	3	$\frac{1}{2}$	$\frac{1}{2}$	0
10052	$\frac{-3}{16}$	$\frac{5}{2}$	8	0	0	$\frac{9}{16}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
10214	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{20}{3}$	$\frac{-1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{-1}{2}$
12014	$\frac{1}{2}$	0	-8	0	0	$\frac{1}{2}$	-2	0	0	0
10016	$\frac{5}{12}$	$\frac{4}{3}$	$\frac{20}{3}$	0	0	$\frac{1}{4}$	1	0	0	0
22022	$\frac{1}{4}$	4	4	0	0	$\frac{3}{4}$	3	1	1	0
21311	1	3	14	$\frac{-1}{2}$	$\frac{-9}{2}$	$\frac{5}{4}$	$\frac{7}{2}$	2	$\frac{1}{2}$	-1
23111	$\frac{-1}{4}$	1	-6	$\frac{3}{4}$	$\frac{27}{4}$	0	$\frac{-3}{2}$	-1	$\frac{1}{2}$	$\frac{3}{4}$
21113	$\frac{3}{4}$	3	6	$\frac{-1}{4}$	$\frac{-9}{4}$	1	$\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{-1}{4}$
21131	$\frac{-3}{8}$	-1	-10	$\frac{11}{8}$	$\frac{27}{8}$	$\frac{-3}{8}$	$\frac{-3}{2}$	-1	$\frac{1}{2}$	$\frac{9}{8}$
20222	0	0	-8	$\frac{3}{2}$	$\frac{-9}{2}$	0	-4	-1	0	$\frac{3}{2}$
20042	$\frac{-3}{8}$	3	4	0	0	$\frac{9}{8}$	3	1	1	0
20024	$\frac{1}{2}$	0	-8	1	-9	$\frac{1}{2}$	-2	0	0	1
31121	$\frac{-1}{4}$	1	-14	2	0	0	$\frac{-3}{2}$	-1	$\frac{1}{2}$	$\frac{3}{2}$
30212	0	4	-8	$\frac{3}{2}$	$\frac{-9}{2}$	1	-4	-1	0,	$\frac{3}{2}$
32012	$\frac{1}{2}$	6	4	0	0	$\frac{3}{2}$	3	1	1	0
30032	$\frac{-1}{4}$	4	-4	0	0	$\frac{3}{2}$	6	$\frac{3}{2}$	$\frac{3}{2}$	0
30014	1	4	-8	1	-9	1	-2	0	0	1
41111	0	5	-14	2	0	$\frac{3}{4}$	$\frac{-3}{2}$	-1	$\frac{1}{2}$	$\frac{3}{2}$
40022	$\frac{1}{2}$	6	-12	0	0	$\frac{3}{2}$	9	2	2	0

50012	2	10	-12	0	0	$\frac{3}{2}$	9	2	2	0

Table 5.7, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,3,4,6\}$. $ijklm$ represents quadratic form corresponding to tuple $(1^i,2^j,3^k,4^l,6^m)$, $i+j+k+l+m=8$, whose generating functions belong to $M_4(48,\chi_8)$, with dimension 28.

$ijklm$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$
00323	0	$\frac{4}{451}$	0	$\frac{78}{451}$	$\frac{32}{451}$	0	$\frac{-624}{451}$	0	$\frac{44746}{4059}$	$\frac{-17944}{4059}$
00143	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{16}{451}$	0	$\frac{180}{451}$	0	$\frac{-8177}{1353}$	$\frac{3544}{1353}$
00341	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{32}{451}$	0	$\frac{360}{451}$	0	$\frac{-13648}{1353}$	$\frac{5992}{451}$
00521	0	$\frac{4}{451}$	0	$\frac{78}{451}$	$\frac{64}{451}$	0	$\frac{-1248}{451}$	0	$\frac{26824}{1353}$	$\frac{-7184}{1353}$
00125	0	$\frac{4}{451}$	0	$\frac{78}{451}$	$\frac{16}{451}$	0	$\frac{-312}{451}$	0	$\frac{23275}{4059}$	$\frac{3704}{4059}$
00161	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{28}{451}$	0	$\frac{-54}{451}$	0	$\frac{11889}{1804}$	$\frac{-9246}{451}$
01412	0	$\frac{-2}{451}$	0	$\frac{84}{451}$	$\frac{32}{451}$	0	$\frac{1344}{451}$	0	$\frac{-44552}{4059}$	$\frac{-3656}{4059}$
03212	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{80}{451}$	0	$\frac{-576}{451}$	0	$\frac{7372}{1353}$	$\frac{-17960}{1353}$
05012	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	0	$\frac{1664}{451}$	0	$\frac{6912}{451}$	0	$\frac{-912}{451}$
01034	0	$\frac{10}{451}$	0	$\frac{72}{451}$	0	$\frac{320}{451}$	0	$\frac{-2304}{451}$	0	$\frac{76}{41}$
01232	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{40}{451}$	0	$\frac{-288}{451}$	0	$\frac{3634}{451}$	$\frac{-12548}{1353}$
03032	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	0	$\frac{832}{451}$	0	$\frac{3456}{451}$	0	$\frac{-1092}{451}$
01052	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	0	$\frac{416}{451}$	0	$\frac{1728}{451}$	0	$\frac{-1182}{451}$
01214	0	$\frac{-2}{451}$	0	$\frac{84}{451}$	$\frac{16}{451}$	0	$\frac{672}{451}$	0	$\frac{-22276}{4059}$ 0	$\frac{-7264}{4059}$
03014	0	$\frac{10}{451}$	0	$\frac{72}{451}$	0	$\frac{640}{451}$	0	$\frac{-4608}{451}$	0	$\frac{-760}{451}$
01016	0	$\frac{-2}{451}$	0	$\frac{84}{451}$	0	$\frac{128}{451}$	0	$\frac{5376}{451}$	0	$\frac{-4096}{1353}$
02123	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{32}{451}$	0	$\frac{360}{451}$	0	$\frac{-7334}{1353}$	$\frac{3544}{1353}$
02141	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{56}{451}$	0	$\frac{-108}{451}$	0	$\frac{2111}{451}$	$\frac{-8344}{451}$
02321	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{64}{451}$	0	$\frac{720}{451}$	0	$\frac{-4739}{451}$	$\frac{5992}{451}$
04121	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{112}{451}$	0	$\frac{-216}{451}$	0	$\frac{1967}{451}$	$\frac{-8344}{451}$
11222	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{80}{451}$	0	$\frac{-576}{451}$	0	$\frac{4562}{451}$	$\frac{-7136}{1353}$
11042	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{52}{451}$	0	$\frac{216}{451}$	0	$\frac{-4713}{902}$	$\frac{-1272}{451}$
11024	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{40}{451}$	0	$\frac{-288}{451}$	0	$\frac{928}{451}$	$\frac{2432}{451}$
13022	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{104}{451}$	0	$\frac{432}{451}$	0	$\frac{-1556}{451}$	$\frac{-1272}{451}$
10313	0	$\frac{4}{451}$	0	$\frac{78}{451}$	$\frac{64}{451}$	0	$\frac{-1248}{451}$	0	$\frac{39431}{4059}$	$\frac{-17944}{4059}$
12113	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{64}{451}$	0	$\frac{720}{451}$	0	$\frac{-8354}{1353}$	$\frac{3544}{1353}$
10115	0	$\frac{4}{451}$	0	$\frac{78}{451}$	$\frac{32}{451}$	0	$\frac{-624}{451}$	0	$\frac{19490}{4059}$	$\frac{3704}{4059}$
10133	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{32}{451}$	0	$\frac{360}{451}$	0	$\frac{-8347}{902}$	$\frac{4188}{451}$
10511	0	$\frac{4}{451}$	0	$\frac{78}{451}$	$\frac{128}{451}$	0	$\frac{-2496}{451}$	0	$\frac{26588}{1353}$	$\frac{-7184}{1353}$
12311	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{128}{451}$	0	$\frac{1440}{451}$	0	$\frac{-5419}{451}$	$\frac{5992}{451}$
14111	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{224}{451}$	0	$\frac{-432}{451}$	0	$\frac{2130}{451}$	$\frac{-8344}{451}$
10331	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{64}{451}$	0	$\frac{720}{451}$	0	$\frac{-39709}{2706}$	$\frac{7796}{451}$
12131	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{112}{451}$	0	$\frac{-216}{451}$	0	$\frac{6189}{902}$	$\frac{-6540}{451}$
10151	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{56}{451}$	0	$\frac{-108}{451}$	0	$\frac{5575}{902}$	$\frac{-7442}{451}$

22121	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{224}{451}$	0	$\frac{-432}{451}$	0	$\frac{3934}{451}$	$\frac{-4736}{451}$
21212	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{160}{451}$	0	$\frac{-1152}{451}$	0	$\frac{12940}{1353}$	$\frac{-7136}{1353}$
23012	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{208}{451}$	0	$\frac{864}{451}$	0	$\frac{-3112}{451}$	$\frac{-1272}{451}$
21032	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{104}{451}$	0	$\frac{432}{451}$	0	$\frac{-4713}{451}$	$\frac{-1272}{451}$
21014	0	$\frac{10}{451}$	0	$\frac{72}{451}$	$\frac{80}{451}$	0	$\frac{-576}{451}$	0	$\frac{1856}{451}$	$\frac{2432}{451}$
20123	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{64}{451}$	0	$\frac{720}{451}$	0	$\frac{-16021}{1353}$	$\frac{21584}{1353}$
20141	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{112}{451}$	0	$\frac{-216}{451}$	0	$\frac{3483}{902}$	$\frac{-4736}{451}$
20321	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{128}{451}$	0	$\frac{1440}{451}$	0	$\frac{-26630}{1353}$	$\frac{9600}{451}$
31022	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{208}{451}$	0	$\frac{864}{451}$	0	$\frac{-6269}{451}$	$\frac{-1272}{451}$
30113	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{128}{451}$	0	$\frac{1440}{451}$	0	$\frac{-15355}{1353}$	$\frac{21584}{1353}$
30311	0	$\frac{-8}{451}$	0	$\frac{90}{451}$	$\frac{256}{451}$	0	$\frac{2880}{451}$	0	$\frac{-9936}{451}$	$\frac{9600}{451}$
32111	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{448}{451}$	0	$\frac{-864}{451}$	0	$\frac{3809}{451}$	$\frac{-4736}{451}$
30131	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{224}{451}$	0	$\frac{-432}{451}$	0	$\frac{2005}{902}$	$\frac{-1128}{451}$
41012	0	$\frac{-26}{451}$	0	$\frac{108}{451}$	$\frac{416}{451}$	0	$\frac{1728}{451}$	0	$\frac{-6224}{451}$	$\frac{-1272}{451}$
40121	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{448}{451}$	0	$\frac{-864}{451}$	0	$\frac{2456}{451}$	$\frac{2480}{451}$
50111	0	$\frac{28}{451}$	0	$\frac{54}{451}$	$\frac{896}{451}$	0	$\frac{-1728}{451}$	0	$\frac{6716}{451}$	$\frac{2480}{451}$

$ijklm$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$	$\alpha_{4,16}$	$\alpha_{4,17}$	$\alpha_{4,18}$	$\alpha_{4,19}$	$\alpha_{4,20}$
00323	$\frac{-234}{451}$	$\frac{-15456}{451}$	$\frac{-7232}{451}$	$\frac{-128}{451}$	$\frac{312}{451}$	$\frac{-2496}{451}$	$\frac{-2582}{369}$	$\frac{1664}{369}$	$\frac{-3682}{123}$	$\frac{2776}{123}$
00143	$\frac{-1962}{451}$	$\frac{17136}{451}$	$\frac{3600}{451}$	$\frac{256}{451}$	$\frac{8028}{451}$	$\frac{-2880}{451}$	$\frac{499}{123}$	$\frac{-344}{123}$	$\frac{589}{41}$	$\frac{-648}{41}$
00341	$\frac{4194}{451}$	$\frac{49608}{451}$	$\frac{7200}{451}$	$\frac{256}{451}$	$\frac{-180}{451}$	$\frac{-2880}{451}$	$\frac{752}{123}$	$\frac{-552}{41}$	$\frac{932}{41}$	$\frac{-2616}{41}$
00521	$\frac{-3174}{451}$	$\frac{-4632}{451}$	$\frac{-14464}{451}$	$\frac{-128}{451}$	$\frac{624}{451}$	$\frac{-89088}{451}$	$\frac{-1448}{123}$	$\frac{664}{123}$	$\frac{-2072}{41}$	$\frac{1800}{41}$
00125	$\frac{-1470}{451}$	$\frac{-4632}{451}$	$\frac{-3616}{451}$	$\frac{-128}{451}$	$\frac{5568}{451}$	$\frac{-2496}{451}$	$\frac{-1373}{369}$	$\frac{-304}{369}$	$\frac{-2005}{123}$	$\frac{-176}{123}$
00161	$\frac{3024}{451}$	$\frac{-96876}{451}$	$\frac{-2720}{451}$	$\frac{-896}{451}$	$\frac{12204}{451}$	$\frac{193104}{451}$	$\frac{-831}{164}$	$\frac{866}{41}$	$\frac{-3585}{164}$	$\frac{3006}{41}$
01412	$\frac{504}{451}$	$\frac{22488}{451}$	$\frac{7200}{451}$	$\frac{64}{451}$	$\frac{-672}{451}$	$\frac{-2688}{451}$	$\frac{2584}{369}$	$\frac{-1160}{369}$	$\frac{3452}{123}$	$\frac{-568}{123}$
03212	$\frac{-216}{451}$	$\frac{-31752}{451}$	$\frac{-3648}{451}$	$\frac{-320}{451}$	$\frac{288}{451}$	$\frac{-2304}{451}$	$\frac{-416}{123}$	$\frac{1168}{123}$	$\frac{-540}{41}$	$\frac{2368}{41}$
05012	0	$\frac{3672}{451}$	0	0	0	$\frac{-6912}{451}$	0	$\frac{-48}{41}$	0	$\frac{192}{41}$
01034	0	$\frac{-144}{451}$	0	$\frac{-480}{451}$	0	$\frac{-1152}{451}$	0	$\frac{-56}{41}$	0	$\frac{-448}{41}$
01232	$\frac{-108}{451}$	$\frac{-31752}{451}$	$\frac{-5432}{451}$	$\frac{-320}{451}$	$\frac{144}{451}$	$\frac{-2304}{451}$	$\frac{-206}{41}$	$\frac{1168}{123}$	$\frac{-844}{41}$	$\frac{1876}{41}$
03032	0	$\frac{216}{41}$	0	$\frac{416}{451}$	0	$\frac{-5184}{451}$	0	$\frac{104}{41}$	0	$\frac{396}{41}$
01052	0	$\frac{1728}{451}$	0	$\frac{624}{451}$	0	$\frac{-4320}{451}$	0	$\frac{98}{41}$	0	$\frac{498}{41}$
01214	$\frac{252}{451}$	$\frac{11664}{451}$	$\frac{3600}{451}$	$\frac{64}{451}$	$\frac{-336}{451}$	$\frac{-2688}{451}$	$\frac{1292}{369}$	$\frac{-832}{369}$	$\frac{1726}{123}$	$\frac{88}{123}$
03014	0	$\frac{-1008}{451}$	0	$\frac{-640}{451}$	0	0	0	$\frac{-64}{41}$	0	$\frac{16}{41}$
01016	0	$\frac{2856}{451}$	0	0	0	$\frac{-5376}{451}$	0	$\frac{-112}{123}$	0	$\frac{368}{41}$
02123	$\frac{-3924}{451}$	$\frac{17136}{451}$	$\frac{3592}{451}$	$\frac{256}{451}$	$\frac{16056}{451}$	$\frac{-2880}{451}$	$\frac{424}{123}$	$\frac{-344}{123}$	$\frac{481}{41}$	$\frac{-648}{41}$
02141	$\frac{-6129}{451}$	$\frac{-48168}{451}$	$\frac{-3636}{451}$	$\frac{-896}{451}$	$\frac{24408}{451}$	$\frac{-1728}{451}$	$\frac{-108}{41}$	$\frac{784}{41}$	$\frac{-1125}{82}$	$\frac{3744}{41}$
02321	$\frac{270}{451}$	$\frac{49608}{451}$	$\frac{7184}{451}$	$\frac{256}{451}$	$\frac{-360}{451}$	$\frac{-2880}{451}$	$\frac{269}{41}$	$\frac{-552}{41}$	$\frac{1085}{41}$	$\frac{-2616}{41}$
04121	$\frac{-12258}{451}$	$\frac{-48168}{451}$	$\frac{-3664}{451}$	$\frac{-896}{451}$	$\frac{48816}{451}$	$\frac{-1728}{451}$	$\frac{-93}{41}$	$\frac{784}{41}$	$\frac{-633}{41}$	$\frac{3744}{41}$
11222	$\frac{-216}{451}$	$\frac{-31752}{451}$	$\frac{-7256}{451}$	$\frac{-320}{451}$	$\frac{288}{451}$	$\frac{-2304}{451}$	$\frac{-248}{41}$	$\frac{1168}{123}$	$\frac{-1032}{41}$	$\frac{1384}{41}$

