ON SOME PROBLEMS INVOLVING NEARLY HOLOMORPHIC MODULAR FORMS AND AN ESTIMATE FOR FOURIER COEFFICIENTS OF HERMITIAN CUSP FORMS

By

ARVIND KUMAR

MATH08201204002

Harish-Chandra Research Institute, Allahabad

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

November, 2017

Homi Bhabha National Institute[1]

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Arvind Kumar entitled "On some problems involving nearly holomorphic modular forms and an estimate for Fourier coefficients of Hermitian cusp forms" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Chairman – Prof. S. D. Adhikari	**Date:** 25/11/17
Guide / Convener – Prof. B. Ramakrishnan	**Date:** 25/11/17
Co-guide - N/A	**Date:**
Examiner - Prof. Eknath Ghate	**Date:** 25/11/17
Member 1- Prof. R. Thangadurai	**Date:** 25/11/17
Member 2- Prof. D. Surya Ramana	**Date:** 25/11/17
Member 3- Prof. P. K. Ratnakumar	**Date:** 25/11/17

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I/We hereby certify that I/we have read this thesis prepared under my/our direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: 25/11/2017
Place: Allahabad

Prof. B. Ramakrishnan
Guide

[1] This page is to be included only for final submission after successful completion of viva voce.

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgment the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

अरविन्द कुमार

Arvind Kumar

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

अरविन्द कुमार

Arvind Kumar

List of Publications arising from the thesis

Journal

1. "On arbitrary products of eigenforms", Arvind Kumar and Jaban Meher, *Acta Arith.*, **2016**, *173, no.3*, 283-295.

2. "The adjoint map of the Serre derivative and special values of shifted Dirichlet series", Arvind Kumar, *J. of Number Theory,* **2017**, *177* 516-527.

Accepted

1. "Estimates for Fourier coefficients of Hermitian cusp forms of degree two", Arvind Kumar and B. Ramakrishnan, *Acta Arith.* (To appear)

Conferences

1. The adjoint map of the Serre derivative and its application, 31st Automorphic Forms Workshop, East Tennessee State University, Johnson City, Tn, USA, 06-09 March 2017.
2. The adjoint map of the Serre derivative and its application, Discussion Meeting on Automorphic forms, NISER, Bhubaneswar, 21-27 Dec 2016.
3. On arbitrary products of eigenforms, Building Bridges: 3rd EU/US Workshop on Automorphic Forms and Related Topics, University of Sarajevo, Bosnia and Herzegovina, 18-22 July 2016.
4. On arbitrary products of eigenforms, Workshop on Automorphic forms, KSOM, Kozhikode, India, 10-16 Feb 2016.

अरविन्द कुमार

Arvind Kumar

Dedicated to

MY MOTHER

&

the best person I know

Shri Prem Chandra Gupta

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my supervisor Prof. B. Ramakrishnan, for all his guidance, affectionate encouragement and endless support. Not only is he a great adviser, who made me into the mathematician I am today, for me he is also truly a life coach, and I would not be the person that I am now if it wasn't for him. He was always there, whenever I needed any academic or non-academic attention, however microscopic it may have been. This thesis is the result of his enterprising supervision.

I thank the members of my Doctoral committee, Prof. S. D. Adhikari, Prof. R. Thangadurai, Prof. D. Surya Ramana and Prof. P. K. Ratnakumar, for their constant support. I also thank the faculty members of HRI for the courses they offered me.

I express my sincere thanks to Prof. S. Boecherer, Prof. E. Ghate, Prof. B. Heim, Prof. D. Lanphier, Prof. Y. Martin and Prof. O. K. Richter, for their assistance, suggestions and mathematical discussions. I would also like to thank Prof. M. Manickam, Dr. S. Gun and Dr. B. Sahu for their guidance and providing ideal conditions at their respective institutes during my visits.

I also thanks a lot to the referees for their careful reading of the thesis and providing many suggestions.

I express my sincere gratitude to Karam Deo bhaiya and my co-author Jaban bhaiya for their advice and useful discussions. I also thanks my other co-authors Abhash bhaiya and especially, Manish who was always ready for discussing maths.

Remembering some wonderful moments during my PhD, I thank Ashutosh, Balesh, Bibek, Divyang, Jay, Juyal, Mallesham, Manna, Pallab, Prem Prakash, Rahul, Senthil, Shailesh, Sudheer, Utkarsh (bhaiya's), Sneh, Sayani, Sumana (di's), Anoop, Anup, Manikandan, Nabin, Pradeep, Pramod and many other friends. Special thanks to Bhuwanesh for helping me in many ways.

I would like to acknowledge the HRI for providing me the ideal atmosphere, required infrastructure and funding. I heartily appreciate to almost all the administrative staff of HRI for their co-operation and help. I thank the CSIR, India for providing partial financial support through SPM fellowship.

I would also like to note my sincere thanks to my teachers who have taught me at different levels, specially S. V. Maurya, Dr. Bankteshwar Tiwari and Dr.

Shantanu Bhattacharya for their love and for encouraging me to pursue higher studies in mathematics. I am indebted to all my friends who have supported me over the past several years, specially Dhananjay, Priyanshu, Ravi, Vivekanand, Abhishek, Anil, Delip, Harendra, Ishan, Manoj, Somveer, Prachi, Vijay, Yogesh, and many others from BHU (I can not list all the names here, but you are always on my mind), who are very close to my heart. I feel blessed to have them all in my life.

Words are not enough to thank my family, especially to my mother, for the encouragement and support, without which I would not have come this far. Last but not the least I thank my companion Moni for sharing all my agony, all my sorrow, all my frustration and finally all my joy. I appreciate her endless support, affection and encouragement, which helped me to keep going.

Abstract

This thesis studies three different problems in the theory of modular forms. The first result shows that apart from the 26 exceptions, the product of finitely many quasimodular eigenforms in not a quasimodular eigenform. This is proved by using a structure theorem which says that a quasimodular eigenform is the derivative of a homomorphic eigenform, or of the Eisenstein series E_2. We prove a similar theorem for nearly homomorphic modular forms using an interesting algebra isomorphism between the space of quasimodular forms and the space of nearly holomorphic eigenforms.

The second problem is to compute the adjoint of the Serre derivative map. We give a remarkable formula for the adjoint of the Serre derivative map with respect to the usual Petersson inner product in terms of special values of certain shifted Dirichlet series attached to modular forms and some applications. To prove our result, we use existing tools of the theory of nearly holomorphic modular forms.

Finally, the third part of the thesis gives estimates for the Fourier coefficients of Hermitian cusp forms over the imaginary quadratic field $\mathbb{Q}(i)$ by proving estimates for the Fourier coefficients of the Hermitian Jacobi forms occurring in the Fourier-Jacobi expansion of the original Hermitian form. This is the most technical part of the thesis and involves several estimates.

Contents

Synopsis

0.1 Introduction

This thesis deals with three problems in the theory of modular forms. The first problem is about the question : "when is an arbitrary product of Hecke eigenforms again an eigenform?". We discuss this problem in the context of nearly holomorphic modular forms and quasimodular forms. The second one is on finding the adjoint map of the Serre derivative map and as an application we find a formula for the Ramanujan tau function in terms of special values of certain shifted Dirichlet series. The third problem is on finding an estimate for Fourier coefficients of Hermitian cusp forms of degree two (for the field $\mathbb{Q}(i)$).

0.2 Background

We denote the space of modular forms and the subspace of cusp forms of weight k for a congruence subgroup Γ of $SL_2(\mathbb{Z})$ by $M_k(\Gamma)$ and $S_k(\Gamma)$, respectively. We write M_k and S_k for the corresponding spaces if Γ is the full modular group. Moreover, for $k \geq 4$, $M_k = \langle E_k \rangle \oplus S_k$, where $E_k(z)$ denotes the normalized Eisenstein series of weight k. Unless otherwise stated we assume that $z =$

$x + iy \in \mathbb{H}$ and $q = e^{2\pi iz}$. We know that for a positive even integer k,

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor function and B_k is the k-th Bernoulli number. In the sequel, Δ_k denotes the unique normalized cusp form of weight k for $SL_2(\mathbb{Z})$ for $k \in \{12, 16, 18, 20, 22, 26\}$ and we write Δ for Δ_{12}, the Ramanujan delta function.

It is well-known that M_k forms a complex vector space and there is a basis consisting entirely of forms called Hecke eigenforms which are simultaneous eigenvectors for all of the Hecke operators. The Fourier coefficients of a Hecke eigenform are particularly important and satisfy some nice arithmetical relations.

0.3 On arbitrary product of eigenforms

Identities among modular forms have attracted the attention of many mathematicians since they imply nice identities among the Fourier coefficients of modular forms. One such identity is the following:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m) \sigma_3(n-m), \tag{1}$$

for $n \geq 1$. Since the vector space M_8 is one dimensional, it follows that $E_4^2 = E_8$ and comparing the n-th Fourier coefficients of both sides yields (1). The identity $E_4^2 = E_8$ can be viewed as an eigenform identity as both E_4 and E_8 are Hecke eigenforms. The set of all modular forms (of all weights) for the full modular group is a graded complex algebra. Having seen an identity as above ($E_4^2 = E_8$), it is quite natural to ask whether the property of being a Hecke eigenform is preserved under multiplication. This problem was first studied independently by W. Duke [13] and E. Ghate [17]. They found that it is indeed quite rare that the product of Hecke eigenforms is again a Hecke eigenform. In fact, they proved that there are only a finite number of examples of this phenomenon, which are forced from dimensional constraints. More precisely, they proved the following theorem.

Theorem A. (W. Duke [13], E. Ghate [17]) *The product of two Hecke eigenforms for $SL_2(\mathbb{Z})$ is an eigenform only in the following cases:*

$$E_4^2 = E_8, \quad E_4E_6 = E_{10}, \quad E_6E_8 = E_4E_{10} = E_{14}, \quad E_4\Delta = \Delta_{16}, \quad E_6\Delta = \Delta_{18},$$
$$E_4\Delta_{16} = E_8\Delta = \Delta_{20}, \quad E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta = \Delta_{22},$$
$$E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta = \Delta_{26}.$$

Both the results in [13], [17] make use of Rankin-Selberg convolutions. In [16], B. Emmons and D. Lanphier extended this result to a product of an arbitrary number of Hecke eigenforms. Instead of products of two eigenforms one can also consider the Rankin-Cohen bracket of eigenforms and pose a similar question. In [34], D. Lanphier and R. Takloo-Bighash considered this problem and in this case also only finitely many cases occur.

Let $D = \frac{1}{2\pi i}\frac{d}{dz}$ be the derivative function. It is known that the Eisenstein series $E_2(z)$ of weight 2 is not a modular form. The same is true if we consider the derivative Df of a modular form f. These functions belong to a different class of forms, called quasimodular forms. For $n \geq 1$, we have the well-known identity proved by Ramanujan

$$n\tau(n) = \tau(n) - 24\sum_{m=1}^{n-1}\tau(m)\sigma(n-m), \tag{2}$$

where $\tau(n)$ is the Ramanujan's tau function and $\sigma(n)$ is the sum of divisors of n. The above relation is obtained by the identity $D\Delta = E_2\Delta$. This can be regarded as an identity in the graded complex algebra of quasimodular forms for the full modular group, where the product of two quasimodular eigenforms results in an eigenform. Therefore, it is interesting to find all such cases. This question was considered by S. Das and J. Meher in [10] and [36] for the full modular group. They showed that there are two extra identities apart from the 16 coming from modular forms.

In 1976, G. Shimura introduced the concept of nearly holomorphic modular forms in order to prove some algebraicity results for special values of Rankin product L-functions. Let $\widehat{M}_k$ denote the space of nearly holomorphic modular forms of weight k. There is a differential operator R_k on $\widehat{M}_k$, which is called the Maass-Shimura operator, defined by

$$R_k(f)(z) = \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{k}{2i\mathrm{Im}(z)}\right) f(z), \tag{3}$$

where $\mathrm{Im}(z)$ stands for the imaginary part of z. The operator R_k takes $\widehat{M}_k$ to $\widehat{M}_{k+2}$. We write $R_k^{(m)} := R_{k+2m-2} \circ \cdots \circ R_{k+2} \circ R_k$ with $R_k^{(0)} = id$ and $R_k^{(1)} = R_k$. In [3], J. Beyerl and et al. considered the problem of product relations among eigenfoms for a subclass of nearly holomorphic modular forms for the group $SL_2(\mathbb{Z})$. More precisely, they examined the problem for the class of nearly holomorphic modular forms which can be written as a Maass-Shimura operator applied on modular forms.

It is important to note that there is an explicit ring isomorphism between the space of quasimodular forms and the space of nearly holomorphic modular forms. However in [36] the author used the multiplicative properties of Hecke eigenforms for his results, whereas in [3], the authors made use of properties of Rankin-Cohen bracket operators and the results of [34] to obtain the main result. In our work, which is presented in Chapter 2, we make use of the isomorphism between the spaces of quasimodular forms and nearly holomorphic modular forms mentioned above. First we characterize the structure of a general nearly holomorphic eigenform. Then we prove that it is sufficient to consider the problem of finding a polynomial eigenforms identity in any one of the spaces. We show that this will correspond to a similar identity in the other space. More precisely, the first main result in chapter 2 is the following.

Theorem 0.3.1 ([32], page no. 290) *In a space of quasimodular or nearly holomorphic modular forms, a polynomial relation among eigenforms in one space gives rise to a corresponding polynomial relation in the other space.*

From Theorem 0.3.1, it follows that the main results of [3] and [36] imply each other.

In [36] and [10], the authors classified all the cases where the product of two quasimodular eigenform results in an eigenform. Using this classification and Theorem 0.3.1, we prove the following theorem.

Theorem 0.3.2 ([32], Theorem 1.2) *The product of two nearly holomorphic eigenforms for $SL_2(\mathbb{Z})$ is never an eigenform except for the following exceptional cases:*

1. *The 16 holomorphic cases presented in [13] and [17]*

2. $(R_4E_4)E_4 = \frac{1}{2}R_8E_8, \quad E_2^*\Delta = R_{12}\Delta,$ *where* $E_2^*(z) = E_2(z) - \frac{3}{\pi \mathrm{Im} z}$.

Thus, Theorem 0.3.2 extends and gives another proof of the main result of [3] together with an extra identity. We now consider the case of products of an arbitrary number of quasimodular eigenforms and characterize all quasimodular eigenforms which can be written as products of finitely many eigenforms. This is obtained in the following theorem.

Theorem 0.3.3 ([32], Theorem 1.3, Theorem 1.4) *The products of finitely many quasimodular eigenforms (resp. nearly holomorphic eigenforms) for $SL_2(\mathbb{Z})$ is never a quasimodular eigenform (resp. nearly holomorphic eigenform) except for the following exceptional cases:*

1. *The 16 holomorphic cases presented in Theorem A.*

2. *Other holomorphic cases which can be obtained trivially from some of the identities presented in Theorem A, namely*

$$E_4^2 E_6 = E_{14}, \quad E_4^2\Delta_{12} = \Delta_{20}, \quad E_4E_6\Delta_{12} = \Delta_{22},$$
$$E_4^2\Delta_{18} = E_4E_6\Delta_{16} = E_4^2E_6\Delta_{12} = E_6E_8\Delta_{12} = E_4E_{10}\Delta_{12} = \Delta_{26}.$$

3. $(DE_4)E_4 = \frac{1}{2}DE_8, \quad E_2\Delta_{12} = D\Delta_{12}$ *(resp.* $(R_4E_4)E_4 = \frac{1}{2}R_8E_8, E_2^*\Delta = R_{12}\Delta$*).*

From Theorem 0.3.1, it is sufficient to consider one of the spaces. Here we consider the case of quasimodular forms. The methods used in the proofs are largely elementary and use the structure theorem of the space of quasimodular forms. To prove our result, several cases are distinguished. The proof essentially shows that the identities given by the distinguished cases cannot hold in general by Fourier coefficient considerations. Technical arguments are then used to show that these cases result in eigenforms only in certain situations which are listed in Theorem 0.3.3.

The contents of chapter 2 are published in [32].

0.4 The Adjoint map of the Serre derivative

In section 1 we observed that both E_2 and Df are not modular forms, if f is a modular form. However, by taking a certain linear combination of Df and $E_2 f$, we get a function which transforms like a modular form. The underlying linear operator ϑ_k on $M_k(\Gamma)$ is defined by

$$\vartheta_k f := Df - \frac{k}{12} E_2 f, \tag{4}$$

which preserves the modular property. It is well-known that $\vartheta_k f$ is a modular form of weight $k+2$ for Γ and the operator ϑ_k is referred to as the *Serre derivative* (or sometimes *the Ramanujan-Serre differential operator*) in the literature. It is an interesting and useful operator because it defines an operator on the space of modular forms for any congruence subgroup with character and also it preserves the space of cusp forms.

Using the properties of Poincaré series and adjoints of linear maps, W. Kohnen [27] constructed the adjoint map of the product map (product by a fixed cusp form), with respect to the Petersson scalar product. After Kohnen's work, similar results have been obtained by many mathematicians for different types of maps and also for other spaces of automorphic forms. These types of results are important because the Fourier coefficients of the image under the adjoint map involve special values of certain shifted Dirichlet series.

In the third chapter of the thesis (which is published in [31]) we find the adjoint of the Serre derivative map ϑ_k with respect to the Petersson inner product. Given a cusp form of weight $k+2$, it gives a construction of a cusp form of weight k with interesting Fourier coefficients. For our purpose, we observe that the Serre derivative can also be written in the form

$$\vartheta_k(f) = R_k f - \frac{k}{12} E_2^* f, \tag{5}$$

where R_k is the Maass-Shimura operator defined by (3) and $E_2^*(z) = E_2(z) - \frac{3}{\pi \mathrm{Im} z}$. The main result of chapter 3 is the following.

Theorem 0.4.1 *[31, Theorem 4.1] Let $k \geq 2$ and ϑ_k^* be the adjoint map of ϑ_k with respect to the Petersson inner product. Then the image of any function $f(z) = \sum_{n\geq 1} a(n)q^n \in S_{k+2}(\Gamma)$ under ϑ_k^* is given by $\vartheta_k^* f(z) = \sum_{m\geq 1} c(m)q^m$, where*

$$c(m) = \frac{1}{\mu_\Gamma} \frac{k(k-1)m^{k-1}}{(4\pi)^2} \left[\frac{(m-\frac{k}{12})}{m^{k+1}} a(m) + 2kL_{f,m}(k+1) \right],$$

and $L_{f,m}(s) = \sum_{n\geq 1} \frac{a(n+m)\sigma(n)}{(n+m)^s}$ is the shifted Dirichlet series associated with f.

The main ingredient in the proof is the theory of nearly holomorphic modular forms and the Rankin unfolding arguments.

An application of Theorem 0.4.1. Let $k \geq 2$ and Γ be a congruence subgroup for which $S_k(\Gamma)$ is a one-dimensional space; we denote a generator of $S_k(\Gamma)$ by $f(z)$. Then applying Theorem 0.4.1, we get $\vartheta_k^* g(z) = \alpha_g f(z)$ for any $g \in S_{k+2}(\Gamma)$, where α_g is a constant. Now equating the m-th Fourier coefficients of both the sides, we get a relation among the special values of the shifted Dirichlet series associated with g and the Fourier coefficients of f. For example, by choosing $k = 10$ and $\Gamma = \Gamma_0(2)$, consider the map $\vartheta_{10}^* : S_{12}(\Gamma_0(2)) \to S_{10}(\Gamma_0(2))$. As $S_{10}(\Gamma_0(2)) = \mathbb{C}\Delta_{10,2}(z)$ and $S_{12}(\Gamma_0(2)) = \mathbb{C}\Delta(z) \oplus \mathbb{C}\Delta(2z)$, applying ϑ_{10}^* on $\Delta(z)$

and $\Delta(2z)$ and using Theorem 0.4.1, we get the following interesting relations.

$$\tau(m) = \frac{-20m^{11}}{(m-\frac{5}{6})} L_{\Delta,m}(11) \tag{6}$$

and

$$\tau_{10,2}(m) = \frac{3(2m)^9 \|\Delta_{10,2}\|^2}{8\pi^2 \|\Delta\|^2} \left[\frac{(m-\frac{10}{12})}{m^{11}} \tau\left(\frac{m}{2}\right) + 20 L_{V_2\Delta,m}(11) \right], \tag{7}$$

where $\tau(n) = 0$ if n is not an integer and $V_2\Delta(\tau) = \Delta(2\tau)$. Here $\tau_{10,2}(m)$ is the m-th Fourier coefficient of unique normalized newform of weight 10 for $\Gamma_0(2)$ and $\|\cdot\|$ denotes the Petersson norm.

0.5 Estimates for Fourier coefficients of Hermitian cusp forms of degree two

The theory of Hermitian Jacobi forms along the lines of the classical Jacobi forms was first considered by K. Haverkamp [19] in his thesis. However, in a recent work, O. Richter and J. Senadheera [39] realized that the Hermitian Jacobi forms are classified into two different classes of forms, one with parity $+1$ and the other with parity -1. It is to be noted that these Hermitian Jacobi forms with parity ± 1 arise in a natural way (like in the case of classical Jacobi forms) via the Fourier Jacobi coefficients of Hermitian modular forms of degree two with character $(\det)^l$, where l varies modulo 2. We remark here that almost all the existing results in the literature consider Hermitian Jacobi forms with parity $+1$, which come with the condition that the weight k is divisible by 4. Using the refined definition of Richter and Senadheera as mentioned above, one can extend all the results for forms with parity -1 as well (one has to assume that $k \equiv 2$

(mod 4) in this case). However, we have not found any result which determines the explicit Fourier expansion of an important class of functions, namely the Poincaré series. One of the aims in Chapter 4 is to obtain the explicit Fourier expansions of Hermitian Jacobi Poincaré series. In this case also their Fourier coefficients involve certain generalized Kloosterman sums and Bessel functions. In [28], W. Kohnen used the estimates of Fourier coefficients of Jacobi cusp forms to obtain estimates for Fourier coefficients of Siegel cusp forms of degree two. Our main objective in this chapter is to adapt this technique in the context of Hermitian Jacobi cusp forms and get estimates for Fourier coefficients of Hermitian cusp forms of degree two on $\Gamma^{(2)} = M_4(\mathcal{O}) \cap U(2)$, where $\mathcal{O}$ denotes the ring of integers in $\mathbb{Q}(i)$ and $U(2)$ is the unitary group of degree 2. It gives improved estimates for the Fourier coefficients when compared with the usual Hecke bound. We now state the main result of this chapter.

Theorem 0.5.1 *[33, Theorem, 5.2] Let $a(T)$ denote the T-th Fourier coefficient of a Hermitian cusp form F of weight k on $\Gamma^{(2)}$ with character $(\det)^l$, then for any $\epsilon > 0$, we have*

$$a(T) \ll_{\epsilon,F} (\min\, T)^{16/19+\epsilon} (\det\, T)^{k/2-3/4+\epsilon}. \tag{8}$$

Moreover, by using the reduction theory, taking $m = \min\, T \ll (\det T)^{1/2}$, we obtain

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-25/76+\epsilon}.$$

The proof of the above theorem is based on appropriate estimates both for the Fourier coefficients of Hermitian Jacobi cusp forms on the Hermitian Jacobi group and for the Petersson norms of the Fourier Jacobi coefficients of the Hermitian cusp form F. The contents of chapter 4 is published in [33].

CHAPTER 1

Background

We introduce basic definitions and properties of different kinds of automorphic forms in this chapter. We only present what is relevant to this thesis, and is by no means a complete overview of the subject. We closely follow [5], [12], [24] and [26].

