SOME TOPICS IN NUMBER THEORY

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. "An analogue of Artin's primitive root conjecture", P. K. Dey and Balesh Kumar, *Integers*, **2016**, *Vol. 16, No. A67*, 1-7.

2. "On Doi-Naganuma lifting", Balesh Kumar and M. Manickam, *Tsukuba J. Math.*, **2016**, *Vol. 40, No. 2*, 125-137.

Others

1. "Some remarks on the coefficients of symmetric power L-functions", Balesh Kumar, J. Meher and S. Pujahari, (Communicated).

2. "On newforms of half-integral weight and Jacobi forms", Balesh Kumar and M. Manickam, (Communicated).

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BALESH KUMAR

DEDICATIONS

To my parents

SMT. ISARAVATI DEVI & SHREE VIJAY SHANKAR PANDEY.

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Contents

Synopsis

This thesis deals with some problems in number theory, especially in the area of modular forms of both integral and half-integral weights, Jacobi forms and Hilbert modular forms of integral weight, symmetric powers of elliptic modular forms. Also, in this thesis, we consider a distribution of quadratic non-residues which are not primitive roots modulo the primes. We divide the thesis into four chapters.

The *first chapter* entitled **"An analogue of Artin's primitive root conjecture"** deals with a lower bound of the density of the set of primes for which a certain finite set of integers which are quadratic non-residues modulo p but not primitive roots modulo p.

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of nonzero integers which are not perfect squares. In 1968, M. Fried [16] proved that there are infinitely many primes p for which a is a quadratic residue modulo p for every $a \in S$. Further, he provided a necessary and sufficient condition for the a_i's to be quadratic non-residues modulo p. In 2011, R. Balasubramanian, F. Luca and R. Thangadurai [3] calculated the exact density of such primes in Fried's result. More recently, S. Wright ([84, 85]) also considered the above result qualitatively. In [40], we consider a similar problem for the non-residues which are not primitive roots modulo prime p. The main result of this chapter is the following:

Theorem 0.0.1 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite set of nonzero integers such that for any nonempty subset T of S, the product of all the elements in T is not* *a perfect square. Let $q > 2$ be the least prime such that $q \nmid a_1 a_2 \ldots a_n$. Then the density of the set of primes p for which a_i's are quadratic non-residues but not primitive roots modulo p, is at least $\frac{1}{2^n (q-1) q^m}$, where m is a non-negative integer with $m \leq n$.*

In order to prove Theorem 0.0.1, we use the 'Chebotarev density theorem' in an efficient way along with the results of Galois theory. We discuss this in detail in chapter 1 of the thesis.

The *second chapter* entitled with **"Sign change in the coefficients of symmetric power L-functions"** deals with the sign change property in sub-sequence of the sequence of the coefficients of the symmetric power L-functions. Also, this chapter deals with certain analytic property of the symmetric power L-functions.

Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(SL_2(\mathbb{Z}))$ be a normalized eigenform, where $q = e^{2\pi i z}$ with z lying in the complex upper-half plane $\mathbb{H}$. Write $a(n) = \frac{a_n}{n^{(k-1)/2}}$. For $\Re(s) > 1$, the L-function attached to the normalized eigenform f is given by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p (1 - a(p)p^{-s} + p^{-2s})^{-1} = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where the product varies over all the prime numbers. By the above, the Hecke polynomials of f factors into

$$1 - a(p)p^{-s} + p^{-2s} = (1 - \alpha(p)p^{-s})(1 - \beta(p)p^{-s})$$

where $a(p) = \alpha(p) + \beta(p) \quad$ and $\quad \alpha(p)\beta(p) = 1$. The Ramanujan conjecture (proved by Deligne [11]) asserts that $|\alpha(p)| = |\beta(p)| = 1$. It is well known that the L-function $L(s,f)$ is analytically continued to the whole complex plane and satisfies a functional equation. For any positive integer r, the r-th symmetric power L-function attached to f is defined as

$$L(s, sym^r f) = \prod_p \prod_{m=0}^{r} \left(1 - \frac{\alpha(p)^{r-m}\beta(p)^m}{p^s}\right)^{-1}, \qquad \Re(s) > 1.$$

For $\Re(s) > 1$, we write the Dirichlet series expansion of $L(s, sym^r f)$ as

$$L(s, sym^r f) = \sum_{n=1}^{\infty} \frac{a_{sym^r f}(n)}{n^s}.$$

It is also known that the series $L(s, sym^r f)$ can be analytically continued to the region $\Re(s) \geq 1$ and it is non-vanishing in that region. Our first result of Chapter 2 is about finding the abscissa of absolute convergence of $L(s, sym^r f)$, where $f \in S_k(SL_2(\mathbb{Z}))$ is a Hecke eigenform. More precisely, in [43], we prove the following result:

Theorem 0.0.2 *The series*

$$L(s, sym^r f) = \sum_{n=1}^{\infty} \frac{a_{sym^r f}(n)}{n^s}$$

has abscissa of absolute convergence 1*.*

To prove Theorem 0.0.2, we use the idea of W. Kohnen [37] to prove the following result on general Dirichlet series.

Theorem 0.0.3 *Let $a(n)$ be a sequence of complex numbers such that $a(n) = O_\epsilon(n^{\alpha+\epsilon})$ for any $\epsilon > 0$, and the series $\sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s}$ has a singularity at $s = \beta \geq 0$, where α, β are real numbers such that $2\alpha + 1 \leq \beta$. Then the series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ has abscissa of absolute convergence $\alpha + 1$.*

By using Theorem 0.0.3 along with the certain analytic properties of Rankin-Selberg convolution $L(s, sym^r f \times sym^r f)$ and by using the method of J. Meher and M. R. Murty [56] and S. Gun and M. R. Murty [23], we get the required proof of Theorem 0.0.2.

Next, we study the sign change of the sequence $\{a_{sym^2 f}(p)\}_{p\in\mathbb{P}}$, where $\mathbb{P}$ denotes the set of primes in $\mathbb{Z}$. Our next result is given below:

Theorem 0.0.4 *Let* $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(SL_2(\mathbb{Z}))$ *be a normalized eigenform. Then the sequence* $\{a_{sym^2 f}(p)\}_{p\in\mathbb{P}}$ *changes sign infinitely often. Moreover, there exists a small positive constant* θ *such that the number of sign changes for* $p \in [x, 2x]$ *is atleast* ax^{θ}, *for some* $a > 0$.

We use the interesting idea of M. R. Murty [60] to prove Theorem 0.0.4. The proof is achieved at the expense of having an estimate of average sum of the coefficients $a_{sym^2 f}(p)$ and $(a_{sym^2 f}(p))^2$ over primes in short intervals. These estimates will be deduced by similar methods adopted by of C. J. Moreno [59]. The details are given in Chapter 2.

The *third chapter* entitled with **"Doi-Naganuma lifting"** deals with the lifting of elliptic modular forms to Hilbert modular forms over a real quadratic field. We describe the results of chapter 3 here.

Let $D > 0$ be a fundamental discriminant of a real quadratic field $F = \mathbb{Q}(\sqrt{D})$ of class number one. Furthur, we assume that $D \equiv 1 \pmod 4$. Let $\mathcal{O}$ be the ring of integers of F. In [12], K. Doi and H. Naganuma constructed a Hecke equivarient lifting from elliptic cusp forms for $SL_2(\mathbb{Z})$ to Hilbert cusp forms for $SL_2(\mathcal{O})$ by using the 'converse theorem' of Weil. Subsequently, using a similar idea, H. Naganuma [64] constructed one such lift from elliptic cusp forms of weight k, level D with the quadratic character $\chi_D = \left(\frac{D}{\cdot}\right)$ (the Kronecker symbol) to Hilbert cusp forms for $SL_2(\mathcal{O})$. In his remarkable work [86], Don Zagier constructed the kernel function for the Doi-Naganuma lift and there by he obtained the adjoint of the lift. Later, S. Kudla [39] mentioned the possibility of an extension of Zagier's type lift for an arbitrary level and a character. In a joint work with M. Manickam [41], we have extended the Doi-Naganuma lifting on the lines of Zagier's work. We treat the case, where the level is a squarefree integer. Let $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$ denote the space of Hilbert cusp forms of weight k for the congruence subgroup $\tilde{\Gamma}_0(N)$ for the Hilbert modular group $SL_2(\mathcal{O})$ and let $S_k(N,\psi)$ denote the space of cusp forms of weight k for $\Gamma_0(N)$ with character ψ (ψ a character mod N). The main result in the third chapter of the thesis is the following:

Theorem 0.0.5 *Let $m \geq 1$ be an integer. Let G_m be the mth Poincare series for the cusp at ∞ of $\Gamma_0(M)$ (M squarefree integer) with the character χ_D. Then we have a linear map $\iota_D : S_k(M,\chi_D) \to S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$ ($N := M/D$) such that $\iota_D(G_m) = \lambda_k\ \omega_m \in S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$ where $\lambda_k = \frac{(-1)^k(k-1)!}{2(2\pi)^k}$ and ω_m is defined by:*

$$\omega_m(z_1,z_2) = \sum_{\substack{a,\ b\in\mathbb{Z},\ \lambda\in\mathfrak{d}^{-1} \\ N(\lambda)-ab=\frac{m}{D} \\ N|a}} \frac{1}{(az_1z_2+\lambda z_1+\lambda' z_2+b)^k}.$$

In the above, the summation varies over all (a,b,λ) satisfying the given conditions; $\mathfrak{d}^{-1}$ denotes the inverse different in F and $N(\lambda)$ denotes the norm of λ. Moreover, ι_D takes Hecke eigenforms to Hecke eigenforms.

We now briefly describe the idea of the proof of Theorem 0.0.5. For each positive integer m, we construct a Hilbert cusp form $\omega_m(z_1,z_2)$ of weight k for the congruence subgroup $\tilde{\Gamma}_0(N)$ of $SL_2(\mathcal{O})$. We study its main properties and compute its Fourier expansion. By means of an identity relating certain finite exponential sums to Kloosterman sums, we find that the Fourier coefficients of $\omega_m(z_1,z_2)$ are closely related to the coefficients of certain linear combinations of Poincaré series of weight k at various cusps of $\Gamma_0(M)$. Then we show that under the mapping ι_D, the mth Poincaré series for the cusp at ∞ of $\Gamma_0(M)$ of weight k is mapped (up to some constant) to $\omega_m(z_1,z_2)$. Using the fact that any cusp form of weight k for $\Gamma_0(M)$ can be uniquely written as a linear combination of Poincaré series for the cusp at ∞ of $\Gamma_0(M)$ of weight k, the above theorem follows. Moreover, we prove that ι_D commutes with the respective Hecke operators.

The *forth and the final chapter* entitled with **"Newforms of half-integral weight and Jacobi forms"** deals with the theory of newforms of half-integral weight and Jacobi forms for certain congruence subgroups.

Let k and N be positive integers with $k \geq 2$ and N odd. Let $\alpha \geq 2$ be an integer and let χ be a Dirichlet character modulo $2^\alpha N$. Let $S_{k+1/2}(2^\alpha N, \chi_0)$ be the space of cusp forms of weight $k + 1/2$ for $\Gamma_0(2^\alpha N)$ with character χ_0 where χ_0 is the even character defined by $\chi_0 = \left(\frac{4\epsilon}{\cdot}\right)\chi$, $\epsilon = \chi(-1)$. Let $S_{2k}(2^{\alpha-2}N, \chi^2)$ be the space of cusp forms of weight $2k$, level $2^{\alpha-2}N$ with character χ^2 equipped with the Petersson inner product. In his inspiring work [34], [35], W. Kohnen initiated the study of the theory of newforms for the plus space $S^+_{k+1/2}(4N, \chi_0)$ along the lines of the theory obtained by Atkin-Lehner [2], where N is an odd and squarefree integer and $\chi^2 = 1$. Using the trace identities (proved by S. Niwa [66]), M. Manickam, B. Ramakrishnan and T. C. Vasudevan [51] had set up the theory of newforms for the full space $S_{k+1/2}(4N, \chi_0)$, where N is an odd and squarefree integer and $\chi^2 = 1$. In a joint work with M. Manickam [42], we set up the theory of newforms for the space of cusp forms $S_{k+1/2}(4N, \chi_0)$ for an odd and squarefree integer N and χ a primitive Dirichlet character modulo N such that χ^2 is also a primitive Dirichlet character modulo N. The main results of the fourth chapter are as follows:

Theorem 0.0.6 *The multiplicity one result holds good for* $S^{new}_{k+1/2}(4N, \chi_0)$ *and the space* $S^{new}_{k+1/2}(4N, \chi_0)$ *is isomorphic to the space* $S^{new}_{2k}(2N, \chi^2)$ *under certain linear combination of Shimura lifts.*

Theorem 0.0.7 *The multiplicity one result holds good for* $S^+_{k+1/2}(4N, \chi_0)$ *and the space* $S^+_{k+1/2}(4N, \chi_0)$ *is isomorphic to the space* $S_{2k}(N, \chi^2)$ *under certain linear combination of Shimura-Kohnen lifts.*

We now briefly describe the idea of the proof of Theorem 0.0.6. The main ingredient is the equality of the dimension of the corresponding spaces $S_{k+1/2}(4N, \chi_0)$

and $S_{2k}(2N, \chi^2)$ derived by Kojima [38]. The equality of the dimension along with the non-vanishing of Shintani lifts on normalised Hecke eigenforms $F \in S_{2k}(2N, \chi^2)$ gives the required multiplicity one result on $S_{k+1/2}(4N, \chi_0)$. The non-vanishing of Shintani lifts on one such F follows by using the Waldspurger's formula derived by M. Manickam and B. Ramakrishnan [49] along with the non-vanishing of the special values $L(F, \overline{\chi}\left(\frac{D}{\cdot}\right), k)$ derived by M. R. Murty and V. K. Murty [63]. By using these facts, we obtain Theorem 0.0.6.

In order to prove Theorem 0.0.7, we observe that all the arguments as above also hold good for the plus space and using the equality of the dimension of the corresponding spaces $S^+_{k+1/2}(4N, \chi_0)$ and $S_{2k}(N, \chi^2)$ derived by Kojima [38], we get the required proof of Theorem 0.0.7.

Denote by $J^{cusp}_{k+1,1}(N, \chi_0)$, (respectively, $J^{*,cusp}_{k+1,1}(N, \chi_0)$) the space of holomorphic Jacobi cusp forms (respectively, skew-holomorphic Jacobi cusp forms) of weight $k+1$, index 1 for $\Gamma_0(N)$ with character χ_0. Assume that χ is a primitive Dirichlet character modulo N with $\chi^2 \neq 1$. In [42], we also prove the following result:

Theorem 0.0.8 *Let N be an odd and squarefree integer, $k \geq 2$ be an odd integer and $\epsilon = \chi(-1) = 1$. Then $J^{cusp,new}_{k+1,1}(N, \chi_0)$ has multiplicity one result.*

Theorem 0.0.9 *Let N be an odd and squarefree integer, $k \geq 2$ be an even integer and $\epsilon = \chi(-1) = -1$. Then $J^{*,cusp,new}_{k+1,1}(N, \chi_0)$ has multiplicity one result.*

In order to prove Theorem 0.0.8, we first prove that the Eichler-Zagier map $\mathcal{Z}_1 : J^{\text{cusp}}_{k+1,1}(N, \chi_0) \longrightarrow S^+_{k+1/2}(4N, \chi_0)$ is an Hecke equivarient isomorphism and preserving the inner product structure. Using this isomorphism, we derive the multiplicity one result for $J^{\text{cusp,new}}_{k+1,1}(N, \chi_0)$.