11042	$\frac{81}{451}$	$\frac{1080}{451}$	$\frac{2680}{451}$	$\frac{832}{451}$	$\frac{-108}{451}$	$\frac{-3456}{451}$	$\frac{309}{82}$	$\frac{256}{41}$	$\frac{633}{41}$	$\frac{600}{41}$
11024	$\frac{-108}{451}$	$\frac{720}{451}$	$\frac{-1824}{451}$	$\frac{-320}{451}$	$\frac{144}{451}$	$\frac{-2304}{451}$	$\frac{-42}{41}$	$\frac{-48}{41}$	$\frac{-188}{41}$	$\frac{-912}{41}$
13022	$\frac{162}{451}$	$\frac{1080}{451}$	$\frac{1752}{451}$	$\frac{832}{451}$	$\frac{-216}{451}$	$\frac{-3456}{451}$	$\frac{104}{41}$	$\frac{256}{41}$	$\frac{405}{41}$	$\frac{600}{41}$
10313	$\frac{-468}{451}$	$\frac{-15456}{451}$	$\frac{-7248}{451}$	$\frac{-128}{451}$	$\frac{624}{451}$	$\frac{-2496}{451}$	$\frac{-2089}{369}$	$\frac{1664}{369}$	$\frac{-3059}{123}$	$\frac{2776}{123}$
12113	$\frac{270}{451}$	$\frac{17136}{451}$	$\frac{3576}{451}$	$\frac{256}{451}$	$\frac{-360}{451}$	$\frac{-2880}{451}$	$\frac{520}{123}$	$\frac{-344}{123}$	$\frac{675}{41}$	$\frac{-648}{41}$
10115	$\frac{-234}{451}$	$\frac{-4632}{451}$	$\frac{-3624}{451}$	$\frac{-128}{451}$	$\frac{312}{451}$	$\frac{-2496}{451}$	$\frac{-1024}{369}$	$\frac{-304}{369}$	$\frac{-1427}{123}$	$\frac{-176}{123}$
10133	$\frac{135}{451}$	$\frac{17136}{451}$	$\frac{5396}{451}$	$\frac{256}{451}$	$\frac{-180}{451}$	$\frac{-2880}{451}$	$\frac{515}{82}$	$\frac{-224}{41}$	$\frac{1055}{41}$	$\frac{-1632}{41}$
10511	$\frac{-936}{451}$	$\frac{-4632}{451}$	$\frac{-14496}{451}$	$\frac{-128}{451}$	$\frac{1248}{451}$	$\frac{-89088}{451}$	$\frac{-1420}{123}$	$\frac{664}{123}$	$\frac{-2094}{41}$	$\frac{1800}{41}$
12311	$\frac{540}{451}$	$\frac{49608}{451}$	$\frac{7152}{451}$	$\frac{256}{451}$	$\frac{-720}{451}$	$\frac{-2880}{451}$	$\frac{333}{41}$	$\frac{-552}{41}$	$\frac{1309}{41}$	$\frac{-2616}{41}$
14111	$\frac{-162}{451}$	$\frac{-48168}{451}$	$\frac{-3720}{451}$	$\frac{-896}{451}$	$\frac{216}{451}$	$\frac{-1728}{451}$	$\frac{-104}{41}$	$\frac{784}{41}$	$\frac{-405}{41}$	$\frac{3744}{41}$
10331	$\frac{8388}{451}$	$\frac{17136}{451}$	$\frac{10792}{451}$	$\frac{256}{451}$	$\frac{-360}{451}$	$\frac{127008}{451}$	$\frac{2147}{246}$	$\frac{-552}{41}$	$\frac{2867}{82}$	$\frac{-3764}{41}$
12131	$\frac{-81}{451}$	$\frac{-48168}{451}$	$\frac{-5468}{451}$	$\frac{-896}{451}$	$\frac{108}{451}$	$\frac{-1728}{451}$	$\frac{-309}{82}$	$\frac{784}{41}$	$\frac{-633}{41}$	$\frac{3252}{41}$
10151	$\frac{-81}{902}$	$\frac{-96876}{451}$	$\frac{-4538}{451}$	$\frac{-896}{451}$	$\frac{54}{451}$	$\frac{193104}{451}$	$\frac{-149}{41}$	$\frac{866}{41}$	$\frac{-2127}{164}$	$\frac{2514}{41}$
22121	$\frac{-162}{451}$	$\frac{-48168}{451}$	$\frac{-7328}{451}$	$\frac{-896}{451}$	$\frac{216}{451}$	$\frac{-1728}{451}$	$\frac{-186}{41}$	$\frac{784}{41}$	$\frac{-774}{41}$	$\frac{2760}{41}$
21212	$\frac{-432}{451}$	$\frac{-31752}{451}$	$\frac{-7296}{451}$	$\frac{-320}{451}$	$\frac{576}{451}$	$\frac{-2304}{451}$	$\frac{-668}{123}$	$\frac{1168}{123}$	$\frac{-916}{41}$	$\frac{1384}{41}$
23012	$\frac{324}{451}$	$\frac{1080}{451}$	$\frac{3504}{451}$	$\frac{832}{451}$	$\frac{-432}{451}$	$\frac{-3456}{451}$	$\frac{208}{41}$	$\frac{256}{41}$	$\frac{810}{41}$	$\frac{600}{41}$
21032	$\frac{162}{451}$	$\frac{1080}{451}$	$\frac{5360}{451}$	$\frac{832}{451}$	$\frac{-216}{451}$	$\frac{-3456}{451}$	$\frac{309}{41}$	$\frac{420}{41}$	$\frac{1266}{41}$	$\frac{600}{41}$
21014	$\frac{-216}{451}$	$\frac{720}{451}$	$\frac{-3648}{451}$	$\frac{-320}{451}$	$\frac{288}{451}$	$\frac{-2304}{451}$	$\frac{-84}{41}$	$\frac{-48}{41}$	$\frac{-376}{41}$	$\frac{-912}{41}$
20123	$\frac{270}{451}$	$\frac{17136}{451}$	$\frac{7184}{451}$	$\frac{256}{451}$	$\frac{-360}{451}$	$\frac{-2880}{451}$	$\frac{971}{123}$	$\frac{-1000}{123}$	$\frac{1413}{41}$	$\frac{-2616}{41}$
20141	$\frac{-12258}{451}$	$\frac{-48168}{451}$	$\frac{-7272}{451}$	$\frac{-896}{451}$	$\frac{108}{451}$	$\frac{-1728}{451}$	$\frac{19}{82}$	$\frac{784}{41}$	$\frac{333}{82}$	$\frac{2760}{41}$
20321	$\frac{8658}{451}$	$\frac{-15336}{451}$	$\frac{14368}{451}$	$\frac{256}{451}$	$\frac{-720}{451}$	$\frac{256896}{451}$	$\frac{1450}{123}$	$\frac{-552}{41}$	$\frac{2088}{41}$	$\frac{-4912}{41}$
31022	$\frac{324}{451}$	$\frac{1080}{451}$	$\frac{7112}{451}$	$\frac{832}{451}$	$\frac{-432}{451}$	$\frac{-3456}{451}$	$\frac{413}{41}$	$\frac{584}{41}$	$\frac{1671}{41}$	$\frac{600}{41}$
30113	$\frac{540}{451}$	$\frac{17136}{451}$	$\frac{7152}{451}$	$\frac{256}{451}$	$\frac{-720}{451}$	$\frac{-2880}{451}$	$\frac{917}{123}$	$\frac{-1000}{123}$	$\frac{1391}{41}$	$\frac{-2616}{41}$
30311	$\frac{1080}{451}$	$\frac{-15336}{451}$	$\frac{14304}{451}$	$\frac{256}{451}$	$\frac{-1440}{451}$	$\frac{256896}{451}$	$\frac{584}{41}$	$\frac{-552}{41}$	$\frac{2618}{41}$	$\frac{-4912}{41}$
32111	$\frac{-324}{451}$	$\frac{-48168}{451}$	$\frac{-7440}{451}$	$\frac{-896}{451}$	$\frac{432}{451}$	$\frac{-1728}{451}$	$\frac{-167}{41}$	$\frac{784}{41}$	$\frac{-687}{41}$	$\frac{2760}{41}$
30131	$\frac{-24516}{451}$	$\frac{49248}{451}$	$\frac{-10936}{451}$	$\frac{-896}{451}$	$\frac{216}{451}$	$\frac{-391392}{451}$	$\frac{325}{82}$	$\frac{620}{41}$	$\frac{1527}{82}$	$\frac{3744}{41}$
41012	$\frac{648}{451}$	$\frac{1080}{451}$	$\frac{7008}{451}$	$\frac{832}{451}$	$\frac{-864}{451}$	$\frac{-3456}{451}$	$\frac{416}{41}$	$\frac{584}{41}$	$\frac{1620}{41}$	$\frac{600}{41}$
40121	$\frac{-24678}{451}$	$\frac{146664}{451}$	$\frac{-14656}{451}$	$\frac{-896}{451}$	$\frac{432}{451}$	$\frac{-781056}{451}$	$\frac{120}{41}$	$\frac{456}{41}$	$\frac{420}{41}$	$\frac{4728}{41}$
50111	$\frac{-648}{451}$	$\frac{146664}{451}$	$\frac{-14880}{451}$	$\frac{-896}{451}$	$\frac{864}{451}$	$\frac{-781056}{451}$	$\frac{-252}{41}$	$\frac{456}{41}$	$\frac{-1374}{41}$	$\frac{4728}{41}$

$ijklm$	$\alpha_{4,21}$	$\alpha_{4,22}$	$\alpha_{4,23}$	$\alpha_{4,24}$	$\alpha_{4,25}$	$\alpha_{4,26}$	$\alpha_{4,27}$	$\alpha_{4,28}$
00323	$\frac{650}{123}$	$\frac{-1520}{123}$	$\frac{23752}{369}$	$\frac{-13888}{369}$	$\frac{4488}{41}$	$\frac{-6976}{41}$	$\frac{-1154}{123}$	$\frac{1016}{123}$
00143	$\frac{-83}{41}$	$\frac{248}{41}$	$\frac{-2984}{123}$	$\frac{3136}{123}$	$\frac{-2184}{41}$	$\frac{3456}{41}$	$\frac{163}{41}$	$\frac{-240}{41}$
00341	$\frac{-84}{41}$	$\frac{904}{41}$	$\frac{-4984}{123}$	$\frac{4544}{41}$	$\frac{-3384}{41}$	$\frac{11328}{41}$	$\frac{244}{41}$	$\frac{-896}{41}$
00521	$\frac{324}{41}$	$\frac{-944}{41}$	$\frac{13648}{123}$	$\frac{-13376}{123}$	$\frac{7664}{41}$	$\frac{-12224}{41}$	$\frac{-660}{41}$	$\frac{776}{41}$
00125	$\frac{407}{123}$	$\frac{-536}{123}$	$\frac{14008}{369}$	$\frac{1856}{369}$	$\frac{2408}{41}$	$\frac{-1728}{41}$	$\frac{-659}{123}$	$\frac{32}{123}$
00161	$\frac{567}{164}$	$\frac{-786}{41}$	$\frac{1266}{41}$	$\frac{-2784}{41}$	$\frac{2574}{41}$	$\frac{-12096}{41}$	$\frac{-827}{164}$	$\frac{758}{41}$
01412	$\frac{-580}{123}$	$\frac{1088}{123}$	$\frac{-20528}{369}$	$\frac{-2240}{369}$	$\frac{-4368}{41}$	$\frac{4800}{41}$	$\frac{1060}{123}$	$\frac{-344}{123}$
03212	$\frac{36}{41}$	$\frac{-392}{41}$	$\frac{2464}{123}$	$\frac{-13760}{123}$	$\frac{1872}{41}$	$\frac{-8256}{41}$	$\frac{-128}{41}$	$\frac{792}{41}$
05012	0	$\frac{456}{41}$	0	$\frac{320}{41}$	0	$\frac{576}{41}$	0	$\frac{-64}{41}$

01034	0	$\frac{80}{41}$	0	$\frac{448}{41}$	0	$\frac{-768}{41}$	0	$\frac{-48}{41}$
01232	$\frac{100}{41}$	$\frac{-556}{41}$	$\frac{1504}{41}$	$\frac{-9824}{123}$	$\frac{2904}{41}$	$\frac{-8256}{41}$	$\frac{-228}{41}$	$\frac{628}{41}$
03032	0	$\frac{180}{41}$	0	0	0	0	0	$\frac{-12}{41}$
01052	0	$\frac{42}{41}$	0	$\frac{-160}{41}$	0	$\frac{-288}{41}$	0	$\frac{14}{41}$
01214	$\frac{-290}{123}$	$\frac{760}{123}$	$\frac{-10264}{369}$	$\frac{-4864}{369}$	$\frac{-2184}{41}$	$\frac{2176}{41}$	$\frac{530}{123}$	$\frac{-16}{123}$
03014	0	$\frac{224}{41}$	0	$\frac{-640}{41}$	0	$\frac{-1152}{41}$	0	$\frac{96}{41}$
01016	0	$\frac{136}{41}$	0	$\frac{-2752}{123}$	0	$\frac{448}{41}$	0	$\frac{96}{41}$
02123	$\frac{-43}{41}$	$\frac{248}{41}$	$\frac{-1868}{123}$	$\frac{3136}{123}$	$\frac{-1908}{41}$	$\frac{3456}{41}$	$\frac{121}{41}$	$\frac{-240}{41}$
02141	$\frac{75}{82}$	$\frac{-1032}{41}$	$\frac{974}{41}$	$\frac{-6720}{41}$	$\frac{1458}{41}$	$\frac{-12096}{41}$	$\frac{-253}{82}$	$\frac{1168}{41}$
02321	$\frac{-127}{41}$	$\frac{904}{41}$	$\frac{-1792}{41}$	$\frac{4544}{41}$	$\frac{-3816}{41}$	$\frac{11328}{41}$	$\frac{283}{41}$	$\frac{-896}{41}$
04121	$\frac{75}{41}$	$\frac{-1032}{41}$	$\frac{1456}{41}$	$\frac{-6720}{41}$	$\frac{1440}{41}$	$\frac{-12096}{41}$	$\frac{-171}{41}$	$\frac{1168}{41}$
11222	$\frac{200}{41}$	$\frac{-720}{41}$	$\frac{2024}{41}$	$\frac{-5888}{123}$	$\frac{3840}{41}$	$\frac{-8256}{41}$	$\frac{-292}{41}$	$\frac{464}{41}$
11042	$\frac{-75}{41}$	$\frac{-96}{41}$	$\frac{-1374}{41}$	$\frac{-320}{41}$	$\frac{-2178}{41}$	$\frac{-576}{41}$	$\frac{212}{41}$	$\frac{40}{41}$
11024	$\frac{100}{41}$	$\frac{-64}{41}$	$\frac{520}{41}$	$\frac{1536}{41}$	$\frac{936}{41}$	$\frac{-384}{41}$	$\frac{-64}{41}$	$\frac{-192}{41}$
13022	$\frac{-27}{41}$	$\frac{-96}{41}$	$\frac{-780}{41}$	$\frac{-320}{41}$	$\frac{-1404}{41}$	$\frac{-576}{41}$	$\frac{137}{41}$	$\frac{40}{41}$
10313	$\frac{685}{123}$	$\frac{-1520}{123}$	$\frac{19952}{369}$	$\frac{-13888}{369}$	$\frac{4056}{41}$	$\frac{-6976}{41}$	$\frac{-955}{123}$	$\frac{1016}{123}$
12113	$\frac{-45}{41}$	$\frac{248}{41}$	$\frac{-3572}{123}$	$\frac{3136}{123}$	$\frac{-2340}{41}$	$\frac{3456}{41}$	$\frac{201}{41}$	$\frac{-240}{41}$
10115	$\frac{445}{123}$	$\frac{-536}{123}$	$\frac{9812}{369}$	$\frac{1856}{369}$	$\frac{2028}{41}$	$\frac{-1728}{41}$	$\frac{-457}{123}$	$\frac{32}{123}$
10133	$\frac{-125}{41}$	$\frac{412}{41}$	$\frac{-2126}{41}$	$\frac{3232}{41}$	$\frac{-3630}{41}$	$\frac{7392}{41}$	$\frac{326}{41}$	$\frac{-568}{41}$
10511	$\frac{402}{41}$	$\frac{-944}{41}$	$\frac{13520}{123}$	$\frac{-13376}{123}$	$\frac{8112}{41}$	$\frac{-12224}{41}$	$\frac{-664}{41}$	$\frac{776}{41}$
12311	$\frac{-131}{41}$	$\frac{904}{41}$	$\frac{-2272}{41}$	$\frac{4544}{41}$	$\frac{-4680}{41}$	$\frac{11328}{41}$	$\frac{361}{41}$	$\frac{-896}{41}$
14111	$\frac{27}{41}$	$\frac{-1032}{41}$	$\frac{452}{41}$	$\frac{-6720}{41}$	$\frac{1404}{41}$	$\frac{-12096}{41}$	$\frac{-55}{41}$	$\frac{1168}{41}$
10331	$\frac{-213}{82}$	$\frac{1232}{41}$	$\frac{-8984}{123}$	$\frac{8480}{41}$	$\frac{-5292}{41}$	$\frac{15264}{41}$	$\frac{853}{82}$	$\frac{-1388}{41}$
12131	$\frac{75}{41}$	$\frac{-1032}{41}$	$\frac{882}{41}$	$\frac{-5408}{41}$	$\frac{2178}{41}$	$\frac{-12096}{41}$	$\frac{-130}{41}$	$\frac{1004}{41}$
10151	$\frac{273}{164}$	$\frac{-786}{41}$	$\frac{605}{41}$	$\frac{-1472}{41}$	$\frac{1827}{41}$	$\frac{-12096}{41}$	$\frac{-383}{164}$	$\frac{594}{41}$
22121	$\frac{150}{41}$	$\frac{-1032}{41}$	$\frac{1272}{41}$	$\frac{-4096}{41}$	$\frac{2880}{41}$	$\frac{-12096}{41}$	$\frac{-178}{41}$	$\frac{840}{41}$
21212	$\frac{236}{41}$	$\frac{-720}{41}$	$\frac{5584}{123}$	$\frac{-5888}{123}$	$\frac{3744}{41}$	$\frac{-8256}{41}$	$\frac{-256}{41}$	$\frac{464}{41}$
23012	$\frac{-54}{41}$	$\frac{-96}{41}$	$\frac{-1560}{41}$	$\frac{-320}{41}$	$\frac{-2808}{41}$	$\frac{-576}{41}$	$\frac{274}{41}$	$\frac{40}{41}$
21032	$\frac{-150}{41}$	$\frac{-96}{41}$	$\frac{-2748}{41}$	$\frac{-320}{41}$	$\frac{-4356}{41}$	$\frac{-576}{41}$	$\frac{424}{41}$	$\frac{40}{41}$
21014	$\frac{200}{41}$	$\frac{-64}{41}$	$\frac{1040}{41}$	$\frac{1536}{41}$	$\frac{1872}{41}$	$\frac{-384}{41}$	$\frac{-128}{41}$	$\frac{-192}{41}$
20123	$\frac{-127}{41}$	$\frac{576}{41}$	$\frac{-8656}{123}$	$\frac{16256}{123}$	$\frac{-4800}{41}$	$\frac{11328}{41}$	$\frac{447}{41}$	$\frac{-896}{41}$
20141	$\frac{-219}{82}$	$\frac{-1032}{41}$	$\frac{-348}{41}$	$\frac{-4096}{41}$	$\frac{-36}{41}$	$\frac{-12096}{41}$	$\frac{191}{82}$	$\frac{840}{41}$
20321	$\frac{-172}{41}$	$\frac{1560}{41}$	$\frac{-13376}{123}$	$\frac{12416}{41}$	$\frac{-7632}{41}$	$\frac{19200}{41}$	$\frac{648}{41}$	$\frac{-1880}{41}$
31022	$\frac{-177}{41}$	$\frac{-96}{41}$	$\frac{-3528}{41}$	$\frac{-320}{41}$	$\frac{-5760}{41}$	$\frac{-576}{41}$	$\frac{561}{41}$	$\frac{40}{41}$
30113	$\frac{-49}{41}$	$\frac{576}{41}$	$\frac{-8128}{123}$	$\frac{16256}{123}$	$\frac{-4680}{41}$	$\frac{11328}{41}$	$\frac{443}{41}$	$\frac{-896}{41}$
30311	$\frac{-262}{41}$	$\frac{1560}{41}$	$\frac{-5200}{41}$	$\frac{12416}{41}$	$\frac{-9360}{41}$	$\frac{19200}{41}$	$\frac{804}{41}$	$\frac{-1880}{41}$
32111	$\frac{177}{41}$	$\frac{-1032}{41}$	$\frac{1232}{41}$	$\frac{-4096}{41}$	$\frac{2808}{41}$	$\frac{-12096}{41}$	$\frac{-151}{41}$	$\frac{840}{41}$
30131	$\frac{-561}{82}$	$\frac{-1524}{41}$	$\frac{-1024}{41}$	$\frac{-10656}{41}$	$\frac{-1548}{41}$	$\frac{-12096}{41}$	$\frac{505}{82}$	$\frac{1496}{41}$

41012	$\frac{-108}{41}$	$\frac{-96}{41}$	$\frac{-3120}{41}$	$\frac{-320}{41}$	$\frac{-5616}{41}$	$\frac{-576}{41}$	$\frac{548}{41}$	$\frac{40}{41}$
40121	$\frac{-192}{41}$	$\frac{-2016}{41}$	$\frac{-80}{41}$	$\frac{-17216}{41}$	$\frac{-144}{41}$	$\frac{-12096}{41}$	$\frac{136}{41}$	$\frac{2152}{41}$
50111	$\frac{354}{41}$	$\frac{-2016}{41}$	$\frac{3120}{41}$	$\frac{-17216}{41}$	$\frac{5616}{41}$	$\frac{-12096}{41}$	$\frac{-384}{41}$	$\frac{2152}{41}$

Table 5.8, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,3,4,6\}$. $ijklm$ represents quadratic form corresponding to tuple $(1^i,2^j,3^k,4^l,6^m)$, $i+j+k+l+m=8$, whose generating functions belong to $M_4(48,\chi_{12})$, with dimension 30.