1.1 Notations

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. For $z \in \mathbb{C}$, $\mathrm{Re}(z)$ denotes the real part of z and $\mathrm{Im}(z)$ denotes the imaginary part of z. For any complex number z and a non-zero real number c, we denote by $e_c(z) = e^{2\pi i z/c}$. If $c = 1$, we simply write $e(z)$ instead of $e_1(z)$. Also the square root of z is defined as follows:

$$\sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{i}{2} arg\ z}, \text{ with } -\pi < arg\ z \leq \pi.$$

We set $z^{\frac{k}{2}} = (\sqrt{z})^k$ for any $k \in \mathbb{Z}$. For integers a, b and c, the notation $a \equiv b(c)$ means that $c|(a-b)$ and for any positive integer ν, $a^\nu || b$ emphasize that $a^\nu | b$ but $a^{\nu+1} \nmid b$

1.2 Modular forms

The group

$$GL_2^+(\mathbb{Q}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Q}, ad - bc > 0 \right\}$$

acts on the Poincaré upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, by the fractional linear transformation as follows. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ and $z \in \mathbb{H}$, we let

$$\gamma z := \frac{az+b}{cz+d} \in \mathbb{H}. \tag{1.1}$$

Then for any integer k and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, the slash operator on functions $f : \mathbb{H} \to \mathbb{C}$ is defined by

$$(f \mid_k \gamma)(z) := (\det \gamma)^{k/2} (cz+d)^{-k} f(\gamma z).$$

We mostly need the action (1.1) for $\gamma \in SL_2(\mathbb{Z})$, the *full modular group*, defined by

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

1.2.1 The congruence subgroup

Let N be a positive integer. The principal congruence subgroup of level N is

$$\begin{aligned}\Gamma(N) &= \operatorname{Ker}\left(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\right) \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1(N), b \equiv c \equiv 0(N) \right\}.\end{aligned}$$

A *congruence subgroup* of $SL_2(\mathbb{Z})$ is any subgroup Γ of $SL_2(\mathbb{Z})$ that contains $\Gamma(N)$ for some $N \in \mathbb{Z}, N > 0$. The smallest such N is called the *level* of Γ.

Besides $\Gamma(N)$, the two most important congruence subgroups are

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1(N), c \equiv 0(N) \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0(N) \right\}.$$

Then, one has

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z}).$$

Unless stated otherwise, we always let $k \in \mathbb{Z}$ and Γ denotes a congruence subgroup of level N.

Cusps: Let $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. We then extend the action of $SL_2(\mathbb{Z})$ on $\widehat{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, the extended upper half-plane, in the following way.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \widehat{\mathbb{H}}$, define

$$\gamma z := \begin{cases} \infty & \text{if } z = -d/c, \\ a/c & \text{if } z = \infty, \\ (az+b)/(cz+d) & \text{otherwise.} \end{cases}$$

A *cusp* of Γ is a Γ-equivalent class of elements in $\mathbb{P}^1(\mathbb{Q})$ under the action of Γ. Note that the group $SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, hence there is only one cusp of $SL_2(\mathbb{Z})$. Since every congruence subgroup Γ has finite index, it follows that there are only finitely many cusps of Γ.

Holomorphicity at the cusps: Assume that $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function which satisfies the *modular transformation property* for Γ, namely $f \mid_k \gamma = f$ for all $\gamma \in \Gamma$ (such an f is said to be *weakly holomorphic modular form* of weight k with respect to Γ). Let D' be the open unit disk in $\mathbb{C}$ with the origin removed. Then $z \mapsto q_N := e_N(z)$ defines a map from $\mathbb{H}$ into D'. Since $f \mid_k \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} = f$, it follows that f is periodic with period N and hence there exists a function $F : D' \to \mathbb{C}$ such that $F(q_N) = f(z)$. If for all $q_N \in D'$, we have the Laurent series expansion of the form

$$f(z) = F(q_N) = \sum_{n \geq 0} a(n) q_N^n, \tag{1.2}$$

then f is said to be holomorphic at ∞. Moreover, if $a(0) = 0$, we say that f vanishes at ∞. Eq. (1.2) is called the Fourier expansion of f at ∞ or the q-expansion of f about ∞, and the numbers $a(n) \in \mathbb{C}$ are called the Fourier coefficients of f.

Since any cusp $s \in \mathbb{P}^1(\mathbb{Q})$ can be written as $s = \gamma_0\infty$, for some $\gamma_0 \in SL_2(\mathbb{Z})$ and therefore holomorphy at s is naturally defined in terms of holomorphy at ∞ via the slash operator. More precisely, f is said to be holomorphic (or vanishes) at the cusp s if $f\mid_k \gamma_0$ is holomorphic (or vanishes) at ∞ (it makes sense as it is seen easily that $f\mid_k \gamma_0$ is a weakly holomorphic modular form of weight k with respect to the congruence subgroup $\gamma_0^{-1}\Gamma\gamma_0$, if f is weakly holomorphic modular form with respect to Γ).

We are ready to define the modular forms.

1.2.2 Definition and examples

Definition 1.2.1 *A modular form of weight k with respect to Γ is a function $f : \mathbb{H} \to \mathbb{C}$ which satisfies*

1. *f is holomorphic on $\mathbb{H}$,*
2. *for all $\gamma \in \Gamma$,*
$$f\mid_k \gamma = f, \tag{1.3}$$
3. *f is holomorphic at all the cusps of Γ.*

Moreover, if f vanishes at all the cusps of Γ, then f is said to be a cusp form of weight k with respect to Γ.

We denote the space of modular forms and the subspace of cusp forms of weight k for Γ by $M_k(\Gamma)$ and $S_k(\Gamma)$, respectively. We simply write M_k and S_k for the corresponding spaces, if Γ is the full modular group $SL_2(\mathbb{Z})$.

Fundamental domain: If we know the value of a modular form f for Γ at one point $z \in \mathbb{H}$, then equation (1.3) tells us the value at all points in the same Γ-orbit of z. So in order to completely determine f, it is enough to know the value at one point from each orbit. This leads to the concept of a *fundamental domain* for Γ, namely an open and connected subset $\mathcal{F} \subset \mathbb{H}$ such that no two distinct points of $\mathcal{F}$ are equivalent under the action of Γ and every point $z \in \mathbb{H}$ is Γ-equivalent to some point in the closure of $\mathcal{F}$.

Proposition 1.2.2 *The set*

$$\mathcal{F}_1 = \left\{ z \in \mathbb{H} \;\middle|\; |z| > 1, |\mathrm{Re}(z)| < \tfrac{1}{2} \right\}$$

is a fundamental domain for the full modular group $SL_2(\mathbb{Z})$.

Remark 1.2.1 *Using Proposition 1.2.2, we can easily determine a fundamental domain of any congruence subgroup Γ from its coset representatives in $SL_2(\mathbb{Z})$.*

Definition 1.2.3 (Petersson inner product) *Let $f, g \in M_k(\Gamma)$ be such that at least one of them is a cusp form. Write $z = x + iy$, then the Petersson inner product of f and g is defined as:*

$$\langle f, g \rangle := \frac{1}{\mu_\Gamma} \int\limits_{\Gamma \backslash \mathbb{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}, \tag{1.4}$$

where $\Gamma \setminus \mathbb{H}$ is a fundamental domain, $\dfrac{dxdy}{y^2}$ is an invariant measure under the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ and μ_Γ denotes the index of Γ in $SL_2(\mathbb{Z})$.

It is well-known that $S_k(\Gamma)$ is a finite-dimensional Hilbert space with respect to the inner product defined by (1.4). For our purpose, we state the following invariance property of the inner product under the slash operator.

Proposition 1.2.4 *[26, Chapter 3, Proposition 46] Let $f, g \in M_k(\Gamma)$ with f or g a cusp form. Let $\gamma \in GL_2^+(\mathbb{Q})$. Then*

$$\langle f \mid_k \gamma, g \mid_k \gamma \rangle = \langle f, g \rangle. \tag{1.5}$$

The following basic examples of modular forms are needed in our discussion.

Example 1. Let k be an even integer greater than 2. The normalized Eisenstein series E_k of weight k for $SL_2(\mathbb{Z})$ is defined as:

$$E_k(z) := \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ (m,n)=1}} \frac{1}{(mz+n)^k}.$$

Then E_k is a modular form of weight k for $SL_2(\mathbb{Z})$ with Fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n, \tag{1.6}$$

where $\sigma_r(n) = \sum\limits_{d|n} d^r$, for any positive integer r and B_k's are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{k\geq 0} B_k \frac{x^k}{k!}.$$

Because we use Eisenstein series in our work, we give the Fourier expansion of the first few Eisenstein series:

$$\begin{aligned}
E_4(z) &= 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \\
E_6(z) &= 1 - 504 \sum_{n\geq 1}^{\infty} \sigma_5(n) q^n, \\
E_8(z) &= 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n, \\
E_{10}(z) &= 1 - 264 \sum_{n\geq 1} \sigma_9(n) q^n, \\
E_{12}(z) &= 1 + \frac{65520}{691} \sum_{n\geq 1} \sigma_{11}(n) q^n, \\
E_{14}(z) &= 1 - 24 \sum_{n\geq 1} \sigma_{13}(n) q^n.
\end{aligned}$$

Remark 1.2.2 *When $k = 2$ we have the Fourier expansion of the Eisenstein* *series* E_2 *of weight* 2 *as*

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma(n) q^n, \tag{1.7}$$

where $\sigma(n) = \sigma_1(n)$. *However,* E_2 *is not a modular form because it has the following transformation property.*

$$(cz+d)^{-2} E_2\left(\frac{az+b}{cz+d}\right) = E_2(z) + \frac{6}{\pi i} \frac{c}{cz+d}, \tag{1.8}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. *Therefore,* E_2 *is not a modular form, it is a quasi-modular form of weight* 2 *and depth* 1 *(the concept of quasimodular forms is introduced in Section 1.4.1).*

Example 2. The *Ramanujan delta function* is defined as

$$\Delta(z) \ := \ \frac{1}{1728}(E_4(z)^3 - E_6(z)^2).$$

It is a cusp form of weight 12 for $SL_2(\mathbb{Z})$ with Fourier expansion

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} \ = \ \sum_{n \geq 1} \tau(n) q^n,$$

where $\tau(n)$ is called the *Ramanujan tau function.*

In fact for $k \geq 16$, it is easy to see that the function ΔE_{k-12} is a cusp form of weight k for the full modular group $SL_2(\mathbb{Z})$.

Example 3. Let k and n be positive integers. The n-th *Poincaré series* of weight k for a congruence subgroup Γ is defined by

$$P_{k,n}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i n z} \,|_k\, \gamma, \tag{1.9}$$

where $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{Z} \right\} \cap \Gamma.$
It is well-known that $P_{k,n} \in S_k(\Gamma)$ for $k > 2$ and it is characterized by the following property which is known as the Petersson coefficient formula.

Lemma 1.2.5 *Let $f \in S_k(\Gamma)$ with Fourier expansion $f(z) = \sum_{m \geq 1} a(m) q^m$. Then*

$$\langle f,\ P_{k,n} \rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a(n).$$

The structure of the vector space M_k is well-known; $M_k = \mathbb{C}E_k \oplus S_k$. In fact, we also have the following dimension formula for the space M_k:

$$\dim M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \equiv 2(12), \\ \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2(12), \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. If S_k is 1-dimensional, let Δ_k denote the unique normalized cusp form. When $k = 12$, we write Δ instead of Δ_{12}. Using the dimension formula, we have

$$\begin{aligned} M_k &= \mathbb{C}E_k, \text{ for } k = 4, 6, 8, 10, 14, \\ S_k &= \mathbb{C}\Delta_k, \text{ for } k = 12, 16, 18, 20, 22, 26. \qquad \text{(note that } \Delta_k = \Delta E_{k-12}) \end{aligned} \tag{1.10}$$

The following familiar result tells about the growth of the Fourier coefficients of a modular form in which the first statement can be easily obtained and the second is due to P. Deligne [11].

Proposition 1.2.6 *Let $a(n)$ be the n-th Fourier coefficient of a modular form $f \in M_k(\Gamma)$. Then for any $\epsilon > 0$, we have*

$$a(n) \ll_\epsilon n^{k-1+\epsilon},$$

and moreover, if f is a cusp form, then

$$a(n) \ll_\epsilon n^{\frac{k-1}{2}+\epsilon}.$$

1.3 Hecke operators

In this section, we define the Hecke operators for modular forms for the full modular group $SL_2(\mathbb{Z})$ and highlight some properties that are relevant to this thesis. We remark here that there is an analogous Hecke theory on modular forms of higher levels.

For any positive integer n, let

$$X_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{Z}, ad = n, 0 \leq b < d \right\}.$$

Definition 1.3.1 *For a positive integer k, we define the n-th Hecke operator T_n acting on functions $f : \mathbb{H} \to \mathbb{C}$ by the formula*

$$T_n f = n^{\frac{k}{2}-1} \sum_{\rho \in X_n} f \mid_k \rho.$$

The above expression means that for any function $f : \mathbb{H} \to \mathbb{C}$, we have

$$(T_n f)(z) = \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right). \tag{1.11}$$

1.3.1 Basic properties

Fixing the weight k, the Hecke operators satisfy the following properties (see [24]):

1. T_n maps periodic functions to periodic functions.
2. Let $f(z) = \sum_{n\geq 0} a(n)q^n$, then $T_n f$ is given by the series $(T_n f)(z) = \sum_{m\geq 0} a_n(m)q^m$, where

$$a_n(m) = \sum_{d|(m,n)} d^{k-1} a(mn/d^2). \tag{1.12}$$

3. For every $m, n \geq 1$, we have

$$T_m T_n = \sum_{d|(m,n)} d^{k-1} T_{mn/d^2}. \tag{1.13}$$

4. Let p be a prime and ν be a positive integer, then

$$T_{p^{\nu+1}} = T_p T_{p^\nu} - p^{k-1} T_{p^{\nu-1}}. \tag{1.14}$$

Next we consider the Hecke operators T_n on the space M_k of modular forms of weight k for the full modular group $SL_2(\mathbb{Z})$ and it turns out that the operators preserve the space. We state this in the following theorem.

Theorem 1.3.2 *[24, Theorem 6.8] The Hecke operators T_n map linearly a modular form to a modular form and a cusp form to a cusp form:*

$$T_n : M_k \to M_k,$$

$$T_n : S_k \to S_k.$$

Note that the Hecke operators T_n acting on the space of cusp forms for the full modular group are normal operators because they are self-adjoint operators, i.e.,

$$\langle T_n f, g\rangle = \langle f, T_n g\rangle \text{ for all } f, g \in S_k.$$

We give the definition of Hecke eigenforms which are the main part of the the second chapter of this thesis.

1.3.2 Hecke eigenforms

Definition 1.3.3 *A modular form $f \in M_k$ is said to be a Hecke eigenform (or an eigenform) if for all $n \in \mathbb{N}$, there are $\lambda_n \in \mathbb{C}$ so that $T_n f = \lambda_n f$. That is, f is the simultaneous eigenvector for all of the Hecke operators.*

If $a(n)$ is the n-th Fourier coefficient of an eigenform f then f is said to be normalized if $a(1) = 1$.

For example, one can easily prove that $T_n E_k = -\frac{B_k}{2k}\sigma_{k-1}(n)E_k$, which shows that the Eisenstein series E_k for each $k \geq 4$ is an eigenform. The small weight cusp forms $\Delta_{12}, \Delta_{16}, \Delta_{18}, \Delta_{20}, \Delta_{22}$ and Δ_{26} are also eigenforms. All of these examples come trivially from the fact that the operator T_n is acting on a 1-dimensional space.

We now state the following result from linear algebra, which is needed to get a basis consisting of Hecke eigenforms for S_k.

Proposition 1.3.4 *Let $\mathcal{S}$ be a finite dimensional Hilbert space over $\mathbb{C}$ and let $\mathcal{T}$ be a commuting family of normal operators $T : \mathcal{S} \to \mathcal{S}$. Then there exist an orthonormal basis of $\mathcal{S}$ which consists of common eigenfunctions of all the operators in $\mathcal{T}$.*

Let $\mathbb{T}$ denote the algebra over $\mathbb{C}$ generated by all the Hecke operators T_n (called the Hecke algebra). By the properties of Hecke operators given in (1.13) and

(1.14), we conclude that $\mathbb{T}$ is a commutative algebra generated by the operators T_p, for primes p. Also theses are normal operators while acting on the finite dimensional Hilbert space S_k. Applying Proposition 1.3.4 to the Hecke algebra $\mathbb{T}$ gives the following result.

Theorem 1.3.5 (Hecke) *Let S_k be the space of cusp forms for the full modular group. Then there exists an orthonormal basis (known as Hecke basis) which consists of eigenfunctions of all the Hecke operators T_n.*

Let $f(z) = \sum_{n\geq 0} a(n)q^n \in M_k$ be a Hecke eigenform, i.e., for all $n = 1, 2, 3, \ldots$.

$$T_n f = \lambda(n) f, \quad \text{for some } \lambda(n) \in \mathbb{C}$$

Using the Fourier coefficients relation (1.12), we get

$$\sum_{d|(m,n)} d^{k-1} a(mn/d^2) = \lambda(n)a(m), \quad \text{for all } m, n \geq 1. \tag{1.15}$$

For $m = 1$, it gives

$$a(n) = \lambda(n)a(1), \tag{1.16}$$

which asserts that if f a normalized Hecke eigenform, then $a(n) = \lambda(n)$, that is the eigenvalues are the Fourier coefficients.

Theorem 1.3.6 *[12, Proposition 5.8.5] A normalized modular form $f(z) = \sum_{n\geq 0} a(n)q^n \in M_k$ is an eigenform if and only if the Fourier coefficients $a(n)$ satisfy the following two conditions:*

(i) $a(m)a(n) = a(mn)$ whenever $(m, n) = 1$.

(ii) Let p be a prime and ν be a positive integer, then

$$a(p^{\nu+1}) = a(p)a(p^{\nu}) - p^{k-1}a(p^{\nu-1}).$$

We state the following properties of a Hecke eigenform, which follow from (1.15) and (1.16).

Lemma 1.3.7 *Let $f(z) = \sum_{n\geq 0} a(n)q^n \in M_k$ be a non-zero Hecke eigenform, then*

1. $a(1) \neq 0$.
2. *The Hecke eigenform f has non-zero constant term in the Fourier expansion if and only if $f \in \mathbb{C}E_k$.*

1.4 Two generalizations of modular forms

The starting point for this section is the observation that the derivative of a modular form is not modular, but nearly is. We now introduce the derivative map D, which is defined by

$$D := \frac{1}{2\pi i}\frac{d}{dz}.$$

Specifically, if f is a modular form of weight k for Γ with the Fourier expansion

$$f(z) = \sum_{n\geq 0} a(n)q^n, \tag{1.17}$$

then the derivative of f

$$Df(z) = \frac{1}{2\pi i}\frac{df}{dz} = q\frac{df}{dq} = \sum_{n\geq 0} na(n)q^n \tag{1.18}$$

satisfies

$$(cz+d)^{-k-2}(Df)\left(\frac{az+b}{cz+d}\right) = Df(z) + \frac{k}{2\pi i} f(z)\frac{c}{(cz+d)}, \tag{1.19}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Note that the factor $2\pi i$ has been included in order to preserve the rationality properties of the Fourier coefficients.

If we had only the first term in (1.19), then Df would be a modular form of weight $k+2$. The presence of the second term, far from being a problem, makes the theory much richer. To deal with it, one can:

- *relax the notion of modularity* to include functions satisfying equations like (1.19): (leading to the theory of quasimodular forms, the holomorphic generalization of modular forms);

- *modify the differentiation operator* D so that it preserves the modularity property: (leading to the theory of nearly holomorphic modular forms, the non-holomorphic generalization of modular forms).

These two approaches will be discussed in the next subsections.

Remark 1.4.1 *One can also make combinations of derivatives of modular forms to get again modular forms, so called The Rankin-Cohen brackets of modular forms. However, we will not use this fact in any essential way and refer to [7] for more details.*

1.4.1 Quasimodular forms

As mentioned earlier, the Eisenstein series E_2 and derivatives of modular forms are not modular forms, although they play an important role in the construction of many differential operators on the space of modular forms. Moreover, both satisfy similar transformation formulas (see (1.8) and (1.19)). In 1995, M. Kaneko and D. Zagier [25] generalized the notion of modular forms, called quasimodular forms, by allowing slightly more general functional equations like (1.19). The Eisenstein series E_2 and derivatives of modular forms are quasimodular forms. We also remark that these forms already appear in several previous works by S. Ramanujan and R. A. Rankin.

Definition 1.4.1 *A holomorphic function f on $\mathbb{H}$ is called a quasimodular form of weight k and depth p for Γ if there exist holomorphic functions f_0, f_1, $f_2, \ldots,$ f_p on $\mathbb{H}$ such that*

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{p} f_j(z) \left(\frac{c}{cz+d}\right)^j, \tag{1.20}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, f_p is not identically vanishing and f_j is holomorphic at each of the cusps of Γ for $0 \leq j \leq p$.

Note that this definition, which is different from the one given in the work of Kaneko and Zagier [25], was proposed by Werner Nahm and presented in [46]. The equivalence between the two definitions is a consequence of Theorem 1.4.8 (see Remark 1.4.5).

We denote the space of quasimodular forms of weight k and depth $\leq p$ for Γ by $\widetilde{M}_k^{\leq p}(\Gamma)$ and the space of all quasimodular forms of weight k by $\widetilde{M}_k(\Gamma) = \bigcup_p \widetilde{M}_k^{\leq p}(\Gamma)$. We define the graded ring of quasimodular forms for Γ by $\widetilde{M}_*(\Gamma) := \bigoplus_k \widetilde{M}_k(\Gamma)$. We simply omit Γ from these notations if it is the full modular group $SL_2(\mathbb{Z})$.

The basic facts about quasimodular forms for the full modular group are summarized in the following proposition, which applies also to other congruence subgroups.

Proposition 1.4.2 *[5, Proposition 20] Let $k \geq 2$ be even and $p \geq 0$ be any integer.*

(i) The differential operator D maps $\widetilde{M}_k^{\leq p}$ into $\widetilde{M}_{k+2}^{\leq p+1}$.

(ii) Every quasimodular form on $SL_2(\mathbb{Z})$ is a polynomial in E_2 with modular coefficients. More precisely, any $f \in \widetilde{M}_k^{\leq p}$ can be written as

$$f(z) = g_0(z) + g_1(z)E_2(z) + \cdots + g_p(z)E_2^p(z), \tag{1.21}$$

where $g_i \in M_{k-2i}$ for $0 \leq i \leq p$.

(iii) The space of quasimodular forms has a decomposition of the form

$$\widetilde{M}_k^{\leq p} = \begin{cases} \bigoplus_{r=0}^{p} D^r M_{k-2r} & \text{if } p < k/2, \\ \bigoplus_{r=0}^{\frac{k}{2}-1} D^r M_{k-2r} \oplus \mathbb{C}D^{\frac{k}{2}-1}E_2 & \text{if } p \geq k/2. \end{cases}$$

Remark 1.4.2 *As a consequence of the second property mentioned above, we see that the graded ring of quasimodular forms is generated by E_2, E_4 and E_6. Also $g_p \in M_{k-2p}$ and since there is no modular form of negative weight, we conclude that if f is non-zero quasimodular form of weight k and exact depth p, then $p \leq k/2$.*

The action of the Hecke operator T_n on a quasimodular form is the same as given in (1.11). For each integer $n \geq 1$, T_n maps $\widetilde{M}_k$ to itself. A quasimodular form is called a quasimodular eigenform (or) simply an eigenform, if it is an eigenvector for each Hecke operator T_n. We state the following commuting relation between the derivative operator D and the Hecke operators [36, Proposition 2.3].

Proposition 1.4.3 *Let $f \in \widetilde{M}_k$. Then*

$$(D^m(T_n f))(z) = \frac{1}{n^m}(T_n(D^m f))(z),$$

for $m \geq 0$. Moreover, $D^m f$ is an eigenform for T_n if and only if f is. In this case, if λ_n is the eigenvalue of T_n corresponding to f then the eigenvalue of T_n corresponding to $D^m f$ is $n^m \lambda_n$.

1.4.2 Nearly holomorphic modular forms

Nearly holomorphic modular forms were introduced by G. Shimura in 1976 [44], for proving algebraicity results of special values of Rankin product L-functions.