The proof of Theorem 0.0.9 follows from a similar argument as done for the holomorphic Jacobi forms.

In the recent work [53], M. Manickam, J. Meher and B. Ramakrishnan studied the theory of newforms for the space of cusp forms of weight $k+1/2$, for $\Gamma_0(2^\alpha N)$, ($\alpha = 3$ or $4, N$ odd and squarefree) with real character and noticed that the space of newforms $S^{new}_{k+1/2}(16N)$ becomes trivial. On the other hand, they also showed that the space of newforms $S^{new}_{k+1/2}(16N, (\frac{8}{\cdot}))$ for $\Gamma_0(16N)$ where $(\frac{8}{\cdot})$ is even quadratic character modulo 8 of conductor 8, is isomorphic to the space $S^{new}_{2k}(8N)$ under a certain linear combinations of Shimura maps. In [42], we set up the theory of newforms for the space of cusp forms of weight $k+1/2$, for $\Gamma_0(32N)$. More precisely, we prove the following result:

Theorem 0.0.10 *We have:* $S^{new}_{k+1/2}(32N) = \{0\}$

and

$$\begin{aligned}
S_{k+1/2}(32N) \quad = \quad & \bigoplus_{rd|N}\{S^{+,new}_{k+1/2}(4d) \oplus S^{+,new}_{k+1/2}(4d) \mid U(4) \oplus S^{+,new}_{k+1/2}(4d) \mid U(4)\mathcal{P}_+ \\
& \oplus S^{+,new}_{k+1/2}(4d) \mid U(8)B(2) \oplus S^{+,new}_{k+1/2}(4d) \mid B(4) \\
& \oplus S^{+,new}_{k+1/2}(4d) \mid U(4)B(4) \oplus S^{+,new}_{k+1/2}(4d) \mid U(8)W(8)B(4) \\
& \oplus S^{+,new}_{k+1/2}(4d) \mid R_{(\frac{8}{\cdot})}\} \mid U(r^2) \\
& \oplus \bigoplus_{rd|N}\{S^{new}_{k+1/2}(4d) \oplus S^{new}_{k+1/2}(4d)|\mathcal{P}_+ \oplus S^{new}_{k+1/2}(4d)|U(2)B(2) \\
& \oplus S^{new}_{k+1/2}(4d) \mid B(4) \oplus S^{new}_{k+1/2}(4d) \mid U(2)W(32) \\
& \oplus S^{new}_{k+1/2}(4d) \mid R_{(\frac{8}{\cdot})}\} \mid U(r^2) \\
& \oplus \bigoplus_{rd|N}\{S^{new}_{k+1/2}(8d) \oplus S^{new}_{k+1/2}(8d)|W(16) \oplus S^{new}_{k+1/2}(8d)|B(4) \\
& \oplus S^{new}_{k+1/2}(8d) \mid R_{(\frac{8}{\cdot})}\} \mid U(r^2) \\
& \oplus \bigoplus_{rd|N}\{S^{new}_{k+1/2}(16d, \left(\frac{8}{\cdot}\right)) \mid B(2) \\
& \oplus S^{new}_{k+1/2}(16d, \left(\frac{8}{\cdot}\right))|\mathcal{P}_+W(32)\}|U(r^2)
\end{aligned}$$

The idea of the proof of Theorem 0.0.10 is as follows. By using the explicit decomposition of eigen classes generated by a normalised newform of weight $2k$, level d (d is a divisior of N) carried out by Atkin-Lehner [2], we decompose the respective eigen classes of weight $k+1/2$, level $32N$ cusp forms and then combining this with an explicit relation among the dimensions of the spaces $S_{k+1/2}(32N)$ and $S_{2k}(8N)$, we prove that the space of newforms $S^{\mathrm{new}}_{k+1/2}(\Gamma_0(32N)) = \{0\}$ (trivial space).

Finally, in [42], we also prove that if $\alpha \geq 6$, the space $S^{new}_{k+1/2}(2^{\alpha}N)$ is non-trivial. We shall discuss these proofs in detail in the thesis.

Chapter 1

An analogue of Artin's primitive root conjecture

1.1 Introduction

C. F. Gauss considered the decimal expansion of the numbers of the form $1/p$ with p prime. In the article [18], Gauss asked why the decimal fraction of $1/7$ has period length 6.

$$\frac{1}{7} = 0.142857\ 142857\ 142857\ldots$$

whereas the period length of $1/11$ is 2. In order to motivate the Artin's primitive root conjecture, let p be a prime ($\neq 2, 5$) and let

$$\frac{1}{p} = 0.\overline{a_1 a_2 \ldots a_l}\ldots$$

be its decimal expansion with period l. Then it is easily seen that

$$\begin{aligned} \frac{1}{p} &= \left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_l}{10^l}\right)\left(1 + \frac{1}{10^l} + \frac{1}{10^{2l}} + \ldots\right) \\ &= \frac{C}{10^l - 1} \end{aligned}$$

for some integer $C \geq 1$ and having l digits. Therefore, we have $10^l - 1 = Cp$ which is equivalent to

$$10^l \equiv 1 (\text{mod } p). \tag{1.1}$$

Thus, the period l satisfies the above congruence and moreover, l is the smallest exponent for which (1.1) is satisfied. Since l is the smallest integer satisfying (1.1), we conclude that 10 has order l (mod p). By Fermat's little theorem, we have $0 < l \leq p-1$. Since by a theorem of Lagrange, $l \mid (p-1)$, the largest period of $1/p$ can occur if and only if 10 has order $p-1 \pmod p$. In such a case, 10 is called *a primitive root* $(\bmod\ p)$. In general, if $p \nmid a$ and the smallest l such that

$$a^l \equiv 1 (\bmod\ p)$$

is $p-1$, then a is called *a primitive root* $(\bmod\ p)$. Moreover, Gauss asked the question of how often 10 is a primitive root $(\bmod\ p)$ as p varies over the primes but made no specific conjecture. In 1927, E. Artin [1] formulated a precise conjecture during a conversation with H. Hasse.

Conjecture 1 *Let a be a non-zero integer other than $1, -1$ or a perfect square.*

1. *Qualitative form: Then there exists infinitely many primes p for which a is a primitive root (mod p).*

2. *Quantitative form: Let h be the largest integer such that $a = a_0^h$, $a_0 \in \mathbb{Z}$ and h is an odd integer. If $N_a(x) := \{p \leq x : p \text{ prime},\ a \text{ is primitive root}(\bmod\ p)\}$, then as $x \to \infty$,*

$$N_a(x) = \prod_{\substack{q \nmid h \\ q \text{ prime}}} \left(1 - \frac{1}{q(q-1)}\right) \prod_{\substack{q \mid h \\ q \text{ prime}}} \left(1 - \frac{1}{q-1}\right) \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \tag{1.2}$$

Write the main term in (1.2) as $A(h)\frac{x}{\log x}$, then $A(h)$ equals the positve rational multiple of

$$A(1) = A = \prod_q \left(1 - \frac{1}{q(q-1)}\right),$$

which is called the Artin's constant.

In the major content of the above introduction, we follow the presentation of M. R. Murty [61]. For Artin's conjecture on primitive roots, we refer to M. R. Murty [61] and for a comprehensive survey, we refer to P. Moree [58].

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of nonzero integers which are not perfect squares. In 1968, M. Fried [16] proved that there are infinitely many primes p for which a is a quadratic residue modulo p for every $a \in S$. Further, he provided a necessary and sufficient condition for the a_i's to be quadratic non-residues modulo p. In 2011, R. Balasubramanian, F. Luca and R. Thangadurai [3] calculated the exact density of such primes in Fried's results. More recently, S. Wright ([84, 85]) also considered the above result qualitatively. In 1976,

K. R. Matthews [55] proved, assuming the truth of the generalized Riemann hypothesis, that given nonzero integers $a_1, a_2, \ldots, a_n$, there exists a real positive constant $C = C(a_1, a_2, \ldots, a_n)$ such that

$$|\{p \le x : \mathrm{ord}_p a_i = p-1, \forall i = 1, 2, \ldots, n\}| = C\frac{x}{\log x} + O\left(\frac{x \log\log x)^{2^n-1}}{(\log x)^2}\right),$$

where $\mathrm{ord}_p(a_i) = \min\{k \in \mathbb{N} : a_i^k \equiv 1 \pmod{p}\}$. Matthews [55] generalized the result of Hooley [25] which confirms Artin's primitive root conjecture, under the assumption of generalized Riemann hypothesis. This conjecture is still unsolved. In this paper, we consider a similar problem for the non-residues which are not primitive roots modulo prime p. It is easy to check that every non-residue modulo prime p is a primitive root modulo p if and only if p is a Fermat prime. Conjecturally, there are only finitely many Fermat primes. Hence for almost all the primes p, the set consists of non-residues modulo p has an element which is not a primitive root modulo p. The distribution of these residues was considered in [22] and [47]. In this chapter, we prove the following theorem:

Theorem 1.1.1 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite set of nonzero integers such that for any nonempty subset T of S, the product of all the elements in T is not a perfect square. Let $q > 2$ be the least prime such that $q \nmid a_1 a_2 \ldots a_n$. Then the density of the set of primes p for which the a_i's are quadratic non-residues but not primitive roots modulo p, is at least $\dfrac{1}{2^n(q-1)q^m}$, where m is a non-negative integer with $m \le n$.*

The contents of this chapter is published in [40].
For the proof of the above theorem, we need the following preliminaries:

1.2 Preliminaries

Lemma 1.2.1 *Let L and M be field extensions of a field K. Then the following conditions are equivalent:*

1. *Each m-tuple $(x_1, \ldots, x_m)$ of elements of L which is linearly independent over K is also linearly independent over M.*

2. *Each n-tuple $(y_1, \ldots, y_n)$ of elements of M which are linearly independent over K is also linearly independent over L.*

Definition 1.2.2 Let L and M be field extensions of a field K. We say that L and M are **linearly disjoint** over K if (1) (or (2)) of the Lemma 1.2.1 holds.

For the definition of linear disjointness, we refer to [17], p. 34.

Lemma 1.2.3 *Let L and M be finite extensions over $\mathbb{Q}$ and let LM be their compositum over $\mathbb{Q}$. Then $[LM:\mathbb{Q}] = [L:\mathbb{Q}][M:\mathbb{Q}]$ if and only if L and M are linearly disjoint over $\mathbb{Q}$.*

For the Lemma 1.2.3, we refer to [17], p. 34.

Lemma 1.2.4 *Let $\{L_i : i \in I\}$ be a linearly disjoint family of Galois extensions over $\mathbb{Q}$ and let $\prod_{i\in I} L_i$ be the compositum of L_i's over $\mathbb{Q}$. Then*

$$Gal\left(\prod_{i\in I} L_i/\mathbb{Q}\right) \cong \prod_{i\in I} Gal(L_i/\mathbb{Q}).$$

For the Lemma 1.2.4, we refer to [17], p. 36.

Lemma 1.2.5 *Let L and M be finite extensions over $\mathbb{Q}$ with $L \cap M = \mathbb{Q}$. If one of them is a normal extension over $\mathbb{Q}$, then L and M are linearly disjoint over $\mathbb{Q}$.*

The Lemma 1.2.5 follows from [9], p. 420.

We need the following results in order to deduce the Theorem 1.1.1.

Theorem 1.2.6 *Let K be a number field. Suppose that there is a $\theta \in K$ such that $\mathcal{O}_K = \mathbb{Z}[\theta]$. Let $F(x)$ be the minimal polynomial of θ over $\mathbb{Z}[x]$. Let p be a rational prime and suppose that*

$$F(x) \equiv F_1(x)^{e_1} \dots F_g(x)^{e_g} \pmod p$$

where each $F_i(x)$ is irreducible in $(\mathbb{Z}/p\mathbb{Z})[x]$ and degree of $F_i(x) = f_i$. Then $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_g^{e_g}$ where $\mathfrak{p}_i = (p, F_i(\theta))$ are prime ideals with $N(\mathfrak{p}_i) = p^{f_i}$. Here $N(\mathfrak{p}_i)$ denotes the norm of the ideal $\mathfrak{p}_i$. Moreover, if $\mathcal{O}_K \neq \mathbb{Z}[\theta]$ but if $p \nmid [\mathcal{O}_K : \mathbb{Z}[\theta]]$ then the same result holds. Also, when K is a Galois extension over $\mathbb{Q}$, in particular, we have $e_1 = \dots = e_g = e$ and $f_1 = \dots = f_g = f$.

The Theorem 1.2.6 gives an important connection between factoring polynomials mod p and factoring ideals in number field. For the Theorem 1.2.6, we refer to [61], p. 65.

Proposition 1.2.7 *Let T be a monic irreducible polynomial of degree n in $\mathbb{Z}[x]$, θ a root of T and $K = \mathbb{Q}(\theta)$. Let $\mathcal{O}_K$ be the ring of integers of K. Denote by*

$d(T)$ (respectively $d(K)$) the discriminant of the polynomial T (respectively of the number field K). We have

1. $d(1, \theta, \ldots, \theta^{n-1}) = d(T)$
2. $d(T) = d(K)[\mathcal{O}_K : \mathbb{Z}[\theta]]^2$

where $d(1, \theta, \ldots, \theta^{n-1})$ denotes the discriminant of $\{1, \theta, \ldots, \theta^{n-1}\}$ in K.

For the Proposition 1.2.7, we refer to [8], p. 166.

Proposition 1.2.8 *Let $K \subset E \subset L$ be number fields and $\mathcal{O}_K \subset \mathcal{O}_E \subset \mathcal{O}_L$ be its ring of integers. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$, $\mathfrak{q}$ a prime ideal of $\mathcal{O}_E$ lying above $\mathfrak{p}$ and $\mathfrak{P}$ a prime ideal of $\mathcal{O}_L$ lying above $\mathfrak{q}$. Then*

$$\begin{aligned} e(\mathfrak{P}|\mathfrak{p}) &= e(\mathfrak{P}|\mathfrak{q})e(\mathfrak{q}|\mathfrak{p}) \\ f(\mathfrak{P}|\mathfrak{p}) &= f(\mathfrak{P}|\mathfrak{q})f(\mathfrak{q}|\mathfrak{p}) \end{aligned}$$

where $e(\mathfrak{P}|\mathfrak{p})$ (respectively $f(\mathfrak{P}|\mathfrak{p})$) is the ramification index (respectively, residual degree)of $\mathfrak{P}$ over $\mathfrak{p}$.