$ijklm$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$
00512	0	$\frac{2}{23}$	$\frac{2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
00134	0	$\frac{3}{92}$	$\frac{-1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0
00332	0	$\frac{3}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
00152	0	$\frac{9}{184}$	$\frac{1}{184}$	0	0	0	0	0	$\frac{1}{23}$	0
00314	0	$\frac{1}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
00116	0	$\frac{1}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
02312	0	$\frac{3}{23}$	$\frac{-1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
02132	0	$\frac{9}{92}$	$\frac{1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0
04112	0	$\frac{9}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
02114	0	$\frac{3}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
03023	0	0	0	0	0	$\frac{36}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	0
01223	0	$\frac{3}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
01043	0	0	0	0	0	$\frac{18}{23}$	$\frac{2}{23}$	0	$\frac{1}{23}$	0
01241	0	$\frac{9}{92}$	$\frac{-1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0
03041	0	0	0	0	0	$\frac{54}{23}$	$\frac{-2}{23}$	0	$\frac{1}{23}$	0
01421	0	$\frac{3}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
03221	0	$\frac{9}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
05021	0	0	0	0	0	$\frac{108}{23}$	$\frac{-4}{23}$	0	$\frac{1}{23}$	0
01025	0	0	0	0	0	$\frac{12}{23}$	$\frac{-4}{23}$	0	$\frac{1}{23}$	0
01061	0	0	0	0	0	$\frac{27}{23}$	$\frac{-1}{23}$	0	$\frac{1}{23}$	0
11213	0	$\frac{3}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
11033	0	$\frac{9}{92}$	$\frac{-1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0
11411	0	$\frac{6}{23}$	$\frac{2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
11015	0	$\frac{3}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
11231	0	$\frac{9}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
11051	0	$\frac{27}{184}$	$\frac{1}{184}$	0	0	0	0	0	$\frac{1}{23}$	0
10322	0	$\frac{3}{23}$	$\frac{-1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
12122	0	$\frac{9}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
10142	0	$\frac{9}{92}$	$\frac{1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0
10124	0	$\frac{3}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
13013	0	$\frac{9}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
13211	0	$\frac{9}{23}$	$\frac{-1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0

15011	0	$\frac{27}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
13031	0	$\frac{27}{92}$	$\frac{1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0
22112	0	$\frac{9}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
20312	0	$\frac{6}{23}$	$\frac{-2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
20132	0	$\frac{9}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
20114	0	$\frac{3}{23}$	$\frac{-1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
21023	0	$\frac{9}{46}$	$\frac{-1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
21041	0	$\frac{27}{92}$	$\frac{1}{92}$	0	0	0	0	0	$\frac{1}{23}$	0
21221	0	$\frac{9}{23}$	$\frac{-1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
23021	0	$\frac{27}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
33011	0	$\frac{27}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
30122	0	$\frac{9}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
31013	0	$\frac{9}{23}$	$\frac{-1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
31211	0	$\frac{18}{23}$	$\frac{-2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
31031	0	$\frac{27}{46}$	$\frac{1}{46}$	0	0	0	0	0	$\frac{1}{23}$	0
40112	0	$\frac{18}{23}$	$\frac{2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
41021	0	$\frac{27}{23}$	$\frac{1}{23}$	0	0	0	0	0	$\frac{1}{23}$	0
51011	0	$\frac{54}{23}$	$\frac{2}{23}$	0	0	0	0	0	$\frac{1}{23}$	0

ijklm	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$	$\alpha_{4,16}$	$\alpha_{4,17}$	$\alpha_{4,18}$	$\alpha_{4,19}$	$\alpha_{4,20}$
00512	0	$\frac{1}{23}$	$\frac{19}{23}$	0	$\frac{22}{23}$	$\frac{22}{23}$	0	16	24	0
00134	0	$\frac{-3}{23}$	$\frac{-1}{46}$	$\frac{-8}{23}$	$\frac{173}{46}$	$\frac{17}{46}$	0	0	0	12
00332	0	$\frac{-3}{23}$	$\frac{-1}{23}$	$\frac{-16}{23}$	$\frac{277}{46}$	$\frac{-35}{46}$	0	-6	6	24
00152	0	$\frac{9}{23}$	$\frac{-5}{92}$	$\frac{-8}{23}$	$\frac{-56}{23}$	$\frac{50}{23}$	0	5	-15	-20
00314	0	$\frac{1}{23}$	$\frac{21}{23}$	0	$\frac{11}{23}$	$\frac{11}{23}$	0	12	12	0
00116	0	$\frac{1}{23}$	$\frac{22}{23}$	0	$\frac{-6}{23}$	$\frac{-6}{23}$	0	4	12	0
02312	0	$\frac{-3}{23}$	$\frac{21}{23}$	$\frac{-32}{23}$	$\frac{93}{23}$	$\frac{-35}{23}$	4	4	4	16
02132	0	$\frac{9}{23}$	$\frac{-5}{46}$	$\frac{-16}{23}$	$\frac{-17}{46}$	$\frac{131}{46}$	4	4	-8	-16
04112	0	$\frac{9}{23}$	$\frac{18}{23}$	$\frac{-32}{23}$	$\frac{6}{23}$	$\frac{62}{23}$	8	4	-4	-16
02114	0	$\frac{-3}{23}$	$\frac{22}{23}$	$\frac{-16}{23}$	$\frac{58}{23}$	$\frac{-6}{23}$	4	4	4	8
03023	0	$\frac{-9}{23}$	0	0	0	0	$\frac{98}{23}$	$\frac{296}{23}$	$\frac{-412}{23}$	$\frac{220}{23}$
01223	0	$\frac{3}{23}$	$\frac{-2}{23}$	$\frac{-8}{23}$	$\frac{-26}{23}$	$\frac{34}{23}$	2	8	-4	-12
01043	0	$\frac{-9}{23}$	0	0	0	0	$\frac{26}{23}$	$\frac{240}{23}$	$\frac{-298}{23}$	$\frac{294}{23}$
01241	0	$\frac{-9}{23}$	$\frac{-2}{23}$	$\frac{-43}{23}$	$\frac{91}{23}$	$\frac{-104}{23}$	3	-8	18	26
03041	0	$\frac{27}{23}$	0	0	0	0	$\frac{86}{23}$	$\frac{-80}{23}$	$\frac{146}{23}$	$\frac{-358}{23}$
01421	0	$\frac{3}{23}$	$\frac{-4}{23}$	$\frac{-16}{23}$	$\frac{-52}{23}$	$\frac{68}{23}$	2	8	-4	-28
03221	0	$\frac{-9}{23}$	$\frac{-4}{23}$	$\frac{-40}{23}$	$\frac{44}{23}$	$\frac{-116}{23}$	6	-8	12	20
05021	0	$\frac{27}{23}$	0	0	0	0	$\frac{126}{23}$	$\frac{24}{23}$	$\frac{108}{23}$	$\frac{-348}{23}$
01025	0	$\frac{3}{23}$	0	0	0	0	$\frac{38}{23}$	$\frac{56}{23}$	$\frac{188}{23}$	$\frac{-140}{23}$

01061	0	$\frac{27}{23}$	0	0	0	0	$\frac{20}{23}$	$\frac{-224}{23}$	$\frac{188}{23}$	$\frac{-340}{23}$
11213	0	$\frac{3}{23}$	$\frac{42}{23}$	$\frac{-16}{23}$	$\frac{-6}{23}$	$\frac{22}{23}$	2	16	8	-8
11033	0	$\frac{-9}{23}$	$\frac{21}{23}$	$\frac{-20}{23}$	$\frac{45}{23}$	$\frac{11}{23}$	2	16	-10	14
11411	0	$\frac{3}{23}$	$\frac{38}{23}$	$\frac{14}{23}$	$\frac{-58}{23}$	$\frac{44}{23}$	0	16	12	-20
11015	0	$\frac{3}{23}$	$\frac{44}{23}$	$\frac{-8}{23}$	$\frac{20}{23}$	$\frac{-12}{23}$	2	8	20	-4
11231	0	$\frac{-9}{23}$	$\frac{19}{23}$	$\frac{-63}{23}$	$\frac{136}{23}$	$\frac{-93}{23}$	3	-8	18	26
11051	0	$\frac{27}{23}$	$\frac{39}{46}$	$\frac{-75}{46}$	$\frac{-87}{46}$	$\frac{-21}{23}$	$\frac{5}{2}$	-10	10	-20
10322	0	$\frac{-3}{23}$	$\frac{21}{23}$	$\frac{-32}{23}$	$\frac{93}{23}$	$\frac{-35}{23}$	0	-8	0	16
12122	0	$\frac{9}{23}$	$\frac{18}{23}$	$\frac{-32}{23}$	$\frac{6}{23}$	$\frac{62}{23}$	4	16	-8	-16
10142	0	$\frac{9}{23}$	$\frac{41}{46}$	$\frac{-16}{23}$	$\frac{-89}{23}$	$\frac{31}{23}$	0	18	-18	-24
10124	0	$\frac{-3}{23}$	$\frac{22}{23}$	$\frac{-16}{23}$	$\frac{58}{23}$	$\frac{-6}{23}$	0	0	0	8
13013	0	$\frac{-9}{23}$	$\frac{42}{23}$	$\frac{-40}{23}$	$\frac{90}{23}$	$\frac{22}{23}$	6	24	-12	12
13211	0	$\frac{-9}{23}$	$\frac{38}{23}$	$\frac{-80}{23}$	$\frac{134}{23}$	$\frac{-94}{23}$	6	0	24	24
15011	0	$\frac{27}{23}$	$\frac{32}{23}$	$\frac{-104}{23}$	$\frac{56}{23}$	$\frac{8}{23}$	10	8	4	-20
13031	0	$\frac{27}{23}$	$\frac{16}{23}$	$\frac{-52}{23}$	$\frac{28}{23}$	$\frac{4}{23}$	6	0	6	-18
22112	0	$\frac{9}{23}$	$\frac{59}{23}$	$\frac{-64}{23}$	$\frac{35}{23}$	$\frac{55}{23}$	8	28	-4	-16
20312	0	$\frac{-3}{23}$	$\frac{65}{23}$	$\frac{-64}{23}$	$\frac{2}{23}$	$\frac{-70}{23}$	4	0	-8	0
20132	0	$\frac{9}{23}$	$\frac{41}{23}$	$\frac{-32}{23}$	$\frac{-149}{46}$	$\frac{55}{46}$	4	30	-14	-24
20114	0	$\frac{-3}{23}$	$\frac{67}{23}$	$\frac{-32}{23}$	$\frac{1}{23}$	$\frac{-35}{23}$	4	4	4	0
21023	0	$\frac{-9}{23}$	$\frac{42}{23}$	$\frac{-40}{23}$	$\frac{90}{23}$	$\frac{22}{23}$	6	24	-12	12
21041	0	$\frac{27}{23}$	$\frac{39}{23}$	$\frac{-75}{23}$	$\frac{-87}{23}$	$\frac{-42}{23}$	7	-4	10	-26
21221	0	$\frac{-9}{23}$	$\frac{38}{23}$	$\frac{-80}{23}$	$\frac{134}{23}$	$\frac{-94}{23}$	6	-8	12	20
23021	0	$\frac{27}{23}$	$\frac{32}{23}$	$\frac{-104}{23}$	$\frac{56}{23}$	$\frac{8}{23}$	10	8	4	-20
33011	0	$\frac{27}{23}$	$\frac{64}{23}$	$\frac{-208}{23}$	$\frac{112}{23}$	$\frac{16}{23}$	18	24	0	-24
30122	0	$\frac{9}{23}$	$\frac{59}{23}$	$\frac{-64}{23}$	$\frac{35}{23}$	$\frac{55}{23}$	12	40	0	-16
31013	0	$\frac{-9}{23}$	$\frac{84}{23}$	$\frac{-80}{23}$	$\frac{180}{23}$	$\frac{44}{23}$	14	40	-16	8
31211	0	$\frac{-9}{23}$	$\frac{76}{23}$	$\frac{-114}{23}$	$\frac{130}{23}$	$\frac{-96}{23}$	12	0	12	12
31031	0	$\frac{27}{23}$	$\frac{55}{23}$	$\frac{-127}{23}$	$\frac{-59}{23}$	$\frac{-38}{23}$	15	12	6	-30
40112	0	$\frac{9}{23}$	$\frac{95}{23}$	$\frac{-128}{23}$	$\frac{254}{23}$	$\frac{110}{23}$	24	48	24	0
41021	0	$\frac{27}{23}$	$\frac{64}{23}$	$\frac{-208}{23}$	$\frac{112}{23}$	$\frac{16}{23}$	26	40	-4	-28
51011	0	$\frac{27}{23}$	$\frac{82}{23}$	$\frac{-370}{23}$	$\frac{454}{23}$	$\frac{124}{23}$	40	80	-20	-20

ijklm	$\alpha_{4,21}$	$\alpha_{4,22}$	$\alpha_{4,23}$	$\alpha_{4,24}$	$\alpha_{4,25}$	$\alpha_{4,26}$	$\alpha_{4,27}$	$\alpha_{4,28}$	$\alpha_{4,29}$	$\alpha_{4,30}$
00512	$\frac{-416}{23}$	0	$\frac{2912}{23}$	$\frac{2912}{23}$	$\frac{-5}{2}$	$\frac{7}{6}$	$\frac{3}{2}$	$\frac{31}{6}$	$\frac{40}{3}$	$\frac{-56}{3}$
00134	$\frac{140}{23}$	$\frac{272}{23}$	$\frac{-1312}{23}$	$\frac{-1072}{23}$	$\frac{3}{4}$	$\frac{-1}{4}$	$\frac{-3}{4}$	$\frac{-7}{4}$	-4	4
00332	$\frac{140}{23}$	$\frac{272}{23}$	$\frac{-2416}{23}$	$\frac{-2176}{23}$	$\frac{3}{2}$	0	$\frac{-3}{2}$	-3	-8	8
00152	$\frac{220}{23}$	$\frac{-544}{23}$	$\frac{1240}{23}$	$\frac{760}{23}$	$\frac{-5}{8}$	$\frac{-7}{8}$	$\frac{5}{8}$	$\frac{3}{8}$	-2	6
00314	$\frac{-232}{23}$	0	$\frac{1440}{23}$	$\frac{1440}{23}$	$\frac{-3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}$	8	-8
00116	$\frac{-48}{23}$	0	$\frac{704}{23}$	$\frac{704}{23}$	-1	$\frac{1}{3}$	0	$\frac{4}{3}$	$\frac{16}{3}$	$\frac{-16}{3}$
02312	$\frac{-136}{23}$	$\frac{640}{23}$	$\frac{-1312}{23}$	$\frac{-1440}{23}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-3}{2}$	$\frac{-3}{2}$	0	0

02132	$\frac{220}{23}$	$\frac{-176}{23}$	$\frac{1792}{23}$	$\frac{944}{23}$	$\frac{-3}{4}$	$\frac{-3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0	0
04112	$\frac{496}{23}$	$\frac{192}{23}$	$\frac{1792}{23}$	$\frac{576}{23}$	-1	-1	0	0	0	0
02114	$\frac{48}{23}$	$\frac{640}{23}$	$\frac{-576}{23}$	$\frac{-704}{23}$	0	0	-1	-1	0	0
03023	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	0	0	0	0	0	0
01223	$\frac{-96}{23}$	$\frac{-320}{23}$	$\frac{1344}{23}$	$\frac{1408}{23}$	-1	0	1	2	4	-4
01043	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	0	0	0	0	0	0
01241	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	1	2	-1	0	-4	-12
03041	$\frac{64}{23}$	$\frac{-1216}{23}$	$\frac{3136}{23}$	$\frac{1024}{23}$	0	0	0	0	0	0
01421	$\frac{-96}{23}$	$\frac{-1056}{23}$	$\frac{3552}{23}$	$\frac{2880}{23}$	-2	0	2	4	8	-8
03221	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	1	2	-1	0	-4	-12
05021	$\frac{64}{23}$	$\frac{-1216}{23}$	$\frac{3136}{23}$	$\frac{1024}{23}$	0	0	0	0	0	0
01025	$\frac{-96}{23}$	$\frac{-320}{23}$	$\frac{1344}{23}$	$\frac{1408}{23}$	0	0	0	0	0	0
01061	$\frac{248}{23}$	$\frac{-1400}{23}$	$\frac{2216}{23}$	$\frac{656}{23}$	0	0	0	0	0	0
11213	$\frac{-96}{23}$	$\frac{-320}{23}$	$\frac{1344}{23}$	$\frac{1408}{23}$	-1	0	1	2	4	-4
11033	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	1	$\frac{-1}{2}$	0	$\frac{-3}{2}$	-6	6
11411	$\frac{-96}{23}$	$\frac{-1056}{23}$	$\frac{3552}{23}$	$\frac{2880}{23}$	-2	0	2	4	8	-8
11015	$\frac{-96}{23}$	$\frac{-320}{23}$	$\frac{1344}{23}$	$\frac{1408}{23}$	0	0	0	0	0	0
11231	$\frac{8}{23}$	$\frac{496}{23}$	$\frac{-2128}{23}$	$\frac{-2080}{23}$	2	$\frac{3}{2}$	-1	$\frac{-3}{2}$	-10	-6
11051	$\frac{248}{23}$	$\frac{-1400}{23}$	$\frac{2216}{23}$	$\frac{656}{23}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	-2	-6
10322	$\frac{416}{23}$	$\frac{-96}{23}$	$\frac{-3520}{23}$	$\frac{-2912}{23}$	$\frac{5}{2}$	$\frac{-1}{2}$	$\frac{-3}{2}$	$\frac{-9}{2}$	-16	16
12122	$\frac{-56}{23}$	$\frac{-544}{23}$	$\frac{1792}{23}$	$\frac{1312}{23}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0	0
10142	$\frac{-56}{23}$	$\frac{-544}{23}$	$\frac{1792}{23}$	$\frac{1312}{23}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0	0
10124	$\frac{232}{23}$	$\frac{-96}{23}$	$\frac{-2048}{23}$	$\frac{-1440}{23}$	$\frac{3}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{-5}{2}$	-8	8
13013	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	0	0	0	0	0	0
13211	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	1	2	-1	0	-4	-12
15011	$\frac{64}{23}$	$\frac{-1216}{23}$	$\frac{3136}{23}$	$\frac{1024}{23}$	0	0	0	0	0	0
13031	$\frac{64}{23}$	$\frac{-1216}{23}$	$\frac{3136}{23}$	$\frac{1024}{23}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	-2	-6
22112	$\frac{-56}{23}$	$\frac{-544}{23}$	$\frac{1792}{23}$	$\frac{1312}{23}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0	0
20312	$\frac{416}{23}$	$\frac{-96}{23}$	$\frac{-3520}{23}$	$\frac{-2912}{23}$	$\frac{5}{2}$	$\frac{-1}{2}$	$\frac{-3}{2}$	$\frac{-9}{2}$	-16	16
20132	$\frac{-332}{23}$	$\frac{-176}{23}$	$\frac{2896}{23}$	$\frac{2048}{23}$	$\frac{-1}{2}$	0	$\frac{5}{2}$	3	4	-12
20114	$\frac{232}{23}$	$\frac{-96}{23}$	$\frac{-2048}{23}$	$\frac{-1440}{23}$	$\frac{3}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{-5}{2}$	-8	8
21023	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	2	-1	0	-3	-12	12
21041	$\frac{64}{23}$	$\frac{-1216}{23}$	$\frac{3136}{23}$	$\frac{1024}{23}$	0	1	2	3	-4	-12
21221	$\frac{192}{23}$	$\frac{128}{23}$	$\frac{-2496}{23}$	$\frac{-2816}{23}$	3	1	-1	-3	-16	0
23021	$\frac{64}{23}$	$\frac{-1216}{23}$	$\frac{3136}{23}$	$\frac{1024}{23}$	0	1	2	3	-4	-12
33011	$\frac{64}{23}$	$\frac{-1216}{23}$	$\frac{3136}{23}$	$\frac{1024}{23}$	0	1	2	3	-4	-12
30122	$\frac{-608}{23}$	$\frac{192}{23}$	$\frac{4000}{23}$	$\frac{2784}{23}$	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	8	-24
31013	$\frac{-176}{23}$	$\frac{864}{23}$	$\frac{-1760}{23}$	$\frac{-1344}{23}$	2	-1	0	-3	-12	12

31211	$\frac{192}{23}$	$\frac{128}{23}$	$\frac{-2496}{23}$	$\frac{-2816}{23}$	3	1	-1	-3	-16	0
31031	$\frac{-304}{23}$	$\frac{-848}{23}$	$\frac{4976}{23}$	$\frac{1760}{23}$	0	$\frac{3}{2}$	3	$\frac{9}{2}$	-6	-18
40112	$\frac{-608}{23}$	$\frac{192}{23}$	$\frac{4000}{23}$	$\frac{2784}{23}$	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	8	-24
41021	$\frac{-672}{23}$	$\frac{-480}{23}$	$\frac{6816}{23}$	$\frac{2496}{23}$	0	2	4	6	-8	-24
51011	$\frac{-672}{23}$	$\frac{-480}{23}$	$\frac{6816}{23}$	$\frac{2496}{23}$	0	2	4	6	-8	-24

Table 5.9, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,3,4,6\}$. $ijklm$ represents quadratic form corresponding to tuple $(1^i,2^j,3^k,4^l,6^m)$, $i+j+k+l+m=8$, whose generating functions belong to $M_4(48,\chi_{24})$, with dimension 28.