Definition 1.4.4 *A nearly holomorphic modular form f of weight k and depth p for Γ is a polynomial in $\frac{1}{\mathrm{Im}(z)}$ of degree p whose coefficients are holomorphic functions on $\mathbb{H}$ with moderate growth* [1] *such that*

$$f \mid_k \gamma(z) = f(z),$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$.

In the literature, nearly holomorphic modular forms are also referred to as almost holomorphic modular forms. Let $\widehat{M}_k^{\leq p}(\Gamma)$ denote the space of nearly holomorphic modular forms of weight k and depth $\leq p$ for Γ and by $\widehat{M}_k(\Gamma) = \bigcup_p \widehat{M}_k^{\leq p}(\Gamma)$, the space of all nearly holomorphic modular forms of weight k. We define the

[1] $|f(z)| \ll \left((|z|^2+1)/\mathrm{Im}(z)\right)^n$, for some $n \geq 1$

graded ring of nearly holomorphic modular forms for Γ by $\widehat{M}_*(\Gamma) := \bigoplus_k \widehat{M}_k(\Gamma)$. As usual, we omit Γ from the notations if Γ is the full modular group $SL_2(\mathbb{Z})$.

Remark 1.4.3 *Note that*

$$E_2^*(z) = E_2(z) - \frac{3}{\pi \mathrm{Im}(z)} \tag{1.22}$$

is a nearly holomorphic modular form of weight 2 *for the full modular group* $SL_2(\mathbb{Z})$.

Definition 1.4.5 *Let* $f \in \widehat{M}_k(\Gamma)$. *Then* f *is called a* **rapidly decreasing function** *at every cusp of* Γ, *if for each* $\alpha \in SL_2(\mathbb{Q})$ *and positive real number* c, *there exist positive constants* A *and* B *depending on* f, α *and* c *such that*

$$|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)| < A y^{-c} \quad \textit{if } y = \mathrm{Im}(z) > B.$$

We call f *a* **slowly increasing function** *at every cusp of* Γ, *if for each* $\alpha \in SL_2(\mathbb{Q})$, *there exist positive constants* A, B *and* c *depending on* f *and* α *such that*

$$|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)| < A y^{c} \quad \textit{if } y = \mathrm{Im}(z) > B.$$

Remark 1.4.4 *For example, a modular form is slowly increasing and a cusp form is rapidly decreasing function. Moreover, the product of a rapidly decreasing function with any nearly holomorphic modular form provides a rapidly decreasing form.*

Let $f, g \in \widehat{M}_k(\Gamma)$ be such that the product fg is a rapidly decreasing function. Write $z = x + iy$, then the (Petersson) inner product of f and g is defined by

$$\langle f, g \rangle := \frac{1}{\mu_\Gamma} \int\limits_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}. \tag{1.23}$$

By abuse of notation, we use the same symbol for the inner product here as in the case of modular forms given in (1.4). The integral is convergent because of the hypothesis and hence the inner product is well-defined.

Definition 1.4.6 *The Maass-Shimura operator R_k on $f \in \widehat{M}_k^{\leq p}(\Gamma)$ is defined by*

$$(R_k f)(z) = \left(\frac{1}{2\pi i} \left(\frac{k}{2i\mathrm{Im}(z)} + \frac{\partial}{\partial z} \right) f \right)(z). \tag{1.24}$$

The operator R_k takes $\widehat{M}_k^{\leq p}(\Gamma)$ to $\widehat{M}_{k+2}^{\leq p+1}(\Gamma)$ (see Proposition 1.4.9). For any positive integer m, we write $R_k^{(m)} := R_{k+2m-2} \circ \cdots \circ R_{k+2} \circ R_k$ with $R_k^{(0)} = id$ and $R_k^{(1)} = R_k$.

For $f \in \widehat{M}_k$, the action of the n-th Hecke operator T_n on f is defined by (1.11). For each integer $n \geq 1$, T_n maps $\widehat{M}_k$ to itself. A nearly holomorphic modular form is called a nearly holomorphic eigenform (or) simply an eigenform if it is an eigenvector for each Hecke operator T_n. We recall the following commuting relation between Maass-Shimura operators and Hecke operators.

Proposition 1.4.7 *[3, Proposition 2.4, 2.5] Let $f \in \widehat{M}_k$. Then*

$$(R_k^{(m)}(T_n f))(z) = \frac{1}{n^m}(T_n(R_k^{(m)} f))(z),$$

for $m \geq 0$. Moreover, $R_k^{(m)} f$ is an eigenform for T_n if and only if f is. In this case, if λ_n is the eigenvalue of T_n corresponding to f, then the eigenvalue of T_n corresponding to $R_k^{(m)} f$ is $n^m \lambda_n$.

1.4.3 Isomorphism between quasimodular forms and nearly holomorphic modular forms

It is easy to see that a nearly holomorphic modular form $f(z) = \sum_{j=0}^{p} \frac{f_j(z)}{\mathrm{Im}(z)^j}$ of weight k and depth p over Γ is uniquely determined by its constant term f_0. Therefore, the map

$$f(z) = \sum_{j=0}^{p} \frac{f_j(z)}{\mathrm{Im}(z)^j} \mapsto f_0(z) \tag{1.25}$$

is well-defined and one to one from $\widehat{M}_k^{\leq p}(\Gamma)$ to $\widetilde{M}_k^{\leq p}(\Gamma)$. To see the automorphy property of f_0, we need to use the modular transformation formula for f and the identity $\frac{1}{\mathrm{Im}(\gamma z)} = \frac{(cz+d)^2}{\mathrm{Im}(z)} - 2ic(cz+d)$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathbb{H}$. Conversely, let $f \in \widetilde{M}_k^{\leq p}(\Gamma)$, then

$$f(z) \mapsto \sum_{j=0}^{p} \frac{f_j(z)}{\mathrm{Im}(z)^j}, \tag{1.26}$$

gives the inverse map from $\widetilde{M}_k^{\leq p}(\Gamma)$ to $\widehat{M}_k^{\leq p}(\Gamma)$. Here the coefficients f_j are associated to f according to (1.20).

The maps given in (1.25) and (1.26) induce a map between the ring of nearly holomorphic modular forms and the ring of quasimodular forms. It turns out that these maps preserve the ring structure and we have the following theorem.

Theorem 1.4.8 *[5, p. 58] The ring of nearly holomorphic modular forms $\widehat{M}_*(\Gamma)$ is canonically isomorphic to the ring $\widetilde{M}_*(\Gamma)$ of quasimodular forms. The explicit maps are given in (1.25) and (1.26).*

Remark 1.4.5 *From the above discussion, we see that a nearly holomorphic modular form is determined by its coefficient f_0 and it is a quasimodular form.*

Using this, one finds that the definition of quasimodular forms given in [25] (namely, as the "constant terms" f_0 of nearly holomorphic modular forms) is indeed equivalent to the one given in Definition 1.4.1.

Observe that under the isomorphism given in Theorem 1.4.8,

$$E_2 \longleftrightarrow E_2^* \text{ and } D^r f \longleftrightarrow R_k^{(r)} f,$$

where r is any positive integer and $f \in M_k(\Gamma)$. It follows immediately that the space of nearly holomorphic modular forms for the full modular group $SL_2(\mathbb{Z})$ has a decomposition similar to Proposition 1.4.2. To get this, we need only to replace the derivative map by the Maass-Shimura operator and E_2 by E_2^* at corresponding places in Proposition 1.4.2 and so we have the following result. Note that similar results hold for any congruence subgroup.

Proposition 1.4.9 *Let $k \geq 2$ be even and $p \geq 0$ be any integer.*

(i) The Maass-Shimura operator R_k maps $\widehat{M}_k^{\leq p}$ into $\widehat{M}_{k+2}^{\leq p+1}$.

(ii) Every nearly holomorphic modular form on $SL_2(\mathbb{Z})$ is a polynomial in E_2^* with modular coefficients. More precisely, any $f \in \widehat{M}_k^{\leq p}$ can be written as

$$f(z) = g_0(z) + g_1(z)E_2^*(z) + \cdots + g_p(z)E_2^{*p}(z),$$

where $g_i \in M_{k-2i}$ for $0 \leq i \leq p$.

(iii) The space of nearly holomorphic forms have a decomposition of the form

$$\widehat{M}_k^{\leq p} = \begin{cases} \bigoplus_{r=0}^{p} R_{k-2r}^{(r)} M_{k-2r} & \text{if } p < k/2, \\ \bigoplus_{r=0}^{\frac{k}{2}-1} R_{k-2r}^{(r)} M_{k-2r} \oplus \mathbb{C} R_2^{(\frac{k}{2}-1)} E_2^* & \text{if } p \geq k/2. \end{cases} \tag{1.27}$$

Remark 1.4.6 *Note that similar to Remark 1.4.2, we see that* $\widehat{M}_* = \mathbb{C}[E_2^*, E_4, E_6]$ *and there is no non-zero nearly holomorphic modular form of weight* k *and exact depth* p *such that* $p > k/2$.

1.5 Hermitian modular forms

Hermitian modular forms are generalizations of Siegel modular forms. We recall some basic facts about Hermitian modular forms of degree 2. We refer to the work of H. Braun [4] and T. Ikeda [23] for more details.

Let $\mathcal{O} = \mathbb{Z}[i]$ be the ring of integers of $\mathbb{Q}(i)$ and $\mathcal{O}^\sharp = \frac{i}{2}\mathcal{O}$ be the inverse different of $\mathbb{Q}(i)|\mathbb{Q}$. Assume that $J = \begin{pmatrix} \mathbf{0} & I_2 \\ -I_2 & \mathbf{0} \end{pmatrix}$, where I_2 denotes the identity matrix and $\mathbf{0}$ denotes the zero matrix of order 2×2. Let $U(2)$ be the unitary group of degree 2, i.e.,

$$U(2) = \{M \in M_4(\mathbb{C}) \mid M^* J M = J\},$$

where M^* denotes the transpose of the complex conjugate of M. Then, the Hermitian modular group of degree 2 over the imaginary quadratic field $\mathbb{Q}(i)$ is defined by

$$\Gamma^{(2)}(\mathcal{O}) = M_4(\mathcal{O}) \cap U(2),$$

which we simply denote by $\Gamma^{(2)}$.

We denote the Hermitian half-space of degree 2 by $\mathbb{H}_2$, which is defined by

$$\mathbb{H}_2 = \left\{ Z = \begin{pmatrix} \tau & z \\ w & \tau' \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; \tfrac{1}{2i}(Z - \overline{Z}) > 0 \right\}.$$

The Hermitian modular group $\Gamma^{(2)}$ acts on $\mathbb{H}_2$ via,

$$MZ = (AZ+B)(CZ+D)^{-1} \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(2)}, Z \in \mathbb{H}_2.$$

The group $\Gamma^{(2)}$ is an arithmetic discrete subgroup of $U(2)$ and it acts discontinuously on $\mathbb{H}_2$. Let k be an integer. If F is a function on $\mathbb{H}_2$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(2)}$, we put

$$(F\mid_k M)(Z) := (\det(CZ+D))^{-k}F(MZ).$$

Let ν be a character of $\Gamma^{(2)}$, which is trivial on $\left\{\begin{pmatrix} I_2 & B \\ \mathbf{0} & I_2 \end{pmatrix} \in \Gamma^{(2)}\right\}$. With these preliminaries, we now give the definition of a Hermitian modular form.

Definition 1.5.1 (Hermitian modular forms) *A holomorphic function F on $\mathbb{H}_2$ is called a Hermitian modular form of weight k with character ν for $\Gamma^{(2)}$, if it satisfies the following condition.*

$$F\mid_k M = \nu(M)F, \quad \text{for all } M \in \Gamma^{(2)}.$$

Remark 1.5.1 *Since we consider the case of degree two Hermitian modular forms, by the Koecher principle, the holomorphicity condition at cusps follow from this modularity condition.*

Recall that a semi-integral Hermitian matrix (over $\mathcal{O}$) is a Hermitian matrix $T \in M_2(\mathcal{O}^\sharp)$ whose diagonal entries are integral. We denote the set of semi positive-definite and semi-integral Hermitian matrices by $\Lambda_2(\mathcal{O})$. In other words,

$$\Lambda_2(\mathcal{O}) = \left\{ T = \begin{pmatrix} n & t \\ \bar{t} & m \end{pmatrix} \;\middle|\; 0 \le n, m \in \mathbb{Z}, t \in \mathcal{O}^\sharp, nm - |t|^2 \ge 0 \right\}.$$

Then, a Hermitian modular form F has a Fourier expansion of the form

$$F(Z) = \sum_{0 \le T \in \Lambda_2(\mathcal{O})} a(T) e(\mathrm{tr}\ TZ), \tag{1.28}$$

where tr A, represents the trace of any square matrix A.

A Hermitian modular form F is called a *Hermitian cusp form* if the sum in (1.28) runs over positive-definite matrices T. We denote by $M_k(\Gamma^{(2)}, \nu)$ (resp. $S_k(\Gamma^{(2)}, \nu)$) the complex vector space consisting of Hermitian modular forms (resp. Hermitian cusp forms) of weight k with character ν for $\Gamma^{(2)}$. If ν is trivial, we drop it from the notation.

We are interested in the particular case when $\nu = \det^l$, where for an integer l, the character $\det^l$ on $\Gamma^{(2)}$ is defined by $M \mapsto (\det M)^l$. It is a Theorem of Braun [4, Theorem I] that $\det(M) \in \{\xi^2 \mid \xi \in \mathcal{O}^\times\} = \{\pm 1\}$. Hence, it is sufficient to consider l modulo 2 and therefore for definiteness, we take l to be either 0 or 1 from now on.

We will discuss the Fourier-Jacobi expansion of Hermitian modular forms after introducing the Hermitian Jacobi forms.

1.6 Hermitian Jacobi forms

M. Eichler and D. Zagier [14] systematically developed the theory of Jacobi forms, which are holomorphic functions of two complex variables that satisfy certain transformation laws under the action of the Jacobi group. Jacobi forms appear naturally in different areas of mathematics and physics and connect different types of automorphic forms. In particular, they appear as Fourier-Jacobi coefficients of Siegel modular forms of degree 2. This link play an important role in proving the Saito-Kurokawa conjecture.

The theory of Hermitian Jacobi forms over an imaginary quadratic field along the lines of the classical Jacobi forms was first considered systematically by K. Haverkamp [19] in his thesis. Before the study of Haverkamp, the theory was studied intrinsically by S. Raghavan and J. Sengupta [38] in their work. Later R. Sasaki [40] and S. Das [8, 9] studied Hermitian Jacobi forms over the Gaussian number field $\mathbb{Q}(i)$, and they established several properties of such Jacobi forms. Hermitian Jacobi forms are holomorphic functions of three complex variables and they also occur as Fourier-Jacobi coefficients of Hermitian modular forms of degree two. We will restrict our attention to the case where the complex quadratic field is the Gaussian number field $\mathbb{Q}(i)$.

However, in a recent work, O. Richter and J. Senadheera [39] (see also [42]) realized that the Hermitian Jacobi forms are classified into two different classes of forms, one with parity $+1$ and the other with parity -1. The work mentioned above by Haverkamp, Sasaki and Das considered only the case of parity $+1$. The study of Hermitian Jacobi forms with parity -1 is also important as discussed in the work of J. D. Martin and J. Senadheera [35]. It is to be noted that these Hermitian Jacobi forms with parity ± 1 arise in a natural way (like in the case of classical Jacobi forms) via the Fourier-Jacobi coefficients of Hermitian modular forms of degree two with character $(\det)^l$, where l varies modulo 2 (see Theorem 1.7.1). We remark here that almost all the existing results in the literature consider Hermitian Jacobi forms with parity $+1$, which come with the condition that the weight k is divisible by 4. Using the refined definition of Richter and Senadheera as mentioned above, one can extend all the results for forms with parity -1 as well (one has to assume that $k \equiv 2 \pmod 4$ in this case).

Recall that $\mathcal{O} = \mathbb{Z}[i]$ is the ring of integers of $\mathbb{Q}(i)$. We will denote the units $\{\pm 1, \pm i\}$ in $\mathcal{O}$ by $\mathcal{O}^\times$. The group $\Gamma_1(\mathcal{O}) = \{\xi M \mid \xi \in \mathcal{O}^\times, M \in SL_2(\mathbb{Z})\}$ acts on $\mathcal{O}^2$ by

$$[\lambda, \mu](\xi M) := [(\bar{\xi}\lambda, \bar{\xi}\mu)M],$$

where $[\lambda, \mu] \in \mathcal{O}^2$ and $\xi M \in \Gamma_1(\mathcal{O})$. The discrete group

$$\Gamma^J = \Gamma^J(\mathcal{O}) := \Gamma_1(\mathcal{O}) \ltimes \mathcal{O}^2 = \{(\xi M, X) \mid \xi M \in \Gamma_1(\mathcal{O}), X \in \mathcal{O}^2\}$$

is called the Hermitian Jacobi group over $\mathcal{O}$, with the group law defined as

$$(\xi_1 M_1, X_1)(\xi_2 M_2, X_2) := (\xi_1\xi_2 M_1 M_2, X_1(\xi_2 M_2) + X_2),$$

for $\xi_1 M_1, \xi_2 M_2 \in \Gamma_1(\mathcal{O}), X_1, X_2 \in \mathcal{O}^2$.

Now we are ready to define the Hermitian Jacobi forms.

Definition 1.6.1 (Hermitian Jacobi forms) *A holomorphic function $\phi : \mathbb{H}\times \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ is called a Hermitian Jacobi form of weight k, index m and parity $\delta(= \pm 1)$, if the function ϕ satisfies the following conditions:*

1. *For each $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \xi \in \mathcal{O}^\times$, we have*

$$\begin{aligned}\phi(\tau, z, w) &= \phi \mid_{k,m,\delta} \xi M(\tau, z, w)\\ &:= \sigma(\xi)\xi^{-k}(c\tau+d)^{-k} e^{-m}\left(\frac{czw}{c\tau+d}\right)\phi\left(M\tau, \frac{\xi z}{c\tau+d}, \frac{\overline{\xi} w}{c\tau+d}\right),\end{aligned} \tag{1.29}$$

where $\sigma(\xi) = 1$ if $\delta = +1$ and $\sigma(\xi) = \xi^2$ if $\delta = -1$.

2. *For each $[\lambda, \mu] \in \mathcal{O}^2$, we have*

$$\begin{aligned}\phi(\tau, z, w) &= \phi \mid_m [\lambda, \mu](\tau, z, w)\\ &:= e^m\left(|\lambda|^2\tau + \overline{\lambda} z + \lambda w\right)\phi\left(\tau, z+\lambda\tau+\mu, w+\overline{\lambda}\tau+\overline{\mu}\right).\end{aligned} \tag{1.30}$$

3. *The function ϕ has a Fourier expansion of the form*

$$\phi(\tau, z, w) = \sum_{n\geq 0}\sum_{\substack{r\in\mathcal{O}^\sharp\\ |r|^2\leq mn}} c(n, r) q^n \zeta^r \zeta'^{\overline{r}}. \tag{1.31}$$

Moreover, we call ϕ a *cusp form* if the Fourier coefficients $c(n, r)$ in (1.31) have the property that $c(n, r) = 0$ for $mn = |r|^2$. Here and subsequently in this section, we let $\tau = u + iv \in \mathbb{H}$, $z = x_1 + iy_1 \in \mathbb{C}$, $w = x_2 + iy_2 \in \mathbb{C}$, $q = e(\tau)$, $\zeta = e(z)$ and $\zeta' = e(w)$. We write $J^{\delta}_{k,m}$ (respectively $J^{\delta,cusp}_{k,m}$) for the finite dimensional vector space of Hermitian Jacobi forms (respectively Hermitian Jacobi cusp forms) of weight k, index m and parity δ.

Definition 1.6.2 (Petersson inner product) *For $\phi, \psi \in J^{\delta}_{k,m}$ such that $\phi\psi$ is cuspidal, we define the inner product of ϕ and ψ as*

$$\langle \phi, \psi \rangle := \int_{\Gamma^J \backslash \mathbb{H} \times \mathbb{C}^2} \phi(\tau, z, w)\overline{\psi(\tau, z, w)} v^k e^{-\pi m \frac{|z-\overline{w}|^2}{v}} dV^J, \tag{1.32}$$

where $dV^J = v^{-4} du\, dv\, dx_1\, dy_1\, dx_2\, dy_2$ is a Γ^J invariant measure.

From now on ξ stands for any element in $\mathcal{O}^{\times}$ and for definiteness, we choose $\{1, i\}$ as representatives for $\mathcal{O}^{\times}/\{\pm 1\}$.

1.6.1 Poincaré series

Let k, m and n be positive integers and $r \in \mathcal{O}^{\sharp}$ be such that $D := 4\left(|r|^2 - mn\right) < 0$. Then we define the (n, r)-th *Poincaré series* of exponential type for Γ^J by

$$P^{k,m,\delta}_{(n,r)}(\tau, z, w) := \sum_{\widetilde{\gamma} \in \Gamma^J_{\infty} \backslash \Gamma^J} e(n\tau + rz + \overline{r}w) \mid_{k,m,\delta} \widetilde{\gamma}(\tau, z, w), \tag{1.33}$$

where $\Gamma^J_{\infty} = \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) \,\middle|\, n \in \mathbb{Z}, \mu \in \mathcal{O} \right\}$ is the stabilizer group of $e(n\tau + rz + \overline{r}w)$.

The (n, r)-th Poincaré series $P^{k,m,\delta}_{(n,r)}$ is analytic for $k > 3$ and satisfies the transformation formulas (1.29) and (1.30) of a Hermitian Jacobi form. It is characterized by the following property which is obtained by using the standard Rankin's unfolding argument.

Lemma 1.6.3 *Let* $\phi(\tau, z, w) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{O}^\sharp \\ |r|^2\le mn}} c(n,r)q^n\zeta^r\zeta'^{\overline{r}} \in J^{\delta}_{k,m}$ *then*

$$\langle \phi, P^{k,m,\delta}_{(n,r)}\rangle = \lambda^{n,r}_{k,m} c(n,r), \tag{1.34}$$

where

$$\lambda^{n,r}_{k,m} = \frac{m^{k-3}\Gamma(k-2)}{(4\pi)^{k-2}(mn-|r|^2)^{k-2}}. \tag{1.35}$$

Remark 1.6.1 *The above lemma has been proved in [9] for $\delta = +1$. However, there is a misprint in [9, Lemma 4.3], where the constant $\lambda^{k,m}_{n,r}$ (in the notation of [9]) should have $k-1$ in place of k in the definition (except for the gamma function).*

1.7 Fourier-Jacobi expansion of Hermitian modular forms

Let F be a Hermitian modular form of weight k with character $(\det)^l$. Let $Z \in \mathbb{H}_2$ and $T \in \Lambda_2(\mathcal{O})$. Writing

$$Z = \begin{pmatrix} \tau & z \\ w & \tau' \end{pmatrix} \text{ and } T = \begin{pmatrix} n & r \\ \overline{r} & m \end{pmatrix},$$

where $\tau, \tau' \in \mathbb{H}, z, w \in \mathbb{C}$ and $0 \le n, m \in \mathbb{Z}, r \in \mathcal{O}^\sharp$ with $nm - |r|^2 \ge 0$, in eq. (1.28), the Fourier expansion of F can be written in the form

$$F(Z) = \sum_{m\ge 0} \phi_m(\tau, z, w)e(m\tau'), \tag{1.36}$$

where

$$\phi_m(\tau, z, w) = \sum_{n\geq 0} \sum_{\substack{r\in\mathcal{O}^\sharp \\ |r|^2\leq mn}} c(n,r) q^n \zeta^r \zeta'^{\overline{r}}.$$

The development in (1.36) is called the Fourier-Jacobi expansion of F and the coefficients ϕ_m are known as the m-th Fourier-Jacobi coefficients of F.

In [19, Theorem 7.1], Haverkamp has shown that the Hermitian-Jacobi coefficients of a Hermitian modular form in $M_k(\Gamma^{(2)})$ are Hermitian Jacobi forms with parity $+1$. The same reasoning applies to a Hermitian modular form with character $\det^l$, to get the following result.