For the Proposition 1.2.8, we refer to [45], p. 24.

Proposition 1.2.9 *Let L be a Galois extension over a field K and M a subextension such that L is the normal closure of M over K. Then a prime ideal $\mathfrak{p}$ of K splits completely in M over K if and only if it splits completely in L over K.*

For the Proposition 1.2.9, we refer to [33], p. 179. As an application of the Proposition 1.2.9, we get the following result:

Corollary 1.2.10 *Let L and M be finite Galois extensions over $\mathbb{Q}$ and let LM be their compositum over $\mathbb{Q}$. Let p be a rational prime. Then p splits completely in both L and M if and only if p splits completely in LM.*

Let K be a finite Galois extension with Galois group $G = G(K|\mathbb{Q})$ and let $\mathcal{O}_K$ be the ring of integers of K. First let us study about some groups associated with the prime ideals of K. If $\mathfrak{p}$ is a prime ideal of K lying over p, then

$$D_{\mathfrak{p}} := \{\sigma \in G : \sigma\mathfrak{p} = \mathfrak{p}\}$$

forms a subgroup of G and is called *the decomposition group* of G. If the index of $D_{\mathfrak{p}}$ is g in G, then it turns out that p splits into g prime ideals in K. Hence the arithmetical information about $D_{\mathfrak{p}}$ gives the information about primes in the ground field, how it splits in the field extensions. If $\sigma \in D_{\mathfrak{p}}$ and

$$\begin{aligned} x &\equiv y \pmod{\mathfrak{p}}, \quad \text{for all } x \in \mathcal{O}_K, \quad \text{then we have} \\ \sigma(x) &\equiv \sigma(y) \pmod{\sigma\mathfrak{p}} \\ \text{which implies that} \quad \sigma(x) &\equiv \sigma(y) \pmod{\mathfrak{p}} \end{aligned}$$

Therefore every $\sigma \in D_{\mathfrak{p}}$ takes congruences class modulo $\mathfrak{p}$ to congruence class modulo $\mathfrak{p}$. This defines an automorphism $\bar{\sigma} \in \text{Aut}\ (\mathcal{O}_K/\mathfrak{p})$. If $\mathfrak{p}$ is a prime ideal of K over p, we have a group homomorphism from

$$D_{\mathfrak{p}} \to \text{Gal}\ (\mathcal{O}_K/\mathfrak{p} \mid \mathbb{Z}/p\mathbb{Z}).$$

This homomorphism turns out to be surjective and the kernel of this surjective map is called the *inertia group* which is denoted and defined by

$$\begin{aligned} I_{\mathfrak{p}} &= Ker\,\{D_{\mathfrak{p}} \to \text{Gal}\ (\mathcal{O}_K/\mathfrak{p} \mid \mathbb{Z}/p\mathbb{Z})\} \\ &= \{\sigma \in D_{\mathfrak{p}} : \bar{\sigma} = 1\} \\ &= \{\sigma \in D_{\mathfrak{p}} : \sigma(x) \equiv x \pmod{\mathfrak{p}} \text{ for all } x \in \mathcal{O}_K\} \end{aligned}$$

Therefore, we get,

$$\text{Gal}\ (\mathcal{O}_K/\mathfrak{p} \mid \mathbb{Z}/p\mathbb{Z}) \cong D_{\mathfrak{p}}/I_{\mathfrak{p}}$$

It is well-known that $\mathcal{O}_K/\mathfrak{p}$ is a finite field extension over $\mathbb{Z}/p\mathbb{Z}$ and therefore the Galois grup Gal $(\mathcal{O}_K/\mathfrak{p} \mid \mathbb{Z}/p\mathbb{Z})$ is a cyclic group. Also it is known that it is generated by $\sigma_{\mathfrak{p}}$ where $\sigma_{\mathfrak{p}}$ is called the *Frobenious automorphism* and is uniquely defined by $\sigma_{\mathfrak{p}}(x) \equiv x^p \pmod{\mathfrak{p}}$, for all $x \in \mathcal{O}_K$.

Under the isomorphism, corresponding to $\sigma_{\mathfrak{p}}$, we have an element in $D_{\mathfrak{p}}/I_{\mathfrak{p}}$ which is denoted by $\left[\frac{K|\mathbb{Q}}{\mathfrak{p}}\right]$ and hence

$$D_{\mathfrak{p}}/I_{\mathfrak{p}} = < \left[\frac{K \mid \mathbb{Q}}{\mathfrak{p}}\right] >$$

If $\sigma \in G$, then $\sigma(\mathcal{O}_K) = \mathcal{O}_K$ and $\sigma\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$ that lies over p. Conversely, if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two primes ideals in K that lies over p, then there exists $\sigma \in G$ such that $\sigma\mathfrak{p}_1 = \mathfrak{p}_2$. Hence, we get,

$$D_{\sigma\mathfrak{p}} = \sigma D_{\mathfrak{p}}\sigma^{-1}, I_{\sigma\mathfrak{p}} = \sigma I_{\mathfrak{p}}\sigma^{-1} \text{ and } \left[\frac{K \mid \mathbb{Q}}{\sigma\mathfrak{p}}\right] = \sigma \left[\frac{K \mid \mathbb{Q}}{\mathfrak{p}}\right] \sigma^{-1}.$$

Note that when the prime p is unramified in K, then it is known that any prime ideal $\mathfrak{p}$ that lies over p, we have $I_{\mathfrak{p}} = \{1\}$ and in this case, we get,

$$D_{\mathfrak{p}} \cong \mathrm{Gal}\ (\mathcal{O}_K/\mathfrak{p} \mid \mathbb{Z}/p\mathbb{Z}).$$

Thus, for all unramified primes p of $\mathbb{Q}$, and prime ideal $\mathfrak{p}$ lies over p, the decomposition group $D_{\mathfrak{p}}$ is cyclic and generated by $\left[\frac{K|\mathbb{Q}}{\mathfrak{p}}\right]$ which can be unique determined by the condition

$$\left[\frac{K \mid \mathbb{Q}}{\mathfrak{p}}\right] x \equiv x^p \pmod{\mathfrak{p}}, \quad \text{for all } x \in \mathcal{O}_K.$$

Since

$$\left[\frac{K \mid \mathbb{Q}}{\sigma\mathfrak{p}}\right] = \sigma \left[\frac{K \mid \mathbb{Q}}{\mathfrak{p}}\right] \sigma^{-1},$$

when $\mathfrak{p}$ runs through all the prime ideals $\mathfrak{p}$ over p, then we see that $\left[\frac{K \mid \mathbb{Q}}{\mathfrak{p}}\right]$ runs through all the elements in the conjugacy class of $\left[\frac{K \mid \mathbb{Q}}{\mathfrak{p}}\right]$. We shall denote this conjugacy class by $\left(\frac{K \mid \mathbb{Q}}{p}\right)$.

Since G is a finite group, the number of conjugacy classes of G is finite. Since all the prime numbers p, except those primes which divide the discriminant of K, are unramified in K, by the Dirichlet Box Principle, one can conclude that one of the conjugacy classes, say, $\mathcal{C}$, of G such that $\mathcal{C} = \left(\frac{K \mid \mathbb{Q}}{p}\right)$ for infinitely many primes p. The following theorem due to Chebotarev asserts much stronger conclusion. In order to state, we first define the notion of *Dirichlet density* for any subset of the set of all prime numbers as follows. For more details, we refer to page 545 of [65].

Definition 1.2.11 Let M be a subset of the set of all prime numbers in $\mathbb{Q}$. The limit

$$d(M) = \lim_{s \to 1+0} \frac{\displaystyle\sum_{p \in M} \frac{1}{p^s}}{\displaystyle\sum_{p} \frac{1}{p^s}},$$

if it exists, is called the **Dirichlet density** of M.

Now we state the Chebotarev density theorem as follows.

Theorem 1.2.12 (Chebotarev density theorem) *Let $K|\mathbb{Q}$ be a finite Galois extension with the Galois group G. Let $\mathcal{C}$ be a given conjugacy class of G and* *let*

$$P = \left\{ p \in \mathbb{Q} : p \text{ is an unramified prime and } \left(\frac{K \mid \mathbb{Q}}{p}\right) = \mathcal{C} \right\}$$

be a subset of the set of all prime numbers. Then

$$d(P) = \frac{|\mathcal{C}|}{|G|}.$$

Lemma 1.2.13 *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite set of nonzero integers. Let α_S be the number of subsets T of S including the empty set such that $|T|$ is even and $\prod_{t \in T} t$ is a perfect square, and let β_S be the number of subsets T of S such that $|T|$ is odd and $\prod_{t\in T} t$ is a perfect square. If $K = \mathbb{Q}(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n})$, then we have $[K : \mathbb{Q}] = 2^{n-k}$, where k is the non-negative integer given by the relation $2^k = \alpha_S + \beta_S$.*

For the Lemma 1.2.13, we refer to [3].

Lemma 1.2.14 *Let $n_1, n_2, \ldots, n_t$ be odd positive integers and let $a_1, a_2, \ldots, a_t$ be nonzero pairwise co-prime integers where a_i is a n_i-powerfree for all $i = 1, 2, \ldots, t$. Then*

$$[\mathbb{Q}(a_1^{1/n_1}, a_2^{1/n_2}, \ldots, a_t^{1/n_t}) : \mathbb{Q}] = n_1 n_2 \ldots n_t.$$

For the Lemma 1.2.14, we refer to [83], p. 114.

Lemma 1.2.15 *Let m be a nonzero squarefree integer. Let*

$$m' = \begin{cases} |m| & \text{if } m \equiv 1 \pmod 4 \\ 4|m| & \text{otherwise.} \end{cases}$$

Then $\mathbb{Q}(\sqrt{m}) \subseteq \mathbb{Q}(\zeta_n)$ if and only if n is a multiple of m'.

The Lemma 1.2.15 can be found in [83], p. 108.

Lemma 1.2.16 *Let $M = \mathbb{Q}(\sqrt{a})$ be a quadratic extension over $\mathbb{Q}$. Then p does not split in M if and only if $\left(\frac{a}{p}\right) \neq 1$, where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.*

For the Lemma 1.2.16, we refer to [14], p. 89.

We compute the degree of the field extension $\mathbb{Q}(\zeta_q, a_1^{1/q}, a_2^{1/q}, \ldots, a_n^{1/q})$ over $\mathbb{Q}$ for any odd prime q. For simplicity, we let $L_{q,n} := \mathbb{Q}(\zeta_q, a_1^{1/q}, a_2^{1/q}, \ldots, a_n^{1/q})$. We know that $L_{q,n}$ is a Galois extension over $\mathbb{Q}$ as it is both normal and separable over $\mathbb{Q}$.

Lemma 1.2.17 *$[L_{q,n} : \mathbb{Q}] = (q-1)q^m$ for some non-negative integer $m \leq n$.*

Proof. Let $\mathbb{P}$ be the set of all prime numbers in $\mathbb{Q}$. For each $i = 1, 2, \ldots, n$, let $\mathbb{P}_i = \{p \in \mathbb{P} : p \mid a_i\}$ and let $\mathcal{P} = \bigcup_{i=1}^{n} \mathbb{P}_i = \{p_1, p_2, \ldots, p_t\}$ be a finite subset of $\mathbb{P}$. Then, clearly,

$$L_{q,n} \subseteq \mathbb{Q}(\zeta_q, p_1^{1/q}, p_2^{1/q}, \ldots, p_t^{1/q}).$$

By letting $L'_{q,t} := \mathbb{Q}(p_1^{1/q}, p_2^{1/q}, \ldots, p_t^{1/q})$ and by Lemma 1.2.14, we see that, $[L'_{q,t} : \mathbb{Q}] = q^t$. Since $[\mathbb{Q}(\zeta_q) : \mathbb{Q}] = (q-1)$, we see that $L'_{q,t} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Since $\mathbb{Q}(\zeta_q)/\mathbb{Q}$ is a Galois extension, by Lemma 1.2.5, we conclude that $\mathbb{Q}(\zeta_q)$ and $L'_{q,t}$ are linearly disjoint over $\mathbb{Q}$. Hence by Lemma 1.2.3, we have $[L'_{q,t}\mathbb{Q}(\zeta_q) : \mathbb{Q}] = q^t(q-1)$.

Since $L_{q,n} \subseteq L'_{q,t}\mathbb{Q}(\zeta_q)$, we see that $[L_{q,n} : \mathbb{Q}] \mid q^t(q-1)$. Also, since $\mathbb{Q}(\zeta_q) \subseteq L_{q,n}$, we have $(q-1) \mid [L_{q,n} : \mathbb{Q}]$. As $[L_{q,n} : \mathbb{Q}] \leq q^n(q-1)$, we conclude that $[L_{q,n} : \mathbb{Q}] = (q-1)q^m$, for some non-negative integer m with $m \leq n$. □

Remark 1.2.18 The following result was proved in [4]. *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of nonzero integers. Then for any odd prime q, $[L_{q,n} : \mathbb{Q}] = (q-1)q^n$, provided, for any nonempty subset T of S, the product of all the elements in T is not a q-th power of an integer.* In particular, if a_i's are pairwise coprime squarefree integers, we get the same degree as above.

In order that a to be a primitive root for a prime p with $(a, p) = 1$, it is necessary and sufficient condition that, for each prime q with $p \equiv 1 (\mathrm{mod}\ q)$, we have $a^{(p-1)/q} \not\equiv 1 (\mathrm{mod}\ p)$. The following fact can be found in M. R. Murty [61].

Proposition 1.2.19 *Let a be a nonzero squarefree integer and let p and q be odd primes. Then, $p \equiv 1 \pmod q$ and $a^{(p-1)/q} \equiv 1 \pmod p$ if and only if p splits completely in $\mathbb{Q}(\zeta_q, a^{1/q})$, where ζ_q is a primitive q-th root of unity.*

Proof. Let α be a primitive root mod p.

Claim 0. Let p and q be distinct odd primes. Then the polynomial $X^q - 1$ has q distinct solutions in $\mathbb{Z}/p\mathbb{Z}$ if and only if p splits completely in $\mathbb{Q}(\zeta_q)$.

Note that the ring of integers $\mathcal{O}_{\mathbb{Q}(\zeta_q)}$ is $\mathbb{Z}[\zeta_q]$ and its discriminant is $(-1)^{\frac{q(q-1)}{2}} q^{q-2}$. Since $p \neq q$ is a prime, p is unramified in $\mathbb{Q}(\zeta_q)$.