$ijklm$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$
00035	0	$\frac{1}{261}$	0	$\frac{32}{261}$	0	$\frac{32}{87}$	0	$\frac{1}{87}$	0	$\frac{-376}{87}$
00071	0	$\frac{1}{261}$	0	$\frac{8}{261}$	0	$\frac{24}{29}$	0	$\frac{3}{29}$	0	$\frac{168}{29}$
00053	0	$\frac{1}{261}$	0	$\frac{-16}{261}$	0	$\frac{16}{29}$	0	$\frac{-1}{29}$	0	$\frac{-1376}{87}$
00017	0	$\frac{1}{261}$	0	$\frac{-64}{261}$	0	$\frac{64}{261}$	0	$\frac{-1}{261}$	0	$\frac{448}{29}$
00413	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{16}{261}$	0	0	$\frac{-1}{261}$	$\frac{1504}{261}$	$\frac{512}{87}$
00233	0	$\frac{1}{261}$	$\frac{-4}{261}$	0	$\frac{4}{87}$	0	0	$\frac{1}{87}$	$\frac{374}{261}$	$\frac{-13400}{261}$
00611	0	$\frac{1}{261}$	$\frac{32}{261}$	0	$\frac{32}{261}$	0	0	$\frac{-1}{261}$	$\frac{3008}{261}$	$\frac{33280}{261}$
00215	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{8}{261}$	0		$\frac{-1}{261}$	$\frac{752}{261}$	$\frac{8224}{261}$
00431	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{16}{261}$	0	0	$\frac{-1}{261}$	$\frac{1504}{261}$	$\frac{16576}{261}$
00251	0	$\frac{1}{261}$	$\frac{2}{261}$	0	$\frac{2}{29}$	0	0	$\frac{-1}{29}$	$\frac{938}{261}$	$\frac{11800}{261}$
02213	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{8}{87}$	0	0	$\frac{1}{87}$	$\frac{-296}{261}$	$\frac{-7136}{261}$
04013	0	$\frac{1}{261}$	0	$\frac{-64}{261}$	0	$\frac{64}{29}$	0	$\frac{-1}{29}$	0	$\frac{32}{29}$
02033	0	$\frac{1}{261}$	0	$\frac{-32}{261}$	0	$\frac{32}{29}$	0	$\frac{-1}{29}$	0	$\frac{824}{261}$
02411	0	$\frac{1}{261}$ 0	$\frac{-16}{261}$	0	$\frac{16}{87}$	0	0	$\frac{1}{87}$	$\frac{-592}{261}$	$\frac{-15488}{261}$
04211	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{-1}{29}$	$\frac{1664}{261}$	$\frac{13888}{261}$
06011	0	$\frac{1}{261}$	0	$\frac{64}{261}$	0	$\frac{192}{29}$	0	$\frac{3}{29}$	0	$\frac{5600}{261}$
02015	0	$\frac{1}{261}$	0	$\frac{64}{261}$	0	$\frac{64}{87}$	0	$\frac{1}{87}$	0	$\frac{704}{261}$
02231	0	$\frac{1}{261}$	$\frac{4}{261}$	0	$\frac{4}{29}$	0	0	$\frac{-1}{29}$	$\frac{2398}{261}$	$\frac{15976}{261}$
04031	0	$\frac{1}{261}$	0	$\frac{32}{261}$	0	$\frac{96}{29}$	0	$\frac{3}{29}$	0	$\frac{392}{87}$
02051	0	$\frac{1}{261}$	0	$\frac{16}{261}$	0	$\frac{48}{29}$	0	$\frac{3}{29}$	0	$\frac{2096}{261}$
01322	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{8}{87}$	0	0	$\frac{-1}{87}$	$\frac{980}{261}$	$\frac{15904}{261}$
03122	0	$\frac{1}{261}$	$\frac{-4}{261}$	0	$\frac{4}{29}$	0	0	$\frac{1}{29}$	$\frac{-1372}{261}$	$\frac{-9296}{261}$
01142	0	$\frac{1}{261}$	$\frac{-2}{261}$	0	$\frac{2}{29}$	0	0	$\frac{1}{29}$	$\frac{-686}{261}$	$\frac{-9296}{261}$
01124	0	$\frac{1}{261}$	$\frac{4}{261}$	0	$\frac{4}{87}$	0	0	$\frac{-1}{87}$	$\frac{-32}{261}$	$\frac{7552}{261}$
11132	0	$\frac{1}{261}$	$\frac{-4}{261}$	0	$\frac{4}{29}$	0	0	$\frac{1}{29}$	$\frac{194}{261}$	$\frac{-13472}{261}$
11114	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{8}{87}$	0	0	$\frac{-1}{87}$	$\frac{980}{261}$	$\frac{7552}{261}$
11312	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{16}{87}$	0	0	$\frac{-1}{87}$	$\frac{1960}{261}$	$\frac{15904}{261}$
10223	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{8}{87}$	0	0	$\frac{1}{87}$	$\frac{226}{261}$	$\frac{-19664}{261}$
12023	0	$\frac{1}{261}$	$\frac{4}{261}$	0	$\frac{4}{29}$	0	0	$\frac{-1}{29}$ 0	$\frac{832}{261}$ 0	$\frac{1360}{261}$
10043	0	$\frac{1}{261}$	$\frac{2}{261}$	0	$\frac{2}{29}$	0	0	$\frac{-1}{29}$	$\frac{-367}{261}$	$\frac{1360}{261}$

10241	0	$\frac{1}{261}$	$\frac{4}{261}$	0	$\frac{4}{29}$	0	0	$\frac{-1}{29}$	$\frac{571}{261}$	$\frac{18064}{261}$
12041	0	$\frac{1}{261}$	$\frac{-2}{261}$	0	$\frac{6}{29}$	0	0	$\frac{3}{29}$	$\frac{1393}{261}$	$\frac{-112}{9}$
10421	0	$\frac{1}{261}$	$\frac{-16}{261}$	0	$\frac{16}{87}$	0	0	$\frac{1}{87}$	$\frac{-592}{261}$	$\frac{-36368}{261}$
12221	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{-1}{29}$	$\frac{3230}{261}$	$\frac{18064}{261}$
14021	0	$\frac{1}{261}$	$\frac{-4}{261}$	0	$\frac{12}{29}$	0	0	$\frac{3}{29}$	$\frac{1220}{261}$	$\frac{-112}{9}$
10025	0	$\frac{1}{261}$	$\frac{-4}{261}$	0	$\frac{4}{87}$	0	0	$\frac{1}{87}$	$\frac{-148}{261}$	$\frac{-2960}{261}$
10061	0	$\frac{1}{261}$	$\frac{-1}{261}$	0	$\frac{3}{29}$	0	0	$\frac{3}{29}$	$\frac{2915}{261}$	$\frac{-184}{9}$
13112	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{1}{29}$	$\frac{388}{261}$	$\frac{-9296}{261}$
22211	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{16}{29}$	0	0	$\frac{-1}{29}$	$\frac{3328}{261}$	$\frac{18064}{261}$
22013	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{-1}{29}$	$\frac{1664}{261}$	$\frac{1360}{261}$
22031	0	$\frac{1}{261}$	$\frac{-4}{261}$	0	$\frac{12}{29}$	0	0	$\frac{3}{29}$	$\frac{2786}{261}$	$\frac{-328}{9}$
20213	0	$\frac{1}{261}$	$\frac{-16}{261}$	0	$\frac{16}{87}$	0	0	$\frac{1}{87}$	$\frac{-592}{261}$	$\frac{-19664}{261}$
20033	0	$\frac{1}{261}$	$\frac{4}{261}$	0	$\frac{4}{29}$	0	0	$\frac{-1}{29}$	$\frac{-734}{261}$	$\frac{11800}{261}$
20411	0	$\frac{1}{261}$	$\frac{-32}{261}$	0	$\frac{32}{87}$	0	0	$\frac{1}{87}$	$\frac{-3272}{261}$	$\frac{-36368}{261}$
24011	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{24}{29}$	0	0	$\frac{3}{29}$	$\frac{2440}{261}$	$\frac{-112}{9}$
20015	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{8}{87}$	0	0	$\frac{1}{87}$	$\frac{-296}{261}$	$\frac{-2960}{261}$
20231	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{-1}{29}$	$\frac{1664}{261}$	$\frac{28504}{261}$
20051	0	$\frac{1}{261}$	$\frac{-2}{261}$	0	$\frac{6}{29}$	0	0	$\frac{3}{29}$	$\frac{5830}{261}$	$\frac{-400}{9}$
21122	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{1}{29}$	$\frac{388}{261}$	$\frac{-17648}{261}$
30023	0	$\frac{1}{261}$	$\frac{8}{261}$	0	$\frac{8}{29}$	0	0	$\frac{-1}{29}$	$\frac{98}{261}$	$\frac{22240}{261}$
30041	0	$\frac{1}{261}$	$\frac{-4}{261}$	0	$\frac{12}{29}$	0	0	$\frac{3}{29}$	$\frac{7223}{261}$	$\frac{-544}{9}$
30221	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{16}{29}$	0	0	$\frac{-1}{29}$	$\frac{5416}{261}$	$\frac{38944}{261}$
32021	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{24}{29}$	0	0	$\frac{3}{29}$	$\frac{4006}{261}$	$\frac{-544}{9}$
31112	0	$\frac{1}{261}$	$\frac{-16}{261}$	0	$\frac{16}{29}$	0	0	$\frac{1}{29}$	$\frac{776}{261}$	$\frac{-17648}{261}$
40013	0	$\frac{1}{261}$	$\frac{16}{261}$	0	$\frac{16}{29}$	0	0	$\frac{-1}{29}$	$\frac{3328}{261}$	$\frac{22240}{261}$
40211	0	$\frac{1}{261}$	$\frac{32}{261}$	0	$\frac{32}{29}$	0	0	$\frac{-1}{29}$	$\frac{10832}{261}$	$\frac{38944}{261}$
42011	0	$\frac{1}{261}$	$\frac{-16}{261}$	0	$\frac{48}{29}$	0	0	$\frac{3}{29}$	$\frac{4880}{261}$	$\frac{-544}{9}$
40031	0	$\frac{1}{261}$	$\frac{-8}{261}$	0	$\frac{24}{29}$	0	0	$\frac{3}{29}$	$\frac{5572}{261}$	$\frac{-616}{9}$
50021	0	$\frac{1}{261}$	$\frac{-16}{261}$	0	$\frac{48}{29}$	0	0	$\frac{3}{29}$	$\frac{704}{261}$	$\frac{-688}{9}$
60011	0	$\frac{1}{261}$	$\frac{-32}{261}$	0	$\frac{96}{29}$	0	0	$\frac{3}{29}$	$\frac{-4856}{261}$	$\frac{-688}{9}$

$ijklm$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$	$\alpha_{4,16}$	$\alpha_{4,17}$	$\alpha_{4,18}$	$\alpha_{4,19}$	$\alpha_{4,20}$
00035	0	$\frac{8}{29}$	0	$\frac{632}{87}$	0	$\frac{352}{29}$	0	$\frac{1328}{29}$	0	$\frac{-4192}{87}$
00071	0	$\frac{420}{29}$	0	$\frac{-1064}{29}$	0	$\frac{-672}{29}$	0	$\frac{-1680}{29}$	0	$\frac{5600}{29}$
00053	0	$\frac{-292}{29}$	0	$\frac{1984}{87}$	0	$\frac{3840}{29}$	0	$\frac{544}{29}$	0	$\frac{-16736}{87}$
00017	0	$\frac{280}{29}$	0	$\frac{-672}{29}$	0	$\frac{-3584}{29}$	0	$\frac{-2016}{29}$	0	$\frac{5824}{29}$
00413	$\frac{512}{87}$	$\frac{1320}{29}$	$\frac{-1432}{261}$	$\frac{-8432}{87}$	$\frac{-10096}{261}$	$\frac{-132608}{261}$	$\frac{-1024}{87}$	-256	$\frac{6128}{87}$	$\frac{227648}{261}$
00233	$\frac{86}{87}$	$\frac{-136}{3}$	$\frac{-40}{261}$	$\frac{20968}{261}$	$\frac{-1184}{87}$	$\frac{12256}{29}$	$\frac{-1012}{87}$	$\frac{15280}{87}$	$\frac{10880}{261}$	$\frac{-198944}{261}$
00611	$\frac{1024}{87}$	$\frac{2944}{29}$	$\frac{-3908}{261}$	$\frac{-16784}{87}$	$\frac{-22280}{261}$	$\frac{-266240}{261}$	$\frac{-2048}{87}$	-544	$\frac{11560}{87}$	$\frac{478208}{261}$

00215	$\frac{256}{87}$	$\frac{624}{29}$	$\frac{-716}{261}$	$\frac{-4256}{87}$	$\frac{-5048}{261}$	$\frac{-65792}{261}$	$\frac{-512}{87}$	-128	$\frac{3064}{87}$	$\frac{110720}{261}$
00431	$\frac{512}{87}$	$\frac{1320}{29}$	$\frac{-1432}{261}$	$\frac{-8432}{87}$	$\frac{-10096}{261}$	$\frac{-132608}{261}$	$\frac{-1024}{87}$	-256	$\frac{6128}{87}$	$\frac{227648}{261}$
00251	$\frac{278}{87}$	$\frac{4804}{87}$	$\frac{-1360}{261}$	$\frac{-21776}{261}$	$\frac{-2968}{87}$	$\frac{-39104}{87}$	$\frac{-1396}{87}$	$\frac{-1328}{87}$	$\frac{7856}{261}$	$\frac{187168}{261}$
02213	$\frac{-176}{87}$	$\frac{-112}{3}$	$\frac{964}{261}$	$\frac{12616}{261}$	$\frac{1112}{87}$	$\frac{7616}{29}$	$\frac{64}{87}$	$\frac{6928}{87}$	$\frac{-1208}{261}$	$\frac{-140480}{261}$
04013	0	$\frac{-56}{29}$	0	$\frac{160}{29}$	0	$\frac{640}{29}$	0	$\frac{-672}{29}$	0	$\frac{-2112}{29}$
02033	0	$\frac{56}{87}$	0	$\frac{272}{261}$	0	$\frac{-496}{87}$	0	$\frac{-3760}{87}$	0	$\frac{560}{261}$
02411	$\frac{-352}{87}$	$\frac{-232}{3}$	$\frac{1928}{261}$	$\frac{25144}{261}$	$\frac{2224}{87}$	$\frac{15968}{29}$	$\frac{128}{87}$	$\frac{15280}{87}$	$\frac{-2416}{261}$	$\frac{-282464}{261}$
04211	$\frac{416}{87}$	$\frac{5152}{87}$	$\frac{-4396}{261}$	$\frac{-25952}{261}$	$\frac{-5608}{87}$	$\frac{-39104}{87}$	$\frac{-1408}{87}$	$\frac{-5504}{87}$	$\frac{16808}{261}$	$\frac{195520}{261}$
06011	0	$\frac{2168}{87}$	0	$\frac{-7480}{261}$	0	$\frac{-16096}{87}$	0	$\frac{-3952}{87}$	0	$\frac{68768}{261}$
02015	0	$\frac{-184}{87}$	0	$\frac{-472}{261}$	0	$\frac{-416}{29}$	0	$\frac{1040}{87}$	0	$\frac{-9952}{261}$
02231	$\frac{730}{87}$	$\frac{5848}{87}$	$\frac{-4808}{261}$	$\frac{-30128}{261}$	$\frac{-7328}{87}$	$\frac{-46064}{87}$	$\frac{-1748}{87}$	$\frac{-9680}{87}$	$\frac{28240}{261}$	$\frac{241456}{261}$
04031	0	$\frac{152}{29}$	0	$\frac{-904}{87}$	0	$\frac{-960}{29}$	0	16	0	$\frac{-64}{87}$
02051	0	$\frac{644}{87}$	0	$\frac{-4504}{261}$	0	$\frac{-6016}{87}$	0	$\frac{-112}{87}$	0	$\frac{19616}{261}$
01322	$\frac{644}{87}$	$\frac{5128}{87}$	$\frac{-1114}{261}$	$\frac{-26504}{261}$	$\frac{-888}{29}$	$\frac{-45712}{87}$	$\frac{308}{87}$	$\frac{-18080}{87}$	$\frac{15272}{261}$	$\frac{248848}{261}$
03122	$\frac{-202}{87}$	$\frac{-3968}{87}$	$\frac{2234}{261}$	$\frac{560}{9}$	$\frac{5000}{87}$	$\frac{992}{3}$	$\frac{740}{87}$	$\frac{6880}{87}$	$\frac{-7144}{261}$	$\frac{-166496}{261}$
01142	$\frac{-275}{87}$	$\frac{-3968}{87}$	$\frac{1378}{261}$	$\frac{560}{9}$	$\frac{2848}{87}$	$\frac{992}{3}$	$\frac{-152}{87}$	$\frac{6880}{87}$	$\frac{-4616}{261}$	$\frac{-166496}{261}$
01124	$\frac{322}{87}$	$\frac{2344}{87}$	$\frac{226}{261}$	$\frac{-11888}{261}$	$\frac{136}{29}$	$\frac{-22048}{87}$	$\frac{676}{87}$	$\frac{-9728}{87}$	$\frac{4504}{261}$	$\frac{119392}{261}$
11132	$\frac{-202}{87}$	$\frac{-4664}{87}$	$\frac{1190}{261}$	$\frac{776}{9}$	$\frac{1172}{87}$	$\frac{1352}{3}$	$\frac{-304}{87}$	$\frac{13144}{87}$	$\frac{164}{261}$	$\frac{-218696}{261}$
11114	$\frac{470}{87}$	$\frac{2344}{87}$	$\frac{-70}{261}$	$\frac{-11888}{261}$	$\frac{-424}{29}$	$\frac{-22048}{87}$	$\frac{308}{87}$	$\frac{-9728}{87}$	$\frac{11096}{261}$	$\frac{119392}{261}$
11312	$\frac{766}{87}$	$\frac{5128}{87}$	$\frac{-1706}{261}$	$\frac{-26504}{261}$	$\frac{-1312}{29}$	$\frac{-45712}{87}$	$\frac{-428}{87}$	$\frac{-18080}{87}$	$\frac{22192}{261}$	$\frac{248848}{261}$
10223	$\frac{172}{87}$	$\frac{-160}{3}$	$\frac{964}{261}$	$\frac{29320}{261}$	$\frac{68}{87}$	$\frac{16896}{29}$	$\frac{64}{87}$	$\frac{23632}{87}$	$\frac{8188}{261}$	$\frac{-257408}{261}$
12023	$\frac{382}{87}$	$\frac{280}{87}$	$\frac{412}{261}$	$\frac{-896}{261}$	$\frac{-716}{87}$	$\frac{-2912}{87}$	$\frac{340}{87}$	$\frac{-5504}{87}$	$\frac{10492}{261}$	$\frac{20128}{261}$
10043	$\frac{-331}{87}$	$\frac{280}{87}$	$\frac{1250}{261}$	$\frac{-896}{261}$	$\frac{2948}{87}$	$\frac{-2912}{87}$	$\frac{1214}{87}$	$\frac{-5504}{87}$	$\frac{-16156}{261}$	$\frac{20128}{261}$
10241	$\frac{-53}{87}$	$\frac{6544}{87}$	$\frac{-110}{261}$	$\frac{-34304}{261}$	$\frac{-20}{87}$	$\frac{-53024}{87}$	$\frac{-182}{87}$	$\frac{-13856}{87}$	$\frac{-8300}{261}$	$\frac{287392}{261}$
12041	$\frac{163}{87}$	$\frac{-1256}{87}$	$\frac{-1214}{261}$	$\frac{2056}{261}$	$\frac{-2492}{87}$	$\frac{10336}{87}$	$\frac{-1178}{87}$	$\frac{6736}{87}$	$\frac{14284}{261}$	$\frac{-69152}{261}$
10421	$\frac{-178}{87}$	$\frac{-328}{3}$	$\frac{2972}{261}$	$\frac{58552}{261}$	$\frac{1876}{87}$	$\frac{30816}{29}$	$\frac{-916}{87}$	$\frac{48688}{87}$	$\frac{716}{261}$	$\frac{-516320}{261}$
12221	$\frac{1112}{87}$	$\frac{6544}{87}$	$\frac{-4396}{261}$	$\frac{-34304}{261}$	$\frac{-8044}{87}$	$\frac{-53024}{87}$	$\frac{-1408}{87}$	$\frac{-13856}{87}$	$\frac{38732}{261}$	$\frac{287392}{261}$
10025	$\frac{326}{87}$	$\frac{-1256}{87}$	$\frac{-340}{261}$	$\frac{2056}{261}$	$\frac{-1852}{87}$	$\frac{10336}{87}$	$\frac{-268}{87}$	$\frac{6736}{87}$	$\frac{17084}{261}$	$\frac{-69152}{261}$
10025	$\frac{86}{87}$	$\frac{8}{3}$	$\frac{1004}{261}$	$\frac{4264}{261}$	$\frac{1252}{87}$	$\frac{1120}{29}$	$\frac{1076}{87}$	$\frac{6928}{87}$	$\frac{-2692}{261}$	$\frac{-15200}{261}$
10061	$\frac{734}{87}$	$\frac{-212}{87}$	$\frac{-5566}{261}$	$\frac{-6296}{261}$	$\frac{-7336}{87}$	$\frac{21472}{87}$	$\frac{-850}{87}$	$\frac{-1616}{87}$	$\frac{36896}{261}$	$\frac{-27392}{261}$
13112	$\frac{118}{87}$	$\frac{-3968}{87}$	$\frac{1858}{261}$	$\frac{560}{9}$	$\frac{952}{87}$	$\frac{992}{3}$	$\frac{436}{87}$	$\frac{6880}{87}$	$\frac{8680}{261}$	$\frac{-166496}{261}$
22211	$\frac{1180}{87}$	$\frac{6544}{87}$	$\frac{-3572}{261}$	$\frac{-34304}{261}$	$\frac{-7040}{87}$	$\frac{-53024}{87}$	$\frac{-728}{87}$	$\frac{-13856}{87}$	$\frac{37792}{261}$	$\frac{287392}{261}$
22013	$\frac{764}{87}$	$\frac{280}{87}$	$\frac{824}{261}$	$\frac{-896}{261}$	$\frac{-1432}{87}$	$\frac{-2912}{87}$	$\frac{680}{87}$	$\frac{-5504}{87}$	$\frac{20984}{261}$	$\frac{20128}{261}$
22031	$\frac{326}{87}$	$\frac{-3344}{87}$	$\frac{-2428}{261}$	$\frac{10408}{261}$	$\frac{-4984}{87}$	$\frac{29824}{87}$	$\frac{-2356}{87}$	$\frac{15088}{87}$	$\frac{28568}{261}$	$\frac{-177728}{261}$
20213	$\frac{-4}{87}$	$\frac{-160}{3}$	$\frac{2972}{261}$	$\frac{29320}{261}$	$\frac{3616}{87}$	$\frac{16896}{29}$	$\frac{2216}{87}$	$\frac{23632}{87}$	$\frac{-6592}{261}$	$\frac{-257408}{261}$
20033	$\frac{-662}{87}$	$\frac{2368}{87}$	$\frac{2500}{261}$	$\frac{-13424}{261}$	$\frac{5896}{87}$	$\frac{-29360}{87}$	$\frac{2428}{87}$	$\frac{-18032}{87}$	$\frac{-32312}{261}$	$\frac{141232}{261}$
20411	$\frac{-1052}{87}$	$\frac{-328}{3}$	$\frac{8032}{261}$	$\frac{58552}{261}$	$\frac{10712}{87}$	$\frac{30816}{29}$	$\frac{2344}{87}$	$\frac{48688}{87}$	$\frac{-44504}{261}$	$\frac{-516320}{261}$
24011	$\frac{652}{87}$	$\frac{-1256}{87}$	$\frac{-680}{261}$	$\frac{2056}{261}$	$\frac{-3704}{87}$	$\frac{10336}{87}$	$\frac{-536}{87}$	$\frac{6736}{87}$	$\frac{34168}{261}$	$\frac{-69152}{261}$