Theorem 1.7.1 *Let ϕ_m be the m-th Fourier-Jacobi coefficient of a Hermitian modular form $F \in M_k(\Gamma^{(2)}, \det^l)$. Then ϕ_m is a Hermitian Jacobi form of weight k, index m and parity δ, where*

$$\delta = \begin{cases} +1 & \text{if } l = 0, \\ -1 & \text{if } l = 1. \end{cases}$$

Remark 1.7.1 *It is important to note that the space $M_k(\Gamma^{(2)}, \det)$ is equally important as the space $M_k(\Gamma^{(2)})$. In [23], Ikeda has constructed a lift from the space of elliptic modular forms to the space of Hermitian modular forms of degree n. In particular, for $n = 2$, he has given an explicit lift from $S_{2k+1}(\Gamma_0(4), \chi)$ to $S_{2k+2}(\Gamma^{(2)}, \det^{k+1})$, where χ is a primitive Dirichlet character modulo 4. Depending on whether $k+1$ is even or odd, this maps to either of the spaces considered above.*

CHAPTER 2

On arbitrary products of eigenforms

This chapter investigates the product relations among eigenforms, in the context of quasimodular forms and nearly holomorphic modular forms. We first characterize all the cases in which products of arbitrary numbers of quasimodular eigenforms for the full modular group $SL_2(\mathbb{Z})$ are again eigenforms. Then, the corresponding results in the space of nearly holomorphic modular forms can be obtained via the isomorphism between the two spaces. The results of this chapter have been published in [32].

2.1 Introduction and overview of previous work

We use the notations as in §1.2.1. Identities among modular forms have attracted the attention of many mathematicians since they imply nice identities among the Fourier coefficients of modular forms. One such identity is the following:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m), \tag{2.1}$$

for $n \geq 1$. Since the vector space M_8 is one dimensional, it follows that $E_4^2 = E_8$ and comparing the n-th Fourier coefficients of both the sides yield (2.1). The identity $E_4^2 = E_8$ can be viewed as an eigenform identity as both E_4 and E_8 are Hecke eigenforms. The set of all modular forms (of all weights) for the full modular group is a graded complex algebra. Having seen an identity as above ($E_4^2 = E_8$), it is quite natural to ask whether the property of being a Hecke eigenform is preserved under multiplication. Note that the property of being an eigenform imposes strict conditions on the q-expansion of a modular form, and the convolution product of q-expansion is unlikely to preserve these conditions. Hence, one may expect that it should happen rarely. This problem was first studied by W. Duke [13] and E. Ghate [17] independently. They found that there are only 16 such cases, when the product of two Hecke eigenforms is again a Hecke eigenform for the full modular group $SL_2(\mathbb{Z})$. All the identities in these cases are forced from dimensional constraints. More precisely, they proved the following theorem.

Theorem 2.1.1 (W. Duke [13], E. Ghate [17]) *The product of two Hecke eigenforms for $SL_2(\mathbb{Z})$ is an eigenform only in the following cases:*

$$E_4^2 = E_8, \quad E_4E_6 = E_{10}, \quad E_6E_8 = E_4E_{10} = E_{14},$$

$$E_4\Delta_{12} = \Delta_{16}, \quad E_6\Delta_{12} = \Delta_{18}, \quad E_4\Delta_{16} = E_8\Delta_{12} = \Delta_{20},$$

$$E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta_{12} = \Delta_{22},$$

$$E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta_{12} = \Delta_{26}.$$

Here the functions $\Delta_k's$ are given in (1.10). Both the results in [13], [17] make use of the Rankin-Selberg convolution.

In the case of higher levels, in [18], Ghate considered the problem of looking at products of (certain) eigenforms of with respect to $\Gamma_1(N)$, for squarefree N with weight greater than or equal to 3. By using the Rankin-Selberg convolution, he proved that all such identities are forced for dimensional reasons. B. A. Emmons [15] extended the search to $\Gamma_0(p)$ and found 8 new cases. Later, M. L. Johnson [21] resolved the problem in all levels and obtained a complete list of 61 eigenform product identities.

However, there is a possibility that products of more than two eigenforms result in an eigenform. In [16], B. Emmons and D. Lanphier extended the above result to a product of an arbitrary number of Hecke eigenforms. They showed that the product of any number of eigenforms is an eigenform only finitely many times and all the cases are again due to dimensional constraints. More precisely, they proved the following theorem.

Theorem 2.1.2 (B. Emmons, D. Lanphier [16]) *The product of finitely many modular Hecke eigenforms for $SL_2(\mathbb{Z})$ is never an eigenform except for the following exceptional cases:*

1. *The 16 cases presented in Theorem 2.1.1.*

2. *Other cases which can be obtained from some of the identities presented in Theorem 2.1.1, namely*

$$E_4^2E_6 = E_{14}, E_4^2\Delta_{12} = \Delta_{20}, E_4E_6\Delta_{12} = \Delta_{22},$$

$$E_4^2\Delta_{18} = E_4E_6\Delta_{16} = E_4^2E_6\Delta_{12} = E_6E_8\Delta_{12} = E_4E_{10}\Delta_{12} = \Delta_{26}.$$

Instead of products of two eigenforms one can also consider the Rankin-Cohen bracket of two eigenforms and pose a similar question. In [34], D.

Lanphier and R. Takloo-Bighash found all the cases (finitely many) where the Rankin-Cohen brackets of two eigenforms gives an eigenform for the full modular group $SL_2(\mathbb{Z})$.

Another formula (proved by Ramanujan) which does not follow from such an identity in modular forms is the following:

$$n\tau(n) = \tau(n) - 24\sum_{m=1}^{n-1}\tau(m)\sigma(n-m) \tag{2.2}$$

for $n \geq 1$, where $\tau(n)$ is the Ramanujan tau function. The above identity follows from the relation $D\Delta = E_2\Delta$ between quasimodular forms for the group $SL_2(\mathbb{Z})$, where $D = \frac{1}{2\pi i}\frac{d}{dz}$ is the differential operator introduced in Section 1.4 of Chapter 1. Similar to the case of modular forms, the relation $D\Delta = E_2\Delta$ can be regarded as an identity in the graded complex algebra of quasimodular forms for the full modular group, where the product of two quasimodular eigenforms results in an eigenform. Therefore, it is of interest to find all such cases. This question was considered by S. Das and J. Meher in [10] and [36] for the full modular group. They showed that there are two extra identities apart from the 16 coming from modular forms.

Theorem 2.1.3 *[10, 36] The product of two quasimodular eigenforms for $SL_2(\mathbb{Z})$ is never an eigenform except for the following exceptional cases:*

1. *The 16 holomorphic cases presented in Theorem 2.1.1.*

2. $(DE_4)E_4 = \frac{1}{2}DE_8, \quad E_2\Delta_{12} = D\Delta_{12}.$

In [3], J. Beryerl et al. considered the problem for a class of nearly holomorphic modular forms for the group $SL_2(\mathbb{Z})$. More precisely, they solved the problem for those nearly holomorphic modular forms which are in the image of modular forms under the Maass-Shimura operators.

Theorem 2.1.4 (Beyerl et al. [3]) *Let $R_k^{(r)} f \in \widehat{M}_{k+2r}$ and $R_l^{(s)} g \in \widehat{M}_{l+2s}$ both be eigenforms. Then $R_k^{(r)} f R_l^{(s)} g$ is not an eigenform aside from the following finitely many exceptional cases:*

1. *The 16 holomorphic cases presented in Theorem 2.1.1.*
2. $(DE_4)E_4 = \frac{1}{2} DE_8$.

In the proof they first expressed a product of nearly holomorphic modular forms in terms of a sum of Rankin-Cohen bracket operators and then used the result regarding the Rankin-Cohen bracket of two eigenforms [34].

2.2 Arbitrary products of quasimodular eigenforms

2.2.1 Preparatory results

We first state the following result [10, Proposition 3.1], which characterizes all quasimodular eigenforms for the full modular group $SL_2(\mathbb{Z})$.

Theorem 2.2.1 (Structure theorem for quasimodular eigenforms) *Let f be a quasimodular eigenform of weight k and depth p for $SL_2(\mathbb{Z})$. If $p < k/2$ then $f = D^p f_p$, where f_p is a modular Hecke eigenform of weight $k - 2p$, and if $p = k/2$ then $f \in \mathbb{C} D^{\frac{k}{2}-1} E_2$.*

Similar to the case of modular forms, we have the following result for quasimodular eigenforms which is easily obtained by using the similar result for modular Hecke eigenforms given in Lemma 1.3.7, after using Theorem 2.2.1.

Lemma 2.2.2 *Let $k \geq 2$ and $f(z) = \sum_{n\geq 0} a(n) q^n \in \widetilde{M}_k$ be a non-zero quasimodular eigenform. Then*

1. $a(1) \neq 0$.

2. *The quasimodular eigenform f has non-zero constant Fourier coefficient if and only if $f \in \mathbb{C}E_k$.*

Next, we recall the following result proved by Das and Meher [10, Lemma 3.5].

Lemma 2.2.3 *For $r \geq 1$ and $h \in M_k$ let $D^r h = h_0 + h_1 E_2 + h_2 E_2^2 + \cdots + h_r E_2^r$ with $h_i \in M_{k+2r-2i}$. Then $h_r = \frac{r!}{12^r}\binom{k+r-1}{r} h$.*

We prove the corresponding result for derivatives of the Eisenstein series E_2.

Lemma 2.2.4 *For $r \geq 1$ let $D^r E_2 = h_0 + h_1 E_2 + \cdots + h_{r+1} E_2^{r+1}$ with $h_i \in M_{2r+2-2i}$. Then $h_{r+1} = \frac{r!}{12^r}$.*

Proof. We apply induction on r to prove the lemma. For $r = 1$, it is due to Ramanujan that

$$DE_2 = \frac{-E_4}{12} + \frac{E_2^2}{12}. \tag{2.3}$$

Now assuming that the lemma is true for r, we shall prove it for $r + 1$. Let

$$D^{r+1}E_2 = D(D^r E_2) = f_0 + f_1 E_2 + \cdots + f_{r+2} E^{r+2},$$

then by using the induction hypothesis and (2.3), we see that

$$f_{r+2} = \tfrac{r!}{12^r}(r+1)\tfrac{1}{12} = \tfrac{(r+1)!}{12^{r+1}},$$

which completes the proof. □

2.2.2 Main result

We consider the case of products of an arbitrary number of quasimodular eigenforms and characterize all quasimodular eigenforms which can be written as products of finitely many quasimodular eigenforms.

Theorem 2.2.5 *The product of finitely many quasimodular eigenforms for $SL_2(\mathbb{Z})$ is never an eigenform except for the following exceptional cases:*

1. *The holomorphic cases presented in* (1) *and* (2) *of Theorem 2.1.2.*

2. $(DE_4)E_4 = \frac{1}{2}DE_8, \quad E_2\Delta_{12} = D\Delta_{12}.$

Proof. By Theorem 2.2.1 and Lemma 2.2.2, we need to find out only in the following cases if the products of eigenforms result in eigenforms.

1. $E_2^a E_{k_1} \dots E_{k_m}$, where each $k_i \geq 4$ and $a + m \geq 2$,

2. $E_2^a E_{k_1} \dots E_{k_m} f$, where each $k_i \geq 4$ and $a + m \geq 1$ and f is a cusp form which is an eigenform,

3. $E_2^a E_{k_1} \dots E_{k_m} D^r E_2$, where each $k_i \geq 4$, $r \geq 1$ and $a + m \geq 1$,

4. $E_2^a E_{k_1} \dots E_{k_m} D^r f$, where each $k_i \geq 4$, $r \geq 1$, $a + m \geq 1$, and f is a cusp form which is an eigenform,

5. $E_2^a E_{k_1} \dots E_{k_m} D^r E_k$, where each $k_i, k \geq 4$, $r \geq 1$ and $a + m \geq 1$.

In the above cases, we assume that the product $E_{k_1} \dots E_{k_m}$ is 1 if $m = 0$.

Case (1)**.** If $a = 0$, then it reduces to the case of a product of Eisenstein series which are modular forms. Then by Theorem 2.1.2, we have all the cases in which the product is again an eigenform and these are listed in the statement of Theorem 2.2.5. If $a \neq 0$, then the constant term of the product is non-zero and the product is a non-modular quasimodular form. By Lemma 2.2.2 this product is an eigenform only when it is a constant multiple of E_2, which can not be true. Therefore, in this case we do not have any desired identity among eigenforms.

Case (2)**.** If $a = 0$, then again it reduces to the modular case and then by Theorem 2.1.2, we have all the cases in which the product is again an eigenform which are listed in the statement of Theorem 2.2.5. If $a \neq 0$, let k be the weight of f. Without loss of generality, we assume that f is normalized, i.e., the coefficient of q in the Fourier expansion of f is 1. Then the depth a of $E_2^a E_{k_1} \ldots E_{k_m} f$ is strictly less than half of its weight $2a + k_1 + \ldots k_m + k$. Thus by Theorem 2.2.1 we have

$$E_2^a E_{k_1} \ldots E_{k_m} f = D^a h,$$

where h is a normalized modular eigenform. Since $D^a h$ is a quasimodular form of depth a, by (1.21) we can write it as a polynomial in E_2 and then comparing the coefficients of E_2^a from both the sides of above identity, after using Lemma 2.2.3, we obtain

$$E_{k_1} \ldots E_{k_m} f = \frac{a!}{12^a} \binom{k_1 + \ldots k_m + k + a - 1}{a} h.$$

Comparing the Fourier coefficients of q from both sides of the above identity, we obtain

$$\frac{a!}{12^a} \binom{k_1 + \ldots k_m + k + a - 1}{a} = 1,$$

and further simplification gives

$$(k_1 + \cdots + k_m + k)(k_1 + \cdots + k_m + k + 1) \ldots (k_1 + \cdots + k_m + k + a - 1) = 12^a.$$

Since $k \geq 12$, the above equality is valid only when $a = 1$ and $m = 0$. Then it implies that $k = 12$ and hence we get the identity

$$E_2 \Delta_{12} = D\Delta_{12},$$

where Δ_{12} is the Ramanujan Delta function. The above identity is listed in the statement of Theorem 2.2.5.

Case (3)**.** If $m = 0$, then using Theorem 2.2.1 and comparing the first coefficients, we get

$$E_2^a D^r E_2 = D^{r+a} E_2.$$

Comparing the coefficients of q^2 from both the sides of the above equality, we get

$$2^r - 8a = 2^{r+a},$$

which is not possible since $r, a \geq 1$.

Hence assume that $m \geq 1$. Then by Theorem 2.2.1, we have

$$E_2^a E_{k_1} \dots E_{k_m} D^r E_2 = D^{r+a+1} g, \tag{2.4}$$

where g is a modular eigenform of weight $k = k_1 + k_2 \dots + k_m$. Writing $D^r E_2$ and $D^{r+a+1} g$ as a polynomial in E_2 and after using Lemma 2.2.3 and Lemma 2.2.4, compare the coefficients of E_2^{r+a+1} from both sides of the above equality, we obtain

$$\frac{r!}{12^r} E_{k_1} \dots E_{k_m} = dg, \tag{2.5}$$

where

$$d = \frac{(r+a+1)!}{12^{r+a+1}} \binom{k+r+a}{r+a+1}.$$

Since g is an eigenform of weight k with non-zero constant Fourier coefficient, by Lemma 2.2.2 we have $g = cE_k$ for some non-zero constant c. Substituting this value of g in (2.5) and equating the constant Fourier coefficients from both the sides, we obtain

$$c = \frac{r!}{12^r d}.$$

Therefore, by (2.4) we have

$$E_2^a E_{k_1} \dots E_{k_m} D^r E_2 = \frac{r!}{12^r d} D^{r+a+1} E_k. \tag{2.6}$$

Comparing the Fourier coefficients of q from both the sides of the above equation, we get the relation

$$-\frac{2k}{B_k} = -24\frac{12^r}{r!}d. \tag{2.7}$$

If $m = 1$, (2.6) gives the identity

$$E_2^a E_k D^r E_2 = \frac{r!}{12^r d} D^{r+a+1} E_k,$$

and comparing the coefficients of q^2 from both the sides, we get

$$\frac{-2k}{B_k}\frac{r!}{12^r d} 2^{r+a+1}\sigma_{k-1}(2) = -24\left(3\cdot 2^r - 24a - \frac{2k}{B_k}\right).$$

Using the fact that $\frac{-2k}{B_k}\frac{r!}{12^r d} = -24$ from (2.7) we arrive at

$$2^{r+a+1}\sigma_{k-1}(2) - 3\cdot 2^r = -\left(24a + 24\frac{12^r}{r!}d\right).$$

Since $k \geq 4$, the above identity is not possible as the left hand side is a positive quantity whereas the right hand side is a negative number.

If $m > 1$, then by (2.5) dg is, up to a constant, a product of two or more Eisenstein series. Thus by Theorem 2.1.2, the possible values of k are $8, 10, 14$. From (2.7) we see that $\frac{-2k}{B_k}$ must be negative. It follows that the choice $k = 8$ is ruled out because $\frac{-2k}{B_k} = 480 > 0$ (for $k = 8$).

If $k = 10$ then $\frac{-2k}{B_k} = -264$. Substituting it in (2.7) and simplifying, we get

$$12^a \cdot 264 \cdot 9! = 2(r+1)(r+2)\dots(r+a+10).$$

From the above identity, we see that the right hand side is divisible by 25 but the left hand side is not divisible by 25. Thus the case $k = 10$ does not arise. Similarly, we get a contradiction if $k = 14$.

Case (4)**.** Without loss of generality, we assume that f is normalized. Suppose that the weight of f is k. By Theorem 2.2.1 we have

$$E_2^a E_{k_1} \dots E_{k_m} D^r f = D^{r+a} g,$$

where g is a normalized modular eigenform of weight $l = k + k_1 + k_2 + \dots + k_m$. Since both f and g are normalized, similar to the previous cases, applying Lemma 2.2.3 to both sides of the above equality and then comparing the coefficients of E_2^{r+a}, we obtain

$$\frac{r!}{12^r}\binom{k+r-1}{r} = \frac{(r+a)!}{12^{r+a}}\binom{l+r+a-1}{r+a}.$$

From the above identity, we get

$$12^a k(k+1)\dots(k+r-1) = l(l+1)\dots(l+r+a-1).$$

Since $l \geq k \geq 12$, the above equality holds only when $a = 0$ and $l = k$. This implies that $a + m = 0$ which contradicts the assumption that $a + m \geq 1$.

Case (5)**.** By Theorem 2.2.1, we have

$$E_2^a E_{k_1} \dots E_{k_m} D^r E_k = D^{r+a} h, \tag{2.8}$$

where h is a modular eigenform of weight $l = k + k_1 + k_2 + \cdots + k_m$. Applying Lemma 2.2.3 for both $D^r E_k$ and $D^{r+a} h$ after writing them in polynomials in E_2 and then comparing the coefficients of E_2^{r+a} from both sides of the above equality, we have

$$d_1 E_{k_1} \dots E_{k_m} E_k = d_2 h, \tag{2.9}$$

where $d_1 = \frac{r!}{12^r}\binom{k+r-1}{r}$ and $d_2 = \frac{(r+a)!}{12^{r+a}}\binom{l+r+a-1}{r+a}$.
We deduce from (2.9) that the constant Fourier coefficient of h is non-zero. Hence by Lemma 2.2.2, $h = cE_l$ for some non-zero constant c. Substituting the value of h in (2.8), we get the identity

$$E_2^a E_{k_1} \dots E_{k_m} D^r E_k = cD^{r+a} E_l. \tag{2.10}$$

From (2.9) we also see that

$$E_{k_1} \dots E_{k_m} E_k = E_l, \tag{2.11}$$

and

$$c = \frac{d_1}{d_2}.$$

If $m = 0$ then $k = l$ and $c = \frac{d_1}{d_2} = 1$. This implies that

$$12^a = (k + r)(k + r + 1) \dots (k + r + a - 1).$$

The above identity will hold only if $a = 1$ and $k + r = 12$. So we are left with the case

$$E_2 D^r E_k = D^{r+1} E_k, \quad \text{where } k + r = 12.$$

By comparing the Fourier coefficients of q^2 on both sides, we see that the above identity cannot be true. Thus $m = 0$ is not possible.

Now let $m \geq 1$. Comparing the coefficients of q from both sides of (2.10) we see that

$$c = \frac{d_1}{d_2} = \frac{\left(\frac{-2k}{B_k}\right)}{\left(\frac{-2l}{B_l}\right)} = \frac{2kB_l}{2lB_k}. \tag{2.12}$$

By Theorem 2.1.2, the possible values of the tuple (k, l) for which (2.11) holds are

$$(4,8),(4,10),(6,10),(4,14),(6,14),(8,14),(10,14).$$

From the above values of (k, l), we see that the values $(4, 10), (4, 14)$ and $(8, 14)$ are ruled out since for these values, $\frac{2kB_l}{2lB_k}$ is negative but $\frac{d_1}{d_2}$ is always positive which contradicts (2.12). So the remaining values of (k, l) to be checked are

$$(4,8),(6,10),(6,14),(10,14).$$

Also from (2.12), we see that

$$\frac{2l}{B_l}\frac{r!}{12^r}\binom{k+r-1}{r} = \frac{2k}{B_k}\frac{(r+a)!}{12^{r+a}}\binom{l+r+a-1}{r+a}.$$

Simplifying the above identity, we arrive at

$$12^a\frac{2l}{B_l}k(k+1)\ldots(l-1) = \frac{2k}{B_k}(k+r)(k+r+1)\ldots(l+r+a-1). \tag{2.13}$$

For $(k, l) = (4, 8)$ we know that $(-\frac{2k}{B_k}, -\frac{2l}{B_l}) = (240, 480)$, and substituting these values in (2.13) we obtain

$$12^a \times 5 \times 6 \times 7 \times 8 = (r+4)(r+5)\ldots(r+a+7). \tag{2.14}$$

If $r + a + 7 \geq 11$ then $r + 4$ must be less than or equal to 11, and hence we get a contradiction, since the left hand side of (2.14) is not divisible by 11 but the right hand side is divisible by 11. Also since $r \geq 1$, we deduce that $r+a+7 \geq 8$.

Thus $1 \leq r+a \leq 3$ and by taking values of r and a for which $r+a = 2, 3$, one sees that it contradicts (2.14). Thus the only case remaining is $r+a=1$ which implies that $r=1$ and $a=0$. Therefore from (2.10), we obtain the identity

$$E_4(DE_4) = \frac{1}{2}DE_8.$$

If $(k,l) = (6,10)$, then $(-\frac{2k}{B_k}, -\frac{2l}{B_l}) = (504, 264)$. Substituting these values in (2.13) we obtain

$$12^a \times 11 \times 6 \times 7 \times 8 \times 9 = 21(r+6)(r+7)\ldots(r+9+a). \tag{2.15}$$

As in the previous case we get $1 \leq r+a \leq 3$. Substituting the possible values of r and a in (2.15), we see that they contradict (2.15). Thus the case $(k,l) = (6,10)$ is not possible. Similarly, we get contradictions for other possible values $(k,l) = (6,14)$ and $(10,14)$. This proves the theorem. □

2.3 Arbitrary products of nearly holomorphic eigenforms

2.3.1 Characterization of nearly holomorphic eigenforms

In this section, we characterize all nearly holomorphic eigenforms for the full modular group $SL_2(\mathbb{Z})$. We first recall a well-known result from linear algebra.

Lemma 2.3.1 *Let T be a linear operator defined on a finite-dimensional vector space over $\mathbb{C}$. Let $f = \sum_{i=1}^{r} c_i f_i$ be such that f and f_i are eigenvectors under T with eigenvalues a and a_i respectively. If all the f_i are linearly independent, then $a = a_i$ for all i.*

We have the following result for nearly holomorphic modular forms [3, Lemma 2.7].