Assume that $X^q - 1$ has q distinct solutions in $\mathbb{Z}/p\mathbb{Z}$. By Theorem 1.2.6, we conclude that p splits completely in $\mathbb{Q}(\zeta_q)$. Conversely, assume that p splits completely in $\mathbb{Q}(\zeta_q)$. Therefore, the ramification index and residual degree for any prime ideal $\mathfrak{p}$ over p is 1. Therefore, $N(\mathfrak{p}) = p$ for all prime ideal $\mathfrak{p}$ over

p and the decomposition group $D_{\mathfrak{p}}$ is trivial. Hence, the Frobenius element $\left(\dfrac{K|\mathbb{Q}}{\mathfrak{p}}\right) = 1$ and

$$\left(\frac{K|\mathbb{Q}}{\mathfrak{p}}\right)(\zeta_q) = \zeta_q \equiv \zeta_q^{N(\mathfrak{p})} = \zeta_q^p \pmod{\mathfrak{p}}.$$

Therefore, we get $1 - \zeta_q^{p-1} \in \mathfrak{p}$ for all prime ideal $\mathfrak{p}$ lies over p. If $\zeta_q^{p-1} \neq 1$, then $\zeta_q^{p-1} = \zeta_q^i$ for some integer $1 \leq i \leq q-1$. Hence,

$$\prod_{j=1}^{q-1}(1-\zeta_q^j) \in \mathfrak{p} \text{ for all } \mathfrak{p}|p \implies \prod_{j=1}^{q-1}(1-\zeta_q^j) \in p\mathcal{O}_{\mathbb{Q}(\zeta_q)}.$$

Since $q = \prod_j (1-\zeta_q^j)$, we see that $q \in p\mathcal{O}_{\mathbb{Q}(\zeta_q)}$, which is a contradiction to $q \neq p$. Therefore, $\zeta_q^{p-1} = 1$ which implies $p \equiv 1 \pmod q$ and hence the polynomial $X^q - 1$ has q distinct solutions, namely, $1, \alpha^{\frac{p-1}{q}}, \ldots, \alpha^{\frac{(q-1)(p-1)}{q}}$ in $\mathbb{Z}/p\mathbb{Z}$.

Claim 1: $p \equiv 1 \pmod q$ if and only if the polynomial $X^q - 1$ has q distinct solutions in $\mathbb{Z}/p\mathbb{Z}$.

Assume that $p \equiv 1 \pmod q$. Then, $1, \alpha^{\frac{p-1}{q}}, \ldots, \alpha^{\frac{(q-1)(p-1)}{q}}$ are the q distinct solutions of the polynomial $X^q - 1$ in $\mathbb{Z}/p\mathbb{Z}$. Conversely, assume that the polynomial $X^q - 1$ has q distinct solutions in $\mathbb{Z}/p\mathbb{Z}$. By Claim 0, we see that p splits completely in $\mathbb{Q}(\zeta_q)$. Then the proof of the converse of Claim 0 asserts that $p \equiv 1 \pmod q$, as desired.

Claim 2: Let p and q be two odd prime numbers such that $p \equiv 1 \pmod q$. Let a be an integer such that $a^{(p-1)/q} \equiv 1 \pmod p$. Then the polynomial $X^q - a$ has q distinct solutions in $\mathbb{Z}/p\mathbb{Z}$.

We can assume that a is an element of $\mathbb{Z}/p\mathbb{Z}$. Since α is a primitive root modulo p, we write $a = \alpha^l$ for some non-negative integer l. In order to prove Claim 2, we first show that the polynomial $X^q - a$ has a solution $\pmod p$, say, γ. Then it follows, by Claim 1, that $\beta_1\gamma, \ldots, \beta_q\gamma$ are all the distinct solutions of $X^q - a \pmod p$, where $\beta_1, \ldots, \beta_q$ are the distinct solutions solutions of $X^q - 1$ in $\mathbb{Z}/p\mathbb{Z}$.

Since $a^{(p-1)/q} \equiv 1 \pmod p$, we get $\alpha^{l(p-1)/q} \equiv 1 \pmod p$. Since the order of α is $p-1$, we conclude that $(p-1)$ divides $l(p-1)/q$ which means q divides l. Then we can write $l = bq$ for some integer b. Take $\gamma = \alpha^b$. Then we see that $\gamma^q = \alpha^{bq} = \alpha^l = a$ and hence γ is a solution of $X^q - a$ in $\mathbb{Z}/p\mathbb{Z}$. This proves Claim 2.

Now, we complete the proof of the proposition. Assume that $p \equiv 1 \pmod q$ and $a^{(p-1)/q} \equiv 1 \pmod p$. Then by Claim 2, the polynomial $X^q - a$ splits completely in $\mathbb{Z}/p\mathbb{Z}$. Since the discriminant of $X^q - a$ is equal to $(-1)^{\frac{q(q-1)}{2}} q^q a^{q-1}$, the prime divisors of the discriminant divides either a or q. Since the prime $p \equiv 1 (\text{mod } q)$, the prime p does not divide either a or q. Therefore by the Proposition 1.2.7, the prime p does not divide the index $[\mathcal{O}_{\mathbb{Q}(a^{1/q})} : \mathbb{Z}[a^{1/q}]]$. Hence, by the Theorem 1.2.6, we conclude that the prime p splits completely in $\mathbb{Q}(a^{1/q})$.

Also, since $p \equiv 1 \pmod q$. By Claim 1, the polynomial $X^q - 1$ splits completely in $\mathbb{Z}/p\mathbb{Z}$. Since p does not divide the discriminant of $X^q - 1$ which is equal to $(-1)^{\frac{q(q-1)}{2}} q^{q-2}$, by the Theorem 1.2.6, we see that p splits completely in $\mathbb{Q}(\zeta_q)$.

Since the field $\mathbb{Q}(\zeta_q, a^{1/q})$ is a Galois extension over $\mathbb{Q}$ and $\mathbb{Q}(\zeta_q, a^{1/q})$ is the normal closure of the subextension $\mathbb{Q}(a^{1/q})$, by the Proposition 1.2.9, the prime p splits completely in $\mathbb{Q}(\zeta_q, a^{1/q})$.

Conversely, assume that p splits completely in $\mathbb{Q}(\zeta_q, a^{1/q})$. Then, by the Theorem 1.2.8, p splits completely in the field $\mathbb{Q}(\zeta_q)$ and hence by Claim 1, $p \equiv 1 \pmod q$. Moreover, since $\mathbb{Q}(\zeta_q, a^{1/q})$ is a Galois extension over $\mathbb{Q}$ and p splits completely in $\mathbb{Q}(\zeta_q, a^{1/q})$, the decomposition group is the trivial subgroup of Galois group of $\mathbb{Q}(\zeta_q, a^{1/q})|\mathbb{Q}$, for every prime ideal $\mathfrak{P}$ over p. In particular, the Frobenious element $\left(\frac{\mathbb{Q}(\zeta_q, a^{1/q})|\mathbb{Q}}{\mathfrak{P}}\right)$ is trivial and

$$\left(\frac{\mathbb{Q}(\zeta_q, a^{1/q})|\mathbb{Q}}{\mathfrak{P}}\right)(a^{1/q}) = a^{1/q} \equiv (a^{1/q})^p \pmod{\mathfrak{P}},$$

for every prime ideal $\mathfrak{P}$ over p. Since $a^{\frac{p-1}{q}} - 1 \in \mathbb{Z}$, we conclude that $a^{\frac{p-1}{q}} \equiv 1 \pmod p$. This proves the proposition. $\square$

We need the following generalization of this result.

Proposition 1.2.20 *Let $a_1, a_2, \ldots, a_n$ be any distinct nonzero integers and let p and q be odd primes. Then, $p \equiv 1 \pmod q$ and $a_i^{(p-1)/q} \equiv 1 \pmod p$ for all $i = 1, 2, \ldots, n$ if and only if p splits completely in $\mathbb{Q}(\zeta_q, a_1^{1/q}, a_2^{1/q}, \ldots, a_n^{1/q})$, where ζ_q is a primitive q-th root of unity.*

Proof. Suppose $p \equiv 1 \pmod q$ and $a_i^{(p-1)/q} \equiv 1 \pmod p$ holds for all $i = 1, 2, \ldots, n$. Then by Lemma 1.2.19, p splits completely in $\mathbb{Q}(\zeta_q, a_i^{1/q})$ for all $i = 1, 2, \ldots, n$. Hence by Corollary 1.2.10, p splits completely in their compositum $\mathbb{Q}(\zeta_q, a_1^{1/q}, a_2^{1/q}, \ldots, a_n^{1/q})$.

Conversely, let us assume that p splits completely in $\mathbb{Q}(\zeta_q, a_1^{1/q}, a_2^{1/q}, \ldots, a_n^{1/q})$. Since it is the compositum of $\mathbb{Q}(\zeta_q, a_1^{1/q})$, $\mathbb{Q}(\zeta_q, a_2^{1/q})$, ... , $\mathbb{Q}(\zeta_q, a_n^{1/q})$, by Corollary 1.2.10, we see that the prime p splits completely in those subfields of $\mathbb{Q}(\zeta_q, a_1^{1/q}, a_2^{1/q}, \ldots, a_n^{1/q})$. Hence by Lemma 1.2.19, we see that $p \equiv 1 \pmod q$ and $a_i^{(p-1)/q} \equiv 1 \pmod p$ for all $i = 1, 2, \ldots, n$. $\square$

1.3 Proof of the Theorem 1.1.1

Let $\mathbb{P}$ be the set of all prime numbers and let $\mathbb{P}_i = \{p \in \mathbb{P} : p \mid a_i\}$ for all $i = 1, 2, \ldots, n$. Then

$$\mathcal{P} = \bigcup_{i=1}^{n} \mathbb{P}_i = \{p_1, p_2, \ldots, p_t\}$$

is a finite subset of $\mathbb{P}$. Let q be the least odd prime such that $q \notin \mathcal{P}$.

Consider the number fields $L_q = \mathbb{Q}(a_1^{1/q}, a_2^{1/q}, \ldots, a_n^{1/q}, \zeta_q)$ and $M_i = \mathbb{Q}(\sqrt{a_i})$ for all $i = 1, 2, \ldots, n$. Since for any nonempty subset T of S, the product of all the elements in T is not a perfect square, we have $[\mathbb{Q}(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}) : \mathbb{Q}] = 2^n$, by Lemma 1.2.13. Also from Lemma 1.2.3, it is clear that the compositum $M_1 \cdots M_{j-1}$ and M_j are linearly disjoint over $\mathbb{Q}$ for $j = 2, 3, \ldots, n$. Hence $\{M_j\}_{j=1}^n$ is a linearly disjoint family over $\mathbb{Q}$.

Let $M = M_1 M_2 \cdots M_n$ be the compositum of M_j's over $\mathbb{Q}$. Since the M_j's are Galois extensions over $\mathbb{Q}$, we see that M is a Galois extension over $\mathbb{Q}$. Since $\{M_j\}_{j=1}^n$ is a linearly disjoint family of Galois extensions over $\mathbb{Q}$, by Lemma 1.2.4, we have

$$\mathrm{Gal}(M/\mathbb{Q}) \cong \mathrm{Gal}(M_1/\mathbb{Q}) \times \mathrm{Gal}(M_2/\mathbb{Q}) \times \cdots \times \mathrm{Gal}(M_n/\mathbb{Q}).$$

Now consider the compositum of L_q and M and let $L = L_q M$.

We claim that $L_q \cap M = \mathbb{Q}$. To see this, assume for a contradiction that $L_q \cap M \neq \mathbb{Q}$. Since any subfield of M containing $\mathbb{Q}$ contains a quadratic extension, we see that $\mathbb{Q}(\sqrt{d}) \subseteq L_q \cap M$, where $d = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$ with $n_i = 0$ or 1 for all $i = 1, 2, \ldots, t$. By Lemma 1.2.15, $\mathbb{Q}(\sqrt{d}) \nsubseteq \mathbb{Q}(\zeta_q)$. Hence, $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\zeta_q)$ are linearly disjoint over $\mathbb{Q}$. Therefore, $[\mathbb{Q}(\sqrt{d}, \zeta_q) : \mathbb{Q}] = 2(q-1)$. Since $\mathbb{Q}(\sqrt{d}, \zeta_q) \subseteq L_q$ and by Lemma 1.2.17, $[L_q : \mathbb{Q}] = q^m(q-1)$ with $m \leq n$, we arrive at a contradiction as $2(q-1) \nmid q^m(q-1)$. So, $L_q \cap M = \mathbb{Q}$.

Since L_q and M both are Galois extensions over $\mathbb{Q}$, by Lemma 1.2.4,

$$\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{Gal}(L_q/\mathbb{Q}) \times \mathrm{Gal}(M/\mathbb{Q}).$$

Thus,

$$\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{Gal}(L_q/\mathbb{Q}) \times \mathrm{Gal}(M_1/\mathbb{Q}) \times \cdots \times \mathrm{Gal}(M_n/\mathbb{Q}).$$

Consider the set

$$\begin{aligned} R = \{p \in \mathbb{P} \quad : \quad & p \text{ splits completely in } L_q \text{ and} \\ & p \text{ does not split in } M_i \text{ for all } i = 1, 2, \ldots, n\}. \end{aligned}$$

Let p be a prime unramified in L. Then $p \in R$ if and only if the Frobenius element $\sigma_p \in \mathrm{Gal}(L/\mathbb{Q})$ is equal to $(1, -1, -1, \ldots, -1)$. This is because the first projection is trivial if and only if p splits completely in L_q, and the $(i+1)$-th projection is non-trivial if and only if p does not split in M_i and hence it is -1 as its Galois group is of order 2. Also, note that when $\sigma_p = (1, -1, -1, \ldots, -1)$, the conjugacy class of σ_p contains only one element which is nothing but σ_p itself. Therefore, by the Chebotarev Density Theorem 1.2.12, the density of R is $\dfrac{1}{[L:\mathbb{Q}]}$.

By Lemma 1.2.3, 1.2.5 and the above claim, we conclude that $[L : \mathbb{Q}] = [L_q : \mathbb{Q}][M : \mathbb{Q}] = 2^n q^m (q-1)$, where m is a non-negative integer with $m \leq n$. Therefore, the density of R is $\dfrac{1}{2^n(q-1)q^m}$.

By Proposition 1.2.20, p splits completely in L_q if and only if $p \equiv 1 \pmod q$ and

$$a_i^{(p-1)/q} \equiv 1 \pmod p \text{ for all } i = 1, 2, \ldots, n.$$

Also, by Lemma 1.2.16, we have that p does not split in M_i if and only if

$$\left(\frac{a_i}{p}\right) = -1 \text{ for all } i = 1, 2, \ldots, n.$$

Therefore, for any prime p in R, we have that, $a_1, a_2, \ldots, a_n$ are quadratic non-residues but not primitive roots modulo p.