20015	$\frac{172}{87}$	$\frac{8}{3}$	$\frac{2008}{261}$	$\frac{4264}{261}$	$\frac{2504}{87}$	$\frac{1120}{29}$	$\frac{2152}{87}$	$\frac{6928}{87}$	$\frac{-5384}{261}$	$\frac{-15200}{261}$
20231	$\frac{68}{87}$	$\frac{9328}{87}$	$\frac{-2308}{261}$	$\frac{-55184}{261}$	$\frac{-1432}{87}$	$\frac{-73904}{87}$	$\frac{680}{87}$	$\frac{-34736}{87}$	$\frac{-4072}{261}$	$\frac{441904}{261}$
20051	$\frac{1468}{87}$	$\frac{-2300}{87}$	$\frac{-11132}{261}$	$\frac{2056}{261}$	$\frac{-14672}{87}$	$\frac{40960}{87}$	$\frac{-1700}{87}$	$\frac{6736}{87}$	$\frac{73792}{261}$	$\frac{-135968}{261}$
21122	$\frac{-56}{87}$	$\frac{-5360}{87}$	$\frac{1858}{261}$	$\frac{992}{9}$	$\frac{1648}{87}$	$\frac{1712}{3}$	$\frac{436}{87}$	$\frac{19408}{87}$	$\frac{2416}{261}$	$\frac{-270896}{261}$
30023	$\frac{-280}{87}$	$\frac{4456}{87}$	$\frac{2912}{261}$	$\frac{-25952}{261}$	$\frac{5180}{87}$	$\frac{-55808}{87}$	$\frac{2768}{87}$	$\frac{-30560}{87}$	$\frac{-21820}{261}$	$\frac{262336}{261}$
30041	$\frac{1631}{87}$	$\frac{-5432}{87}$	$\frac{-12346}{261}$	$\frac{18760}{261}$	$\frac{-17164}{87}$	$\frac{49312}{87}$	$\frac{-2878}{87}$	$\frac{23440}{87}$	$\frac{88076}{261}$	$\frac{-286304}{261}$
30221	$\frac{1354}{87}$	$\frac{12112}{87}$	$\frac{-8792}{261}$	$\frac{-76064}{261}$	$\frac{-10868}{87}$	$\frac{-94784}{87}$	$\frac{316}{87}$	$\frac{-55616}{87}$	$\frac{47188}{261}$	$\frac{596416}{261}$
32021	$\frac{652}{87}$	$\frac{-5432}{87}$	$\frac{-2768}{261}$	$\frac{18760}{261}$	$\frac{-6836}{87}$	$\frac{49312}{87}$	$\frac{-2624}{87}$	$\frac{23440}{87}$	$\frac{45652}{261}$	$\frac{-286304}{261}$
31112	$\frac{410}{87}$	$\frac{-5360}{87}$	$\frac{3194}{261}$	$\frac{992}{9}$	$\frac{1904}{87}$	$\frac{1712}{3}$	$\frac{1916}{87}$	$\frac{19408}{87}$	$\frac{13184}{261}$	$\frac{-270896}{261}$
40013	$\frac{1528}{87}$	$\frac{4456}{87}$	$\frac{1648}{261}$	$\frac{-25952}{261}$	$\frac{-2864}{87}$	$\frac{-55808}{87}$	$\frac{1360}{87}$	$\frac{-30560}{87}$	$\frac{41968}{261}$	$\frac{262336}{261}$
40211	$\frac{3752}{87}$	$\frac{12112}{87}$	$\frac{-16540}{261}$	$\frac{-76064}{261}$	$\frac{-25912}{87}$	$\frac{-94784}{87}$	$\frac{-1456}{87}$	$\frac{-55616}{87}$	$\frac{140312}{261}$	$\frac{596416}{261}$
42011	$\frac{1304}{87}$	$\frac{-5432}{87}$	$\frac{-1360}{261}$	$\frac{18760}{261}$	$\frac{-7408}{87}$	$\frac{49312}{87}$	$\frac{-1072}{87}$	$\frac{23440}{87}$	$\frac{68336}{261}$	$\frac{-286304}{261}$
40031	$\frac{652}{87}$	$\frac{-9608}{87}$	$\frac{-4856}{261}$	$\frac{43816}{261}$	$\frac{-9968}{87}$	$\frac{46528}{87}$	$\frac{-4712}{87}$	$\frac{48496}{87}$	$\frac{57136}{261}$	$\frac{-478400}{261}$
50021	$\frac{-1306}{87}$	$\frac{-13784}{87}$	$\frac{12212}{261}$	$\frac{68872}{261}$	$\frac{7556}{87}$	$\frac{43744}{87}$	$\frac{-6292}{87}$	$\frac{73552}{87}$	$\frac{-16228}{261}$	$\frac{-670496}{261}$
60011	$\frac{-2612}{87}$	$\frac{-13784}{87}$	$\frac{32776}{261}$	$\frac{68872}{261}$	$\frac{27640}{87}$	$\frac{43744}{87}$	$\frac{-4232}{87}$	$\frac{73552}{87}$	$\frac{-78392}{261}$	$\frac{-670496}{261}$

$ijklm$	$\alpha_{4,21}$	$\alpha_{4,22}$	$\alpha_{4,23}$	$\alpha_{4,24}$	$\alpha_{4,25}$	$\alpha_{4,26}$	$\alpha_{4,27}$	$\alpha_{4,28}$
00035	0	$\frac{-9544}{87}$	0	$\frac{16384}{87}$	0	$\frac{332}{87}$	0	$\frac{328}{87}$
00071	0	$\frac{2744}{29}$	0	$\frac{1792}{29}$	0	$\frac{-196}{29}$	0	$\frac{-196}{29}$
00053	0	$\frac{-1664}{87}$	0	$\frac{-12160}{87}$	0	$\frac{1336}{87}$	0	$\frac{1316}{87}$
00017	0	$\frac{2912}{29}$	0	$\frac{-3584}{29}$	0	$\frac{-448}{29}$	0	$\frac{-336}{29}$
00413	$\frac{1640}{87}$	$\frac{3760}{9}$	0	$\frac{-132608}{261}$	$\frac{-512}{87}$	$\frac{-16576}{261}$	$\frac{512}{87}$	$\frac{-15536}{261}$
00233	$\frac{3404}{261}$	$\frac{-75800}{261}$	$\frac{-12512}{261}$	$\frac{99200}{261}$	$\frac{-382}{261}$	$\frac{13396}{261}$	$\frac{-386}{261}$	$\frac{13400}{261}$
00611	$\frac{2236}{87}$	$\frac{7504}{9}$	0	$\frac{-266240}{261}$	$\frac{-1024}{87}$	$\frac{-33280}{261}$	$\frac{-1024}{87}$	$\frac{-32240}{261}$
00215	$\frac{820}{87}$	$\frac{1888}{9}$	0	$\frac{-65792}{261}$	$\frac{-256}{87}$	$\frac{-8224}{261}$	$\frac{-256}{87}$	$\frac{-7184}{261}$
00431	$\frac{1640}{87}$	$\frac{3760}{9}$	0	$\frac{-132608}{261}$	$\frac{-512}{87}$	$\frac{-16576}{261}$	$\frac{-512}{87}$	$\frac{-15536}{261}$
00251	$\frac{244}{9}$	$\frac{36064}{261}$	$\frac{20896}{261}$	$\frac{-78400}{261}$	$\frac{-958}{261}$	$\frac{-11792}{261}$	$\frac{-962}{261}$	$\frac{-11812}{261}$
02213	$\frac{-500}{261}$	$\frac{-42392}{261}$	$\frac{-8320}{261}$	$\frac{65792}{261}$	$\frac{280}{261}$	$\frac{8176}{261}$	$\frac{272}{261}$	$\frac{9224}{261}$
04013	0	$\frac{-1248}{29}$	0	$\frac{-3072}{29}$	0	$\frac{144}{29}$	0	$\frac{240}{29}$
02033	0	$\frac{4064}{261}$	0	$\frac{-44672}{261}$	0	$\frac{-28}{261}$	0	$\frac{-128}{261}$
02411	$\frac{-1000}{261}$	$\frac{-71624}{261}$	$\frac{-16640}{261}$	$\frac{132608}{261}$	$\frac{560}{261}$	$\frac{16528}{261}$	$\frac{544}{261}$	$\frac{17576}{261}$
04211	$\frac{364}{9}$	$\frac{19360}{261}$	$\frac{16768}{261}$	$\frac{-61696}{261}$	$\frac{-1744}{261}$	$\frac{-11792}{261}$	$\frac{-1760}{261}$	$\frac{-10768}{261}$
06011	0	$\frac{9992}{261}$	0	$\frac{21760}{261}$	0	$\frac{-4288}{261}$	0	$\frac{-3608}{261}$
02015	0	$\frac{-14872}{261}$	0	$\frac{32512}{261}$	0	$\frac{80}{261}$	0	$\frac{1096}{261}$
02231	$\frac{380}{9}$	$\frac{52768}{261}$	$\frac{16736}{261}$	$\frac{-95104}{261}$	$\frac{-2438}{261}$	$\frac{-14924}{261}$	$\frac{-2446}{261}$	$\frac{-14944}{261}$
04031	0	$\frac{-8}{3}$	0	$\frac{12544}{87}$	0	$\frac{-4}{87}$	0	$\frac{-56}{87}$
02051	0	$\frac{10664}{261}$	0	$\frac{28864}{261}$	0	$\frac{-1528}{261}$	0	$\frac{-1580}{261}$
01322	$\frac{830}{261}$	$\frac{93928}{261}$	$\frac{12544}{261}$	$\frac{-131584}{261}$	$\frac{-1012}{261}$	$\frac{-16424}{261}$	$\frac{-1016}{261}$	$\frac{-16432}{261}$
03122	$\frac{-3478}{261}$	$\frac{-34640}{261}$	$\frac{-10400}{261}$	$\frac{62720}{261}$	$\frac{1340}{261}$	$\frac{9808}{261}$	$\frac{1330}{261}$	$\frac{9824}{261}$
01142	$\frac{-434}{261}$	$\frac{-34640}{261}$	$\frac{-13552}{261}$	$\frac{62720}{261}$	$\frac{670}{261}$	$\frac{9808}{261}$	$\frac{665}{261}$	$\frac{9824}{261}$

01124	$\frac{-1934}{261}$	$\frac{45904}{261}$	$\frac{6272}{261}$	$\frac{-64768}{261}$	$\frac{16}{261}$	$\frac{-8072}{261}$	$\frac{14}{261}$	$\frac{-8080}{261}$
11132	$\frac{698}{261}$	$\frac{-68048}{261}$	$\frac{-14576}{261}$	$\frac{96128}{261}$	$\frac{296}{261}$	$\frac{13984}{261}$	$\frac{286}{261}$	$\frac{12956}{261}$
11114	$\frac{-214}{261}$	$\frac{45904}{261}$	$\frac{4192}{261}$	$\frac{-64768}{261}$	$\frac{-490}{261}$	$\frac{-8072}{261}$	$\frac{-494}{261}$	$\frac{-8080}{261}$
11312	$\frac{4270}{261}$	$\frac{93928}{261}$	$\frac{8384}{261}$	$\frac{-131584}{261}$	$\frac{-1502}{261}$	$\frac{-16424}{261}$	$\frac{-1510}{261}$	$\frac{-16432}{261}$
10223	$\frac{-500}{261}$	$\frac{-109208}{261}$	$\frac{-8320}{261}$	$\frac{132608}{261}$	$\frac{280}{261}$	$\frac{18616}{261}$	$\frac{272}{261}$	$\frac{17576}{261}$
12023	$\frac{-16}{9}$	$\frac{19360}{261}$	$\frac{32}{261}$	$\frac{-61696}{261}$	$\frac{-350}{261}$	$\frac{-1352}{261}$	$\frac{-358}{261}$	$\frac{-2416}{261}$
10043	$\frac{-152}{9}$	$\frac{19360}{261}$	$\frac{4192}{261}$	$\frac{-61696}{261}$	$\frac{869}{261}$	$\frac{-1352}{261}$	$\frac{865}{261}$	$\frac{-2416}{261}$
10241	$\frac{92}{9}$	$\frac{86176}{261}$	$\frac{25088}{261}$	$\frac{-128512}{261}$	$\frac{-89}{261}$	$\frac{-18056}{261}$	$\frac{-97}{261}$	$\frac{-19120}{261}$
12041	$\frac{6208}{261}$	$\frac{-11384}{261}$	$\frac{-6208}{261}$	$\frac{53504}{261}$	$\frac{-923}{261}$	$\frac{4264}{261}$	$\frac{-937}{261}$	$\frac{3272}{261}$
10421	$\frac{-1000}{261}$	$\frac{-205256}{261}$	$\frac{-16640}{261}$	$\frac{266240}{261}$	$\frac{1082}{261}$	$\frac{35320}{261}$	$\frac{1066}{261}$	$\frac{34280}{261}$
12221	$\frac{364}{9}$	$\frac{86176}{261}$	$\frac{16768}{261}$	$\frac{-128512}{261}$	$\frac{-2788}{261}$	$\frac{-18056}{261}$	$\frac{-2804}{261}$	$\frac{-19120}{261}$
14021	$\frac{4064}{261}$	$\frac{-11384}{261}$	$\frac{-4064}{261}$	$\frac{53504}{261}$	$\frac{-802}{261}$	$\frac{4264}{261}$	$\frac{-830}{261}$	$\frac{3272}{261}$
10025	$\frac{-3904}{261}$	$\frac{-42392}{261}$	$\frac{4192}{261}$	$\frac{65792}{261}$	$\frac{662}{261}$	$\frac{1912}{261}$	$\frac{658}{261}$	$\frac{872}{261}$
10061	$\frac{9368}{261}$	$\frac{5320}{261}$	$\frac{-9368}{261}$	$\frac{36800}{261}$	$\frac{-2419}{261}$	$\frac{5308}{261}$	$\frac{-2426}{261}$	$\frac{4316}{261}$
13112	$\frac{-1214}{261}$	$\frac{-34640}{261}$	$\frac{-12448}{261}$	$\frac{62720}{261}$	$\frac{70}{261}$	$\frac{9808}{261}$	$\frac{50}{261}$	$\frac{9824}{261}$
22211	$\frac{332}{9}$	$\frac{86176}{261}$	$\frac{16832}{261}$	$\frac{-128512}{261}$	$\frac{-2444}{261}$	$\frac{-18056}{261}$	$\frac{-2476}{261}$	$\frac{-19120}{261}$
22013	$\frac{-32}{9}$	$\frac{19360}{261}$	$\frac{64}{261}$	$\frac{-61696}{261}$	$\frac{-700}{261}$	$\frac{-1352}{261}$	$\frac{-716}{261}$	$\frac{-2416}{261}$
22031	$\frac{12416}{261}$	$\frac{-44792}{261}$	$\frac{-12416}{261}$	$\frac{86912}{261}$	$\frac{-1846}{261}$	$\frac{11572}{261}$	$\frac{-1874}{261}$	$\frac{9536}{261}$
20213	$\frac{-8308}{261}$	$\frac{-109208}{261}$	$\frac{64}{261}$	$\frac{132608}{261}$	$\frac{1604}{261}$	$\frac{18616}{261}$	$\frac{1588}{261}$	$\frac{17576}{261}$
20033	$\frac{-304}{9}$	$\frac{52768}{261}$	$\frac{8384}{261}$	$\frac{-95104}{261}$	$\frac{1738}{261}$	$\frac{-10748}{261}$	$\frac{1730}{261}$	$\frac{-12856}{261}$
20411	$\frac{-16616}{261}$	$\frac{-205256}{261}$	$\frac{128}{261}$	$\frac{266240}{261}$	$\frac{4252}{261}$	$\frac{35320}{261}$	$\frac{4220}{261}$	$\frac{34280}{261}$
24011	$\frac{8128}{261}$	$\frac{-11384}{261}$	$\frac{-8128}{261}$	$\frac{53504}{261}$	$\frac{-1604}{261}$	$\frac{4264}{261}$	$\frac{-1660}{261}$	$\frac{3272}{261}$
20015	$\frac{-7808}{261}$	$\frac{-42392}{261}$	$\frac{8384}{261}$	$\frac{65792}{261}$	$\frac{1324}{261}$	$\frac{1912}{261}$	$\frac{1316}{261}$	$\frac{872}{261}$
20231	$\frac{76}{9}$	$\frac{152992}{261}$	$\frac{25120}{261}$	$\frac{-195328}{261}$	$\frac{-700}{261}$	$\frac{-27452}{261}$	$\frac{-716}{261}$	$\frac{-29560}{261}$
20051	$\frac{18736}{261}$	$\frac{-28088}{261}$	$\frac{-18736}{261}$	$\frac{70208}{261}$	$\frac{-4838}{261}$	$\frac{12616}{261}$	$\frac{-4852}{261}$	$\frac{10580}{261}$
21122	$\frac{-1214}{261}$	$\frac{-101456}{261}$	$\frac{-12448}{261}$	$\frac{129536}{261}$	$\frac{592}{261}$	$\frac{18160}{261}$	$\frac{572}{261}$	$\frac{16088}{261}$
30023	$\frac{-320}{9}$	$\frac{86176}{261}$	$\frac{8416}{261}$	$\frac{-128512}{261}$	$\frac{1388}{261}$	$\frac{-20144}{261}$	$\frac{1372}{261}$	$\frac{-23296}{261}$
30041	$\frac{24944}{261}$	$\frac{-78200}{261}$	$\frac{-24944}{261}$	$\frac{120320}{261}$	$\frac{-5761}{261}$	$\frac{18880}{261}$	$\frac{-5789}{261}$	$\frac{15800}{261}$
30221	$\frac{332}{9}$	$\frac{219808}{261}$	$\frac{16832}{261}$	$\frac{-262144}{261}$	$\frac{-4010}{261}$	$\frac{-36848}{261}$	$\frac{-4042}{261}$	$\frac{-40000}{261}$
32021	$\frac{16480}{261}$	$\frac{-78200}{261}$	$\frac{-16480}{261}$	$\frac{120320}{261}$	$\frac{-2648}{261}$	$\frac{18880}{261}$	$\frac{-2704}{261}$	$\frac{15800}{261}$
31112	$\frac{-5038}{261}$	$\frac{-101456}{261}$	$\frac{-8192}{261}$	$\frac{129536}{261}$	$\frac{662}{261}$	$\frac{18160}{261}$	$\frac{622}{261}$	$\frac{16088}{261}$
40013	$\frac{-64}{9}$	$\frac{86176}{261}$	$\frac{128}{261}$	$\frac{-128512}{261}$	$\frac{-1400}{261}$	$\frac{-20144}{261}$	$\frac{-1432}{261}$	$\frac{-23296}{261}$
40211	$\frac{844}{9}$	$\frac{219808}{261}$	$\frac{256}{261}$	$\frac{-262144}{261}$	$\frac{-9064}{261}$	$\frac{-36848}{261}$	$\frac{-9128}{261}$	$\frac{-40000}{261}$
42011	$\frac{16256}{261}$	$\frac{-78200}{261}$	$\frac{-16256}{261}$	$\frac{120320}{261}$	$\frac{-3208}{261}$	$\frac{18880}{261}$	$\frac{-3320}{261}$	$\frac{15800}{261}$
40031	$\frac{24832}{261}$	$\frac{-145016}{261}$	$\frac{-24832}{261}$	$\frac{187136}{261}$	$\frac{-3692}{261}$	$\frac{24100}{261}$	$\frac{-3748}{261}$	$\frac{19976}{261}$
50021	$\frac{16256}{261}$	$\frac{-211832}{261}$	$\frac{-16256}{261}$	$\frac{253952}{261}$	$\frac{1490}{261}$	$\frac{29320}{261}$	$\frac{1378}{261}$	$\frac{24152}{261}$
60011	$\frac{-896}{261}$	$\frac{-211832}{261}$	$\frac{896}{261}$	$\frac{253952}{261}$	$\frac{7156}{261}$	$\frac{29320}{261}$	$\frac{6932}{261}$	$\frac{24152}{261}$

Table 5.10, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,4,8\}$. $ijkl$ represents quadratic form corresponding to tuple $(1^i,2^j,4^k,8^l)$, $i+j+k+l+m=8$, whose generating functions belong to $M_4(32,\chi_0)$, with dimension 16.