Lemma 2.3.2 *Let $k > l$ and $f \in M_k$, $g \in M_l$ be eigenforms, then for $m \geq 0$, $R_l^{(\frac{k-l}{2}+m)} g$ and $R_k^{(m)} f$ do not have the same set of eigenvalues with respect to the Hecke operators.*

Theorem 2.3.3 (Structure theorem for nearly holomorphic eigenforms) *Let f be a nearly holomorphic eigenform of weight k and depth p for the full modular group $SL_2(\mathbb{Z})$. If $p < k/2$ then $f = R_{k-2p}^{(p)} f_p$, where f_p is a modular form of weight $k - 2p$ which is an eigenform, and if $p = k/2$ then $f \in \mathbb{C} R_2^{(\frac{k}{2}-1)} E_2^*$.*

Proof. Let f be a nearly holomorphic modular form of weight k and depth p for the group $SL_2(\mathbb{Z})$. If $p < k/2$, then by Proposition 1.4.9 we have

$$f = \sum_{r=0}^{p} R_{k-2r}^{(r)} f_r, \tag{2.16}$$

where $f_r \in M_{k-2r}$. Since the depth of f is p, $R_{k-2p}^{(p)} f_p$ is not identically equal to zero. Also each f_r can be written as

$$f_r = \sum_{j=1}^{d_r} b_{rj} h_{rj} + \beta_r E_{k-2r}, \tag{2.17}$$

where $b_{rj}, \beta_r \in \mathbb{C}$, d_r is the dimension of S_{k-2r} and the set $\{h_{rj} \mid 1 \leq j \leq d_r\}$ is a Hecke basis of S_{k-2r} for each $0 \leq r \leq p$. By the hypothesis of the theorem, f is an eigenform. Therefore, by Lemma 2.3.1 and Lemma 2.3.2, we obtain

$$f = R_{k-2p}^{(p)} f_p = \sum_{j=1}^{d_p} b_{pj} R_{k-2p}^{(p)} h_{pj} + \beta_p R_{k-2p}^{(p)} E_{k-2p}.$$

By using the bounds for the eigenvalues of modular eigenforms in Theorem 1.2.6

and Proposition 1.4.7, the n-th Hecke eigenvalue of $R^{(p)}_{k-2p}h_{pj}$ is $O(n^{\frac{k-1}{2}+\epsilon})$ for any $\epsilon > 0$. Also the n-th eigenvalue of $R^{(p)}_{k-2p}E_{k-2p}$ is $n^p\sigma_{k-2p-1}(n)$. Since we know that $\sigma_l(n) \ll n^l$ for $l > 1$, we have

$$n^{k-p-1} \leq n^p\sigma_{k-2p-1}(n) \leq Cn^{k-p-1},$$

where C is some positive constant, and hence there exist positive integers n such that the eigenvalues with respect to the Hecke operator T_n for $R^{(p)}_{k-2p}E_{k-2p}$ and $R^{(p)}_{k-2p}h_{pj}$ are different for each j. Then by Lemma 2.3.1, we have either $f = \sum_{j=1}^{d_p} b_{pj}R^{(p)}_{k-2p}h_{pj}$ or $f = R^{(p)}_{k-2p}E_{k-2p}$. In the case $f = \sum_{j=1}^{d_p} b_{pj}R^{(p)}_{k-2p}h_{pj}$, we again apply Lemma 2.3.1 and use the fact that there are infinitely many n such that the eigenvalues of T_n with respect to any two h_{pj} are different. Next, we consider the case when $p = k/2$. In this case, we write f as

$$f = \sum_{r=0}^{\frac{k}{2}-1} R^{(r)}_{k-2r}f_r + \alpha R_2^{(\frac{k}{2}-1)}E_2^*,$$

where $f_r \in M_{k-2r}$ and $\alpha \in \mathbb{C}$ is non-zero.
The eigenvalue of T_n with respect to $R_2^{(\frac{k}{2}-1)}E_2^*$ is $n^{\frac{k}{2}-1}\sigma(n)$. Also for $n > 1$, we have

$$n^{\frac{k}{2}} < n^{\frac{k}{2}-1}\sigma(n) \leq n^{\frac{k}{2}}(\log n + 1).$$

Again using Lemma 2.3.1 and comparing the eigenvalues as in the case when $p < k/2$, we conclude that $f = \alpha R_2^{(\frac{k}{2}-1)}E_2^*$. This proves the theorem. □

Using the above result, we then extend the result given in [3] to all nearly holomorphic modular forms for the group $SL_2(\mathbb{Z})$.

2.3.2 Correspondence of polynomial relations among eigenforms

From Theorem 1.4.8, we recall that the map defined by

$$f(z) = \sum_{j=0}^{p} \frac{f_j(z)^j}{\mathrm{Im}(z)} \mapsto f_0(z), \tag{2.18}$$

induces a ring isomorphism between the graded ring of nearly holomorphic modular forms and the graded ring of quasimodular forms, i.e.,

$$\widehat{M}_* \cong \widetilde{M}_*.$$

Therefore, any polynomial relation among nearly holomorphic modular forms gives rise to a corresponding polynomial relation in quasimodular forms (namely, the constant term of the polynomial) and vice versa.

Also for $f \in M_k$, we see that

$$R_k^{(r)} f \longleftrightarrow D^r f$$

$$R_2^{(r)} E_2^* \longleftrightarrow D^r E_2,$$

under the map given in (2.18). Therefore, we deduce that $R_k^{(r)} f$ is a nearly holomorphic eigenform if and only if $D^r f$ is a quasimodular eigenform. In view of the above ring isomorphism, by using the structure theorem of quasimodular eigenforms and nearly holomorphic eigenforms in Theorem 2.2.1 and Theorem 2.3.3 respectively, we conclude the following result.

Proposition 2.3.4 *The explicit map given in (2.18) is a Hecke equivariant ring isomorphism between $\widehat{M}_*$ and $\widetilde{M}_*$.*

An immediate application of Proposition 2.3.4 is the following theorem.

Theorem 2.3.5 *In the space of quasimodular and nearly holomorphic modular forms, a polynomial relation among eigenforms in one space gives rise to the corresponding polynomial relation in the other space.*

2.3.3 Main statement

In this section, we provide all the cases in which a nearly holomorphic eigenform can be written as products of finitely many nearly holomorphic eigenforms.

Theorem 2.3.6 *The product of finitely many nearly holomorphic eigenforms for $SL_2(\mathbb{Z})$ is never an eigenform except for the following exceptional cases:*

1. *The holomorphic cases presented in* (1) *and* (2) *of Theorem 2.1.2.*

2. $(R_4E_4)E_4 = \frac{1}{2}R_8E_8, \quad E_2^*\Delta_{12} = R_{12}\Delta_{12}.$

Proof. By Theorem 2.3.5, the statement of Theorem 2.2.5 gives the required result. □

Corollary 2.3.7 *The product of two nearly holomorphic eigenforms for $SL_2(\mathbb{Z})$ is never an eigenform except for the following exceptional cases:*

1. *The 16 holomorphic cases presented in Theorem 2.1.1.*

2. $(R_4E_4)E_4 = \frac{1}{2}R_8E_8, \quad E_2^*\Delta_{12} = R_{12}\Delta_{12}.$

Remark 2.3.1 *This theorem is an extension of [3, Theorem 3.1] and gives another proof of the main result of [3].*

CHAPTER 3

The adjoint of the Serre derivative map

In this chapter we compute the adjoint of the Serre derivative map with respect to the Petersson scalar product by using existing tools of nearly holomorphic modular forms. The Fourier coefficients of a cusp form of integer weight k, constructed using this method, involve special values of certain shifted Dirichlet series associated with a given cusp form f of weight $k+2$. We also give some applications, including a formula for the Ramanujan tau function in terms of the special values of the shifted Dirichlet series associated to the Ramanujan delta function. The results of this chapter have been published in [31].

3.1 Introduction

Using the properties of Poincaré series and adjoints of linear maps, W. Kohnen [27] obtained the adjoint map of the product map (product by a fixed cusp form), with respect to the Petersson scalar product. After Kohnen's work, similar results have been obtained by many authors for different types of maps and also for other spaces of automorphic forms. These type of results are important because the Fourier coefficients of the image under the adjoint map involve special values of certain shifted Dirichlet series.

In this chapter we consider the Serre derivative map, which is defined in terms of the derivative map D and the Eisenstein series E_2 (see (3.1) for the precise definition) and obtain the adjoint of this map. The weight-k Serre derivative map is denoted as ϑ_k and it maps the space $M_k(\Gamma)$ into $M_{k+2}(\Gamma)$ and preserves the space of cusp forms (Theorem 3.2.1).

In our present case also, the Fourier coefficients of the image of f under the adjoint of the Serre derivative map involve special values of certain shifted Dirichlet series associated with the Fourier coefficients of f. As an application, we get an asymptotic bound for the special values of these shifted Dirichlet series (eq. (3.10)). As another application, we also give a formula for the Ramanujan tau function in terms of the special values of the shifted Dirichlet series associated to the Ramanujan delta function.

We employ the theory of nearly holomorphic modular forms since quasimodular forms do not satisfy the modular transformation property and hence it is not possible to define the Petersson inner product in the usual way for the space of quasimodular forms. To transfer the problem from quasimodular forms to nearly holomorphic modular forms, we use the isomorphism given in Theorem 1.4.8. It is to be noted that the Petersson inner product is well-defined in the space of nearly holomorphic modular forms. Therefore, sometimes it is convenient to switch our problems from quasimodular forms to nearly holomorphic modular forms and vice versa. By the means of the Maass-Shimura operator R_k and E_2^*, we first transform the definition of the Serre derivative in the context of nearly holomorphic modular forms and then we compute the adjoint map explicitly.

3.2 Set up

3.2.1 The Serre derivative

As mentioned in the introduction, we define the *Serre derivative* map on $M_k(\Gamma)$ by

$$\vartheta_k f := Df - \frac{k}{12} E_2 f. \tag{3.1}$$

Then $\vartheta_k f$ is a modular form of weight $k+2$ for Γ. The operator ϑ_k is sometimes called the *the Ramanujan-Serre differential operator* in the literature. We first observe the mapping property of ϑ_k.

Theorem 3.2.1 *Let k be a non-negative integer. Then the weight k-operator ϑ_k maps $M_k(\Gamma)$ to $M_{k+2}(\Gamma)$ and $S_k(\Gamma)$ to $S_{k+2}(\Gamma)$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Using the transformation formula of E_2 in (1.8), a straightforward calculation gives

$$(\vartheta_k f) \mid_{k+2} \gamma(z) = D(f \mid_k \gamma)(z) - \frac{k}{12} E_2(z)(f \mid_k \gamma)(z), \tag{3.2}$$

from which the first assertion follows that $\vartheta_k f \in M_{k+2}(\Gamma)$, if $f \in M_k(\Gamma)$. From (3.2), it is also clear that ϑ_k preserve the cuspidal condition by noting the action of the differential map D on the q-expansion. $\square$

Remark 3.2.1 *Similar to (3.1), we can also define the weight $\frac{k}{2}$-operator $\vartheta_{k/2}$ for an odd positive integer k. Then proceeding as above, we see that $\vartheta_{k/2}$ maps* *a modular form (resp. cusp form) of weight* $\frac{k}{2}$ *to a modular form (resp. cusp form) of weight* $\frac{k}{2}+2$.

3.2.2 The nearly holomorphic setting

By Proposition 1.4.9, we have the fact that the operator R_k takes $\widehat{M}_k(\Gamma)$ into $\widehat{M}_{k+2}(\Gamma)$, so sometimes it is called the Maass-raising operator. There is another operator, which is known as the Maass-lowering operator defined by,

$$L_k := -y^2 \frac{\partial}{\partial \overline{z}} : \widehat{M}_{k+2}(\Gamma) \to \widehat{M}_k(\Gamma),$$

The operator L_k annihilates any holomorphic function. In [43, Theorem 6.8], it has been shown that the operator L_k is the adjoint of R_k with respect to the inner product (1.23), whenever the product of the functions is rapidly decreasing. More precisely,

$$\langle f, R_{k-2}g \rangle = \langle L_k f, g \rangle, \tag{3.3}$$

where f and g are nearly holomorphic modular forms of weight k and $k-2$ respectively such that fg is rapidly decreasing function. We now state an interesting application of this assertion, which plays a crucial role in the proof of our main result.

Lemma 3.2.2 *Let* Γ *be a congruence subgroup and let* $f \in S_{k+2}(\Gamma)$. *Then* $\langle f, R_k g \rangle = 0$ *for any* $g \in M_k(\Gamma)$.

Remark 3.2.2 *We observe that the Serre derivative map can also be written in the form*

$$\vartheta_k f = R_k f - \frac{k}{12} E_2^* f, \tag{3.4}$$

by using the definitions of E_2^* *and* R_k *given in* (1.22) *and* (1.24) *respectively.*

This form is quite useful for computing the Petersson inner product as $R_k f$ and $E_2^* f$ are nearly holomorphic modular forms, where the inner product is defined by (1.23).

3.2.3 A shifted Dirichlet series

Let $f(z) = \sum_{n\geq 0} a(n)q^n$ and $g(z) = \sum_{n\geq 0} b(n)q^n$. For $m \geq 0$, define a shifted Dirichlet series of Rankin type as in [27] by

$$L_{f,g,m}(s) = \sum_{n\geq 1} \frac{a(n+m)\overline{b(n)}}{(n+m)^s} \quad (s \in \mathbb{C}). \tag{3.5}$$

If the coefficients $a(n)$ and $b(n)$ satisfy an appropriate bound, then $L_{f,g,m}(s)$ converges absolutely in some half-plane.

For $f \in S_k(\Gamma)$ and a non-negative integer m, consider a shifted Dirichlet series of Rankin type associated with f and E_2, defined by

$$L_{f,m}(s) := -\frac{1}{24} L_{f,E_2,m}(s) = \sum_{n\geq 1} \frac{a(n+m)\sigma(n)}{(n+m)^s}. \tag{3.6}$$

Then by Proposition 1.2.6, it is absolutely convergent for $\mathrm{Re}(s) > \frac{k+3}{2}$. It can be shown that $L_{f,m}(s)$ has a meromorphic continuation to $\mathbb{C}$. A slightly different shifted Dirichlet series of this kind associated with two modular forms was first introduced by A. Selberg in [41]. Recently, in [20], J. Hoffstein and T. A. Hulse rigorously investigated the meromorphic continuation of a variant of Selberg's shifted Dirichlet series and multiple shifted Dirichlet series.

3.3 Main theorem

From Theorem 3.2.1 we know that $\vartheta_k : S_k(\Gamma) \to S_{k+2}(\Gamma)$ is a $\mathbb{C}$-linear map of finite dimensional Hilbert spaces and hence has an adjoint map $\vartheta_k^* : S_{k+2}(\Gamma) \to S_k(\Gamma)$, such that

$$\langle \vartheta_k^* f, g\rangle = \langle f, \vartheta_k g\rangle, \qquad \forall\ f \in S_{k+2}(\Gamma),\ g \in S_k(\Gamma).$$

In the main result we exhibit the Fourier coefficients of $\vartheta_k^* f$ for $f \in S_{k+2}(\Gamma)$. Its m-th Fourier coefficient involves special values of the shifted Dirichlet series $L_{f,m}(s)$. Now we shall state the main theorem of this chapter.

Theorem 3.3.1 *Let $k \geq 2$ be an integer. The image of any function $f(z) = \sum\limits_{n\geq 1} a(n)q^n \in S_{k+2}(\Gamma)$ under ϑ_k^* is given by*

$$\vartheta_k^* f(z) = \sum_{m\geq 1} c(m)q^m,$$

where

$$c(m) = \frac{k(k-1)m^{k-1}}{\mu_\Gamma (4\pi)^2}\left[\frac{(m-\frac{k}{12})}{m^{k+1}}a(m) + 2kL_{f,m}(k+1)\right].$$

We need the following Lemma to prove the above theorem.

Lemma 3.3.2 *Using the same notation as in Theorem 3.3.1, the following sum of integrals*

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Gamma\backslash\mathbb{H}}\left|f(z)\overline{E_2^*(z)}\ \overline{e^{2\pi imz}\ |_k\ \gamma}\ y^{k+2}\right|\frac{dxdy}{y^2} \tag{3.7}$$

converges.

Proof. Since f is a cusp form of weight k, the function fE_2^* is a nearly holomorphic modular form of weight $k+2$ and is rapidly decreasing at every cusp.

Therefore, for some positive constant M, we have

$$\left| y^{\frac{k}{2}+1} f(z) E_2^*(z) \right| \leq M, \quad \forall\ z \in \mathbb{H}.$$

Changing the variable z to $\gamma^{-1}z$ and using the standard Rankin unfolding argument, the sum in (3.7) equals

$$\begin{aligned}
\int_{\Gamma_\infty\backslash\mathbb{H}} \left| f(z)\overline{E_2^*(z)}\ \overline{e^{2\pi imz}}\ y^{k+2} \right| \frac{dxdy}{y^2} &= \int_{\Gamma_\infty\backslash\mathbb{H}} \left| y^{\frac{k}{2}+1} f(z) E_2^*(z) \right| e^{-2\pi my} y^{\frac{k}{2}+1} \frac{dxdy}{y^2} \\
&\leq M \int_0^\infty \int_0^1 e^{-2\pi my}\ y^{\frac{k}{2}-1} dxdy \\
&= M \frac{\Gamma(\frac{k}{2})}{(2\pi m)^{\frac{k}{2}}},
\end{aligned}$$

which gives the required convergence. □

Proof of Theorem 3.3.1. Since $\vartheta_k^* f(z) = \sum_{m\geq 1} c(m) q^m$, using Lemma 1.2.5 we get

$$c(m) = \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \langle \vartheta_k^* f, P_{k,m} \rangle = \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \langle f, \vartheta_k P_{k,m} \rangle.$$

By considering the above inner product in the space of nearly holomorphic modular forms and using (3.4), we get

$$\begin{aligned}
\langle f, \vartheta_k P_{k,m} \rangle &= \langle f, R_k P_{k,m} - \frac{k}{12} E_2^* P_{k,m} \rangle \\
&= \langle f, R_k P_{k,m} \rangle - \frac{k}{12} \langle f, E_2^* P_{k,m} \rangle \\
&= -\frac{k}{12} \langle f, E_2^* P_{k,m} \rangle. \qquad \text{(using Lemma 3.2.2)}
\end{aligned}$$

Hence,

$$c(m) = -\frac{k}{12} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \langle f, E_2^* P_{k,m} \rangle. \tag{3.8}$$

Now consider,

$$\begin{aligned}\langle f, E_2^* P_{k,m}\rangle &= \frac{1}{\mu_\Gamma}\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{E_2^*(z)P_{k,m}}\; y^{k+2}\,\frac{dxdy}{y^2}\\ &= \frac{1}{\mu_\Gamma}\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{E_2^*(z)}\;\overline{\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e^{2\pi imz}\mid_k \gamma}\; y^{k+2}\,\frac{dxdy}{y^2}.\end{aligned}$$

Using the convergence proved in Lemma 3.3.2, we can interchange summation and integration in the above expression. Using Rankin's unfolding argument, the integral in the above expression can be written as

$$\begin{aligned}&\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{E_2^*(z)}\;\overline{e^{2\pi imz}}\; y^k\; dxdy\\ &= \int_0^\infty\int_0^1\sum_{s\geq 1} a(s)e^{2\pi is(x+iy)}\overline{\left(1-\frac{3}{\pi y}-24\sum_{t\geq 1}\sigma(t)e^{2\pi it(x+iy)}\right)}\,\overline{e^{2\pi im(x+iy)}}y^k dxdy\\ &= \sum_{s\geq 1} a(s)\int_0^\infty\int_0^1\left(1-\frac{3}{\pi y}\right)e^{-2\pi y(s+m)}y^k e^{2\pi ix(s-m)}dxdy\\ &\qquad -24\sum_{t\geq 1}\sigma(t)\sum_{s\geq 1}a(s)\int_0^\infty\int_0^1 e^{-2\pi y(t+m-s)}y^k e^{2\pi ix(s-t-m)}dxdy \qquad (3.9)\\ &= a(m)\int_0^\infty\left(1-\frac{3}{\pi y}\right)e^{-4\pi my}y^k dy - 24\sum_{t\geq 1}a(t+m)\sigma(t)\int_0^\infty e^{-4\pi y(t+m)}y^k dy\\ &= \frac{\Gamma(k)}{\pi(4\pi m)^k}\left(\frac{k}{4m}-3\right)a(m) - 24\frac{\Gamma(k+1)}{(4\pi)^{k+1}}\sum_{t\geq 1}\frac{a(t+m)\sigma(t)}{(t+m)^{k+1}}\\ &= \frac{\Gamma(k)}{(4\pi)^{k+1}}\left[\frac{(k-12m)}{m^{k+1}}a(m) - 24kL_{f,m}(k+1)\right].\end{aligned}$$

By Proposition 1.2.6, interchanging the sum and integral in (3.9) is justified. Hence

$$\langle f, E_2^* P_{k,m}\rangle = \frac{1}{\mu_\Gamma}\frac{\Gamma(k)}{(4\pi)^{k+1}}\left[\frac{(k-12m)}{m^{k+1}}a(m) - 24kL_{f,m}(k+1)\right].$$

This proves the theorem. □

Remark 3.3.1 *It is worth pointing out that Lemma 3.2.2 holds for forms of half-integral weight. So in view of Remark 3.2.1 and using the same techniques as in the proof of Theorem 3.3.1, one can explicitly find the map*

$$\vartheta^*_{k/2} : S_{\frac{k}{2}+2}(\Gamma) \longrightarrow S_{\frac{k}{2}}(\Gamma),$$

where k is an odd positive integer and $\Gamma = \Gamma_0(N), N \in 4\mathbb{N}$. It gives a construction of cusp forms of half-integral weight whose coefficients involve special values of shifted Dirichlet series of Rankin type.

3.4 Applications

3.4.1 An asymptotic bound for $L_{f,m}(k+1)$

Let $f(z) = \sum_{n\geq 1} a(n)q^n \in S_{k+2}(\Gamma)$ and $\vartheta^*_k f(z) = \sum_{n\geq 1} c(n)q^n$. Then from Theorem 3.3.1, for any $m \geq 1$, we get

$$c(m) = \frac{k(k-1)m^{k-1}}{\mu_\Gamma (4\pi)^2}\left[\frac{(m-\frac{k}{12})}{m^{k+1}}a(m) + 2kL_{f,m}(k+1)\right].$$

Therefore, we have

$$L_{f,m}(k+1) = \frac{1}{2k}\left[\frac{\mu_\Gamma (4\pi)^2}{k(k-1)}\frac{1}{m^{k-1}}c(m) - \frac{(m-\frac{k}{12})}{m^{k+1}}a(m)\right].$$

Therefore, in view of the Fourier coefficients bound given in Proposition 1.2.6, a direct calculation gives the following asymptotic bound:

$$L_{f,m}(k+1) \ll m^{\frac{1-k}{2}}, \tag{3.10}$$

where the implied constant depends on f.

3.4.2 Values of $L_{f,m}(k+1)$ in terms of the Fourier coefficients

Let $k \geq 2$ and Γ be a congruence subgroup for which $S_k(\Gamma)$ is a 1-dimensional space generated by $f(z)$. Then applying Theorem 3.3.1, we get $\vartheta_k^* g(z) = \alpha_g f(z)$ for any $g \in S_{k+2}(\Gamma)$, where α_g is a constant. Now equating the m-th Fourier coefficients both the sides, we get a relation among the special values of the shifted Dirichlet series associated with g and the Fourier coefficients of f. In the following, we illustrate this with one example.