Since the set R is contained in the set of primes for which $a_1, a_2, \ldots, a_n$ are quadratic non-residues but not primitive roots modulo p, the theorem follows. $\square$

Chapter 2

Sign change in the coefficients of symmetric power L-functions

2.1 Introduction

Let S_k be the space of cusp forms of integral weight k for the full modular group $SL_2(\mathbb{Z})$. Suppose that

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k$$

is a normalized eigenform, where $q = e^{2\pi i z}$ with z lies in the complex upper-half plane $\mathbb{H}$. It is well known that for $n \geq 1$, a_n are real algebraic numbers lying in a number field K_f, where the number field depends only on the form f. Let

$$a(n) = \frac{a_n}{n^{\frac{k-1}{2}}}.$$

The Ramanujan-Petersson conjecture, proved by Deligne [11], is the fact that

$$|a(n)| \leq \sigma_0(n),$$

where $\sigma_0(n) = \sum_{d|n} 1$ is the divisor function. For $\Re(s) > 1$, the L-function attached to the normalized eigenform f is given by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p (1 - a(p)p^{-s} + p^{-2s})^{-1} = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where the product varies over all the prime numbers. By the above, the Hecke polynomials of f factors into

$$1 - a(p)p^{-s} + p^{-2s} = (1 - \alpha(p)p^{-s})(1 - \beta(p)p^{-s})$$

where $a(p) = \alpha(p) + \beta(p)$ and $\alpha(p)\beta(p) = 1$. The Ramanujan-Petersson conjecture (proved by Deligne [11]) asserts that $|\alpha(p)| = |\beta(p)| = 1$. It is well known that the L-function $L(s, f)$ is analytically continued to the whole complex plane having a certain functional equation. For any positive integer r, the r-th symmetric power L-function attached to f is defined as

$$L(s, sym^r f) = \prod_p \prod_{m=0}^{r} \left(1 - \frac{\alpha(p)^{r-m}\beta(p)^m}{p^s}\right)^{-1} \tag{2.1}$$

for $\Re(s) > 1$. We write the Dirichlet series expansion as

$$L(s, sym^r f) = \sum_{n=1}^{\infty} \frac{a_{sym^r f}(n)}{n^s}, \qquad \Re(s) > 1. \tag{2.2}$$

By the works of S. Gelbart and H. Jacqet [19] ($r = 2$), H. H. Kim [29], H. H. Kim and F. Shahidi [30, 31] ($r = 3, 4$), it is known that for $r \leq 4$, the symmetric power L-functions $L(s, sym^r f)$ can be analytically continued to the entire complex plane. Also by combaining these results with Rankin-Selberg theory, H. H. Kim and F. Shahidi [30, 31] established a functional equation and the meromorphic continuation of $L(s, sym^r f)$ to $\mathbb{C}$ for $r = 5, \ldots, 9$ and the holomorphy and non-vanishing of $L(s, sym^r f)$ in the half-plane $\Re(s) \geq 1$, for $r = 5, \ldots, 8$.

The study of the analytic properties of symmetric power L-functions are important, as they are related to the Sato-Tate conjecture. In fact, if each $L(s, sym^r f)$ extends analytically to $\Re(s) \geq 1$ and does not vanish there, then by the Tauberian theorem, the Sato-Tate conjecture follows. For the Sato-Tate conjecture and the variety of applications of the analytic properties of symmetric power L-functions, we refer to [6]. It is also known (see [69, section 4], [24, 44]) that for each integer $r \geq 1$, the series $L(s, sym^r f)$ can be analytically continued to the region $\Re(s) \geq 1$ and it is non-vanishing in that region.

The Rankin-Selberg convolution of L-functions attached to $sym^r f$ and $sym^t f$

is defined as

$$L(s, sym^r f \times sym^t f) := \prod_p \prod_{m=0}^{r} \prod_{l=0}^{t} \left(1 - \frac{\alpha(p)^{r-m}\beta(p)^m\alpha(p)^{t-l}\beta(p)^l}{p^s}\right)^{-1} \quad (2.3)$$

for $\Re(s) > 1$. We write the Dirichlet series expansion as

$$L(s, sym^r f \times sym^t f) = \sum_{n=1}^{\infty} \frac{a_{sym^r f \times sym^t f}(n)}{n^s}, \quad \Re(s) > 1. \quad (2.4)$$

Based on the work of Cogdell and Michel [5], Lau and Wu [46] have shown that for $r = 2, 3, 4$, the Rankin-Selberg convolution $L(s, sym^r f \times sym^r f)$ has analytic continuation to the whole complex plane with a simple pole at $s = 1$ and it satisfies a functional equation as predicted by the Langlands program.

The contents of this chapter is published in [43]. Our first result of this chapter is about finding the abscissa of absolute convergence of symmetric power L-series attached to any eigenform $f \in S_k$. More precisely, we prove the following result:

Theorem 2.1.1 *The series*

$$L(s, sym^r f) = \sum_{n=1}^{\infty} \frac{a_{sym^r f}(n)}{n^s}$$

has abscissa of absolute convergence 1*.*

To prove the above result, we prove the following more general result on Dirichlet series.

Theorem 2.1.2 *Let $a(n)_{n\geq 1}$ be a sequence of complex numbers such that $a(n) = O_\epsilon(n^{\nu+\epsilon})$ for any $\epsilon > 0$, and the series $\sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s}$ has a singularity at $s = \nu' \geq 0$, for some real numbers ν and ν' satisfying $2\nu + 1 \leq \nu'$. Then the series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ has abscissa of absolute convergence $\nu + 1$.*

We exploit an idea of Kohnen [37] to prove the above result on general Dirichlet series.

Finally we prove a sign change property of the coefficients of symmetric square L-functions attached to a normalised eigenform. In fact, we obtained a quantitative result on the number of sign changes over primes in short intervals. More precisely, we prove the following theorem:

Theorem 2.1.3 *There exists δ with $0 < \delta < 1$ such that the number sign changes of the sequence $a_{sym^2f}(p)$ for a prime number $p \in [x, 2x]$ is at least ax^δ for some $a > 0$ and for all sufficiently large x. In particular, the sequence $a_{sym^2f}(p)$ changes sign infinitely often.*

In order to prove Theorem 2.1.3, we use an interesting idea of M. R. Murty [60]. The proof is achieved by an application of an estimate of average sum of the coefficients $a_{sym^2f}(p)$ and $(a_{sym^2f}(p))^2$ over primes in short intervals. These estimates are deduced by the methods adopted by C. J. Moreno [59].

2.2 Preliminaries

In this section, we define the L-functions (in a certain sense) denoted by $L(s, F)$, where F is usually attached to some interesting arithmetic object. For the definition and more information about L-functions, we refer to Chapter 5 of [28].

Definition 2.2.1 We say that $L(s, F)$ is an L-function if we have the following conditions:

1. A Dirichlet series with Euler product of degree $d \geq 1$,

$$L(s, F) = \sum_{n=1}^{\infty} \lambda_F(n) n^{-s} = \prod_p (1 - \alpha_1(p)p^{-s})^{-1} \cdots (1 - \alpha_d(p)p^{-s})^{-1} \quad (2.5)$$

   with $\lambda_F(1) = 1$, $\lambda_F(n) \in \mathbb{C}$, $\alpha_i(p) \in \mathbb{C}$. The series and the Euler products must be absolutely convergent for $\Re(s) > 1$. The $\alpha_i(p)$, $1 \leq i \leq d$, are called the *local roots* or *local parameters* of $L(s, F)$ at p, and they satisfy

$$|\alpha_i(p)| < p, \qquad \text{for all } p.$$

2. A gamma factor

$$\gamma(s, F) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s + \kappa_j}{2}\right)$$

   where the numbers $\kappa_j \in \mathbb{C}$ are called the *local parameters* of $L(s, F)$ at infinity. We assume that these numbers are either real or come in conjugate pairs. Moreover, we also assume that $\Re(\kappa_j) > -1$.

3. An integer $q(F) \geq 1$, called the *conductor* of $L(s, F)$ if $\alpha_i(p) \neq 0$ for $p \nmid q(F)$ and $1 \leq i \leq d$.

From these, the completed function defined by

$$\Lambda(s, F) = q(F)^{\frac{s}{2}} \gamma(s, F) L(s, F)$$

is holomorphic in the half plane $\Re(s) > 1$, yet it must admit analytic continuation to a meromorphic function for $s \in \mathbb{C}$ of order 1 with at most poles at $s = 0$ and $s = 1$. Moreover, it must satisfy the functional equation

$$\Lambda(s, F) = \epsilon(F) \Lambda(1 - s, \bar{F})$$

where $\bar{F}$ is an object associated with F (*the dual of* F) for which $\lambda_{\bar{F}}(n) = \bar{\lambda}_F(n)$, $\gamma(s, \bar{F}) = \gamma(s, F)$, $q(\bar{F}) = q(F)$ and $\epsilon(F)$ is a complex number of absolute value 1, called the *root number* of $L(s, F)$.

Definition 2.2.2 $L(s, F)$ is said to be *self-dual* if $\bar{F} = F$.

If an L-function is self dual then the Dirichlet series of the L-function has real coefficients. It turns out that for a self-dual L-function, the root number is real, hence $\epsilon(F) = \pm 1$. It is then called *the sign* of the functional equation.

Definition 2.2.3 The *analytic conductor* of $L(s, F)$ is defined by

$$\mathfrak{q}(s, F) = q(F) \prod_{j=1}^{d} (|s + \kappa_j| + 3) \tag{2.6}$$

where $q(F)$ is the conductor of $L(s, F)$ and $\kappa_j \in \mathbb{C}$ are the local parameters of $L(s, F)$ at infinity.

Ramanujan-Petersson conjecture: The L-function $L(s, F)$ is said to satisfy the *Ramanujan-Petersson conjecture* if for any i, we have $|\alpha_i(p)| = 1$ for all $p \nmid q(F)$ and $|\alpha_i(p)| \leq 1$ otherwise.

Now we state the following results which are needed to prove the theorems of this chapter. For the following proposition, we refer to Theorem 5.8 of [28].

Proposition 2.2.4 *Let $L(s, F)$ be an L-function of degree d in the sense of Definition 2.2.1. Let $N(T, F)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s, F)$ such that $0 \leq \beta \leq 1$ and $|\gamma| \leq T$. We have*

$$N(T, F) = \frac{T}{\pi} \log \frac{qT^d}{(2\pi e)^d} + O(\log \mathfrak{q}(iT, F))$$

for $T \geq 1$ with an absolute implied constant. Here q is the conductor of $L(s,F)$ and $\mathfrak{q}(s,F)$ is the analytic conductor of $L(s,F)$.

We denote by

$$-\frac{L'}{L}(s,F) = \sum_{n\geq 1} \Lambda_F(n) n^{-s}$$

the expansion of the logarithmic derivative of an L-function in Dirichlet series supported on prime powers. In terms of the local roots $\alpha_i(p)$ of the Euler product (2.5), we have

$$\Lambda_F(p^k) = \sum_{j=1}^{d} \alpha_j(p)^k \log p \tag{2.7}$$

and

$$\Lambda_F(p) = \sum_{j=1}^{d} \alpha_j(p) \log p = \lambda_F(p) \log p.$$

Let us denote the partial sum by

$$\psi(F,x) = \sum_{n\leq x} \Lambda_F(n)$$

which is essentially the sum of $\lambda_F(p)\log p$ over primes. More precisely, we have

$$\psi(F,x) = \sum_{n\leq x} \Lambda_F(n) = \sum_{p\leq x} \lambda_F(p) \log p + \sum_{\substack{n\leq x \\ n=p^m, m\geq 2}} \Lambda_F(n). \tag{2.8}$$

If $L(s,F)$ satisfies the Ramanujan-Petersson conjecture, then by (2.7), we have $|\Lambda_F(p^k)| \leq d\log p$. Therefore, the above expression turns out to be

$$\psi(F,x) = \sum_{n\leq x} \Lambda_F(n) = \sum_{p\leq x} \lambda_F(p) \log p + O(\sqrt{x}\log x). \tag{2.9}$$

Now we state a proposition which is mentioned as Exercise 7 on p. 112 of [28].

Proposition 2.2.5 *Assume that $L(s,F)$ satisfies the Ramanujan-Petersson conjecture. Then we have the following approximate expansion*

$$\psi(F,x) = Rx - \sum_{|\gamma|\leq T} \frac{x^\rho - 1}{\rho} + O\left(\frac{x}{T}(\log x)\log(x^d \mathfrak{q}(F))\right)$$

where $\rho = \beta + i\gamma$ runs over the zeros of $L(s, F)$ in the critical strip of height up to T, $1 \leq T \leq x$, R is the residue of $L(s, F)$ at $s = 1$, $\mathfrak{q}(F) := \mathfrak{q}(0, F)$ is the analytic conductor of $L(s, F)$ and the implied constant is absolute.

By the above proposition and (2.9), we have

$$\sum_{p \leq x} \lambda_F(p) \log p = Rx - \sum_{|\gamma| \leq T} \frac{x^\rho - 1}{\rho} + O\left(\frac{x}{T}(\log x) \log(x^d \mathfrak{q}(F))\right) \tag{2.10}$$

where $\rho = \beta + i\gamma$ runs over the zeros of $L(s, F)$ in the critical strip of height up to T, $1 \leq T \leq \sqrt{x}$.

If we take F to be the symmetric power attached to a normalized Hecke eigenform f in S_k, which is usually known as the *symmetric power L-function*. Then this is an L-function in the sense of Definition 2.2.1, under the assumption of certain conjecture. More precisely, we have:

Symmetric power L-functions: The symmetric power L-functions denoted by $L(s, sym^r f)$ and defined by (2.1) has the following analytic properties:

1. $L(s, sym^r f)$ has an Euler product of degree $r+1$ defined by (2.1) and (2.2). The $\alpha(p)^{r-m}$ and $\beta(p)^m$, $0 \leq m \leq r$, are called *the local roots or local parameters* of $L(s, sym^r f)$ at p, and they satisfy $|\alpha(p)^{r-m}| = |\beta(p)^m| = 1$.

2. A gamma factor
$$\gamma(s, sym^r f) = \pi^{-(r+1)s/2} \prod_{0 \leq j \leq r} \Gamma\left(\frac{s + \kappa_j}{2}\right)$$
where the numbers $\kappa_j \in \mathbb{C}$ are called the *local parameters* of $L(s, sym^r f)$ at infinity.

The symmetric power L-function, $L(s, sym^r f)$, is expected to satisfy the following properties which is stated in Chapter 13, p. 252 of [27].

Conjecture 2 *The symmetric power L-functions are entire. In fact, the completed function defined by*

$$\Lambda_{sym^r f}(s) = \gamma(s, sym^r f) L(s, sym^r f)$$

is entire and it satisfies a functional equation of the type

$$\Lambda_{sym^r f}(s) = \epsilon \Lambda_{sym^r f}(1 - s).$$

where ϵ is a complex number of absolute value one.

The above conjecture is still open for $r \geq 5$. The case $r = 2$ was first established by G. Shimura [77] over $\mathbb{Q}$. The case $r = 3$ and 4 are due to H. H. Kim [29], H. H. Kim and F. Shahidi [30, 31]. Hence, for $r \leq 4$, the symmetric power L-function is a true L-function in the sense of Definition 2.2.1. In particular, $L(s, sym^r f)$ is self dual for $r \leq 4$.

Hoheisel Phenomenon: In this section, we shall briefly describe the Hoheisel property about the Dirichlet series. We shall follow the presentations and notations as given in [59].
Let

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad s = \sigma + it,$$

be a Dirichlet series with nonnegative coefficients and $N_\varphi(\sigma, T)$ denotes the number of zeros $\rho = \beta + i\gamma$ of $\varphi(s)$ with $\beta \geq \sigma$ and $|\gamma| \leq T$.