$ijkl$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$	$\alpha_{4,16}$
1016	$\frac{1}{15360}$	$\frac{-3}{5120}$	$\frac{17}{1920}$	$\frac{-1}{120}$	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{1}{64}$	0	$\frac{31}{64}$	0	2	$\frac{31}{64}$	0	$\frac{13}{8}$	$\frac{3}{4}$	$\frac{-5}{8}$
1034	$\frac{1}{7680}$	$\frac{-3}{2560}$	$\frac{17}{960}$	$\frac{-1}{60}$	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{1}{32}$	0	$\frac{15}{32}$	0	4	$\frac{15}{32}$	0	$\frac{3}{2}$	1	$\frac{-1}{2}$
1052	$\frac{1}{3840}$	$\frac{-3}{1280}$	$\frac{17}{480}$	$\frac{-1}{30}$	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{1}{16}$	0	$\frac{7}{16}$	0	4	$\frac{7}{16}$	0	$\frac{3}{2}$	1	$\frac{-1}{2}$
1115	$\frac{1}{7680}$	$\frac{-1}{7680}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{11}{32}$	$\frac{3}{4}$	2	$\frac{5}{8}$	1	$\frac{11}{8}$	$\frac{1}{4}$	$\frac{-3}{8}$
1133	$\frac{1}{3840}$	$\frac{-1}{3840}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{7}{16}$	$\frac{1}{2}$	4	$\frac{1}{2}$	1	$\frac{5}{4}$	$\frac{1}{2}$	$\frac{-1}{4}$
1151	$\frac{1}{1920}$	$\frac{-1}{1920}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{3}{8}$	1	4	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{-1}{2}$
1214	$\frac{1}{3840}$	$\frac{-1}{3840}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{3}{16}$	$\frac{3}{2}$	4	$\frac{3}{4}$	2	1	0	0
1232	$\frac{1}{1920}$	$\frac{-1}{1920}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{3}{8}$	1	4	$\frac{1}{2}$	2	1	0	0
1313	$\frac{1}{1920}$	$\frac{-1}{1920}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{1}{8}$	2	8	$\frac{3}{4}$	3	$\frac{1}{2}$	0	$\frac{1}{2}$
1331	$\frac{1}{960}$	$\frac{-1}{960}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{1}{4}$	2	4	$\frac{1}{2}$	2	1	0	0
1412	$\frac{1}{960}$	$\frac{-1}{960}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{1}{4}$	2	12	$\frac{1}{2}$	4	0	0	1
1511	$\frac{1}{480}$	$\frac{-1}{480}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{1}{2}$	2	12	0	4	0	0	1
2006	$\frac{1}{7680}$	$\frac{-1}{7680}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{31}{32}$	$\frac{7}{4}$	2	$\frac{31}{32}$	$\frac{7}{4}$	$\frac{13}{4}$	$\frac{3}{2}$	$\frac{-5}{4}$
2024	$\frac{1}{3840}$	$\frac{-1}{3840}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{1}{16}$	0	$\frac{15}{16}$	$\frac{3}{2}$	4	$\frac{15}{16}$	2	3	2	-1
2042	$\frac{1}{1920}$	$\frac{-1}{1920}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{1}{8}$	0	$\frac{7}{8}$	1	4	$\frac{7}{8}$	2	3	2	-1
2105	$\frac{1}{3840}$	$\frac{-1}{3840}$	$\frac{1}{120}$	$\frac{-17}{120}$	$\frac{1}{15}$	$\frac{16}{15}$	0	$\frac{1}{4}$	$\frac{11}{16}$	$\frac{5}{2}$	6	$\frac{5}{4}$	$\frac{11}{4}$	$\frac{11}{4}$	$\frac{1}{2}$	$\frac{-3}{4}$
2123	$\frac{1}{1920}$	$\frac{-1}{1920}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{7}{8}$	2	8	1	3	$\frac{5}{2}$	1	$\frac{-1}{2}$
2141	$\frac{1}{960}$	$\frac{-1}{960}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{3}{4}$	2	4	1	2	3	2	-1
2204	$\frac{1}{1920}$	$\frac{-1}{1920}$	$\frac{1}{60}$	$\frac{-17}{60}$	$\frac{1}{5}$	$\frac{16}{15}$	0	0	$\frac{3}{8}$	3	12	$\frac{3}{2}$	4	2	0	0
2222	$\frac{1}{960}$	$\frac{-1}{960}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{3}{4}$	2	12	1	4	2	0	0
2303	$\frac{1}{960}$	$\frac{-1}{960}$	$\frac{1}{60}$	$\frac{-17}{60}$	$\frac{1}{5}$	$\frac{16}{15}$	0	$\frac{-1}{2}$	$\frac{1}{4}$	3	20	$\frac{3}{2}$	$\frac{11}{2}$	1	0	1
2321	$\frac{1}{480}$	$\frac{-1}{480}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{1}{2}$	2	12	1	4	2	0	0

2402	$\frac{1}{480}$	$\frac{-1}{480}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	-1	$\frac{1}{2}$	2	28	1	7	0	0	2
2501	$\frac{1}{240}$	$\frac{-1}{240}$	$\frac{-1}{30}$	$\frac{17}{30}$	$\frac{-3}{5}$	$\frac{16}{15}$	0	-1	1	0	28	0	7	0	0	2
3014	$\frac{1}{1920}$	$\frac{1}{640}$	$\frac{-17}{480}$	$\frac{1}{30}$	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{1}{16}$	0	$\frac{9}{8}$	$\frac{9}{2}$	4	$\frac{27}{16}$	6	4	2	-1
3032	$\frac{1}{960}$	$\frac{1}{320}$	$\frac{-17}{240}$	$\frac{1}{15}$	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{1}{8}$	0	$\frac{5}{4}$	3	4	$\frac{11}{8}$	6	4	2	-1
3113	$\frac{1}{960}$	$\frac{-1}{960}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	1	5	16	$\frac{7}{4}$	7	3	1	0
3131	$\frac{1}{480}$	$\frac{-1}{480}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	1	4	4	$\frac{3}{2}$	6	4	2	-1
3212	$\frac{1}{480}$	$\frac{-1}{480}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	1	4	28	$\frac{3}{2}$	8	2	0	1
3311	$\frac{1}{240}$	$\frac{-1}{240}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	1	2	28	1	8	2	0	1
4004	$\frac{1}{960}$	$\frac{1}{320}$	$\frac{-13}{240}$	$\frac{-13}{60}$	$\frac{1}{5}$	$\frac{16}{15}$	0	0	$\frac{3}{4}$	9	12	3	12	4	0	0
4022	$\frac{1}{480}$	$\frac{1}{160}$	$\frac{-17}{120}$	$\frac{2}{15}$	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	$\frac{3}{2}$	6	12	2	12	4	0	0
4103	$\frac{1}{480}$	$\frac{-1}{480}$	$\frac{1}{60}$	$\frac{-17}{60}$	$\frac{1}{5}$	$\frac{16}{15}$	0	$\frac{-1}{2}$	$\frac{1}{2}$	9	36	3	$\frac{27}{2}$	2	0	2
4121	$\frac{1}{240}$	$\frac{-1}{240}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	1	6	12	2	12	4	0	0
4202	$\frac{1}{240}$	$\frac{-1}{240}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	-1	1	6	60	2	15	0	0	4
4301	$\frac{1}{120}$	$\frac{-1}{120}$	$\frac{-1}{30}$	$\frac{17}{30}$	$\frac{-3}{5}$	$\frac{16}{15}$	0	-1	2	0	60	0	15	0	0	4
5012	$\frac{1}{240}$	$\frac{1}{240}$	$\frac{-17}{120}$	$\frac{2}{15}$	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{-1}{4}$	0	$\frac{3}{2}$	10	44	$\frac{11}{4}$	20	2	-4	3
5111	$\frac{1}{120}$	$\frac{-1}{120}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	0	0	1	6	44	2	20	2	-4	3
6002	$\frac{1}{120}$	$\frac{-1}{120}$	0	0	$\frac{-1}{15}$	$\frac{16}{15}$	$\frac{-1}{2}$	-1	1	14	124	$\frac{7}{2}$	31	-4	-8	10
6101	$\frac{1}{60}$	$\frac{-1}{60}$	$\frac{-1}{30}$	$\frac{17}{30}$	$\frac{-3}{5}$	$\frac{16}{15}$	0	-1	2	0	124	0	31	-4	-8	10

Table 5.11, corresponding to qudratic forms, consisting of sum of 8 squares with coefficients in $\{1,2,4,8\}$. $ijkl$ represents quadratic form corresponding to tuple $(1^i,2^j,4^k,8^l)$, $i+j+k+l+m=8$, whose generating functions belong to $M_4(32,\chi_8)$, with dimension 16.

$ijkl$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$	$\alpha_{4,7}$	$\alpha_{4,8}$	$\alpha_{4,9}$	$\alpha_{4,10}$	$\alpha_{4,11}$	$\alpha_{4,12}$	$\alpha_{4,13}$	$\alpha_{4,14}$	$\alpha_{4,15}$	$\alpha_{4,16}$
1007	$\frac{1}{88}$	$\frac{-1}{88}$	$\frac{2}{11}$	$\frac{1}{88}$	$\frac{-1}{11}$	$\frac{16}{11}$	$\frac{1}{88}$	$\frac{1}{88}$	$\frac{43}{176}$	$\frac{43}{22}$	$\frac{8}{11}$	$\frac{129}{176}$	$\frac{-43}{44}$	$\frac{-4}{11}$	$\frac{43}{44}$	$\frac{43}{44}$
1025	0	0	$\frac{2}{11}$	$\frac{1}{44}$	$\frac{-2}{11}$	$\frac{32}{11}$	0	$\frac{1}{44}$	$\frac{7}{22}$	$\frac{43}{22}$	$\frac{12}{11}$	$\frac{29}{44}$	$\frac{-14}{11}$	$\frac{20}{11}$	$\frac{43}{44}$	$\frac{14}{11}$
1043	0	0	$\frac{2}{11}$	$\frac{1}{22}$	$\frac{-4}{11}$	$\frac{64}{11}$	0	$\frac{1}{22}$	$\frac{17}{44}$	$\frac{21}{11}$	$\frac{20}{11}$	$\frac{25}{44}$	$\frac{-17}{11}$	$\frac{24}{11}$	$\frac{21}{22}$	$\frac{17}{11}$
1061	0	0	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{-8}{11}$	$\frac{128}{11}$	0	$\frac{1}{11}$	$\frac{3}{11}$	$\frac{20}{11}$	$\frac{-8}{11}$	$\frac{7}{11}$	$\frac{-12}{11}$	$\frac{32}{11}$	$\frac{10}{11}$	$\frac{12}{11}$
1106	0	0	$\frac{2}{11}$	$\frac{1}{44}$	0	0	$\frac{1}{44}$	0	$\frac{3}{44}$	$\frac{5}{2}$	$\frac{48}{11}$	$\frac{10}{11}$	1	$\frac{-28}{11}$	$\frac{43}{44}$	$\frac{37}{22}$
1124	0	0	$\frac{2}{11}$	$\frac{1}{22}$	0	0	0	0	$\frac{3}{22}$	2	$\frac{48}{11}$	$\frac{9}{11}$	1	$\frac{16}{11}$	1	2
1142	0	0	$\frac{2}{11}$	$\frac{1}{11}$	0	0	0	0	$\frac{3}{11}$	2	$\frac{48}{11}$	$\frac{7}{11}$	0	$\frac{16}{11}$	1	2
1205	$\frac{-1}{44}$	$\frac{1}{44}$	$\frac{2}{11}$	$\frac{1}{22}$	0	0	$\frac{1}{44}$	0	$\frac{-3}{88}$	$\frac{67}{22}$	$\frac{92}{11}$	$\frac{89}{88}$	$\frac{59}{22}$	$\frac{-28}{11}$	$\frac{43}{44}$	$\frac{59}{22}$
1223	0	0	$\frac{2}{11}$	$\frac{1}{11}$	0	0	0	0	$\frac{1}{44}$	2	$\frac{92}{11}$	$\frac{39}{44}$	3	$\frac{16}{11}$	1	3
1241	0	0	$\frac{2}{11}$	$\frac{2}{11}$	0	0	0	0	$\frac{1}{22}$	2	$\frac{48}{11}$	$\frac{17}{22}$	2	$\frac{16}{11}$	1	2
1304	$\frac{-1}{22}$	$\frac{1}{22}$	$\frac{2}{11}$	$\frac{1}{11}$	0	0	0	0	$\frac{-3}{44}$	$\frac{34}{11}$	$\frac{136}{11}$	$\frac{45}{44}$	$\frac{48}{11}$	$\frac{16}{11}$	1	4
1322	0	0	$\frac{2}{11}$	$\frac{2}{11}$	0	0	0	0	$\frac{1}{22}$	2	$\frac{136}{11}$	$\frac{17}{22}$	4	$\frac{16}{11}$	1	4
1403	$\frac{-1}{22}$	$\frac{1}{22}$	$\frac{2}{11}$	$\frac{2}{11}$	0	0	$\frac{-1}{22}$	0	$\frac{-1}{22}$	$\frac{23}{11}$	$\frac{180}{11}$	$\frac{10}{11}$	$\frac{70}{11}$	$\frac{104}{11}$	$\frac{23}{22}$	$\frac{62}{11}$
1421	0	0	$\frac{2}{11}$	$\frac{4}{11}$	0	0	0	0	$\frac{1}{11}$	2	$\frac{136}{11}$	$\frac{6}{11}$	4	$\frac{16}{11}$	1	4
1502	0	0	$\frac{2}{11}$	$\frac{4}{11}$	0	0	$\frac{-1}{11}$	0	$\frac{1}{11}$	0	$\frac{224}{11}$	$\frac{6}{11}$	8	$\frac{192}{11}$	$\frac{12}{11}$	$\frac{80}{11}$
1601	$\frac{1}{11}$	$\frac{-1}{11}$	$\frac{2}{11}$	$\frac{8}{11}$	0	0	$\frac{-1}{11}$	0	$\frac{4}{11}$	$\frac{-24}{11}$	$\frac{224}{11}$	$\frac{-2}{11}$	$\frac{80}{11}$	$\frac{192}{11}$	$\frac{12}{11}$	$\frac{80}{11}$
2015	0	0	$\frac{2}{11}$	$\frac{1}{22}$	0	0	0	$\frac{1}{22}$	$\frac{7}{11}$	5	$\frac{92}{11}$	$\frac{29}{22}$	0	$\frac{-28}{11}$	$\frac{43}{22}$	$\frac{28}{11}$
2033	0	0	$\frac{2}{11}$	$\frac{1}{11}$	0	0	0	$\frac{1}{11}$	$\frac{17}{22}$	4	$\frac{92}{11}$	$\frac{25}{22}$	0	$\frac{16}{11}$	$\frac{21}{11}$	$\frac{34}{11}$
2051	0	0	$\frac{2}{11}$	$\frac{2}{11}$	0	0	0	$\frac{2}{11}$	$\frac{6}{11}$	4	$\frac{48}{11}$	$\frac{14}{11}$	0	$\frac{16}{11}$	$\frac{20}{11}$	$\frac{24}{11}$
2114	0	0	$\frac{2}{11}$	$\frac{1}{11}$	0	0	0	0	$\frac{3}{11}$	5	$\frac{136}{11}$	$\frac{18}{11}$	3	$\frac{16}{11}$	2	4
2132	0	0	$\frac{2}{11}$	$\frac{2}{11}$	0	0	0	0	$\frac{6}{11}$	4	$\frac{136}{11}$	$\frac{14}{11}$	2	$\frac{16}{11}$	2	4
2213	0	0	$\frac{2}{11}$	$\frac{2}{11}$	0	0	0	0	$\frac{1}{22}$	4	$\frac{180}{11}$	$\frac{39}{22}$	6	$\frac{104}{11}$	2	6

2231	0	0	$\frac{2}{11}$	$\frac{4}{11}$	0	0	0	0	$\frac{1}{11}$	4	$\frac{136}{11}$	$\frac{17}{11}$	4	$\frac{16}{11}$	2	4
2312	0	0	$\frac{2}{11}$	$\frac{4}{11}$	0	0	0	0	$\frac{1}{11}$	2	$\frac{224}{11}$	$\frac{17}{11}$	8	$\frac{192}{11}$	2	8
2411	0	0	$\frac{2}{11}$	$\frac{8}{11}$	0	0	0	0	$\frac{2}{11}$	0	$\frac{224}{11}$	$\frac{12}{11}$	8	$\frac{192}{11}$	2	8
3005	$\frac{-1}{44}$	$\frac{1}{44}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{4}{11}$	$\frac{-64}{11}$	$\frac{1}{44}$	$\frac{1}{22}$	$\frac{53}{88}$	$\frac{201}{22}$	$\frac{252}{11}$	$\frac{205}{88}$	$\frac{115}{22}$	$\frac{-124}{11}$	$\frac{129}{44}$	$\frac{115}{22}$
3023	0	0	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{8}{11}$	$\frac{-128}{11}$	0	$\frac{1}{11}$	$\frac{35}{44}$	$\frac{68}{11}$	$\frac{236}{11}$	$\frac{89}{44}$	$\frac{67}{11}$	0	$\frac{32}{11}$	$\frac{67}{11}$
3041	0	0	$\frac{2}{11}$	$\frac{4}{11}$	$\frac{16}{11}$	$\frac{-256}{11}$	0	$\frac{2}{11}$	$\frac{13}{22}$	$\frac{70}{11}$	$\frac{160}{11}$	$\frac{45}{22}$	$\frac{46}{11}$	$\frac{-16}{11}$	$\frac{31}{11}$	$\frac{46}{11}$
3104	$\frac{-1}{22}$	$\frac{1}{22}$	$\frac{2}{11}$	$\frac{2}{11}$	0	0	0	0	$\frac{9}{44}$	$\frac{100}{11}$	$\frac{312}{11}$	$\frac{117}{44}$	$\frac{92}{11}$	$\frac{16}{11}$	3	8
3122	0	0	$\frac{2}{11}$	$\frac{4}{11}$	0	0	0	0	$\frac{13}{22}$	6	$\frac{312}{11}$	$\frac{45}{22}$	8	$\frac{16}{11}$	3	8
3203	$\frac{-1}{22}$	$\frac{1}{22}$	$\frac{2}{11}$	$\frac{4}{11}$	0	0	$\frac{-1}{22}$	0	0	$\frac{67}{11}$	$\frac{356}{11}$	$\frac{59}{22}$	$\frac{136}{11}$	$\frac{280}{11}$	$\frac{67}{22}$	$\frac{128}{11}$
3221	0	0	$\frac{2}{11}$	$\frac{8}{11}$	0	0	0	0	$\frac{2}{11}$	6	$\frac{312}{11}$	$\frac{23}{11}$	8	$\frac{16}{11}$	3	8
3302	0	0	$\frac{2}{11}$	$\frac{8}{11}$	0	0	$\frac{-1}{11}$	0	$\frac{2}{11}$	0	$\frac{400}{11}$	$\frac{23}{11}$	16	$\frac{544}{11}$	$\frac{34}{11}$	$\frac{168}{11}$
3401	$\frac{1}{11}$	$\frac{-1}{11}$	$\frac{2}{11}$	$\frac{16}{11}$	0	0	$\frac{-1}{11}$	0	$\frac{6}{11}$	$\frac{-68}{11}$	$\frac{400}{11}$	$\frac{10}{11}$	$\frac{168}{11}$	$\frac{544}{11}$	$\frac{34}{11}$	$\frac{168}{11}$
4013	0	0	$\frac{2}{11}$	$\frac{4}{11}$	$\frac{16}{11}$	$\frac{-256}{11}$	0	0	$\frac{1}{11}$	$\frac{92}{11}$	$\frac{468}{11}$	$\frac{39}{11}$	$\frac{200}{11}$	$\frac{72}{11}$	4	12
4031	0	0	$\frac{2}{11}$	$\frac{8}{11}$	$\frac{32}{11}$	$\frac{-512}{11}$	0	0	$\frac{2}{11}$	$\frac{96}{11}$	$\frac{360}{11}$	$\frac{34}{11}$	$\frac{136}{11}$	$\frac{-48}{11}$	4	8
4112	0	0	$\frac{2}{11}$	$\frac{8}{11}$	0	0	0	0	$\frac{2}{11}$	6	$\frac{576}{11}$	$\frac{34}{11}$	20	$\frac{192}{11}$	4	16
4211	0	0	$\frac{2}{11}$	$\frac{16}{11}$	0	0	0	0	$\frac{4}{11}$	4	$\frac{576}{11}$	$\frac{24}{11}$	16	$\frac{192}{11}$	4	16
5003	$\frac{-1}{22}$	$\frac{1}{22}$	$\frac{2}{11}$	$\frac{8}{11}$	$\frac{16}{11}$	$\frac{-256}{11}$	$\frac{-1}{22}$	$\frac{-2}{11}$	$\frac{-31}{22}$	$\frac{115}{11}$	$\frac{820}{11}$	$\frac{63}{11}$	$\frac{402}{11}$	$\frac{424}{11}$	$\frac{115}{22}$	$\frac{258}{11}$
5021	0	0	$\frac{2}{11}$	$\frac{16}{11}$	$\frac{32}{11}$	$\frac{-512}{11}$	0	$\frac{-4}{11}$	$\frac{-7}{11}$	$\frac{118}{11}$	$\frac{712}{11}$	$\frac{46}{11}$	$\frac{268}{11}$	$\frac{-48}{11}$	$\frac{9}{11}$	$\frac{172}{11}$
5102	0	0	$\frac{2}{11}$	$\frac{16}{11}$	0	0	$\frac{-1}{11}$	0	$\frac{-7}{11}$	0	$\frac{928}{11}$	$\frac{46}{11}$	40	$\frac{896}{11}$	$\frac{56}{11}$	$\frac{344}{11}$

Table 5.12, corresponding to qudratic forms $(x_1^2+x_1x_2+x_2^2)+c_1(x_3^2+x_3x_4+x_4^2)+c_2(x_5^2+x_5x_6+x_6^2)+c_3(x_7^2+x_7x_8+x_8^2)$, where $c_1 \leq c_2 \leq c_3$ and $c_1, c_2, c_3 \in \{1,2,4,8\}$. The tuples $(1, c_1, c_2, c_3)$ are given explicitly in table and the associated generating function belong to the space $M_4(24, \chi_0)$, with dimension 16.