Let us take $f(z) = \Delta_{k,N}(z)$, which is the unique normalized cusp form with Fourier coefficients $\tau_{k,N}(n)$ in the 1-dimensional space $S_k(\Gamma_0(N))$. Note that $\Delta_{12,1}(z) = \Delta(z)$, whose Fourier coefficients are $\tau(n)$, the Ramanujan tau function. For a positive integer t, we introduce the V-operator acting on a function f (defined on $\mathbb{C}$) by

$$V_t f(z) := f(tz).$$

It is known that V_t is a linear operator from $S_k(\Gamma_0(N))$ into $S_k(\Gamma_0(Nt))$. Note that $S_{10}(\Gamma_0(2)) = \mathbb{C}\Delta_{10,2}(z)$ and $S_{12}(\Gamma_0(2)) = \mathbb{C}\Delta(z) \oplus \mathbb{C}V_2\Delta(z)$. Now considering the map $\vartheta_{10} : S_{10}(\Gamma_0(2)) \to S_{12}(\Gamma_0(2))$, a direct computation shows that

$$\vartheta_{10}\Delta_{10,2}(z) = \frac{1}{6}\Delta(z) + \frac{128}{3}V_2\Delta(z). \tag{3.11}$$

Let $\vartheta_{10}^*\Delta(z) = \alpha\Delta_{10,2}(z)$ and $\vartheta_{10}^* V_2\Delta(z) = \beta\Delta_{10,2}(z)$, for some $\alpha, \beta \in \mathbb{C}$. By using the property of the adjoint map and (3.11), we have

$$\alpha\|\Delta_{10,2}\|^2 = \langle \alpha\Delta_{10,2}, \Delta_{10,2}\rangle = \langle \vartheta_{10}^*\Delta, \Delta_{10,2}\rangle = \langle \Delta, \vartheta_{10}\Delta_{10,2}\rangle = \langle \Delta, \frac{1}{6}\Delta + \frac{128}{3}V_2\Delta\rangle$$

$$= \frac{1}{6}\|\Delta\|^2 + \frac{128}{3}\langle \Delta, V_2\Delta\rangle. \tag{3.12}$$

Similarly,

$$\beta\|\Delta_{10,2}\|^2 = \frac{128}{3}\|V_2\Delta\|^2 + \frac{1}{6}\langle \Delta, V_2\Delta\rangle. \tag{3.13}$$

From [6, eq. 49], we know that $\langle \Delta, V_2\Delta\rangle = -\frac{1}{256}\|\Delta\|^2$. Using this in (3.12), we get $\alpha\|\Delta_{10,2}\|^2 = 0$, i.e., $\alpha = 0$. This gives

$$\vartheta_{10}^*\Delta(z) = 0. \tag{3.14}$$

Now applying Theorem 3.3.1, for each $m \geq 1$ we have

$$\frac{(m - \frac{10}{12})}{m^{11}}\tau(m) + 20L_{\Delta,m}(11) = 0,$$

from which we get

$$\tau(m) = \frac{-20m^{11}}{(m - \frac{5}{6})}L_{\Delta,m}(11). \tag{3.15}$$

Note that $V_2\Delta = 2^{-6}\Delta \mid_{12} \gamma_2$, where $\gamma_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore from Proposition 1.2.4, we get

$$\|V_2\Delta\|^2 = \langle V_2\Delta, V_2\Delta\rangle = 2^{-12}\langle \Delta \mid_{12} \gamma_2, \Delta \mid_{12} \gamma_2\rangle = 2^{-12}\|\Delta\|^2$$

and using this in (3.13), we get the value of β as

$$\beta = \frac{5}{2^9}\frac{\|\Delta\|^2}{\|\Delta_{10,2}\|^2}. \tag{3.16}$$

Using Theorem 3.3.1 in the identity $\Delta_{10,2}(z) = \frac{1}{\beta}\vartheta_{10}^{*}V_2\Delta(z)$, we get

$$\tau_{10,2}(m) = \frac{15m^9}{8\beta\pi^2}\left[\frac{(m-\frac{10}{12})}{m^{11}}\tau\left(\frac{m}{2}\right) + 20L_{V_2\Delta,m}(11)\right], \tag{3.17}$$

where we assume that $\tau(n) = 0$ if n is not an integer.

In particular, for odd m, we have

$$\tau_{10,2}(m) = \frac{3840m^9}{\pi^2}\frac{\|\Delta_{10,2}\|^2}{\|\Delta\|^2}L_{V_2\Delta,m}(11), \tag{3.18}$$

where $L_{V_2\Delta,m}(11) = \sum_{\substack{n\geq 1\\ n:\text{odd}}} \frac{\tau(\frac{m+n}{2})\sigma(n)}{(m+n)^{11}}$.

Remark 3.4.1 *From (3.15), we see that for any $m \geq 1$ there exists $n \geq 1$ such that $\tau(m)$ and $\tau(m+n)$ are of opposite sign. In other words, it follows that the Ramanujan tau function $\tau(m)$ and $L_{\Delta,m}(11)$ both exhibit infinitely many sign changes. We can also find the values of $L_{\Delta,m}(11)$ for each $m \geq 1$. In particular, we have $L_{\Delta,1}(11) = -\frac{1}{120}$. Moreover, the conjecture that $\tau(n) \neq 0$ for all n due to Lehmer is equivalent to the non-vanishing of the special value $L_{\Delta,m}(11)$. From (3.15), we also observe the rationality of $L_{\Delta,m}(11)$, for each $m \geq 1$, as the coefficient field of Δ is $\mathbb{Q}$.*

In general for any $f \in S_{k+2}(\Gamma)$ and $m \geq 1$, using a similar method, we can write $L_{f,m}(k+1)$ as a linear combination of m-th Fourier coefficients of f and elements from a fixed basis of $S_k(\Gamma)$. Then analogous observations can be made as in Remark 3.4.1.

CHAPTER 4

Estimates for Fourier coefficients of Hermitian cusp forms of degree two

In this chapter we determine the Fourier series development of Hermitian Jacobi Poincaré series and obtain bounds for its Fourier coefficients. This gives rise to estimates for Fourier coefficients of Hermitian Jacobi cusp forms, in general. Then, by following the method of Kohnen [28], we obtain estimates for Fourier coefficients of Hermitian cusp forms of degree two with respect to $\mathbb{Q}(i)$. *The content of this chapter have been published in [33].*

4.1 Introduction

In [28], W. Kohnen evaluated a bound for the Fourier coefficients of Jacobi cusp forms and as a consequence, he obtained an estimate for Fourier coefficients of Siegel cusp forms of degree two. Our main objective in this chapter is to adopt this technique in the context of Hermitian Jacobi cusp forms and get estimates for Fourier coefficients of Hermitian cusp forms of degree two on $\Gamma^{(2)} = M_4(\mathcal{O}) \cap U(2)$, where $\mathcal{O}$ denotes the ring of integers in $\mathbb{Q}(i)$ and $U(2)$ is the unitary group of degree two defined in section 1.5. It gives an improved estimate for Fourier coefficients when compared with the usual Hecke bound. More precisely, in Theorem 4.5.3, we show that the T-th Fourier coefficient $a(T)$ of a Hermitian cusp form of weight k on $\Gamma^{(2)}$ with character $\det^l$ satisfies the estimate:

$$a(T) \ll_{\epsilon,F} (\min\, T)^{16/19+\epsilon}(\det T)^{k/2-3/4+\epsilon},$$

where $\min\, T$ denotes the least positive integer represented by T. Using reduction theory we have $\min\, T \ll (\det T)^{1/2}$, so we obtain an improved estimate for $a(T)$ as:

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-25/76+\epsilon}.$$

The idea of the proof is to use the Fourier-Jacobi expansion of Hermitian modular forms, whose Fourier coefficients are Hermitian Jacobi forms. So in order to get a bound for the Fourier coefficients of a Hermitian modular form one can use the bound for the Fourier coefficients of Hermitian Jacobi forms, the latter is obtained by the estimation of Hermitian Jacobi Poincaré series. One of the aims of this chapter is to obtain the Fourier expansion of Hermitian Jacobi Poincaré series. These Fourier coefficients involve certain generalized Kloosterman sums and Bessel functions, which are similar to the case of classical Jacobi forms. As mentioned above, bounding the Fourier coefficients of Hermitian Jacobi Poincaré series enables us to estimate the Fourier coefficients of a general Hermitian Jacobi cusp form.

This chapter is organized as follows. In §2, we give a Kohnen-Skoruppa type Dirichlet series for Hermitian modular forms, in §3 we get the Fourier expansion of Hermitian Jacobi Poincaré series, in §4 we use the Fourier expansion of Poincaré series to get estimates for Fourier coefficients of Hermitian Jacobi cusp forms and finally in §5, we use the method of Kohnen to obtain the required estimate for Fourier coefficients of a Hermitian modular form.

We remark that though we have considered the imaginary quadratic field $\mathbb{Q}(i)$ in this chapter, we expect that our method can be carried over to get similar results in the case of any imaginary quadratic field.

However, we would also like to remark about the generalized estimate in the higher degree. The theory of classical Jacobi forms as developed by Eichler and Zagier [14] was generalized to higher degree by C. Ziegler [47]. Also the work of W. Kohnen and N. -P. Skoruppa [30] on certain Dirichlet series involving the Fourier-Jacobi coefficients in the degree two case was generalized for higher degree by T. Yamazaki [45]. These generalizations played a key role in the work of S. Böcherer and W. Kohnen [2] for obtaining a generalized estimate for the Fourier coefficients of Siegel modular forms of higher degree, which extended the work of Kohnen [29] for higher degree. So, the method adopted by Böcherer and Kohnen can be extended to the case of Hermitian modular forms of higher degree, if one has similar generalizations as done by Ziegler and Yamazaki in the case of Hermitian Jacobi forms and Hermitian modular forms. Moreover, if one uses a weaker (Hecke-type) estimate for the Petersson products of Fourier-Jacobi coefficients as mentioned in [2, p. 501], even getting a weaker estimate in the general degree case would require some elements of a systematic theory of higher degree Hermitian Jacobi forms (like the work of Ziegler), which is lacking. Due to these reasons, we restricted ourselves only to the degree 2 case.

4.2 Kohnen-Skoruppa type Dirichlet series

Let $F, G \in S_k(\Gamma^{(2)}, \det^l)$ be two Hermitian modular forms with Fourier-Jacobi coefficients ϕ_m and ψ_m respectively as given by (1.36). Following [38], we associate a Dirichlet series $D_{F,G}(s)$ of the Kohnen-Skoruppa type defined by

$$D_{F,G}(s) = \zeta_{\mathbb{Q}(i)}(s-k+3)\zeta(2s-2k+4)\sum_{m\geq 1}\langle \phi_m, \psi_m\rangle m^{-s}, \qquad (4.1)$$

where $\zeta_K(s)$ denotes the Dedekind zeta function associated to the number field K. Note that the inner product $\langle \cdot, \cdot \rangle$ used here is as in (1.32).

In [38, Theorem 1], the analytic properties of the Dirichlet series $D_{F,G}(s)$ for $F, G \in S_k(\Gamma^{(2)})$, have been studied. In fact, the same proof goes through in the case of Hermitian modular forms with character, from which we get the following result.

Proposition 4.2.1 *The Dirichlet series $D_{F,G}(s)$ associated to F, G in $S_k(\Gamma^{(2)}, \det^l)$ can be continued meromorphically to the entire s-plane. The function*

$$D^*_{F,G}(s) := (4\pi)^{-s}\Gamma(s)\Gamma(s-k+2)\Gamma(s-k+3)D_{F,G}(s)$$

is holomorphic in s except for possible simple poles at $s = k, k-1, k-2, k-3$ and satisfies the functional equation

$$D^*_{F,G}(s) = D^*_{F,G}(2k-3-s).$$

4.3 Fourier expansion of Hermitian Jacobi Poincaré series

In this section we determine the Fourier expansion of the Hermitian Jacobi Poincaré series $P^{k,m,\delta}_{(n,r)}$, which is defined in (1.33). For this purpose, first we consider a function $F_{k,m,p}$ on $\mathbb{H} \times \mathbb{C} \times \mathbb{C}$ defined by

$$F_{k,m,p}(\tau, z, w) := \sum_{\substack{\alpha \in \mathbb{Z} \\ \eta \in \mathcal{O}}} (\tau+\alpha)^{-k} e^{-m}\left(\frac{p + (z+\eta)(w+\overline{\eta})}{\tau+\alpha}\right),$$

where p is any positive number.

4.3.1 Two lemmas

To obtain the Fourier development of the function $F_{k,m,p}$, we need the following lemmas.

Lemma 4.3.1 *[19, Lemma 2.7] Let $c \in \mathbb{R}$, $r, q \in \mathbb{C}$ with $\mathrm{Im}(q) > 0$. Then we have*

$$\int_{\mathrm{Im}(z)=c} e(qz^2 + rz)dz = (-2iq)^{-\frac{1}{2}} e^{-\pi i \frac{r^2}{2q}}.$$

Note that we have slightly changed the notations when compared with [19].

To state our next lemma, we first recall the Bessel function. The Bessel function of the first kind and order α is defined by the series

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{n \geq 0} \frac{(-1)^n (z/2)^{2n}}{\Gamma(n+1)\Gamma(\alpha+n+1)} \qquad (z, \alpha \in \mathbb{C}, \alpha \neq -1, -2, -3, \dots),$$

which is convergent absolutely and uniformly in any closed domain of z and in any bounded domain of α. It is the solution of Bessel's equation

$$z^2\frac{d^y}{dz^2}+z\frac{dy}{dz}+(z^2-\alpha^2)=0,$$

which is non-singular at $z=0$. The function $J_\alpha(z)$ is therefore an analytic function of z for any z, except for the branch point $z=0$ if α is not an integer. For $a\in\mathbb{R}$ and non-negative integer α, the Bessel functions have the following integral representations (cf. [1, p. 14]).

$$J_\alpha(az)=\frac{a^\alpha}{2\pi i}\int_C t^{\alpha-1}e^{\frac{z}{2}(t-a^2t^{-1}})dt, \tag{4.2}$$

where C is any simple closed contour in the t-plane around the origin. For more details on Bessel function, we refer to [1].

Lemma 4.3.2 *Let $b,c,p\in\mathbb{R}$ and $c,p>0$. Then we have*

$$\int_{\mathrm{Re}(w)=c} w^{-(k-1)}e^{2\pi(bw-\frac{p}{w})}dw=\begin{cases}0 & b\le 0,\\ 2\pi i\sqrt{\frac{p}{b}}^{(-k+2)}J_{k-2}\left(4\pi\sqrt{bp}\right) & b>0,\end{cases}$$

where $J_\alpha(x)$ is the Bessel function (of first kind) of order α.

Proof. Denote the integral in the statement of the lemma by $\mathcal{I}$.

Case (i): $b=0$.

Let $\gamma_{\rho,c}$ denote the closed contour and γ_ρ denotes the semi-circular arc of radius ρ centred at $(c,0)$ as illustrated in the following Figure 1.

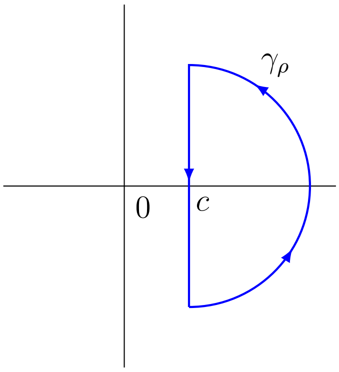

Figure 1: $\gamma_{\rho,c}$

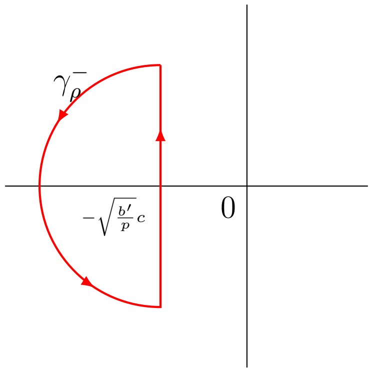

Figure 2: $\gamma^-_{\rho,-\sqrt{\frac{b'}{p}}c}$

Since the integrand $w^{-(k-1)}e^{2\pi(bw-\frac{p}{w})}$ does not have any pole in the region bounded by $\gamma_{\rho,c}$, by Cauchy's residue theorem, we have

$$\int_{\gamma_{\rho,c}} w^{-(k-1)}e^{-2\pi\frac{p}{w}}dw = 0,$$

or equivalently

$$\lim_{\rho\to\infty}\int_{\gamma_\rho} w^{-(k-1)}e^{-2\pi\frac{p}{w}}dw - \mathcal{I} = 0$$

To prove $\mathcal{I} = 0$, it is sufficient to show that

$$\lim_{\rho\to\infty}\int_{\gamma_\rho} \left|w^{-(k-1)}e^{-2\pi\frac{p}{w}}\right| |dw| = 0. \tag{4.3}$$

Writing $w = c + \rho e^{i\theta} \in \gamma_\rho$ and observe that $\left|e^{-2\pi\frac{p}{w}}\right| = e^{-2\pi p\frac{\mathrm{Re}(w)}{|w|^2}} \leq e^{-2\pi p}$, the left hand side of (4.3) becomes

$$\begin{aligned}\lim_{\rho\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-2\pi p}}{|c+\rho e^{i\theta}|^{(k-1)}}|i\rho e^{i\theta}d\theta| &= e^{-2\pi p}\lim_{\rho\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho}{(c^2+\rho^2+2c\rho\cos\theta)^{(k-1)/2}}d\theta \\ &\leq e^{-2\pi p}\lim_{\rho\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho}{(\rho^2)^{(k-1)/2}}d\theta \\ &= 0,\end{aligned}$$

which completes the proof in this case.

Case (ii): $b < 0$. Let $b = -b'$, for some $b' > 0$.

Changing the variable $w = -\sqrt{\frac{b'}{p}}z$, the integral $\mathcal{I}$ can be written as

$$\mathcal{I} = -\left(\sqrt{\frac{p}{b'}}\right)^{(-k+2)} \int_{\mathrm{Re}(z)=-\sqrt{\frac{b'}{p}}c} z^{-k+1} e^{2\pi\sqrt{pb'}\left(z+\frac{1}{z}\right)} dz.$$

Now consider the closed contour $\gamma^-_{\rho,-\sqrt{\frac{b'}{p}}c}$ and the semi-circular curve γ^-_ρ of radius ρ centered at $-\sqrt{\frac{b'}{p}}c$, as shown in the above Figure 2.

For $z \in \gamma^-_\rho$, we see that $\mathrm{Re}(z) \leq -\sqrt{\frac{b'}{p}}c$ and $\frac{\mathrm{Re}(z)}{|z|^2} \leq -1$. Hence

$$\left| e^{2\pi\sqrt{pb'}\left(z+\frac{1}{z}\right)} \right| = e^{2\pi\sqrt{pb'}\left(\mathrm{Re}(z)+\frac{\mathrm{Re}(z)}{|z|^2}\right)} \leq e^{-2\pi b'c} e^{-2\pi\sqrt{pb'}}.$$

Since the function $z^{-k+1}e^{2\pi\sqrt{pb'}\left(z+\frac{1}{z}\right)}$ has no poles in the region bounded by $\gamma^-_{\rho,-\sqrt{\frac{b'}{p}}c}$, proceeding as in the previous case we get that the integral is zero in this case also.

Case(iii): $b > 0$.

Changing the variable $w = \sqrt{\frac{b}{p}}z$, we have

$$\mathcal{I} = \left(\sqrt{\frac{p}{b}}\right)^{(-k+2)} \int_{\mathrm{Re}(z)=\sqrt{\frac{b}{p}}c} z^{-(k-2)-1} e^{2\pi\sqrt{pb}\left(z-\frac{1}{z}\right)} dz. \tag{4.4}$$

Using (4.2), the integral on the right hand side of above equation is equal to $2\pi i J_{k-2}\left(4\pi\sqrt{bp}\right)$. The proof is now complete. □

4.3.2 Expansion of the function $F_{k,m,p}$

We now prove the following proposition which gives the Fourier development of the function $F_{k,m,p}$.

Proposition 4.3.3 *For $p > 0$, we have*

$$F_{k,m,p}(\tau,z,w) = 2^{-k+2}m^{-k+1}i^{-k}\pi \sum_{\substack{n'\in\mathbb{Z}, r'\in\mathcal{O}^\sharp \\ D'=4\left(|r'|^2-n'm\right)<0}} \sqrt{\frac{p}{|D'|}}^{(-k+2)} J_{k-2}\left(2\pi\sqrt{|D'|p}\right) q^{n'}\zeta^{r'}\zeta'^{\overline{r'}}.$$

Proof. Writing

$$F_{k,m,p}(\tau,z,w) = \sum_{\alpha,\beta,\gamma\in\mathbb{Z}} (\tau+\alpha)^{-k} e^{-m}\left(\frac{p+(z+\beta+i\gamma)(w+\beta-i\gamma)}{\tau+\alpha}\right)$$

and using the Poisson summation formula we get,

$$F_{k,m,p}(\tau,z,w) = \sum_{\alpha',\beta',\gamma'\in\mathbb{Z}} \int_\alpha\int_\beta\int_\gamma (\tau+\alpha)^{-k} e^{-m}\left(\frac{p+(z+\beta+i\gamma)(w+\beta-i\gamma)}{\tau+\alpha}\right)$$
$$\times\, e\left(-(\alpha'\alpha+\beta'\beta+\gamma'\gamma)\right) d\gamma\ d\beta\ d\alpha.$$

Substituting $\alpha-\tau$ for α, $\beta-\frac{z}{2}-\frac{w}{2}$ for β and $\gamma+\frac{iz}{2}-\frac{iw}{2}$ for γ, we get

$$F_{k,m,p}(\tau,z,w) = \sum_{\alpha',\beta',\gamma'\in\mathbb{Z}} \int\limits_{\mathrm{Im}(\alpha)=v}\int_\beta\int_\gamma \alpha^{-k} e^{-m}\left(\frac{p+(\beta+i\gamma)(\beta-i\gamma)}{\alpha}\right)$$
$$\times\, e\left(-(\alpha'\alpha+\beta'\beta+\gamma'\gamma)\right) d\gamma\ d\beta\ d\alpha\ e\left(\alpha'\tau+z\left(\frac{\beta'+i\gamma'}{2}\right)+w\left(\frac{\beta'-i\gamma'}{2}\right)\right).$$

Here the integration over α,β and γ are over the straight line in $\mathbb{C}$ with fixed imaginary parts, which is positive for α because $\mathrm{Im}(\alpha)=v>0$. Now splitting the integral, we get

$$F_{k,m,p}(\tau,z,w) = \sum_{\alpha',\beta',\gamma'\in\mathbb{Z}} \int_{\mathrm{Im}(\alpha)=v} \alpha^{-k} e^{-1}\left(\frac{mp}{\alpha}+\alpha\alpha'\right)\int_\beta e\left(-\frac{m\beta^2}{\alpha}-\beta\beta'\right)d\beta$$
$$\times \int_\gamma e\left(-\frac{m\gamma^2}{\alpha}-\gamma\gamma'\right)d\gamma\ d\alpha q^{\alpha'}\zeta^{\frac{\beta'+i\gamma'}{2}}\zeta'^{\frac{\beta'-i\gamma'}{2}}$$

$$=\frac{1}{2mi}\sum_{\alpha',\beta',\gamma'\in\mathbb{Z}}\int_{\mathrm{Im}(\alpha)=v}\alpha^{-k+1}e\left(-\frac{\alpha}{m}\left(m\alpha'-\frac{\beta'^2+\gamma'^2}{4}\right)-\frac{mp}{\alpha}\right)d\alpha$$
$$q^{\alpha'}\zeta^{\frac{\beta'+i\gamma'}{2}}\zeta'^{\frac{\beta'-i\gamma'}{2}}.$$