We say that $\varphi(s)$ has *the Hoheisel Property* if the following four properties hold:

1. **Explicit Formula:** Let $\psi(x) = \sum_{p \leq x} a_p \log p$. Then we have

   $$\psi(x) = Rx - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log Tx)^2\right)$$

   with $R = 0$ or 1 and the sum is over the zeros $\rho = \beta + i\gamma$ of $\varphi(s)$ with $|\gamma| \leq T \leq x^{1/2}$ and $\beta \geq 0$.

2. **The zero free region:** $\varphi(s) \neq 0$ in the region $\sigma \geq 1 - a/\log(2 + |t|)$, for some $a > 0$.

3. **Log free zero density estimate:**

   $$N_\varphi(\sigma, T) \ll T^{c(1-\sigma)}, \text{ for some } c > 0$$

   holds uniformly for all σ with $\frac{1}{2} \leq \sigma \leq 1$, when $T \to \infty$.

4. **Zero density estimate:** $N_\varphi(0, T) \ll T \log T$.

One of the main theorems proved in [59] in this direction is as follows

Theorem 2.2.6 *([59]) If the Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has the Hoheisel property with $R = 1$, then there exists a positive constant $\nu < 1$ such that*

$$\sum_{x \leq p \leq x+h} a_p \log p \geq c_0 h,$$

is true for some $c_0 > 0$ and for any $h = x^\theta$ with $\nu < \theta < 1$.

When $R = 0$, we prove the following upper bound.

Theorem 2.2.7 *If the Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has the Hoheisel property with $R = 0$ with no restriction on the coefficients (that is, a_n's may be positive or negative), then for any $\epsilon > 0$, there exists a positive constant $\nu < 1$ such that*

$$\sum_{x \leq p \leq x+h} a_p \log p \leq \epsilon h$$

is true for any $h = x^\theta$ with $\nu < \theta < 1$. In other words, we have,

$$\sum_{x \leq p \leq x+h} a_p \log p = o(h).$$

The above result is stated as Lemma 4 in [60] for Hecke cusp forms. Here, we present the proof of theorem 2.2.7 following a method in [59].

Proof. By the Hoheisel property (1), we have

$$\frac{\psi(x+h) - \psi(x)}{h} = -\sum_{\substack{|\gamma| \leq T \\ \beta \geq 0}} \frac{(x+h)^\rho - x^\rho}{h\rho} + O\left(\frac{x}{hT}(\log Tx)^2\right)$$

Now, note that

$$\frac{(x+h)^\rho - x^\rho}{h\rho} = \frac{1}{h}\int_x^{x+h} y^{\rho-1} dy \ll x^{\beta-1},$$

where $\beta = \Re(\rho)$. Hence, we get,

$$\frac{\psi(x+h) - \psi(x)}{h} = O\left(\sum_{\substack{|\gamma| \leq T \\ \beta \geq 0}} x^{\beta-1}\right) + O\left(\frac{x}{hT}(\log Tx)^2\right). \tag{2.11}$$

First note that
$$N_\varphi(\sigma,T)=\sum_{\substack{\rho=\beta+i\gamma\\ \beta\ge\sigma\\ |\gamma|\le T}}1=\sum_{\substack{\rho=\beta+i\gamma\\ \beta\ge 0\\ |\gamma|\le T}}\delta_\beta(\sigma), \tag{2.12}$$
where
$$\delta_\beta(\sigma)=\begin{cases}1 \text{ if } \sigma\le\beta\\ 0 \text{ if } \sigma>\beta.\end{cases}$$
Now, first evaluate the sum in (2.11), using the Stieltjes integration as follows

$$\begin{aligned}\sum_{\substack{|\gamma|\le T\\ \beta\ge 0}}(x^{\beta-1}-x^{-1}) &= \sum_{\substack{|\gamma|\le T\\ \beta\ge 0}}\int_0^\beta x^{\sigma-1}(\log x)d\sigma\\ &= \sum_{j=1}^{\lambda(T)}\sum_{\substack{\beta\\ \rho=\beta+i\gamma_j}}\int_0^\beta x^{\sigma-1}(\log x)d\sigma\\ &= \sum_{j=1}^{\lambda(T)}\sum_{\substack{\beta\\ \rho=\beta+i\gamma_j}}\int_0^1 \delta_\beta(\sigma)x^{\sigma-1}(\log x)d\sigma. \end{aligned}\tag{2.13}$$

Now, by interchanging the order of summation and integration in (2.13) and using (2.12), we obtain

$$\sum_{\substack{|\gamma|\le T\\ \beta\ge 0}}(x^{\beta-1}-x^{-1})=\int_0^1 N_\varphi(\sigma,T)x^{\sigma-1}(\log x)d\sigma.$$

Using the Hoheisel properties (3) and (4) and observing that $N_\varphi(\sigma,T)=0$, for all $\sigma\ge 1-A/\log T:=1-\eta(T)$, we get,

$$\begin{aligned}\sum_{\substack{|\gamma|\le T\\ \beta\ge 0}}x^{\beta-1} &= \sum_{\substack{|\gamma|\le T\\ \beta\ge 0}}\frac{1}{x}+\int_0^{1-\eta(T)}N_\varphi(\sigma,T)x^{\sigma-1}(\log x)d\sigma\\ &= \frac{N_\varphi(0,T)}{x}+\int_0^{1-\eta(T)}N_\varphi(\sigma,T)x^{\sigma-1}(\log x)d\sigma\\ &= O\left(\frac{T\log T}{x}\right)+O\left(\int_0^{1-\eta(T)}(T^c/x)^{1-\sigma}(\log x)d\sigma\right)\end{aligned}\tag{2.14}$$

Put $T=x^\alpha$ for some small positive number α to be chosen suitably later. Then the integral in (2.14) becomes

$$\int_0^{1-\eta(x^\alpha)}x^{(\alpha c-1)(1-\sigma)}\log x\ d\sigma \quad = \quad \frac{x^{-(1-\alpha c)\eta(x^\alpha)}-x^{-(1-\alpha c)}}{1-\alpha c}. \tag{2.15}$$

Note that, whenever $\alpha c \neq 1$, we get,

$$\lim_{\alpha\to 0} \frac{x^{-(1-\alpha c)\eta(x^\alpha)} - x^{-(1-\alpha c)}}{1-\alpha c} = -\frac{1}{x},$$

as $\eta(1) = \infty$. Now, we choose α sufficiently small such that $\alpha c < 1$ and for sufficiently large x, by (2.14) and by (2.15), we get,

$$\left| \sum_{|\gamma|\leq T} x^{\beta-1} \right| \leq \epsilon. \tag{2.16}$$

Therefore, from equation (2.16), we get

$$\frac{\psi(x+h)-\psi(x)}{h} \leq \epsilon + \frac{K(\log x)^2}{h} x^{1-\alpha}, \tag{2.17}$$

for some absolute constant K. Since $\log x = o(x^\delta)$, for all $\delta > 0$, take $\nu = 1-\alpha+\delta$, for some suitable $\delta > 0$. Then for $\theta > \nu$ and for sufficiently large x, we get,

$$\psi(x+x^\theta) - \psi(x) \leq \epsilon x^\theta.$$

This completes the proof of the theorem. □

Now, we present the following lemma which is needed to prove Theorem 2.1.3.

Lemma 2.2.8 *The symmetric-square L-function $L(s, sym^2 f)$ satisfies the Hoheisel property with $R = 0$.*

Proof.
1. The required zero free region of $L(s, sym^2 f)$ follows by the classical method in analytic number theory. This has been also stated in p. 438 of [60].

2. The log free zero density estimate follows from Corollary 1.2 of [67].

3. The zero density estimate follows from Proposition 2.2.4.

4. Now, we prove that $L(s, sym^2 f)$ satisfies the Hoheisel property (1) which is the explicit formula. Since $L(s, sym^2 f)$ satisfies Ramanujan-Petersson conjecture, by (2.10) with $R = 0$, we have

$$\psi(sym^2 f, x) := \sum_{p\leq x} a_{sym^2 f}(p)\log(p) = -\sum_{|\gamma|\leq T} \frac{x^\rho - 1}{\rho} + O\left(\frac{x}{T}(\log x)(\log x^3 q_1)\right) \tag{2.18}$$

where $\rho = \beta + i\gamma$ runs over the zeros of $L(s, sym^2 f)$ in the critical strip of height up to T, with $1 \leq T \leq \sqrt{x}$, q_1 is the analytic conductor of $L(s, sym^2 f)$ and the implied constant is absolute. Now we estimate the sum $\sum_{|\gamma| \leq T} \frac{1}{\rho}$. Since

$$\left| \sum_{|\gamma| \leq T} \frac{1}{\rho} \right| \leq \sum_{|\gamma| \leq T} \frac{1}{|\rho|} \leq \sum_{|\gamma| \leq T} \frac{1}{|\gamma|},$$

it is enough to estimate $\sum_{0 < \gamma \leq T} \frac{1}{\gamma}$. If $N(t)$ denotes the number of zeros of $L(s, sym^2 f)$ in the critical strip with ordinates less than t, as it was done in p. 111 of [10] for the Riemann zeta function, here we have

$$\sum_{0 < \gamma < T} \frac{1}{\gamma} = \int_0^T t^{-1} dN(t) = \frac{1}{T} N(T) + \int_0^T t^{-2} N(t) dt.$$

By the zero density estimate for $L(s, sym^2 f)$, we have $N(t) \ll t \log t$, for large t. Thus from the above identity, we deduce that

$$\sum_{0 < \gamma \leq T} \frac{1}{\gamma} \ll (\log T)^2,$$

and therefore

$$\sum_{|\gamma| \leq T} \frac{1}{\rho} \ll (\log T)^2.$$

Substituting the above estimate in (2.18), we get the required explicit formula:

$$\psi(sym^2 f, x) := \sum_{p \leq x} a_{sym^2 f}(p) \log p = - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left((\frac{x}{T} (\log x)(\log x^3 q_1) \right). \tag{2.19}$$

This proves the lemma. □

We also prove the following lemma in order to prove Theorem 2.1.3.

Lemma 2.2.9 *The Rankin-Selberg L-function $L(s, sym^2 f \times sym^2 f)$ satisfies the Hoheisel property with $R = 1$.*

Proof. 1. Since f is self dual, therefore, by Theorem B of [72], $L(s, sym^2 f \times sym^2 f)$ does not have any exceptional zero. Thus by Lemma 2.1 of [67], we get the required zero free region for $L(s, sym^2 f \times sym^2 f)$.

2. The required log free zero density estimate follows from Theorem 1.1 of [67], since $sym^2 f$ is self dual and satisfies Ramanujan-Petersson conjecture.

3. The zero density estimate follows from Proposition 2.2.4.

4. Now using the similar method as we have done in the case of $L(s, sym^2 f)$ in (2.19), we get the following explicit formula for $L(s, sym^2 f \times sym^2 f)$:

$$\begin{aligned}\psi(sym^2 f \times sym^2 f, x) &:= \sum_{p \leq x} a_{sym^2 f \times sym^2 f}(p) \log p \\ &= x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left((\frac{x}{T} (\log x)(\log x^9 q_2) \right)\end{aligned}$$

where $\rho = \beta + i\gamma$ runs over the zeros of $L(s, sym^2 f \times sym^2 f)$ in the critical strip of height up to T , with $1 \leq T \leq x$, q_2 is the analytic conductor of $L(s, sym^2 f \times sym^2 f)$ and the implied constant is absolute.

This proves the lemma. □

The analytic properties of Dirichlet series has interesting consequences in number theory. Now, we mention some analytic properties of the Dirichlet series which is needed in order to prove the theorems of this chapter. For more information about the following analytic properties, we refer to Chapter 1 of [57].

Abscissa of convergence: Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be any Dirichlet series. Then $\alpha(s)$ has an *abscissa of convergence* σ_c with the property that $\alpha(s)$ converges for all $s = \sigma + it$ with $\sigma > \sigma_c$ and for no s with $\sigma < \sigma_c$.

In extreme cases a Dirichlet series may converge throughout the plane ($\sigma_c = -\infty$), or nowhere ($\sigma_c = \infty$). When the abscissa of convergence is finite, the series may converge everywhere on the line $\sigma_c + it$, it may converge at some but not all points on this line, or nowhere on the line.

Several equivalent definition of the abscissa of convergence can be found in the literature. Here we present a version of the abscissa of convergence which has been stated as a fact in [37]. For the sake of completeness, we prove it in the following proposition.

Proposition 2.2.10 *Let*

$$L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

be a Dirichlet series with $b(n) \in \mathbb{C}$. *If the series* $\sum_{n=1}^{\infty} b(n)$ *is divergent, then, the* *abscissa of convergence, say, σ_c (which is finite) of $L(s)$ is given by*

$$\sigma_c = \inf \left\{ t \in \mathbb{R} \mid \sum_{n \le x} b(n) = O_t(x^t), \forall x \ge 1 \right\}. \tag{2.20}$$

Proof. Since the series $\sum_{n=1}^{\infty} b(n)$ is divergent, the abscissa of convergence, say, σ_c of $L(s)$ is ≥ 0. Let t_0 be a real positive value for which the series $L(t_0)$ converges. Let $s_x = \sum_{n \le x} b(n)$, $c_n = b_n n^{-t_0}$ and $C_n = c_1 + \ldots + c_n$ with $C_0 = 0$ so that C_n is bounded, say $|C_n| \le C$. Then

$$\begin{aligned} s_N &= \sum_{n=1}^{N} c_n n^{t_0} = \sum_{n=1}^{N} (C_n - C_{n-1}) n^{t_0} \\ &= \sum_{n=1}^{N-1} C_n \{ n^{t_0} - (n+1)^{t_0} \} + C_N N^{t_0}. \end{aligned}$$

Hence

$$|s_N| \le C \sum_{n=1}^{N-1} \{ (n+1)^{t_0} - n^{t_0} \} + C N^{t_0}.$$

Note that

$$(n+1)^{t_0} - n^{t_0} = t_0 \int_n^{n+1} \frac{du}{u^{t_0+1}} = O(n^{-t_0-1}). \tag{2.21}$$

Therefore the above becomes

$$s_N = O(N^{t_0}).$$

Thus we have seen that if the series $L(t_0)$ converges, then $s_N = O(N^{t_0})$.

Conversely, if $s_N = O_t(N^t)$ then we show that the series $L(s)$ converges for $\sigma > t$ where $s = \sigma + i\gamma$. We consider the partial sums

$$\begin{aligned} \sum_{n=M+1}^{N} \frac{b(n)}{n^s} &= \sum_{n=M+1}^{N} \frac{s_n - s_{n-1}}{n^s} \\ &= \sum_{n=M+1}^{N} s_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) + \frac{s_N}{(N+1)^s} - \frac{s_M}{(M+1)^s} \\ &= \sum_{n=M+1}^{N} O(n^{t-\sigma-1}) + O(N^{t-\sigma}) + O(M^{t-\sigma}), \qquad \text{by (2.21)} \\ &= O(M^{t-\sigma}) = o(1) \end{aligned}$$

provided $\sigma > t$. Therefore, by Cauchy's general principle of convergence for series, the Dirichlet series $L(s)$ converges if $\sigma > t$. Hence, the abscissa of convergence is given by

$$\sigma_c = \inf \left\{ t \in \mathbb{R} \mid \sum_{n \leq x} b(n) = O_t(x^t) \right\}.$$

This proves the proposition. □

Landau's theorem: Let σ_c be the abscissa of convergence of the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$, where $a_n \in \mathbb{C}$. The following theorem, usually refered to as **Landau's theorem** describes a situation in which the line of convergence always contains a singularity. The Landau's theorem is particularly important in practice because it is the basis of the proofs of the many oscillation theorems. For more information, we refer to p. 16 of [57].