$(1,c_1,c_2,c_3)$	$\beta 4,1$	$\beta 4,2$	$\beta 4,3$	$\beta 4,4$	$\beta 4,5$	$\beta 4,6$	$\beta 4,7$	$\beta 4,8$	$\beta 4,9$	$\beta 4,10$	$\beta 4,11$	$\beta 4,12$	$\beta 4,13$	$\beta 4,14$	$\beta 4,15$	$\beta 4,16$
(1,1,1,2)	$\frac{3}{40}$	$\frac{-1}{5}$	$\frac{-27}{40}$	0	$\frac{9}{5}$	0	0	0	0	0	0	0	0	0	0	0
(1,1,1,4)	$\frac{3}{100}$	$\frac{-9}{100}$	$\frac{27}{100}$	$\frac{4}{25}$	$\frac{-81}{100}$	0	$\frac{36}{25}$	0	$\frac{54}{5}$	$\frac{432}{5}$	0	0	0	0	0	0
(1,1,1,8)	$\frac{3}{160}$	$\frac{-9}{160}$	$\frac{-27}{160}$	$\frac{9}{80}$	$\frac{81}{160}$	$\frac{-1}{5}$	$\frac{-81}{80}$	$\frac{9}{5}$	0	0	0	$\frac{-27}{4}$	$\frac{243}{4}$	0	81	$\frac{81}{4}$
(1,1,2,2)	$\frac{1}{50}$	$\frac{2}{25}$	$\frac{9}{50}$	0	$\frac{18}{25}$	0	0	0	$\frac{36}{5}$	0	0	0	0	0	0	0
(1,1,2,4)	$\frac{1}{80}$	$\frac{1}{16}$	$\frac{-9}{80}$	$\frac{-1}{5}$	$\frac{-9}{16}$	0	$\frac{9}{5}$	0	0	0	0	0	0	9	0	0
(1,1,2,8)	$\frac{1}{200}$	$\frac{1}{40}$	$\frac{9}{200}$	$\frac{-9}{100}$	$\frac{9}{40}$	$\frac{4}{25}$	$\frac{-81}{100}$	$\frac{36}{25}$	$\frac{9}{5}$	$\frac{144}{5}$	$\frac{1152}{5}$	9	81	0	0	0
(1,1,4,4)	$\frac{1}{200}$	$\frac{3}{200}$	$\frac{9}{200}$	$\frac{2}{25}$	$\frac{27}{200}$	0	$\frac{18}{25}$	0	$\frac{54}{5}$	$\frac{216}{5}$	0	0	0	0	0	0
(1,1,4,8)	$\frac{1}{320}$	$\frac{3}{320}$	$\frac{-9}{320}$	$\frac{1}{16}$	$\frac{-27}{320}$	$\frac{-1}{5}$	$\frac{-9}{16}$	$\frac{9}{5}$	0	0	0	$\frac{9}{4}$	$\frac{-81}{4}$	$\frac{9}{4}$	27	$\frac{27}{4}$
(1,1,8,8)	$\frac{1}{800}$	$\frac{3}{800}$	$\frac{9}{800}$	$\frac{3}{200}$	$\frac{27}{800}$	$\frac{2}{25}$	$\frac{27}{200}$	$\frac{18}{25}$	$\frac{36}{5}$	$\frac{234}{5}$	$\frac{576}{5}$	$\frac{9}{2}$	$\frac{81}{2}$	0	0	0
(1,2,2,2)	$\frac{1}{40}$	$\frac{-3}{20}$	$\frac{-9}{40}$	0	$\frac{27}{20}$	0	0	0	0	0	0	0	0	0	0	0
(1,2,2,4)	$\frac{1}{100}$	$\frac{-7}{100}$	$\frac{9}{100}$	$\frac{4}{25}$	$\frac{-63}{100}$	0	$\frac{36}{25}$	0	$\frac{18}{5}$	$\frac{72}{5}$	0	0	0	0	0	0
(1,2,2,8)	$\frac{1}{160}$	$\frac{-7}{160}$	$\frac{-9}{160}$	$\frac{9}{80}$	$\frac{63}{160}$	$\frac{-1}{5}$	$\frac{-81}{80}$	$\frac{9}{5}$	0	0	0	$\frac{-9}{4}$	$\frac{81}{4}$	0	9	$\frac{27}{4}$
(1,2,4,4)	$\frac{1}{160}$	$\frac{-1}{32}$	$\frac{-9}{160}$	$\frac{-1}{10}$	$\frac{9}{32}$	0	$\frac{9}{10}$	0	0	0	0	0	0	$\frac{9}{2}$	0	0
(1,2,4,8)	$\frac{1}{400}$	$\frac{-1}{80}$	$\frac{9}{400}$	$\frac{-1}{20}$	$\frac{-9}{80}$	$\frac{4}{25}$	$\frac{-9}{20}$	$\frac{36}{25}$	$\frac{9}{10}$	$\frac{27}{5}$	$\frac{72}{5}$	$\frac{9}{2}$	$\frac{81}{2}$	0	0	0
(1,2,8,8)	$\frac{1}{640}$	$\frac{-1}{128}$	$\frac{-9}{640}$	$\frac{-3}{160}$	$\frac{9}{128}$	$\frac{-1}{10}$	$\frac{27}{160}$	$\frac{9}{10}$	0	0	0	$\frac{-9}{8}$	$\frac{81}{8}$	$\frac{27}{8}$	$\frac{9}{2}$	$\frac{27}{8}$
(1,4,4,4)	$\frac{1}{400}$	$\frac{-9}{400}$	$\frac{9}{400}$	$\frac{3}{25}$	$\frac{-81}{400}$	0	$\frac{27}{25}$	0	$\frac{27}{5}$	$\frac{54}{5}$	0	0	0	0	0	0
(1,4,4,8)	$\frac{1}{640}$	$\frac{-9}{640}$	$\frac{-9}{640}$	$\frac{7}{80}$	$\frac{81}{640}$	$\frac{-1}{5}$	$\frac{-63}{80}$	$\frac{9}{5}$	0	0	0	$\frac{9}{8}$	$\frac{-81}{8}$	$\frac{9}{8}$	0	$\frac{27}{8}$
(1,4,8,8)	$\frac{1}{1600}$	$\frac{-9}{1600}$	$\frac{9}{1600}$	$\frac{1}{40}$	$\frac{-81}{1600}$	$\frac{2}{25}$	$\frac{9}{40}$	$\frac{18}{25}$	$\frac{18}{5}$	$\frac{36}{5}$	$\frac{36}{5}$	$\frac{9}{4}$	$\frac{81}{4}$	0	0	0
(1,8,8,8)	$\frac{1}{2560}$	$\frac{-9}{2560}$	$\frac{-9}{2560}$	$\frac{9}{320}$	$\frac{81}{2560}$	$\frac{-3}{20}$	$\frac{-81}{320}$	$\frac{27}{20}$	0	0	0	$\frac{27}{32}$	$\frac{-243}{32}$	$\frac{81}{32}$	0	$\frac{81}{32}$

Table 5.13, corresponding to qudratic forms $a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+b_1(x_5^2+x_5x_6+x_6^2)+b_2(x_7^2+x_7x_8+x_8^2)$, where $a_1 \le a_2 \le a_3 \le a_4$, $b_1 \le b_2$ and a_i's $\in \{1,2,3\}$, b_i's $\in \{1,2,4\}$. The tuples $(a_1a_2a_3a_4, b_1b_2)$ are given explicitly in table and the associated generating function belong to the space $M_4(24,\chi_0)$, with dimension 16.

$(a_1a_2a_3a_4,b_1b_2)$	$\gamma4,1$	$\gamma4,2$	$\gamma4,3$	$\gamma4,4$	$\gamma4,5$	$\gamma4,6$	$\gamma4,7$	$\gamma4,8$	$\gamma4,9$	$\gamma4,10$	$\gamma4,11$	$\gamma4,12$	$\gamma4,13$	$\gamma4,14$	$\gamma4,15$	$\gamma4,16$
(1111,11)	$\frac{7}{75}$	$\frac{-7}{100}$	$\frac{-9}{25}$	$\frac{-28}{75}$	$\frac{27}{100}$	0	$\frac{36}{25}$	0	$\frac{-72}{5}$	$\frac{-288}{5}$	0	0	0	12	0	0
(1111,12)	$\frac{13}{300}$	$\frac{-13}{200}$	$\frac{9}{100}$	$\frac{26}{75}$	$\frac{-27}{200}$	0	$\frac{18}{25}$	0	$\frac{48}{5}$	$\frac{96}{5}$	0	0	0	-6	0	0
(1111,14)	$\frac{7}{300}$	0	$\frac{-9}{100}$	$\frac{-28}{75}$	0	0	$\frac{36}{25}$	0	$\frac{-18}{5}$	$\frac{72}{5}$	0	0	0	12	0	0
(1111,22)	$\frac{7}{300}$	0	$\frac{-9}{100}$	$\frac{-28}{75}$	0	0	$\frac{36}{25}$	0	$\frac{12}{5}$	$\frac{-48}{5}$	0	0	0	0	0	0
(1111,24)	$\frac{13}{1200}$	$\frac{-13}{400}$	$\frac{9}{400}$	$\frac{26}{75}$	$\frac{-27}{400}$	0	$\frac{18}{25}$	0	$\frac{12}{5}$	$\frac{96}{5}$	0	0	0	3	0	0
(1111,44)	$\frac{7}{1200}$	$\frac{7}{400}$	$\frac{-9}{400}$	$\frac{-28}{75}$	$\frac{-27}{400}$	0	$\frac{36}{25}$	0	$\frac{18}{5}$	$\frac{72}{5}$	0	0	0	3	0	0
(1122,11)	$\frac{7}{150}$	$\frac{-7}{150}$	$\frac{-9}{50}$	$\frac{7}{300}$	$\frac{9}{50}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{-36}{5}$	-48	$\frac{-768}{5}$	-3	-81	6	36	9
(1122,12)	$\frac{13}{600}$	$\frac{-13}{600}$	$\frac{9}{200}$	$\frac{-13}{600}$	$\frac{-9}{200}$	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$	$\frac{24}{5}$	12	$\frac{96}{5}$	$\frac{15}{2}$	$\frac{81}{2}$	-3	-6	$\frac{-9}{2}$
(1122,14)	$\frac{7}{600}$	$\frac{-7}{600}$	$\frac{-9}{200}$	$\frac{7}{300}$	$\frac{9}{200}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{-9}{5}$	6	$\frac{-48}{5}$	$\frac{-3}{2}$	$\frac{-81}{2}$	6	0	$\frac{9}{2}$
(1122,22)	$\frac{7}{600}$	$\frac{-7}{600}$	$\frac{-9}{200}$	$\frac{7}{300}$	$\frac{9}{200}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{6}{5}$	-12	$\frac{-288}{5}$	0	0	0	12	0
(1122,24)	$\frac{13}{2400}$	$\frac{-13}{2400}$	$\frac{9}{800}$	$\frac{-13}{600}$	$\frac{-9}{800}$	$\frac{26}{75}$	$\frac{-9}{200}$	$\frac{18}{25}$	$\frac{6}{5}$	12	$\frac{96}{5}$	0	0	$\frac{3}{2}$	-6	0
(1122,44)	$\frac{7}{2400}$	$\frac{-7}{2400}$	$\frac{-9}{800}$	$\frac{7}{300}$	$\frac{9}{800}$	$\frac{-28}{75}$	$\frac{-9}{100}$	$\frac{36}{25}$	$\frac{9}{5}$	6	$\frac{-48}{5}$	0	0	$\frac{3}{2}$	0	0
(1133,11)	$\frac{2}{75}$	$\frac{-1}{30}$	$\frac{6}{25}$	$\frac{8}{75}$	$\frac{-3}{10}$	0	$\frac{24}{25}$	0	$\frac{48}{5}$	$\frac{288}{5}$	0	0	0	0	0	0
(1133,12)	$\frac{1}{60}$	$\frac{-1}{120}$	$\frac{-3}{20}$	$\frac{-2}{15}$	$\frac{3}{40}$	0	$\frac{6}{5}$	0	0	0	0	0	0	6	0	0
(1133,14)	$\frac{1}{150}$	$\frac{-1}{75}$	$\frac{3}{50}$	$\frac{8}{75}$	$\frac{-3}{25}$	0	$\frac{24}{25}$	0	$\frac{42}{5}$	$\frac{168}{5}$	0	0	0	0	0	0
(1133,22)	$\frac{1}{150}$	$\frac{-1}{75}$	$\frac{3}{50}$	$\frac{8}{75}$	$\frac{-3}{25}$	0	$\frac{24}{25}$	0	$\frac{12}{5}$	$\frac{48}{5}$	0	0	0	0	0	0
(1133,24)	$\frac{1}{240}$	$\frac{1}{240}$	$\frac{-3}{80}$	$\frac{-2}{15}$	$\frac{-3}{80}$	0	$\frac{6}{5}$	0	0	0	0	0	0	3	0	0
(1133,44)	$\frac{1}{600}$	$\frac{-1}{120}$	$\frac{3}{200}$	$\frac{8}{75}$	$\frac{-3}{40}$	0	$\frac{24}{25}$	0	$\frac{18}{5}$	$\frac{48}{5}$	0	0	0	0	0	0
(2222,11)	$\frac{7}{400}$	$\frac{91}{1200}$	$\frac{-27}{400}$	$\frac{-7}{100}$	$\frac{-117}{400}$	$\frac{-28}{75}$	$\frac{27}{100}$	$\frac{36}{25}$	$\frac{-36}{5}$	$\frac{-312}{5}$	$\frac{-768}{5}$	-3	-81	9	36	9
(2222,12)	$\frac{13}{800}$	$\frac{-247}{2400}$	$\frac{27}{800}$	$\frac{13}{200}$	$\frac{-171}{800}$	$\frac{26}{75}$	$\frac{27}{200}$	$\frac{18}{25}$	$\frac{18}{5}$	$\frac{84}{5}$	$\frac{96}{5}$	$\frac{15}{2}$	$\frac{81}{2}$	$\frac{-9}{2}$	-6	$\frac{-9}{2}$
(2222,14)	$\frac{7}{800}$	$\frac{-133}{2400}$	$\frac{-27}{800}$	$\frac{7}{100}$	$\frac{171}{800}$	$\frac{-28}{75}$	$\frac{-27}{100}$	$\frac{36}{25}$	$\frac{-18}{5}$	$\frac{-24}{5}$	$\frac{-48}{5}$	$\frac{-3}{2}$	$\frac{-81}{2}$	$\frac{9}{2}$	0	$\frac{9}{2}$

(2222, 22)	0	$\frac{7}{75}$	0	$\frac{-7}{100}$	$\frac{-9}{25}$	$\frac{-28}{75}$	$\frac{27}{100}$	$\frac{36}{25}$	0	$\frac{-72}{5}$	$\frac{-288}{5}$	0	0	0	12	0
(2222, 24)	0	$\frac{13}{300}$	0	$\frac{-13}{200}$	$\frac{9}{100}$	$\frac{26}{75}$	$\frac{-27}{200}$	$\frac{18}{25}$	0	$\frac{48}{5}$	$\frac{96}{5}$	0	0	0	-6	0
(2222, 44)	0	$\frac{7}{300}$	0	0	$\frac{-9}{100}$	$\frac{-28}{75}$	0	$\frac{36}{25}$	0	$\frac{12}{5}$	$\frac{-48}{5}$	0	0	0	0	0
(2233, 11)	$\frac{1}{75}$	$\frac{-1}{75}$	$\frac{3}{25}$	$\frac{-1}{150}$	$\frac{-3}{25}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{24}{5}$	48	$\frac{768}{5}$	10	18	0	-24	-6
(2233, 12)	$\frac{1}{120}$	$\frac{-1}{120}$	$\frac{-3}{40}$	$\frac{1}{120}$	$\frac{3}{40}$	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$	0	-12	-96	$\frac{-7}{2}$	$\frac{-9}{2}$	3	6	$\frac{9}{2}$
(2233, 14)	$\frac{1}{300}$	$\frac{-1}{300}$	$\frac{3}{100}$	$\frac{-1}{150}$	$\frac{-3}{100}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{21}{5}$	18	$\frac{48}{5}$	$\frac{5}{2}$	$\frac{63}{2}$	0	-12	$\frac{-3}{2}$
(2233, 22)	$\frac{1}{300}$	$\frac{-1}{300}$	$\frac{3}{100}$	$\frac{-1}{150}$	$\frac{-3}{100}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{6}{5}$	12	$\frac{288}{5}$	1	-9	0	0	-3
(2233, 24)	$\frac{1}{480}$	$\frac{-1}{480}$	$\frac{-3}{160}$	$\frac{1}{120}$	$\frac{3}{160}$	$\frac{-2}{15}$	$\frac{-3}{40}$	$\frac{6}{5}$	0	0	0	-2	-18	$\frac{3}{2}$	6	0
(2233, 44)	$\frac{1}{1200}$	$\frac{-1}{1200}$	$\frac{3}{400}$	$\frac{-1}{150}$	$\frac{-3}{400}$	$\frac{8}{75}$	$\frac{-3}{50}$	$\frac{24}{25}$	$\frac{9}{5}$	6	$\frac{48}{5}$	$\frac{-1}{2}$	$\frac{9}{2}$	0	0	$\frac{-3}{2}$
(3333, 11)	$\frac{1}{75}$	$\frac{-1}{100}$	$\frac{-7}{25}$	$\frac{-4}{75}$	$\frac{21}{100}$	0	$\frac{28}{25}$	0	$\frac{24}{5}$	$\frac{96}{5}$	0	0	0	4	0	0
(3333, 12)	$\frac{1}{300}$	$\frac{-1}{200}$	$\frac{13}{100}$	$\frac{2}{75}$	$\frac{-39}{200}$	0	$\frac{26}{25}$	0	$\frac{16}{5}$	$\frac{32}{5}$	0	0	0	2	0	0
(3333, 14)	$\frac{1}{300}$	0	$\frac{-7}{100}$	$\frac{-4}{75}$	0	0	$\frac{28}{25}$	0	$\frac{6}{5}$	$\frac{-24}{5}$	0	0	0	4	0	0
(3333, 22)	$\frac{1}{300}$	0	$\frac{-7}{100}$	$\frac{-4}{75}$	0	0	$\frac{28}{25}$	0	$\frac{-4}{5}$	$\frac{16}{5}$	0	0	0	0	0	0
(3333, 24)	$\frac{1}{1200}$	$\frac{-1}{400}$	$\frac{13}{400}$	$\frac{2}{75}$	$\frac{-39}{400}$	0	$\frac{26}{25}$	0	$\frac{4}{5}$	$\frac{32}{5}$	0	0	0	-1	0	0
(3333, 44)	$\frac{1}{1200}$	$\frac{1}{400}$	$\frac{-7}{400}$	$\frac{-4}{75}$	$\frac{-21}{400}$	0	$\frac{28}{25}$	0	$\frac{-6}{5}$	$\frac{-24}{5}$	0	0	0	1	0	0

Table 5.14, corresponding to qudratic forms $a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+b_1(x_5^2+x_5x_6+x_6^2)+b_2(x_7^2+x_7x_8+x_8^2)$, where $a_1 \le a_2 \le a_3 \le a_4$, $b_1 \le b_2$ and a_i's $\in \{1,2,3\}$, b_i's $\in \{1,2,4\}$. The tuples $(a_1a_2a_3a_4, b_1b_2)$ are given explicitly in table and the associated generating function belong to the space $M_4(24,\chi_8)$, with dimension 14.