In the last line, we have used Lemma 4.3.1. Put $r'=\frac{\beta'+i\gamma'}{2}$ and note that the complex number r' runs over $\mathcal{O}^\sharp$ as β' and γ' range over $\mathbb{Z}$. Replacing α' by n', we get

$$F_{k,m,p}(\tau,z,w)=\frac{1}{2mi}\sum_{\substack{n'\in\mathbb{Z},r'\in\mathcal{O}^\sharp\\ D'=4\left(|r'|^2-n'm\right)}}\int_{\mathrm{Im}(\alpha)=v}\alpha^{-(k-1)}e^{2\pi\left(\frac{D'}{4}\frac{i\alpha}{m}+p\frac{m}{i\alpha}\right)}d\alpha q^{n'}\zeta^{r'}\zeta'^{\overline{r'}}$$
$$=\frac{-(im)^{-k+2}}{2mi}\sum_{\substack{n'\in\mathbb{Z},r'\in\mathcal{O}^\sharp\\ D'=4\left(|r'|^2-n'm\right)}}\int_{\mathrm{Re}(s)=\frac{v}{m}}s^{-(k-1)}e^{2\pi\left(-\frac{D'}{4}s-\frac{p}{s}\right)}ds q^{n'}\zeta^{r'}\zeta'^{\overline{r'}},$$

where we have changed the variable α by ims in the last expression. In view of Lemma 4.3.2, we see that the above integral survives only when $D'<0$ and substituting its value, we finally get

$$F_{k,m,p}(\tau,z,w)=\frac{-(im)^{-k+2}}{2mi}2\pi i\sum_{\substack{n'\in\mathbb{Z},r'\in\mathcal{O}^\sharp\\ D'=4\left(|r'|^2-n'm\right)<0}}\sqrt{\frac{p}{|D'|/4}}^{(-k+2)}J_{k-2}\left(2\pi\sqrt{|D'|p}\right)q^{n'}\zeta^{r'}\zeta'^{\overline{r'}}.$$

This completes the proof. □

4.3.3 Fourier expansion

We are now ready to get the Fourier expansion of the Poincaré series $P^{k,m,\delta}_{(n,r)}$. Note that

$$\Gamma^J_\infty\backslash\Gamma^J=\{(\xi M,[\lambda,0]\xi M)\mid \xi\in\mathcal{O}^\times/\{\pm1\},M\in\Gamma_\infty\backslash\Gamma,\lambda\in\mathcal{O}\},$$

and hence from the definition (1.33), we have

$$
\begin{aligned}
P^{k,m,\delta}_{(n,r)}(\tau,z,w) &= \sum_{\substack{\xi\in\mathcal{O}^\times/\{\pm 1\}\\ M\in\Gamma_\infty\backslash\Gamma,\lambda\in\mathcal{O}}} e(n\tau+rz+\overline{r}w)\mid_m(\lambda,0)\mid_{k,m,\delta}\xi M(\tau,z,w)\\
&= \sum_{\substack{\xi\in\mathcal{O}^\times/\{\pm 1\}\\ (c,d)=1,\lambda\in\mathcal{O}}} \sigma(\xi)\xi^{-k}(c\tau+d)^{-k}e^{-m}\left(\frac{czw}{c\tau+d}\right)\\
\times e&\left((m|\lambda|^2+r\lambda+\overline{r\lambda}+n)\left(\frac{a\tau+b}{c\tau+d}\right)+(m\overline{\lambda}+r)\frac{\xi z}{c\tau+d}+(m\lambda+\overline{r})\frac{\overline{\xi}w}{c\tau+d}\right).
\end{aligned}
\tag{4.5}
$$

We split up the sum into two parts, one with $c=0$ and the other with $c\neq 0$. Denote the series corresponding to $c=0$ by I_0. In the second part with $c\neq 0$, substituting $m\overline{\lambda}+r=\mu$ and using the identity $\frac{a\tau+b}{c\tau+d}=\frac{a}{c}-\frac{1}{c(c\tau+d)}$ $(c\neq 0)$, we get

$$
\begin{aligned}
P^{k,m,\delta}_{(n,r)}(\tau,z,w) &= I_0+\sum_{\substack{\xi\in\mathcal{O}^\times/\{\pm 1\}\\ c\neq 0,(c,d)=1}}\sigma(\xi)\xi^{-k}(c\tau+d)^{-k}\\
&\times\sum_{\substack{\mu\in\mathcal{O}^\sharp\\ \mu\equiv r(m\mathcal{O})}} e\left(\left(\frac{|\mu|^2-D/4}{m}\right)\left(\frac{a}{c}-\frac{1}{c(c\tau+d)}\right)+\frac{\mu\xi z}{c\tau+d}+\frac{\overline{\mu\xi}w}{c\tau+d}-\frac{mczw}{c\tau+d}\right)\\
&= I_0+\sum_{\substack{\xi\in\mathcal{O}^\times/\{\pm 1\}\\ c\neq 0,(c,d)=1}}\sigma(\xi)\xi^{-k}(c\tau+d)^{-k}\sum_{\substack{\mu\in\mathcal{O}^\sharp\\ \mu\equiv r(m\mathcal{O})}} e\left(\left(\frac{|\mu|^2-D/4}{m}\right)\frac{a}{c}+\frac{D}{4cm(c\tau+d)}\right.\\
&\left.-\frac{cm\left(\xi z-\frac{\overline{\mu}}{cm}\right)\left(\overline{\xi}w-\frac{\mu}{cm}\right)}{c\tau+d}\right),
\end{aligned}
$$

where $D=4(|r|^2-nm)$. In the above expression, the terms with $c<0$ give $(-1)^k$ times the contribution of the terms with $c>0$, with z and w replaced by $-z$ and $-w$ respectively. So, if I_1 denotes the above series for $c>0$ and I_1'

denotes the same series with z and w replaced by their negatives, we obtain

$$P_{(n,r)}^{k,m,\delta}(\tau,z,w) = I_0 + I_1 + (-1)^k I_1'. \tag{4.6}$$

In the expression for I_1, making the change of variables $d \mapsto d + \alpha c$, with $\alpha \in \mathbb{Z}, d(c)^*$ and $\mu \mapsto \lambda + mc\eta$, with $\eta \in \mathcal{O}, \lambda \in \mathcal{O}^\sharp/mc\mathcal{O}, \lambda \equiv r(m\mathcal{O})$, we have

$$\begin{aligned} I_1 &= \sum_{\substack{\xi\in\mathcal{O}^\times/\{\pm1\}\\ c\geq 1}} \sigma(\xi)(\xi c)^{-k} \sum_{\substack{d(c)^*, dd^*\equiv 1(c)\\ \alpha\in\mathbb{Z}}} \left(\tau + \frac{d}{c} + \alpha\right)^{-k} \sum_{\substack{\eta\in\mathcal{O},\lambda\in\mathcal{O}^\sharp/mc\mathcal{O}\\ \lambda\equiv r(m\mathcal{O})}} e\Bigg(\left(\frac{|\lambda|^2 - D/4}{m}\right)\frac{a}{c} \\ &\quad + \frac{D}{4c^2m\left(\tau + \frac{d}{c} + \alpha\right)} - \frac{m\left(\xi z - \frac{\overline{\lambda}}{cm} - \overline{\eta}\right)\left(\overline{\xi} w - \frac{\lambda}{cm} - \eta\right)}{\left(\tau + \frac{d}{c} + \alpha\right)}\Bigg) \\ &= \sum_{\substack{\xi\in\mathcal{O}^\times/\{\pm1\}\\ c\geq 1}} \sigma(\xi)(\xi c)^{-k} \sum_{\substack{d(c)^*, dd^*\equiv 1(c)\\ \lambda\in\mathcal{O}^\sharp/mc\mathcal{O},\lambda\equiv r(m\mathcal{O})}} e\left(\left(\frac{|\lambda|^2 - D/4}{m}\right)\frac{d^*}{c}\right) F_{k,m,\frac{|D|}{4m^2c^2}}\left(\tau + \frac{d}{c}, \xi z - \frac{\overline{\lambda}}{cm}, \overline{\xi} w - \frac{\lambda}{cm}\right). \end{aligned} \tag{4.7}$$

By using the Fourier expansion of $F_{k,m,p}$ (derived in Proposition 4.3.3) in the above eq. (4.7), we have

$$\begin{aligned} I_1 &= 2^{-k+2} m^{-k+1} i^{-k} \pi \sum_{\substack{\xi\in\mathcal{O}^\times/\{\pm1\}\\ c\geq 1}} \sigma(\xi)(\xi c)^{-k} \sum_{\substack{d(c)^*, dd^*\equiv 1(c)\\ \lambda\in\mathcal{O}^\sharp/mc\mathcal{O},\lambda\equiv r(m\mathcal{O})}} e\left(\left(\frac{|\lambda|^2 - D/4}{m}\right)\frac{d^*}{c}\right) \\ &\quad \sum_{\substack{n'\in\mathbb{Z}, r'\in\mathcal{O}^\sharp\\ D'=4\left(|r'|^2 - n'm\right)<0}} \sqrt{\frac{D}{D'}}^{(-k+2)} \left(\frac{1}{2mc}\right)^{-k+2} J_{k-2}\left(\frac{\pi}{mc}\sqrt{DD'}\right) e^{2\pi i n'\left(\tau + \frac{d}{c}\right)} e^{2\pi i r'\left(\xi z - \frac{\overline{\lambda}}{cm}\right)} e^{2\pi i \overline{r'}\left(\overline{\xi} w - \frac{\lambda}{cm}\right)} \\ &= \sum_{\substack{n'\in\mathbb{Z}, r'\in\mathcal{O}^\sharp\\ D'=4\left(|r'|^2 - n'm\right)<0}} \sqrt{\frac{D}{D'}}^{(-k+2)} \frac{\pi i^{-k}}{m} \sum_{\substack{\xi\in\mathcal{O}^\times/\{\pm1\}\\ c\geq 1}} \sigma(\xi)\xi^{-k} J_{k-2}\left(\frac{\pi}{mc}\sqrt{DD'}\right) H_{m,c,\xi}(n,r,n',r') q^{n'} \zeta^{r'} \zeta'^{\overline{r'}}, \end{aligned}$$

where

$$H_{m,c,\xi}(n,r,n',r') = \frac{1}{c^2} \sum_{\substack{d(c)^* \\ dd^* \equiv 1(c)}} e_c\left(nd^* + n'd\right) \sum_{\substack{\lambda \in \mathcal{O}^\sharp / mc\mathcal{O} \\ \lambda \equiv r(m\mathcal{O})}} e_c\left(\frac{(|\lambda|^2 - |r|^2)d^* - (\xi\lambda\overline{r'} + \overline{\xi\lambda}r')}{m}\right) \tag{4.8}$$

is a generalized Kloosterman sum.

Our next objective is to evaluate the expansion of I_0. A straightforward calculation gives

$$\begin{aligned} I_0 &= \sum_{\substack{\xi \in \mathcal{O}^\times/\{\pm 1\} \\ \lambda \in \mathcal{O}}} \sigma(\xi)\xi^{-k} q^{\left(m|\lambda|^2 + r\lambda + \overline{r\lambda} + n\right)} \left(\zeta^{\xi\left(m\overline{\lambda}+r\right)}\zeta'^{\overline{\xi}(m\lambda+\overline{r})} + (-1)^k \zeta^{-\xi\left(m\overline{\lambda}+r\right)}\zeta'^{-\overline{\xi}(m\lambda+\overline{r})}\right) \\ &= \sum_{\substack{n' \in \mathbb{Z}, r' \in \mathcal{O}^\sharp \\ D' = 4\left(|r'|^2 - n'm\right) < 0}} \sum_{\xi \in \mathcal{O}^\times/\{\pm 1\}} \sigma(\xi)\xi^{-k} \left(\delta_m(n,r,n',r') + (-1)^k \delta_m(n,r,n',-r')\right) q^{n'}\zeta^{r'}\zeta'^{\overline{r'}}, \end{aligned}$$

where $\delta_m(n,r,n',r') = \begin{cases} 1 & D = D', r' \equiv r(m\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$

Substituting the expressions for I_0, I_1 and I_1' in (4.6), we get the following theorem giving the Fourier expansion of the Poincaré series.

Theorem 4.3.4 *The Fourier expansion of the (n,r)-th Poincaré series $P^{k,m,\delta}_{(n,r)}$ is given by*

$$\begin{aligned} P^{k,m,\delta}_{(n,r)}(\tau,z,w) = \sum_{\substack{n' \in \mathbb{Z}, r' \in \mathcal{O}^\sharp \\ D' = n'm - |r'|^2 > 0}} \Bigg(& \sum_{\xi \in \mathcal{O}^\times/\{\pm 1\}} \sigma(\xi)\xi^{-k}\delta^{\pm}_m(n,r,n',r') \\ & + \sqrt{\frac{D}{D'}}^{(-k+2)} \frac{\pi i^{-k}}{m} \sum_{\substack{\xi \in \mathcal{O}^\times/\pm 1 \\ c \geq 1}} J_{k-2}\left(\frac{\pi}{mc}\sqrt{DD'}\right) H^{\pm}_{m,c,\xi}(n,r,n',r') \Bigg) q^{n'}\zeta^{r'}\zeta'^{\overline{r'}}, \end{aligned}$$

where $\delta^{\pm}_m(n,r,n',r') = \delta_m(n,r,n',r') + (-1)^k\delta_m(n,r,n',-r')$

and $\quad H^{\pm}_{m,c,\xi}(n,r,n',r') = H_{m,c,\xi}(n,r,n',r') + (-1)^k H_{m,c,\xi}(n,r,n',-r').$

4.4 Bounds for Fourier coefficients of Hermitian Jacobi cusp form

In this section we shall get an improved Hecke bound for Fourier coefficients of Hermitian Jacobi cusp forms. To do this, we make use of an estimate for the Fourier coefficients of the Poincaré series $P^{k,m,\delta}_{(n,r)}$. We closely follow the method adopted by Kohnen [29].

4.4.1 Estimation for the generalized Kloosterman sum

Throughout this section, for a given $d(c)^*$, d^* denotes an integer such that $dd^* \equiv 1(c)$ and we shall drop this condition. From (4.8), we have

$$H_{m,c,\xi}(n,r,n,\pm r) = \frac{1}{c^2} \sum_{d(c)^*} e_c\left(n(d+d^*)\right) \sum_{\substack{\lambda \in \mathcal{O}^\sharp/mc\mathcal{O} \\ \lambda \equiv r(m\mathcal{O})}} e_c\left(\frac{(|\lambda|^2 - |r|^2)d^* \mp 2\mathrm{Re}\,(\xi\lambda\overline{r})}{m}\right). \tag{4.9}$$

Writing $\lambda = r + m\beta$ with $\beta = a + ib \in \mathcal{O}$, so that we have $r + m\beta \in \mathcal{O}^\sharp/mc\mathcal{O}$. Put

$$\begin{aligned} H^{\pm}_{m,c,\xi}(n,r) &:= c^2 H_{m,c,\xi}(n,r,n,\pm r) \\ &= \sum_{d(c)^*} e_c\left(n(d+d^*)\right) \sum_{r+m\beta \in \mathcal{O}^\sharp/mc\mathcal{O}} e_c\left((m|\beta|^2 + 2\mathrm{Re}(\beta\overline{r}))d^* \mp 2\mathrm{Re}\left(\xi|r|^2 + \xi\beta\overline{r}\right)\right). \end{aligned}$$

If $r = \frac{r_1+ir_2}{2}$, then the condition on $r+m\beta$ gives $-\frac{r_1}{2m} \leq a < c - \frac{r_1}{2m}$ and $\frac{r_2}{2m} \leq b < c - \frac{r_2}{2m}$. Hence, we get

$$\begin{aligned} H^{\pm}_{m,c,1}(n,r) &= e_c\left(\mp\frac{2|r|^2}{m}\right)\sum_{d(c)^*} e_c\left(n(d+d^*)\right)\sum_{a(c)} e_c\left(md^*a^2 + r_1(d^*\mp 1)a\right) \\ &\qquad \times \sum_{b(c)} e_c\left(md^*b^2 + r_2(d^*\mp 1)b\right) \qquad (4.10) \\ &= e_c\left(\mp\frac{2|r|^2}{m}\right) H'^{\pm}_{m,c,1}(n,r) \end{aligned}$$

and

$$\begin{aligned} H^{\pm}_{m,c,i}(n,r) = \sum_{d(c)^*} e_c\left(n(d+d^*)\right)\sum_{a(c)} e_c\left(md^*a^2 + (r_1d^*\mp r_2)a\right) \\ \times \sum_{b(c)} e_c\left(md^*b^2 + (r_2d^*\pm r_1)b\right). \qquad (4.11) \end{aligned}$$

Now, we shall obtain the following estimate for $H^{\pm}_{m,c,\xi}(n,r)$.

Lemma 4.4.1 *For any $\epsilon > 0$, we have*

$$H^{\pm}_{m,c,\xi}(n,r) \ll_\epsilon (m,c)^{\frac{1}{2}} c^{\frac{3}{2}+\epsilon}(D,c). \qquad (4.12)$$

Proof. Note that the last two sums in the expressions (4.10) and (4.11) are nothing but the generalized quadratic Gauss sum defined by

$$G(\alpha,\beta,c) := \sum_{l(c)} e_c(\alpha l^2 + \beta l),$$

which is multiplicative in c in the following sense:

$$G(\alpha,\beta,c_1c_2) = G(\alpha c_1,\beta,c_2)G(\alpha c_2,\beta,c_1), \quad \text{where } (c_1,c_2)=1. \qquad (4.13)$$

Let $c = c_1c_2$ such that $(c_1, c_2) = 1$. Assume that $c_1c_1^* \equiv 1(c_2)$ and $c_2c_2^* \equiv 1(c_1)$. By using (4.13), we easily see that

$$H'^{\pm}_{m,c_1c_2,1}(n,r) = H'^{\pm}_{mc_2,c_1,1}(nc_2^*, r)H'^{\pm}_{mc_1,c_2,1}(nc_1^*, r) \tag{4.14}$$

and

$$H^{\pm}_{m,c_1c_2,i}(n,r) = H^{\pm}_{mc_2,c_1,i}(nc_2^*, r)H^{\pm}_{mc_1,c_2,i}(nc_1^*, r). \tag{4.15}$$

In view of the multiplicative relations in (4.14) and (4.15) and from the definition of $H^{\pm}_{m,c,\xi}(n,r)$, it is sufficient to prove Lemma 4.4.1 when c is a prime power (say) p^ν for some prime p and $\nu \geq 1$ and for $\xi = 1$ or i. We assume that p is an odd prime (the case $p = 2$ can be done similarly). Let $p^\mu || m, p^{\rho_1} || r_1$ and $p^{\rho_2} || r_2$ for some integers $\mu, \rho_1, \rho_2 \geq 0$.

Case(i): $\mu \geq \nu$.

Note that $\sum\limits_{l(p^\nu)} e_{p^\nu}(\alpha l) = \begin{cases} p^\nu & \text{if } \alpha \equiv 0(p^\nu), \\ 0 & \text{otherwise.} \end{cases}$

Therefore,

$$\begin{aligned} &H^{\pm}_{m,p^\nu,1}(n,r) \\ &= e_{p^\nu}\left(\mp\frac{2|r|^2}{m}\right) \sum_{d(p^\nu)^*} e_{p^\nu}\left(n(d+d^*)\right) \sum_{a(p^\nu)} e_{p^\nu}\left(r_1(d^* \mp 1)a\right) \sum_{b(p^\nu)} e_{p^\nu}\left(r_2(d^* \mp 1)b\right) \\ &= e_{p^\nu}\left(\mp\frac{2|r|^2}{m}\right) p^{2\nu} \sum_{\substack{d(p^\nu)^*, r_1(d^*\mp 1)\equiv 0(p^\nu) \\ r_2(d^*\mp 1)\equiv 0(p^\nu)}} e_{p^\nu}\left(n(d+d^*)\right). \end{aligned} \tag{4.16}$$

Similarly,

$$H^{\pm}_{m,p^\nu,i}(n,r) = p^{2\nu} \sum_{\substack{d(p^\nu)^*, r_1 d^* \mp r_1 \equiv 0(p^\nu) \\ r_2 d^* \pm r_1 \equiv 0(p^\nu)}} e_{p^\nu}\left(n(d+d^*)\right). \tag{4.17}$$

Since the number of elements d in $(\mathbb{Z}/p^\nu\mathbb{Z})^*$ which satisfy the congruences in (4.16) and (4.17) is less than or equal to $p^{\min(\nu,\rho_1,\rho_2)} \leq p^{\min(\mu,\nu,2\rho_1,2\rho_2)} \leq (D, p^\nu)$, it follows that

$$|H^{\pm}_{m,p^\nu,\xi}(n,r)| \leq p^{\nu/2} p^{3\nu/2} (D, p^\nu) = (m, p^\nu)^{\frac{1}{2}} (p^\nu)^{\frac{3}{2}} (D, p^\nu),$$

which proves the lemma in this case.

Case(ii): $\mu < \nu$.

Consider the sum $\sum_{a(p^\nu)} e_{p^\nu}\left(md^*a^2 + r_1(d^* \mp 1)a\right)$. The sum is invariant under $a \mapsto a + p^{\nu-\mu}$, except for a multiple of $e_{p^\mu}\left(r_1(d^* \mp 1)a\right)$. Hence the sum vanishes unless $p^\mu | r_1(d^* \mp 1)$. In that case

$$\begin{aligned}
\sum_{a(p^\nu)} e_{p^\nu}\left(md^*a^2 + r_1(d^* \mp 1)a\right) &= \sum_{a(p^\nu)} e_{p^{\nu-\mu}}\left(\frac{md^*}{p^\mu}a^2 + \frac{r_1(d^* \mp 1)}{p^\mu}a\right) \\
&= p^\mu \sum_{a(p^{\nu-\mu})} e_{p^{\nu-\mu}}\left(\frac{md^*}{p^\mu}a^2 + \frac{r_1(d^* \mp 1)}{p^\mu}a\right) \\
&= p^\mu (p^{\nu-\mu})^{\frac{1}{2}} \left(\frac{-4}{p^{\nu-\mu}}\right)^{\frac{1}{2}} \left(\frac{\frac{4md^*}{p^\mu}}{p^{\nu-\mu}}\right) e_{p^{\nu-\mu}}\left(-\left(\frac{4md^*}{p^\mu}\right)^* \left(\frac{r_1(d^* \mp 1)}{p^\mu}\right)^2\right).
\end{aligned}$$

Using this in (4.10), we get

$$\begin{aligned}
H^{\pm}_{m,p^\nu,1}(n,r) =& p^{\nu+\mu} \left(\frac{-4}{p^{\nu-\mu}}\right) e_{p^\nu}\left(\mp\frac{2|r|^2}{m}\right) \\
&\times \sum_{\substack{d(p^\nu)^*, r_1(d^* \mp 1) \equiv 0(p^\mu) \\ r_2(d^* \mp 1) \equiv 0(p^\mu)}} e_{p^\nu}\left(n(d+d^*)\right) e_{p^{\nu-\mu}}\left(-\left(\frac{4md^*}{p^\mu}\right)^* \left(\frac{(r_1^2+r_2^2)(d^* \mp 1)^2}{p^{2\mu}}\right)\right)
\end{aligned}$$

$$= p^{\nu+\mu}\left(\frac{-4}{p^{\nu-\mu}}\right)\sum_{\substack{d(p^\nu)^*, r_1(d^*\mp 1)\equiv 0(p^\mu)\\ r_2(d^*\mp 1)\equiv 0(p^\mu)}} e_{p^{\nu+\mu}}\left(\left(n-\left(\frac{m}{p^\mu}\right)^*\frac{|r|^2}{p^\mu}\right)(d+d^*)\right)$$

$$= p^{\nu}\left(\frac{-4}{p^{\nu-\mu}}\right)\sum_{\substack{d(p^{\nu+\mu})^*, r_1(d^*\mp 1)\equiv 0(p^\mu)\\ r_2(d^*\mp 1)\equiv 0(p^\mu)}} e_{p^{\nu+\mu}}\left(-m_1 D(d+d^*)\right). \tag{4.18}$$

Similarly for $\xi = i$, we get

$$H^{\pm}_{m,p^\nu,i}(n,r) = p^{\nu}\left(\frac{-4}{p^{\nu-\mu}}\right)\sum_{\substack{d(p^{\nu+\mu})^*, r_1 d^*\mp r_2\equiv 0(p^\mu)\\ r_2 d^*\pm r_1\equiv 0(p^\mu)}} e_{p^{\nu+\mu}}\left(-m_1 D(d+d^*)\right). \tag{4.19}$$

Note that in (4.18) and (4.19), m_1 denotes the inverse of $\frac{4m}{p^\mu}(p^{\nu+\mu})$. By estimating the number of summands in the expressions (4.18) and (4.19), we get

$$|H^{\pm}_{m,p^\nu,\xi}(n,r)| \le p^\nu p^{\nu+\mu} \le (p^\mu)^{\frac{1}{2}}(p^\nu)^{\frac{3}{2}}(D,p^\nu), \qquad \text{provided } p^\nu | D,$$

which is the required estimate. For the remaining part, we consider the case $p^\lambda || D$, for some integer $\lambda \ge 0$ with $\lambda < \nu$. Assume that $\kappa = \max\{\mu-\rho_1, \mu-\rho_2, 0\}$ and $\alpha = \nu+\mu-\lambda$. The congruence conditions on d in the sum in (4.19), imply that either $\kappa = 0$ or $\rho_1 = \rho_2$. In the latter case the congruence condition on d can be written as $d^* \equiv \pm {r'_1}^* r'_2 (p^\kappa)$ and $d^* \equiv \mp {r'_2}^* r'_1 (p^\kappa)$, where $r_1 = r'_1 p^{\rho_1}$ and $r_2 = r'_2 p^{\rho_1}$. Note also that the sum in (4.19) survives only when ${r'_1}^2 + {r'_2}^2 \equiv 0(p^\kappa)$. Hence from (4.18) and (4.19), we conclude that

$$H^{\pm}_{m,p^\nu,\xi}(n,r) = p^{\nu}\left(\frac{-4}{p^{\nu-\mu}}\right)\sum_{\substack{d(p^{\nu+\mu})^*\\ (d^*\mp l)\equiv 0(p^\kappa)}} e_{p^\alpha}\left(-\frac{m_1 D}{p^\lambda}(d+d^*)\right) = p^{\nu+\lambda}\left(\frac{-4}{p^{\nu-\mu}}\right) S_{\pm l},$$

where

$$S_{\pm l} = \sum_{\substack{d(p^\alpha)^*\\ (d^*\mp l)\equiv 0(p^\kappa)}} e_{p^\alpha}\left(\Delta(d+d^*)\right), \tag{4.20}$$

with $\Delta = -\frac{m_1 D}{p^\lambda}$ and $l = \begin{cases} 1 & \text{if } \xi = 1, \\ {r'_1}^* r'_2(p^\kappa) & \text{if } \xi = i. \end{cases}$

To complete the proof, it is sufficient to show that

$$S_{\pm l} \ll_\epsilon (p^\mu)^{\frac{1}{2}} (p^\nu)^{\frac{1}{2}+\epsilon}. \tag{4.21}$$

The number of summands in $S_{\pm l}$ is $\leq p^{\alpha-\kappa} = p^{\frac{\nu+\mu}{2}} p^{\frac{\alpha}{2}-\kappa-\frac{\lambda}{2}}$. Hence if $\frac{\alpha}{2} \leq \kappa + \frac{\lambda}{2}$ then $|S_{\pm l}| \leq p^{\frac{\mu+\nu}{2}}$ and we are done. Hence we need to consider the case

$$\frac{\alpha}{2} > \kappa + \frac{\lambda}{2}. \tag{4.22}$$

If $\alpha = 1$ then $\kappa + \frac{\lambda}{2} = 0$ and hence $\nu = 1, \mu = 0$. Then we have $S_{\pm l} = \sum\limits_{d(p)^*} e_p\left(\Delta(d + d^*)\right)$, which is the Kloosterman sum and hence $|S_{\pm l}| \ll p^{\frac{1}{2}}$.
So, let us assume that $\alpha \geq 2$.