Theorem 2.2.11 *Let $\phi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series whose abscissa of convergence σ_c is finite. If $a_n \geq 0$ for all n, then the point σ_c is a singularity of the function $\phi(s)$.*

The Wiener-Ikehara Tauberian theorem: The following theorem gives an asymptotic behavior of the summatory function of an arithmetic sequence. We refer to p. 7 of [63] and p. 43 of [62] for the following theorem.

Theorem 2.2.12 *Let $\phi(s) = \displaystyle\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Suppose there exists a Dirichlet series $\varphi(s) = \displaystyle\sum_{n=1}^{\infty} b_n n^{-s}$ with positive real coefficients such that*

1. *$|a_n| \leq b_n$, for all n*
2. *the series $\varphi(s)$ converges for $\Re(s) > 1$*
3. *the function $\varphi(s)$ (respectively $\phi(s)$) can be extended to a meromorphic function in the region $\Re(s) \geq 1$ having no poles except (respectively except possibly) for a simple pole at $s = 1$ with residue $R \geq 0$ (respectively r).*

Then

$$A(x) := \sum_{n \leq x} a_n = rx + o(x), \quad \text{as } x \to \infty.$$

In particular, if $\phi(s)$ is holomorphic at $s = 1$, then $r = 0$ and $A(x) = o(x)$ as $x \to \infty$.

First we define certain product of two Dirichlet series. We decompose it into product of the Rankin-Selberg convolution of these two Dirichlet series and a Dirichlet series which converges absolutely for $\Re(s) > \frac{1}{2}$. This is proved using a method in [23]. In [23] (Theorem 6), the result has been stated for automorphic L-functions. However, the proof of Theorem 6 in [23] uses only the Deligne's bound. Since Deligne's bound is known in the cases of symmetric power L-functions attached to holomorphic cusp forms, the proof goes through in this case also. More Precisely, we have the following proposition:

Proposition 2.2.13 *For any positive integers r and t, let*

$$L(s, sym^r f) = \sum_{n=1}^{\infty} \frac{a_{sym^r f}(n)}{n^s} \text{ and } L(s, sym^t f) = \sum_{n=1}^{\infty} \frac{a_{sym^t f}(n)}{n^s}.$$

Then

$$\sum_{n=1}^{\infty} \frac{a_{sym^r f}(n) a_{sym^t f}(n)}{n^s} = L(s, sym^r f \times sym^t f) g(s),$$

for some absolutely convergent Dirichlet series $g(\tau)$ for all τ with $\Re(\tau) > \frac{1}{2}$.

Proof. By the definition of symmetric power L-functions, we have

$$\sum_{n=0}^{\infty} a_{sym^r f}(p^n) T^n = \prod_{j=0}^{r} (1 - \alpha(p)^{r-j} \beta(p)^j T)^{-1} = \sum_{i=0}^{r} \frac{c_i}{1 - \alpha(p)^{r-i} \beta(p)^i T} \quad (2.22)$$

and

$$\sum_{n=0}^{\infty} a_{sym^t f}(p^n) T^n = \prod_{j=0}^{t} (1 - \alpha(p)^{t-j} \beta(p)^j T)^{-1} = \sum_{j=0}^{t} \frac{e_j}{1 - \alpha(p)^{t-j} \beta(p)^j T} \quad (2.23)$$

where c_i and e_j are suitable absolute constants. Note that to get the above equalities, we have used the method of partial fractions. In fact, c_i and e_j are rational functions in $\alpha(p)$ and $\beta(p)$. Then

$$a_{sym^r f}(p^n) = \sum_{i=1}^{r} c_i (\alpha(p)^{r-i} \beta(p)^i)^n \quad (2.24)$$

and

$$a_{sym^t f}(p^n) = \sum_{j=1}^{t} e_j (\alpha(p)^{t-j} \beta(p)^j)^n. \quad (2.25)$$

Claim 0: We have

$$\sum_{n=0}^{\infty} a_{sym^r f}(p^n) a_{sym^t f}(p^n) T^n = \frac{R(T)}{\prod_{i=0}^{r}\prod_{j=0}^{t}(1-\alpha(p)^{r-i}\beta(p)^i\alpha(p)^{t-j}\beta(p)^j T)}$$

for some polynomial $R(T) \in \mathbb{C}[T]$. Moreover, the polynomial $R(T)$ is of degree less than rt and its coefficients are bounded (the bound depending only on r and t).

In order to prove the claim, by (2.24) and (2.25), we consider

$$\begin{aligned}\sum_{n=0}^{\infty} a_{sym^r f}(p^n) a_{sym^t f}(p^n) T^n &= \sum_{n=0}^{\infty}\left[\sum_{i=1}^{r} c_i(\alpha(p)^{r-i}\beta(p)^i)^n\right]\left[\sum_{j=1}^{t} e_j(\alpha(p)^{t-j}\beta(p)^j)^n\right]T^n \\ &= \sum_{i,j} c_i e_j \sum_{n=1}^{\infty}(\alpha(p)^{r-i}\beta(p)^i)^n(\alpha(p)^{t-j}\beta(p)^j)^n T^n \\ &= \sum_{i,j}\frac{c_i e_j}{1-\alpha(p)^{r-i}\beta(p)^i\alpha(p)^{t-j}\beta(p)^j T} \\ &= \frac{R(T)}{\prod_{i=0}^{r}\prod_{j=0}^{t}(1-\alpha(p)^{r-i}\beta(p)^i\alpha(p)^{t-j}\beta(p)^j T)},\end{aligned}$$

for some polynomial $R(T) \in \mathbb{C}[T]$. It is also clear from the above that $R(T)$ is a polynomial of degree less than rt whose coefficients are the rational functions in $\alpha(p)$ and $\beta(p)$. By the Ramanujan-Petersson bound, we have $|\alpha(p)| = 1 = |\beta(p)|$. Hence the coefficients of $R(T)$ are absolutely bounded (the bound depending only on r and t). This proves the claim.

Also, since $a_{sym^r f}(1) = 1 = a_{sym^t f}(1)$, we have $a_{sym^r f}(1)a_{sym^t f}(1) = 1 = R(0)$.

Claim 1: We have $R'(0) = 0$. In particular, the coefficient of T in $R(T)$ vanishes.

Observe that $a_{sym^r f}(1) = 1 = a_{sym^t f}(1)$ implies that

$$\sum_{i=1}^{r} c_i = 1 = \sum_{j=1}^{t} e_j. \tag{2.26}$$

By comparing the coefficients of T in (2.22) and (2.23), we get

$$S_1 := \sum_{i=1}^{r}\alpha(p)^{r-i}\beta(p)^i = \sum_{i=1}^{r} c_i\alpha(p)^{r-i}\beta(p)^i \tag{2.27}$$

and

$$S_2 := \sum_{j=1}^{t} \alpha(p)^{t-j}\beta(p)^j = \sum_{j=1}^{t} e_j\alpha(p)^{t-j}\beta(p)^j \tag{2.28}$$

We define

$$D(T) := \prod_{i,j}(1-\alpha(p)^{r-i}\beta(p)^i\alpha(p)^{t-j}\beta(p)^jT)$$

and

$$F(T) = \sum_{n=0}^{\infty} a_{sym^r f}(p^n)a_{sym^t f}(p^n)T^n.$$

Then by Claim 0, we have $R(T) = F(T)D(T)$ and hence

$$R'(0) = (D(T)F(T))'|_{T=0} = D'(0)F(0) + D(0)F'(0).$$

Therefore, by (2.24), (2.25) and (2.26), we get,

$$D(0) = 1 = \sum_{i,j} c_ie_j = F(0).$$

Also by (2.27) and (2.28), we get

$$D'(0) = -\sum_{i,j}\alpha(p)^{r-i}\beta(p)^i\alpha(p)^{t-j}\beta(p)^j = -S_1S_2$$

and

$$F'(0) = \sum_{i,j} c_i\alpha(p)^{r-i}\beta(p)^ie_j\alpha(p)^{t-j}\beta(p)^j = S_1S_2.$$

Therefore, we get, $R'(0) = -S_1S_2 + S_1S_2 = 0$, as desired and hence the claim.

Thus, $R(T)$ is a polynomial satisfying $R(0) = 1$ and $R'(0) = 0$. This means that $R(p^{-s})$ does not contain the term p^{-s} and hence $\prod_p R(p^{-s})$ converges absolutely for all s with $\Re(s) > 1/2$. In this way, we can identify the local factors of the Rankin-Selberg series attached to $sym^r f \times sym^t f$ and the function $g(s) = \prod_p R(p^{-s})$ converges for all s with $\Re(s) > 1/2$. This proves the theorem.□

Next, we shall study the asymptotic behaviour of the average of the coefficients of Rankin-Selberg L-functions attached to symmetric power L-functions over primes. Similar results has already been considered in Lemma 2.5 of [56] where the coefficients are of the Rankin-Selberg L-functions attached to irreducible cuspidal automorphic representations of GL_m. The automorphicity of symmetric power L-functions is not known in general. However, we use the known analytic properties of symmetric power L-functions to prove our result. More precisely, we have the following lemma:

Lemma 2.2.14 *We have*

$$\sum_{p\leq x} \frac{a_{sym^r f\times sym^r f}(p)}{p} = \log\log x + o(\log\log x)$$

and hence the series

$$\sum_{p \text{ prime}} \frac{a_{sym^r f\times sym^r f}(p)}{p}$$

diverges.

Proof. To prove the assertion, we prove that

$$\sum_{p\leq x} a_{sym^r f\times sym^r f}(p) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right), \tag{2.29}$$

where the sum runs through all primes $p \leq x$. Indeed, to prove (2.29), we first show that

$$\sum_{n\leq x} a_{sym^r f\times sym^r f}(n)\Lambda(n) = x + o(x), \tag{2.30}$$

where $\Lambda(n)$ is the von Mangoldt function and then, by partial summation, we get (2.29). In order to prove (2.30), we apply Tauberian theorem to the following Dirichlet series

$$-\frac{L'(s, sym^r f \times sym^r f)}{L(s, sym^r f \times sym^r f)} = \sum_{n=1}^{\infty} \frac{a_{sym^r f\times sym^r f}(n)\Lambda(n)}{n^s}.$$

For each positive integer l, it is known (see section 4 of [69]) that $L(s, sym^l f)$ can be analytically continued to all s with $\Re(s) \geq 1$ and it is non-vanishing in that region. Since

$$L(s, sym^r f \times sym^r f) = \zeta(s)\prod_{l=1}^{r} L(s, sym^{2l} f),$$

we conclude that $L(s, sym^r f \times sym^r f)$ can be analytically continued to all s with $\Re(s) \geq 1$ with only one singularity at $s = 1$ (which is a simple pole with residue 1) and it is non-vanishing for all s with $\Re(s) \geq 1$. Thus the series

$$-\frac{L'(s, sym^r f \times sym^r f)}{L(s, sym^r f \times sym^r f)} = \sum_{n=1}^{\infty} \frac{a_{sym^r f\times sym^r f}(n)\Lambda(n)}{n^s}.$$

converges absolutely for all s with $\Re(s) > 1$ and it is analytically continued for $\Re(s) \geq 1$ except for a simple pole at $s = 1$ with residue 1. Hence, by Theorem 2.2.12 (Tauberian Theorem), we get (2.30) and hence (2.29).

Now by letting $A(x) = \sum_{p \leq x} a_{sym^r f \times sym^r f}(p)$ and by partial summation formula, we get

$$\sum_{p \leq x} \frac{a_{sym^r f \times sym^r f}(p)}{p} = \frac{A(x)}{x} + \int_2^x \frac{A(t)}{t^2} dt.$$

Hence, by (2.29), we get

$$\sum_{p \leq x} \frac{a_{sym^r f \times sym^r f}(p)}{p} = \frac{1}{\log x} + o\left(\frac{1}{\log x}\right) + \int_2^x \left(\frac{1}{t \log t} + o\left(\frac{1}{t \log t}\right)\right) dt,$$

and on simplification, we get,

$$\sum_{p \leq x} \frac{a_{sym^r f \times sym^r f}(p)}{p} = \log \log x + o(\log \log x).$$

This proves the Lemma. $\square$

The following proposition calculates the singularity of a Dirichlet series whose coefficients are non-negative. This is achieved by an application of Landau's theorem along with the Proposition 2.2.13 and Lemma 2.2.14. More precisely, we prove the following result:

Proposition 2.2.15 *The series*

$$\sum_{n=1}^{\infty} \frac{|a_{sym^r f}(n)|^2}{n^s}$$

has a singularity at $s = 1$.

Proof. By Proposition 2.2.13, for any positive integer r, we have

$$\sum_{n=1}^{\infty} \frac{|a_{sym^r f}(n)|^2}{n^s} = L(s, sym^r f \times sym^r f) h(s), \tag{2.31}$$

for some absolutely convergent Dirichlet series $h(\tau)$ for all τ with $\Re(\tau) > \frac{1}{2}$. Suppose, on the contrary, that the series

$$\sum_{n=1}^{\infty} \frac{|a_{sym^r f}(n)|^2}{n^s}$$

does not have any singularity at $s = 1$. Since this Dirichlet series has non-negative coefficients, by Theorem 2.2.11, (Landau's theorem) there exists a real number σ such that this series

$$\sum_{n=1}^{\infty} \frac{|a_{sym^r f}(n)|^2}{n^s}$$

is convergent for all s with $\Re(s) > \sigma$ and it has a singularity at $s = \sigma$. Since the series

$$L(s, sym^r f \times sym^r f) = \sum_{n=1}^{\infty} \frac{a_{sym^r f \times sym^r f}(n)}{n^s}$$

is absolutely convergent for $\Re(s) > 1$ and the series $h(s)$ is absolutely convergent for $\Re(s) > 1/2$, by (2.31), we conclude that $\sigma < 1$. To get a contradiction, we prove that the series

$$\sum_{n=1}^{\infty} \frac{|a_{sym^r f}(n)|^2}{n}$$

is divergent. For any prime p, we know that

$$|a_{sym^r f}(p)|^2 = a_{sym^r f \times sym^r f}(p).$$

Therefore,

$$\sum_{p \text{ prime}} \frac{a_{sym^r f \times sym^r f}(p)}{p} = \sum_{p \text{ prime}} \frac{|a_{sym^r f}(p)|^2}{p} \leq \sum_{n=1}^{\infty} \frac{|a_{sym^r f}(n)|^2}{n}.$$

Since, by Lemma 2.2.14, the series $\displaystyle\sum_{p \text{ prime}} \frac{a_{sym^r f \times sym^r f}(p)}{p}$ diverges, we get the series on the right hand side of the above inequality diverges. This proves the proposition. $\square$

We denote by

$$\pi(x) = \sum_{p \leq x} 1$$

which counts the number of primes up to x. We now state a theorem due to M. N. Huxely [26] which estimates the number of primes in the interval $(x, x+y]$ uniformly for all y with $x^\theta \leq y \leq x$, for any fixed θ with $7/12 < \theta \leq 1$. More precisely, we have

Theorem 2.2.16 *Let θ be a real number such that $7/12 < \theta \leq 1$. Then*

$$\pi(x+y) - \pi(x) \sim \frac{y}{\log x}$$

holds true for all y with $x^\theta \le y \le x$ and for all large enough x.