$a_1a_2a_3a_4, b_1b_2$	$\gamma 4,1$	$\gamma 4,2$	$\gamma 4,3$	$\gamma 4,4$	$\gamma 4,5$	$\gamma 4,6$	$\gamma 4,7$	$\gamma 4,8$	$\gamma 4,9$	$\gamma 4,10$	$\gamma 4,11$	$\gamma 4,12$	$\gamma 4,13$	$\gamma 4,14$
1112, 11	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{6656}{451}$	$\frac{27648}{451}$	$\frac{168}{451}$	$\frac{11448}{451}$	$\frac{-2496}{451}$	$\frac{-17280}{451}$	$\frac{24}{41}$	$\frac{936}{41}$	$\frac{144}{41}$	$\frac{-384}{41}$	$\frac{4032}{41}$	$\frac{-48}{41}$
1112, 12	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{3584}{451}$	$\frac{-6912}{451}$	$\frac{480}{451}$	$\frac{-2052}{451}$	$\frac{-2688}{451}$	$\frac{1728}{451}$	$\frac{-60}{41}$	$\frac{216}{41}$	$\frac{-108}{41}$	$\frac{-2112}{41}$	$\frac{-1440}{41}$	$\frac{288}{41}$
1112, 14	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{1664}{451}$	$\frac{6912}{451}$	$\frac{-912}{451}$	$\frac{3672}{451}$	0	$\frac{-6912}{451}$	$\frac{-48}{41}$	$\frac{-54}{41}$	$\frac{702}{41}$	$\frac{1632}{41}$	$\frac{576}{41}$	$\frac{-228}{41}$
1112, 22	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{1664}{451}$	$\frac{6912}{451}$	$\frac{-912}{451}$	$\frac{3672}{451}$	0	$\frac{-6912}{451}$	$\frac{-48}{41}$	$\frac{684}{41}$	$\frac{-36}{41}$	$\frac{-2304}{41}$	$\frac{576}{41}$	$\frac{264}{41}$
1112, 24	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{896}{451}$	$\frac{-1728}{451}$	$\frac{-66}{41}$	$\frac{-108}{451}$	$\frac{-1344}{451}$	$\frac{-864}{451}$	$\frac{-42}{41}$	$\frac{-90}{41}$	$\frac{306}{41}$	$\frac{336}{41}$	$\frac{-576}{41}$	$\frac{-36}{41}$
1112, 44	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{416}{451}$	$\frac{1728}{451}$	$\frac{-1182}{451}$	$\frac{1728}{451}$	$\frac{624}{451}$	$\frac{-4320}{451}$	$\frac{-66}{41}$	$\frac{252}{41}$	$\frac{288}{41}$	$\frac{-816}{41}$	$\frac{-288}{41}$	$\frac{96}{41}$
1222, 11	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{3328}{451}$	$\frac{13824}{451}$	$\frac{8468}{451}$	$\frac{6264}{451}$	$\frac{-15264}{451}$	$\frac{-10368}{451}$	$\frac{-352}{41}$	$\frac{-708}{41}$	$\frac{516}{41}$	$\frac{3584}{41}$	$\frac{13536}{41}$	$\frac{-660}{41}$
1222, 12	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{1792}{451}$	$\frac{-3456}{451}$	$\frac{9598}{451}$	$\frac{-756}{451}$	$\frac{-16224}{451}$	0	$\frac{-458}{41}$	$\frac{-1956}{41}$	$\frac{168}{41}$	$\frac{2800}{41}$	$\frac{5040}{41}$	$\frac{-420}{41}$
1222, 14	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{832}{451}$	$\frac{3456}{451}$	$\frac{-6955}{451}$	$\frac{216}{41}$	$\frac{7632}{451}$	$\frac{-5184}{451}$	$\frac{473}{41}$	$\frac{1749}{41}$	$\frac{57}{41}$	-56	-144	$\frac{357}{41}$
1222, 22	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{832}{451}$	$\frac{3456}{451}$	$\frac{10634}{451}$	$\frac{216}{41}$	$\frac{-14016}{451}$	$\frac{-5184}{451}$	$\frac{-634}{41}$	$\frac{-2310}{41}$	$\frac{426}{41}$	112	288	$\frac{-750}{41}$
1222, 24	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{448}{451}$	$\frac{-864}{451}$	$\frac{-3182}{451}$	$\frac{216}{451}$	$\frac{6096}{451}$	$\frac{-1296}{451}$	$\frac{166}{41}$	$\frac{474}{41}$	$\frac{6}{41}$	$\frac{-896}{41}$	$\frac{-3384}{41}$	$\frac{156}{41}$
1222, 44	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{208}{451}$	$\frac{864}{451}$	$\frac{2381}{451}$	$\frac{1404}{451}$	$\frac{-2880}{451}$	$\frac{-3888}{451}$	$\frac{-151}{41}$	$\frac{-681}{41}$	$\frac{219}{41}$	$\frac{1400}{41}$	$\frac{2520}{41}$	$\frac{-219}{41}$
1233, 11	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{2560}{451}$	$\frac{-18432}{451}$	$\frac{488}{451}$	$\frac{-6192}{451}$	$\frac{-1600}{451}$	$\frac{6912}{451}$	$\frac{-112}{41}$	$\frac{-480}{41}$	$\frac{432}{41}$	$\frac{704}{41}$	$\frac{-3456}{41}$	$\frac{-24}{41}$
1233, 12	$\frac{-8}{451}$	$\frac{90}{451}$	$\frac{1024}{451}$	$\frac{11520}{451}$	$\frac{-1280}{451}$	$\frac{5220}{451}$	$\frac{-256}{451}$	$\frac{-8640}{451}$	$\frac{-20}{41}$	$\frac{312}{41}$	$\frac{348}{41}$	$\frac{-512}{41}$	$\frac{1440}{41}$	$\frac{24}{41}$
1233, 14	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{640}{451}$	$\frac{-4608}{451}$	$\frac{-760}{451}$	$\frac{-1008}{451}$	$\frac{-640}{451}$	0	$\frac{-64}{41}$	$\frac{-66}{41}$	$\frac{306}{41}$	$\frac{-640}{41}$	$\frac{-1152}{41}$	$\frac{96}{41}$
1233, 22	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{640}{451}$	$\frac{-4608}{451}$	$\frac{-760}{451}$	$\frac{-1008}{451}$	$\frac{-640}{451}$	0	$\frac{-64}{41}$	$\frac{180}{41}$	$\frac{60}{41}$	$\frac{-640}{41}$	$\frac{-1152}{41}$	$\frac{96}{41}$
1233, 24	$\frac{-8}{451}$	$\frac{90}{451}$	$\frac{256}{451}$	$\frac{2880}{451}$	$\frac{-1238}{451}$	$\frac{180}{41}$	$\frac{128}{451}$	$\frac{-4320}{451}$	$\frac{-50}{41}$	$\frac{330}{41}$	$\frac{150}{41}$	-16	0	$\frac{72}{41}$
1233, 44	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{160}{451}$	$\frac{-1152}{451}$	$\frac{-1072}{451}$	$\frac{288}{451}$	$\frac{-400}{451}$	$\frac{-1728}{451}$	$\frac{-52}{41}$	$\frac{222}{41}$	$\frac{90}{41}$	$\frac{-976}{41}$	$\frac{-576}{41}$	$\frac{126}{41}$

Table 5.15, corresponding to qudratic forms $a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+b_1(x_5^2+x_5x_6+x_6^2)+b_2(x_7^2+x_7x_8+x_8^2)$, where $a_1 \leq a_2 \leq a_3 \leq a_4$, $b_1 \leq b_2$ and a_i's $\in \{1,2,3\}$, b_i's $\in \{1,2,4\}$. The tuples $(a_1a_2a_3a_4, b_1b_2)$ are given explicitly in table and the associated generating function belong to the space $M_4(24,\chi_{12})$, with dimension 16.

$(a_1a_2a_3a_4,b_1b_2)$	$\gamma_{4,1}$	$\gamma_{4,2}$	$\gamma_{4,3}$	$\gamma_{4,4}$	$\gamma_{4,5}$	$\gamma_{4,6}$	$\gamma_{4,7}$	$\gamma_{4,8}$	$\gamma_{4,9}$	$\gamma_{4,10}$	$\gamma_{4,11}$	$\gamma_{4,12}$	$\gamma_{4,13}$	$\gamma_{4,14}$	$\gamma_{4,15}$	$\gamma_{4,16}$
(1113, 11)	$\frac{1}{23}$	$\frac{288}{23}$	$\frac{32}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{84}{23}$	$\frac{720}{23}$	$\frac{336}{23}$	$\frac{864}{23}$	0	0	0	0
(1113, 12)	$\frac{1}{23}$	$\frac{144}{23}$	$\frac{-16}{23}$	$\frac{-9}{23}$	0	0	0	0	$\frac{156}{23}$	$\frac{-48}{23}$	$\frac{-168}{23}$	$\frac{-456}{23}$	0	0	0	0
(1113, 14)	$\frac{1}{23}$	$\frac{72}{23}$	$\frac{8}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{186}{23}$	$\frac{600}{23}$	$\frac{-228}{23}$	$\frac{372}{23}$	0	0	0	0
(1113, 22)	$\frac{1}{23}$	$\frac{72}{23}$	$\frac{8}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{48}{23}$	$\frac{48}{23}$	$\frac{48}{23}$	$\frac{96}{23}$	0	0	0	0
(1113, 24)	$\frac{1}{23}$	$\frac{36}{23}$	$\frac{-4}{23}$	$\frac{-9}{23}$	0	0	0	0	$\frac{114}{23}$	$\frac{84}{23}$	$\frac{-120}{23}$	$\frac{-156}{23}$	0	0	0	0
(1113, 44)	$\frac{1}{23}$	$\frac{18}{23}$	$\frac{2}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{108}{23}$	$\frac{156}{23}$	$\frac{-162}{23}$	$\frac{42}{23}$	0	0	0	0
(1223, 11)	0	$\frac{144}{23}$	$\frac{16}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{162}{23}$	$\frac{264}{23}$	$\frac{-1188}{23}$	$\frac{420}{23}$	$\frac{192}{23}$	$\frac{864}{23}$	$\frac{-4704}{23}$	$\frac{-5760}{23}$
(1223, 12)	0	$\frac{72}{23}$	$\frac{-8}{23}$	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{120}{23}$	$\frac{96}{23}$	$\frac{336}{23}$	$\frac{-384}{23}$	$\frac{-240}{23}$	$\frac{-912}{23}$	$\frac{4368}{23}$	$\frac{2784}{23}$
(1223, 14)	0	$\frac{36}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{144}{23}$	$\frac{480}{23}$	$\frac{-504}{23}$	$\frac{312}{23}$	$\frac{-360}{23}$	$\frac{1416}{23}$	$\frac{-1944}{23}$	$\frac{-1344}{23}$
(1223, 22)	0	$\frac{36}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{6}{23}$	$\frac{-72}{23}$	$\frac{-228}{23}$	$\frac{36}{23}$	$\frac{192}{23}$	$\frac{-240}{23}$	$\frac{-1392}{23}$	$\frac{-1344}{23}$
(1223, 24)	0	$\frac{18}{23}$	$\frac{-2}{23}$	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{30}{23}$	$\frac{24}{23}$	$\frac{84}{23}$	$\frac{-96}{23}$	$\frac{36}{23}$	$\frac{-360}{23}$	$\frac{504}{23}$	$\frac{576}{23}$
(1223, 44)	0	$\frac{9}{23}$	$\frac{1}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{36}{23}$	$\frac{120}{23}$	$\frac{-126}{23}$	$\frac{78}{23}$	$\frac{-84}{23}$	$\frac{312}{23}$	$\frac{-840}{23}$	$\frac{-240}{23}$
(1333, 11)	$\frac{1}{23}$	$\frac{96}{23}$	$\frac{-32}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{260}{23}$	$\frac{352}{23}$	$\frac{-544}{23}$	$\frac{-352}{23}$	0	0	0	0
(1333, 12)	$\frac{1}{23}$	$\frac{48}{23}$	$\frac{16}{23}$	$\frac{3}{23}$	0	0	0	0	$\frac{116}{23}$	$\frac{160}{23}$	$\frac{-40}{23}$	$\frac{104}{23}$	0	0	0	0
(1333, 14)	$\frac{1}{23}$	$\frac{24}{23}$	$\frac{-8}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{170}{23}$	$\frac{16}{23}$	$\frac{-292}{23}$	$\frac{-340}{23}$	0	0	0	0
(1333, 22)	$\frac{1}{23}$	$\frac{24}{23}$	$\frac{-8}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{32}{23}$	$\frac{16}{23}$	$\frac{-16}{23}$	$\frac{-64}{23}$	0	0	0	0
(1333, 24)	$\frac{1}{23}$	$\frac{12}{23}$	$\frac{4}{23}$	$\frac{3}{23}$	0	0	0	0	$\frac{26}{23}$	$\frac{76}{23}$	$\frac{32}{23}$	$\frac{116}{23}$	0	0	0	0
(1333, 44)	$\frac{1}{23}$	$\frac{6}{23}$	$\frac{-2}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{44}{23}$	$\frac{-68}{23}$	$\frac{-22}{23}$	$\frac{-130}{23}$	0	0	0	0

Table 5.16, corresponding to qudratic forms $a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+b_1(x_5^2+x_5x_6+x_6^2)+b_2(x_7^2+x_7x_8+x_8^2)$, where $a_1 \le a_2 \le a_3 \le a_4$, $b_1 \le b_2$ and a_i's $\in \{1,2,3\}$, b_i's $\in \{1,2,4\}$. The tuples $(a_1a_2a_3a_4, b_1b_2)$ are given explicitly in table and the associated generating function belong to the space $M_4(24,\chi_{24})$, with dimension 14.

$a_1a_2a_3a_4, b_1b_2$	$\gamma 4,1$	$\gamma 4,2$	$\gamma 4,3$	$\gamma 4,4$	$\gamma 4,5$	$\gamma 4,6$	$\gamma 4,7$	$\gamma 4,8$	$\gamma 4,9$	$\gamma 4,10$	$\gamma 4,11$	$\gamma 4,12$	$\gamma 4,13$	$\gamma 4,14$
1123, 11	$\frac{1}{261}$	$\frac{256}{29}$	$\frac{-256}{261}$	$\frac{-1}{29}$	$\frac{1808}{87}$	$\frac{656}{29}$	$\frac{-2056}{87}$	$\frac{-3808}{29}$	$\frac{-4144}{29}$	$\frac{736}{3}$	$\frac{472}{3}$	$\frac{-41984}{87}$	$\frac{-1096}{87}$	$\frac{-968}{87}$
1123, 12	$\frac{1}{261}$	$\frac{128}{29}$	$\frac{128}{261}$	$\frac{1}{29}$	$\frac{208}{87}$	$\frac{-32}{29}$	$\frac{-284}{87}$	$\frac{-368}{29}$	$\frac{1048}{29}$	$\frac{-6224}{87}$	$\frac{-7100}{87}$	$\frac{21248}{87}$	$\frac{8}{3}$	$\frac{500}{87}$
1123, 14	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{-64}{261}$	$\frac{-1}{29}$	$\frac{-84}{29}$	$\frac{-114}{29}$	$\frac{450}{29}$	$\frac{1800}{29}$	$\frac{-324}{29}$	$\frac{-4200}{29}$	$\frac{-2814}{29}$	$\frac{-3072}{29}$	$\frac{318}{29}$	$\frac{414}{29}$
1123, 22	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{-64}{261}$	$\frac{-1}{29}$	$\frac{264}{29}$	$\frac{60}{29}$	$\frac{-420}{29}$	$\frac{-1680}{29}$	$\frac{-1368}{29}$	$\frac{2064}{29}$	$\frac{1884}{29}$	$\frac{-3072}{29}$	$\frac{-204}{29}$	$\frac{-108}{29}$
1123, 24	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{32}{261}$	$\frac{1}{29}$	$\frac{860}{87}$	$\frac{218}{29}$	$\frac{-970}{87}$	$\frac{-2296}{29}$	$\frac{-628}{29}$	$\frac{7976}{87}$	$\frac{-778}{87}$	$\frac{4288}{87}$	$\frac{-622}{87}$	$\frac{-10}{3}$
1123, 44	$\frac{1}{261}$	$\frac{16}{29}$	$\frac{-16}{261}$	$\frac{-1}{29}$	$\frac{16}{87}$	$\frac{-176}{29}$	$\frac{244}{87}$	$\frac{592}{29}$	$\frac{-152}{29}$	$\frac{-6992}{87}$	$\frac{-3404}{87}$	$\frac{-1024}{87}$	$\frac{292}{87}$	$\frac{620}{87}$
2223, 11	$\frac{1}{261}$	$\frac{128}{29}$	$\frac{-128}{261}$	$\frac{-1}{29}$	$\frac{5480}{261}$	$\frac{1472}{87}$	$\frac{-15016}{261}$	$\frac{-2992}{87}$	$\frac{-23584}{87}$	$\frac{79664}{261}$	$\frac{102248}{261}$	$\frac{-194048}{261}$	$\frac{-116}{9}$	$\frac{-3704}{261}$
2223, 12	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{64}{261}$	$\frac{1}{29}$	$\frac{-160}{261}$	$\frac{-640}{87}$	$\frac{7172}{261}$	$\frac{-5656}{87}$	$\frac{12320}{87}$	$\frac{-50824}{261}$	$\frac{-47284}{261}$	$\frac{96640}{261}$	$\frac{1076}{261}$	$\frac{964}{261}$
2223, 14	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{-32}{261}$	$\frac{-1}{29}$	$\frac{824}{261}$	$\frac{-466}{87}$	$\frac{4970}{261}$	$\frac{-1888}{87}$	$\frac{-628}{87}$	$\frac{-24496}{261}$	$\frac{-10030}{261}$	$\frac{-44672}{261}$	$\frac{494}{261}$	$\frac{394}{261}$
2223, 22	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{-32}{261}$	$\frac{-1}{29}$	$\frac{824}{261}$	$\frac{1100}{87}$	$\frac{-9124}{261}$	$\frac{2288}{87}$	$\frac{-10024}{87}$	$\frac{50672}{261}$	$\frac{32252}{261}$	$\frac{-44672}{261}$	$\frac{-1072}{261}$	$\frac{-1172}{261}$
2223, 24	$\frac{1}{261}$	$\frac{16}{29}$	$\frac{16}{261}$	$\frac{1}{29}$	$\frac{-748}{261}$	$\frac{518}{87}$	$\frac{-2470}{261}$	$\frac{2936}{87}$	$\frac{-1156}{87}$	$\frac{11192}{261}$	$\frac{-8830}{261}$	$\frac{21088}{261}$	$\frac{578}{261}$	$\frac{562}{261}$
2223, 44	$\frac{1}{261}$	$\frac{8}{29}$	$\frac{-8}{261}$	$\frac{-1}{29}$	$\frac{-340}{261}$	$\frac{224}{87}$	$\frac{-604}{261}$	$\frac{1520}{87}$	$\frac{-1936}{87}$	$\frac{5840}{261}$	$\frac{-6388}{261}$	$\frac{-7328}{261}$	$\frac{284}{261}$	$\frac{244}{261}$
2333, 11	$\frac{1}{261}$	$\frac{256}{87}$	$\frac{256}{261}$	$\frac{1}{87}$	$\frac{-13360}{261}$	$\frac{-3520}{87}$	$\frac{20816}{261}$	$\frac{12608}{29}$	$\frac{20960}{87}$	$\frac{-219712}{261}$	$\frac{-113968}{261}$	$\frac{133120}{261}$	$\frac{15464}{261}$	$\frac{16384}{261}$
2333, 12	$\frac{1}{261}$	$\frac{128}{87}$	$\frac{-128}{261}$	$\frac{-1}{87}$	$\frac{11168}{261}$	$\frac{2312}{87}$	$\frac{-14212}{261}$	$\frac{-27920}{87}$	$\frac{-14968}{87}$	$\frac{129296}{261}$	$\frac{45212}{261}$	$\frac{-64256}{261}$	$\frac{-9856}{261}$	$\frac{-8948}{261}$
2333, 14	$\frac{1}{261}$	$\frac{64}{87}$	$\frac{64}{261}$	$\frac{1}{87}$	$\frac{3836}{261}$	$\frac{338}{87}$	$\frac{-4126}{261}$	$\frac{-2968}{29}$	$\frac{-2092}{87}$	$\frac{21368}{261}$	$\frac{-10174}{261}$	$\frac{32512}{261}$	$\frac{-2530}{261}$	$\frac{-1514}{261}$
2333, 22	$\frac{1}{261}$	$\frac{64}{87}$	$\frac{64}{261}$	$\frac{1}{87}$	$\frac{-5560}{261}$	$\frac{-1228}{87}$	$\frac{6836}{261}$	$\frac{4688}{29}$	$\frac{7304}{87}$	$\frac{-72592}{261}$	$\frac{-24268}{261}$	$\frac{32512}{261}$	$\frac{5300}{261}$	$\frac{6316}{261}$
233, 324	$\frac{1}{261}$	$\frac{32}{87}$	$\frac{-32}{261}$	$\frac{-1}{87}$	$\frac{-940}{261}$	$\frac{-274}{87}$	$\frac{842}{261}$	$\frac{2584}{87}$	$\frac{-76}{87}$	$\frac{-12808}{261}$	$\frac{2666}{261}$	$\frac{-14528}{261}$	$\frac{878}{261}$	$\frac{1882}{261}$
2333, 44	$\frac{1}{261}$	$\frac{16}{87}$	$\frac{16}{261}$	$\frac{1}{87}$	$\frac{1088}{261}$	$\frac{128}{87}$	$\frac{-2140}{261}$	$\frac{-1120}{29}$	$\frac{-808}{87}$	$\frac{11168}{261}$	$\frac{5204}{261}$	$\frac{7360}{261}$	$\frac{-1156}{261}$	$\frac{-4}{9}$

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