Subcase (i): α is even, i.e., $\alpha = 2\beta$ for some $\beta \geq 1$.

Writing $d = u + vp^\beta(p^{2\beta})$, where u (resp. v) running over $(\mathbb{Z}/p^\beta\mathbb{Z})^*$ (resp. $\mathbb{Z}/p^\beta\mathbb{Z}$), we have $d^* = u^* - u^{*2}vp^\beta(p^{2\beta})$, where $uu^* \equiv 1(p^{2\beta})$. In view of (4.22), we can write

$$\begin{aligned} S_{\pm l} &= \sum_{u(p^\beta)^*, u^* \equiv \pm l(p^\kappa)} e_{p^\alpha}\left(\Delta(u + u^*)\right) \sum_{v(p^\beta)} e_{p^\beta}\left(-\Delta v({u^*}^2 - 1)\right) \\ &= p^\beta \sum_{\substack{u(p^\beta)^*, u^* \equiv \pm l(p^\kappa) \\ {u^*}^2 \equiv 1(p^\beta)}} e_{p^\alpha}\left(\Delta(u + u^*)\right). \end{aligned}$$

Since the congruence ${u^*}^2 \equiv 1(p^\beta)$ has two solutions ± 1 in $\mathbb{Z}/p^\beta\mathbb{Z}$, whence

$$|S_{\pm l}| \leq 2p^\beta \leq 2(p^\mu)^{\frac{1}{2}}(p^\nu)^{\frac{1}{2}}.$$

Subcase (ii): α is odd, say $\alpha = 2\beta + 1$ for some $\beta \geq 1$.

Write $d = u + vp^{\beta+1}(p^{2\beta+1})$, where $u \in (\mathbb{Z}/p^{\beta+1}\mathbb{Z})^*$ and $v \in \mathbb{Z}/p^{\beta}\mathbb{Z}$. Then $d^* = u^* - u^{*2}vp^{\beta+1}(p^{2\beta+1})$, with $uu^* \equiv 1(p^{2\beta+1})$. From (4.22) and following similar arguments, we get

$$S_{\pm l} = p^{\beta} \sum_{\substack{u(p^{\beta+1})^*, u^* \equiv \pm l(p^{\kappa}) \\ u^{*2} \equiv 1(p^{\beta})}} e_{p^{\alpha}}\left(\Delta(u + u^*)\right).$$

The solutions to $u^{*2} \equiv 1(p^{\beta})$ in $(\mathbb{Z}/p^{\beta+1}\mathbb{Z})$ are given by $u = \pm 1 + tp^{\beta}$ with $t \in \mathbb{Z}/p\mathbb{Z}$ and then $u^* = \pm 1 - tp^{\beta} \pm t^2 p^{2\beta}(p^{2\beta+1})$. Therefore,

$$S_{\pm l} = \begin{cases} p^{\beta}\left(e_{p^{\alpha}}(2\Delta)\sum_{t(p)} e_p(\Delta t^2) + e_{p^{\alpha}}(-2\Delta)\sum_{t(p)} e_p(-\Delta t^2)\right) & \kappa = 0, \\ p^{\beta} e_{p^{\alpha}}(\pm 2\Delta)\sum_{t(p)} e_p(\pm \Delta t^2)\delta_{l1} & \kappa \neq 0, \end{cases}$$

where δ_{l1} is the standard Kronecker symbol. In all the cases, we have

$$S_{\pm l} \ll_{\epsilon} p^{\beta} p^{\frac{1}{2}+\epsilon} \ll_{\epsilon} (p^{\mu})^{\frac{1}{2}} (p^{\nu})^{\frac{1}{2}+\epsilon}.$$

The proof of Lemma 4.4.1 is now complete. $\square$

4.4.2 The final estimate

The following proposition gives an estimate for the Fourier coefficients of a Hermitian Jacobi cusp form.

Proposition 4.4.2 *Let* $\phi(\tau, z, w) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^{\sharp} \\ mn - |r|^2 \geq 0}} c(n, r) q^n \zeta^r \zeta'^{\overline{r}} \in J_{k,m}^{\delta,\mathrm{cusp}}$ *and* $\epsilon > 0$.

Then

$$c(n, r) \ll_{\epsilon,k} \|\phi\| \left(m + |D|^{1/2+\epsilon}\right)^{1/2} \left(\frac{|D|}{m}\right)^{\frac{k-2}{2}}, \tag{4.23}$$

where $\|\cdot\|$ denotes the Petersson norm.

Proof. From Lemma 1.6.3

$$c(n,r) = \frac{1}{\lambda_{k,m}^{n,r}} \langle \phi, P_{(n,r)}^{k,m,\delta} \rangle,$$

where $\lambda_{k,m}^{n,r} = \dfrac{m^{k-3}\Gamma(k-2)}{(4\pi)^{k-2}(mn-|r|^2)^{k-2}}$. By the Cauchy-Schwartz inequality, we get

$$|c(n,r)|^2 \leq \left(\lambda_{k,m}^{n,r}\right)^{-2} \|\phi\|^2 \|P_{(n,r)}^{k,m,\delta}\|^2.$$

Since $\|P_{(n,r)}^{k,m,\delta}\|^2$ gives (up to the factor $\lambda_{k,m}^{n,r}$) the (n,r)-th Fourier coefficient of $P_{(n,r)}^{k,m,\delta}$, we have

$$|c(n,r)|^2 \leq \frac{1}{\lambda_{k,m}^{n,r}} \|\phi\|^2 b_{n,r}(P_{(n,r)}^{k,m,\delta}), \tag{4.24}$$

where $b_{n,r}(P_{(n,r)}^{k,m,\delta})$ is the (n,r)-th Fourier coefficient of the Poincaré series $P_{(n,r)}^{k,m,\delta}$. By Theorem 4.3.4, we have

$$b_{n,r}(P_{(n,r)}^{k,m,\delta}) = \sum_{\xi \in \mathcal{O}^\times/\{\pm 1\}} \sigma(\xi)\xi^{-k}\delta_m(n,r,n,\pm r) + \frac{\pi i^{-k}}{m} \sum_{c\geq 1} H_{m,c,\xi}(n,r,n,\pm r) J_{k-2}\left(\frac{\pi}{mc}|D|\right),$$

and using the estimate given by Lemma 4.4.1, we obtain

$$b_{n,r}(P_{(n,r)}^{k,m,\delta}) \ll 1 + m^{-1/2} \sum_{c\geq 1} c^{-1/2+\epsilon}(D,c) \left| J_{k-2}\left(\frac{\pi}{mc}|D|\right)\right|. \tag{4.25}$$

Using the well-known estimate for the Bessel functions given by,

$$J_{k-2}(x) \ll \min\{x^{-1/2}, x^{k-2}\} \qquad (x>0)$$

[1, p. 4 and 74], we find in a standard manner that

$$\sum_{n\geq 1} n^{-1/2+\epsilon} \left|J_{k-2}\left(\tfrac{A}{n}\right)\right| \ll A^{1/2+\epsilon} \qquad (A>0,\ \text{small } \epsilon>0).$$

Therefore, write t for (D,c) and α for c/t, we find that for any $\epsilon > 0$

$$\begin{aligned}\sum_{c\geq 1} c^{-1/2+\epsilon}(D,c)\left|J_{k-2}\left(\frac{\pi}{mc}|D|\right)\right| &\ll \sum_{t|D} t \sum_{\alpha\geq 1} (\alpha t)^{-1/2+\epsilon}\left|J_{k-2}\left(\frac{\pi}{m\alpha}|D/t|\right)\right| \\ &\ll \sum_{t|D} t^{1/2+\epsilon}\left(\frac{|D/t|}{m}\right)^{1/2+\epsilon} \\ &\ll m^{-1/2}|D|^{1/2+2\epsilon}.\end{aligned}$$

Using this in (4.25) and substituting in (4.24), we get the required estimate. □

Theorem 4.4.3 (Bound for Fourier coefficients of Hermitian Jacobi forms) *Let $\phi \in J^{\delta}_{k,m}$ with Fourier coefficients $c(n,r)$, then*

$$c(n,r) \ll |D|^{k-2}$$

and, if ϕ is a cusp form, then

$$c(n,r) \ll |D|^{\frac{k}{2}-\frac{3}{4}+\epsilon},$$

for any $\epsilon > 0$, where the implied constant depends on ϵ and ϕ.

Proof. If ϕ is the Eisenstein series, it follows from the work of Haverkamp [19, eq. 46]. If ϕ is a cusp form, the theorem follows from Proposition 4.4.2. □

4.5 Estimates for Fourier coefficients of Hermitian cusp forms

Let F be a Hermitian cusp form of integer weight $k > 3$ with character $\det^l$ on $\Gamma^{(2)}$, with Fourier-Jacobi coefficients ϕ_m $(m \geq 1)$. The aim of this section is to give a bound for the Fourier coefficients $a(T)$ given by (1.28). For this we need the following estimate of Fourier-Jacobi coefficients of Hermitian cusp forms (uniformly in m), which is very similar to the case of Siegel cusp forms. As mentioned in the introduction, we follow the method of Kohnen as in [28].

Proposition 4.5.1 *Let $F \in S_k(\Gamma^{(2)}, \det^l)$ and ϕ_m be the m-th Fourier-Jacobi coefficient of F, as given in (1.36). Then we have*

$$\|\phi_m\| \ll_{\epsilon,F} m^{k/2-3/19+\epsilon},$$

for any $\epsilon > 0$.

To prove this proposition, we make use of the special case of the modified version of Landau's Hauptsatz as given in [28, p. 238].

Lemma 4.5.2 *(Perron's formula) Suppose $\xi(s) = \sum_{n\geq 1} c(n)n^{-s}$ is a Dirichlet series with non-negative coefficients which converges for $\mathrm{Re}(s) > \sigma_0$, has a meromorphic continuation to $\mathbb{C}$ with finitely many poles and satisfies a functional equation*

$$\xi^*(s) = \pm\xi^*(\delta - s),$$

where

$$\xi^*(s) := A^{-s}\prod_{j=1}^{J}\Gamma(a_j s + b_j)\xi(s) \qquad (A > 0, a_j > 0, b_j \in \mathbb{R}).$$

Suppose that

$$\kappa := (2\sigma_0 - \delta)\sum_{j=1}^{J} a_j - \frac{1}{2} > 0.$$

Then

$$\sum_{n\leq x} c(n) = \sum_{\text{all poles}} \mathrm{Res}\left(\frac{\xi(s)}{s}x^s\right) + O_\eta(x^\eta)$$

for any $\eta > \eta_0 := (\delta + \sigma_0(\kappa - 1))/(\kappa + 1)$.

Proof of Proposition 4.5.1. The Dirichlet series

$$D_F(s) := D_{F,F}(s) = \zeta_{\mathbb{Q}(i)}(s-k+3)\zeta(2s-2k+4)\sum_{m\geq 1}\|\phi_m\|^2 m^{-s}, \tag{4.26}$$

converges for $\mathrm{Re}(s) > k$, which has been proved in [38]. In the notation of Lemma 4.5.2, in this case, we have $\delta = 2k-3, \sigma_0 = k, J = 4, a_1 = a_2 = a_3 = 1$. Hence $\kappa = \frac{17}{2}, \eta_0 = k - \frac{6}{19}$.

Fix $\epsilon > 0$. Writing $D_F(s) = \sum_{n\geq 1} c(n)n^{-s}$ we deduce from Lemma 4.5.2 that

$$\sum_{n\leq x} c(n) = C(x) + O_\epsilon(x^{k-6/19+\epsilon}),$$

where $C(x)$ is the sum of the residues of $D_F(s)x^s/s$. Taking $x = m$ and $x = m-1$ and subtracting we get $c(m) \ll_{\epsilon,F} m^{k-6/19+\epsilon}$. Inverting the product of zeta functions on the right hand side of (4.26), it follows that

$$\|\phi_m\|^2 = \sum_{d_1 d_2^2 d_3 | m} \mu(d_1)d_1^{k-3}\mu(d_2)d_2^{2k-4}\mu(d_3)\chi_{-1}(d_3)d_3^{k-3}c\left(\frac{m}{d_1 d_2^2 d_3}\right) \ll_{\epsilon,F} m^{k-6/19+\epsilon},$$

which completes the proof of Proposition 4.5.1. □

Theorem 4.5.3 *For any* $\epsilon > 0$, *we have*

$$a(T) \ll_{\epsilon,F} (\min T)^{16/19+\epsilon}(\det T)^{k/2-3/4+\epsilon}. \tag{4.27}$$

Proof. From [22, Theorem 4.1.5], we know that a positive-definite Hermitian matrix T can be written as

$$T = T'[U] := U^* T' U,$$

for some unitary matrix $U \in M_2(\mathbb{C})$ and a real diagonal positive-definite matrix T'. Since both sides of expression (4.27) are invariant if T is replaced by $T[U]$ and hence we may assume that

$$T = \begin{pmatrix} n & t \\ \bar{t} & m \end{pmatrix}, \text{ with } m = \min T.$$

We put $D = 4(|t|^2 - mn)$. Note that by definition $a(T)$ is the (n, r)-th Fourier coefficient of the m-th Fourier-Jacobi coefficient of F. Therefore, combining Proposition 4.4.2 and Proposition 4.5.1, we find that

$$\begin{aligned} a(T) &\ll m^{k/2-3/19+\epsilon}(m + |D|^{1/2+\epsilon})^{1/2} \left(\frac{|D|}{m}\right)^{\frac{k-2}{2}} \\ &\ll m^{16/19+\epsilon}|D|^{k/2-3/4+\epsilon}, \end{aligned}$$

which completes the proof. □

Corollary 4.5.4 *For any $\epsilon > 0$, we have*

$$a(T) \ll_{\epsilon,F} (\det T)^{k/2-25/76+\epsilon}. \tag{4.28}$$

Proof. By Theorem 4.5.3, we have

$$a(T) \ll_{\epsilon,F} (\min T)^{16/19+\epsilon}(\det T)^{k/2-3/4+\epsilon},$$

for any $\epsilon > 0$. By reduction theory it is known that $\min T \ll (\det T)^{1/2}$. Substituting this in the above, we get the required estimate for $a(T)$. □

Bibliography

[1] J. Bateman, Higher Transcendental Functions II. New York-Toronto-London, *McGraw-Hill*, 1953.

[2] S. Böcherer and W. Kohnen, *Estimates for Fourier coefficients of Siegel cusp forms*, Math. Ann. **297** (3) (1993), 499–517.

[3] J. Beyerl, K. James, C. Trentacoste and H. Xue, *Products of nearly holomorphic eigenforms*, Ramanujan J. **27** (2012), 377–386.

[4] H. Braun, *Hermitian modular functions III*, Ann. of Math. **53** (2) (1951), 143–160.

[5] J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier, The 1-2-3 of Modular Forms, *Springer*, Berlin, 2008.

[6] F. Chiera and K. Vankov, *On special values of spinor L-functions of Siegel cusp eigenforms of genus 3*, arXiv:0805.2114v1, 2008.

[7] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*, Math. Ann. **217** (1975), no. 3, 271–285.

[8] S. Das, *Note on Hermitian Jacobi forms*, Tsukuba J. Math. **34** (2010), no. 1, 59–78.

[9] S. Das, *Some aspects of Hermitian Jacobi forms,* Arch. Math. (Basel) **95** (5) (2010), 423–437.

[10] S. Das and J. Meher, *On quasimodular eigenforms*, Int. J. Number Theory **11** (2015), 835–842.

[11] P. Deligne, *La conjecture de Weil. I* (French), Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.

[12] F. Diamond and J. Shurman, A first course in modular forms, Graduate Texts in Mathematics, 228, *Springer-Verlag*, New York, 2005.

[13] W. Duke, *When is the product of two Hecke eigenforms an eigenform?*, In Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, (1999), 737–741.

[14] M. Eichler and D. Zagier, The theory of Jacobi forms, Progress in Mathematics, 55 *Birkhäuser Boston*, Inc., Boston, MA, 1985.

[15] B. A. Emmons, *Products of Hecke eigenforms*, J. Number Theory, **115** (2005), no. 2, 381–393.

[16] B. A. Emmons and D. Lanphier, *Products of an arbitrary number of Hecke eigenforms*, Acta Arith. **130** (2007), 311–319.

[17] E. Ghate, *On monomial relations between Eisenstein series*, J. Ramanujan Math. Soc. **15** (2000), 71–79.

[18] E. Ghate, *On products of eigenforms*, Acta Arith. **102** (2002), no. 1, 27–44.

[19] K. Haverkamp, *Hermitesche Jacobiformen*, Schriftenreihe des Mathematischen Instituts der Universität Münster, 3. Serie **15**, 1995.

[20] J. Hoffstein and T. A. Hulse, *Multiple Dirichlet series and shifted convolution*, J. Number Theory **161** (2016), 417–533.

[21] M. L. Johnson, *Hecke eigenforms as products of eigenforms*, J. Number Theory **133** (2013), no. 7, 2339–2362.

[22] C. Johnson and H. Roger, Matrix analysis Second edition, *Cambridge University Press*, Cambridge (2013).

[23] T. Ikeda, *On the lifting of Hermitian modular forms*, Compositio Math. **144** (5) (2008), 1107–1154.

[24] H. Iwaniec, Topics in classical automorphic forms, Graduate Studies in Mathematics, 17, *American Mathematical Society*, Providence, RI, 1997.

[25] M. Kaneko and D. Zagier, *A generalized Jacobi theta function and quasimodular forms,* The moduli space of curves (Texel Island, 1994), Progr. Math., 129, BirkhÃďuser Boston, Boston, MA (1995) 165–172.

[26] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, 2nd edn., Graduate Texts in Mathematics **97**, *Springer-Verlag*, New York, 1993.

[27] W. Kohnen, *Cusp forms and special value of certain Dirichlet series*, Math. Z. **207** (1991), 657–660.

[28] W. Kohnen, *Estimates for Fourier coefficients of Siegel cusp forms of degree two*, Compositio Math. **87** (2) (1993), 231–240.

[29] W. Kohnen, *Estimates for Fourier coefficients of Siegel cusp forms of degree two II*, Nagoya Math. J. **128** (1992), 171–176.

[30] W. Kohnen and N.-P. Skoruppa, *A certain Dirichlet series attached to Siegel modular forms of degree two*, Invent. Math. 95 (1989), no. 3, 541–558.

[31] A. Kumar, *The adjoint map of the Serre derivative and special values of shifted Dirichlet series*, J. Number Theory **177** (2017), 516–527.

[32] A. Kumar and J. Meher, *On arbitrary products of eigenforms*, Acta Arith. **173** (2016), no. 3, 283–295.

[33] A. Kumar and B. Ramakrishnan, *Estimates for Fourier coefficients of Hermitian cusp forms of degree two*, Acta Arith., To appear.

[34] D. Lanphier and R. Takloo-Bighash, *On Rankin-Cohen brackets of eigenforms*, J. Ramanujan Math. Soc. **19** (4) (2004), 253–259.

[35] J. D. Martin and J. Senadheera, *Differential operators for Hermitian Jacobi forms and Hermitian modular forms*, Ramanujan J. **42** (2017), no. 2, 443–451.

[36] J. Meher, *Some remarks on Rankin-Cohen brackets of eigenforms*, Int. J. Number Theory **8** (2012), 2059–2068.

[37] M. R. Murty, Problems in Analytic Number Theory, 2nd ed., Grad. Texts in Math., vol. 206, Springer, New York, 2008.

[38] S. Raghavan and J. Sengupta, *A Dirichlet series for Hermitian modular forms of degree 2*, Acta Arith. **58** (2) (1991), 181–201.

[39] O. Richter and J. Senadheera, *Hermitian Jacobi forms and $U(p)$ congruences*, Proc. Amer. Math. Soc. **143** (10) (2015), 4199–4210.

[40] R. Sasaki, *Hermitian Jacobi forms of index one*, Tsukuba J. Math. **31** (2007), no. 2, 301–325.

[41] A. Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Sympos. Pure Math., VIII, Amer. Math. Soc., Providence, RI, (1965) 1–15.

[42] J. Senadheera, *Hermitian Jacobi forms and congruences*, Ph. D. thesis, University of North Texas (2014).

[43] G. Shimura, Modular forms: Basics and Beyond, Springer Monographs in Mathematics, *Springer*, New York, 2012.

[44] G. Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), no. 6, 783–804.

[45] T. Yamazaki, *Rankin-Selberg method for Siegel cusp forms*, Nagoya Math. J. **120** (1990), 35–49.

[46] D. Zagier, *Cours at Collége de France*, 2000–2001, in press.

[47] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.