2.3 Proof of Theorem 2.1.1

The idea of the proof is to apply the Theorem 2.1.2. To prove that the abscissa of absolute convergence is 1 for the series $L(s, sym^r f) = \sum_{n=1}^{\infty} \frac{a_{sym^r f}(n)}{n^s}$, we first consider

$$L(s, sym^r f \times sym^r f) = \zeta(s) \prod_{l=1}^{r} L(s, sym^{2l} f), \tag{2.32}$$

by the Euler product expansion. For each positive integer l, the function $L(s, sym^l f)$ is analytically continued to all s with $\Re(s) \ge 1$. Therefore, the function $L(s, sym^r f \times sym^r f)$ can be analytically continued to all s with $\Re(s) \ge 1$ with only one singularity (which is a simple pole) at $s = 1$. Since Ramanujan-Petersson conjecture is true for symmetric power L-functions attached to a holomorphic cusp forms, we see that

$$a_{sym^r f}(n) \ll n^\epsilon.$$

for any $\epsilon > 0$. Therefore, by Proposition 2.2.15, we know that the series

$$\sum_{n=1}^{\infty} \frac{|a_{sym^r f}(n)|^2}{n^s}$$

has a singularity at $s = 1$. Therefore, by Theorem 2.1.2 with $\nu = 0$ and $\nu' = 1$, we get, the series $\sum_{n=1}^{\infty} \frac{a_{sym^r f}(n)}{n^s}$ has the abscissa of absolute convergence 1. This proves the theorem. □

2.4 Proof of Theorem 2.1.2

Since, by hypothesis, $a(n) = O_\epsilon\left(n^{\nu+\epsilon}\right)$ for any $\epsilon > 0$, the series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is absolutely convergent for all s with $\Re(s) > \nu + 1$. Thus, the abscissa of absolute convergence, σ_a, of the series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is less than or equal to $\nu + 1$.

Suppose, on the contrary, that $\sigma_a < \nu + 1$. Note that the series $\sum_{n=1}^{\infty} |a(n)|$ is divergent. For, if the series $\sum_{n=1}^{\infty}|a(n)|$ is convergent, then, as

$$\sum_{n=1}^{\infty}|a(n)|^2 \leq \left(\sum_{n=1}^{\infty}|a(n)|\right)^2 < \infty,$$

the series $\sum_{n=1}^{\infty}\frac{|a(n)|^2}{n^s}$ converges for all s with $\Re(s) \geq 0$, which is a contradiction to the hypothesis that this series has a singularity at $s = \nu' > 0$. Hence the series $\sum_{n=1}^{\infty}|a(n)|$ is divergent. Therefore, by (2.20) (by Proposition 2.2.10), there exists a positive real number c such that

$$\sum_{n\leq x}|a(n)| = O(x^{\nu+1-c}). \tag{2.33}$$

Since, by hypothesis, $a(n) = O_\epsilon\left(n^{\nu+\epsilon}\right)$ for any $\epsilon > 0$, we also see that

$$\sum_{n\leq x}|a(n)|^3 = O_\epsilon\left(x^{3\nu+1+3\epsilon}\right).$$

Now, by Cauchy-Schwarz inequality, we get,

$$\sum_{n\leq x}|a(n)|^2 \leq \left(\sum_{n\leq x}|a(n)|\right)^{\frac{1}{2}}\left(\sum_{n\leq x}|a(n)|^3\right)^{\frac{1}{2}} \ll_\epsilon x^{\frac{\nu+1-c}{2}}x^{\frac{3\nu+1+3\epsilon}{2}} \ll_\epsilon x^{2\nu+1-\frac{c}{2}+\frac{3\epsilon}{2}}$$

is true for any $\epsilon > 0$. Therefore, by (2.20), the abscissa of convergence of the series $\sum_{n=1}^{\infty}\frac{|a(n)|^2}{n^s}$ is less than $2\nu+1$, which is a contradiction to the hypothesis that it has a singularity at $s = \nu' \geq 2\nu+1$, the series $\sum_{n=1}^{\infty}\frac{|a(n)|^2}{n^s}$ has a singularity at $s = \nu' > 0$. This proves the theorem. $\square$

2.5 Proof of Theorem 2.1.3

Suppose that the assertion is not true. that is, for any given $1 > \delta > 0$, there are infinitely many x such that the number of sign changes in the sequence $\{a_{sym^2 f}(p)\}_{p\in[x,2x]}$ is atmost ax^δ.

Claim: For any given $0 < \nu < 1$, there exists θ with $\nu < \theta < 1$ such that there is no sign change in the sequence $\{a_{sym^2 f}(p)\}_{p\in[x,x+x^\theta]}$, for infinitely many x.

For, let $0 < \delta < \frac{1-\nu}{2}$. Then there exists infinitely many x such that the number of sign changes in the sequence $\{a_{sym^2 f}(p)\}$ with $p \in [x, 2x]$ is atmost ax^{δ}. Since the interval $[x, 2x]$ can be broken into disjoint subintervals of size $ax^{1-2\delta}$ and the number of such intervals is $O(x^{2\delta})$, there exists $\theta \geq 1 - 2\delta$ (we can take $\theta = 1 - 2\delta + \epsilon$ with $\epsilon < \delta$) such that there is no sign change in the sequence $\{a_{sym^2 f}(p)\}$, $p \in [y, y + y^{\theta}]$, for some y with $x \leq y \leq y + y^{\theta} \leq 2x$. Note that $\delta < \frac{1-\nu}{2}$ implies that $\nu < 1 - 2\delta < 1 - 2\delta + \epsilon < 1$. Therefore we get, $\nu < \theta < 1$.

In order to get a contradiction, we exploit the claim. By Lemma 2.2.8, we see that $L(s, sym^2 f)$ satisfies all the hypothesis of Theorem 2.2.7. Therefore, by Theorem 2.2.7, there exists a positive constant $\nu_1 < 1$ such that for any α with $\nu_1 < \alpha < 1$, we have

$$\sum_{y \leq p < y + y^{\alpha}} a_{sym^2 f}(p) \log p = o(y^{\alpha}), \quad \forall y \gg 0. \tag{2.34}$$

Also, by Theorem 2.2.6, there exists a positive number $\nu_2 < 1$ such that for any $\nu_2 < \beta < 1$, we have

$$\sum_{y \leq p < y + y^{\beta}} |a_{sym^2 f}(p)|^2 \log p \gg y^{\beta} \tag{2.35}$$

The above equation is true, since

$$|a_{sym^2 f}(p)|^2 = a_{sym^2 f \times sym^2 f}(p)$$

and by Lemma 2.2.9, the Dirichlet series $L(s, sym^2 f \times sym^2 f)$ satisfies all the hypothesis of Theorem 2.2.6.

Now, we apply (2.34) and (2.35) to our situation as follows. Let $\nu = max\{\frac{3}{4}, \nu_1, \nu_2\}$. Since $\nu < 1$, by claim, there exist θ with $\nu < \theta < 1$ and infinitely many x for which the sequence $\{a_{sym^2 f}(p)\}_{p \in [x, x + x^{\theta}]}$ is non negative. Therefore, by (2.34), we get

$$\sum_{x \leq p < x + x^{\theta}} a_{sym^2 f}(p) \log p = o(x^{\theta}).$$

Since, by Theorem 2.2.16, there are primes in $[y, y + y^{\theta}]$ for all large enough y, by (2.35), we get

$$\sum_{x \leq p < x + x^{\theta}} |a_{sym^2 f}(p)|^2 \log p \gg x^{\theta},$$

for infinitely many x. Since

$$a_{sym^2 f}(p) = a(p^2) = a(p)^2 - 1$$

and $a(p) \leq 2$ (as the coefficients $a(p)$ are real), we see that

$$a_{sym^2 f}(p) \leq 3.$$

Therefore, we get

$$x^\theta \ll \sum_{x \leq p < x + x^\theta} |a_{sym^2 f}(p)|^2 \log p \ll \sum_{x \leq p < x + x^\theta} a_{sym^2 f}(p) \log p = o(x^\theta).$$

holds true for infinitely many values of x, which is a contradiction. Thus, the assertion is true. $\square$

Chapter 3

Doi-Naganuma lifting

3.1 Introduction

Any integer $D \neq 0$ with $D \equiv 0, 1 \pmod 4$ is called *a discriminant.* If $D = 1$ or D is the discriminant of a quadratic field, then D is said to be *fundamental.* For the definition and terminology of fundamental discriminant, we refer to p. 52 of [28]. Let $D > 0$ be the fundamental discriminant of a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ and $\mathcal{O}$ be the ring of integers of K. Then the well known identity: $\zeta_K(s) = \zeta(s)L(s, \chi)$ captures information about rationals inside K. Assume that K is of class number one and $D \equiv 1 (\text{mod } 4)$.

In 1969, K. Doi and H. Naganuma [12] have asked the analogue of the above to the case of elliptic modular forms. More precisely, they have shown that given a normalised Hecke eigenform f of even weight k for the full modular group $SL_2(\mathbb{Z})$, how to construct a normalised Hilbert eigenform $\hat{f} \in S_k(SL_2(\mathcal{O}))$, defined by its Fourier expansion so that the standard L-function attached to $\hat{f}$ satisfies

$$L(s, \hat{f}) = L(s, f)L(s, f \otimes \chi_D).$$

The existence of $\hat{f}$ is obtained by proving the 'converse theorem' of Weil in the case of Hilbert modular forms, which essentially says that $\hat{f} \in S_k(SL_2(\mathcal{O}))$ if for each grossencharacter ξ of K, the twisted L-function $L(s, \hat{f} \otimes \xi)$ has sufficiently nice analytic properties, namely, an analytic continuation to the whole complex plane, a functional equation and a property of being bounded in vertical strips. Subsequently, using similar ideas, H. Naganuma [64] constructed a lifting from an elliptic cusp forms of weight k, level D with character χ_D $(= \left(\frac{D}{\cdot}\right)$, Kronecker symbol) to the Hilbert cusp forms for $SL_2(\mathcal{O})$.

In [86], Don Zagier derived the adjoint of the Doi-Naganuma lift by computing its explicit action on Poincaré series. More precisely, he considered the space $S_k(D, \chi_D)$ of all cusp forms of weight k, level D, character $\chi_D = \left(\frac{D}{\cdot}\right)$, ($D > 0$ is a fundamental discriminant) and proved that the mth Poincaré series in $S_k(D, \chi_D)$ maps into an explicit Hilbert cusp form ω_m in $S_k^{\mathcal{H}}(SL_2(\mathcal{O}))$ - the space of Hilbert cusp forms of weight k, level 1 associated with the real quadratic field of discriminant D. He proved that Hecke-eigenforms correspond to each other under the Doi-Naganuma lift.

In this chapter, we prove that for each fundamental discriminant D, there exist a Hecke-equivarient map ι_D, which maps the space $S_k(M, \chi_D)$ into the space $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(M/D))$, the space of Hilbert cusp forms of weight k, level M/D, where M is a squarefree positive integer divisible by D. We prove that ι_D takes the mth Poincaré series in $S_k(M, \chi_D)$ into a similar kind of Hilbert cusp form ω_m in $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(M/D))$ and then we prove that it is an Hecke equivarient map.

It is to be noted that in [39], Kudla had mentioned the possibility of an extension lift of Zagier's type for an arbitrary level and character. In our theorem we treat the case where the level is a squarefree integer M and for each positive squarefree divisor $D \equiv 1 (\bmod 4)$ of M, we construct appropriate Hilbert cusp form ω_m and prove our results as done by Zagier [86]. The contents of this chapter is published in [41]. We now state the main theorem of this chapter:

Theorem 3.1.1 *Let M be a squarefree integer. For any integer $m \geq 1$, let G_m^{∞} be the mth Poincaré series for the cusp at ∞ of $\Gamma_0(M)$ with the character χ_D which is characterized in terms of the Petersson inner product by the formula*

$$\langle f, G_m^{\infty} \rangle = \frac{(k-2)!}{(4\pi m)^{k-1}} a_m(f)$$

for all $f \in S_k(M, \chi_D)$ with the Fourier expansion at the cusp ∞ given by

$$f(z) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z}.$$

Then, for each fundamental discriminant D dividing M, we have a linear map

$$\iota_D : S_k(M, \chi_D) \to S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$$

with $N := M/D$ such that

$$\iota_D(G_m^{\infty}) = \lambda_k \; \omega_m \in S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N)),$$

where $\lambda_k = \dfrac{(-1)^k(k-1)!}{2(2\pi)^k}$ *and* ω_m *is the Hilbert modular form defined by:*

$$\omega_m(z_1, z_2) = \sum_{\substack{a,\ b\in\mathbb{Z},\ \lambda\in\mathfrak{d}^{-1}\\ \mathcal{N}(\lambda)-ab=\frac{m}{D}\\ N|a}} \frac{1}{(az_1z_2 + \lambda z_1 + \lambda' z_2 + b)^k}.$$

In the above, the summation varies over all the tuples (a, b, λ) *satisfying the given conditions;* $\mathfrak{d}^{-1}$ *denotes the inverse different in* K *and* $\mathcal{N}(\lambda)$ *denotes the norm of* λ. *Moreover,* ι_D *takes Hecke eigenforms to Hecke eigenforms.*

We now briefly describe the idea of the proof of Theorem 3.1.1. For each positive integer m, we construct a Hilbert cusp form $\omega_m(z_1, z_2)$ of weight k for the congruence subgroup $\tilde{\Gamma}_0(N)$ of $SL_2(\mathcal{O})$. We study its main properties and compute its Fourier expansion. By means of an identity relating certain finite exponential sums to Kloosterman sums, we find that the Fourier coefficients of $\omega_m(z_1, z_2)$ are closely related to the coefficients of certain linear combinations of Poincaré series of weight k at various cusps of $\Gamma_0(M)$. Then we show that under the mapping ι_D, the mth Poincaré series for the cusp at ∞ of $\Gamma_0(M)$ of weight k is mapped (up to some constant) to $\omega_m(z_1, z_2)$. Using the fact that any cusp form of weight k for $\Gamma_0(M)$ can be uniquely written as a linear combination of Poincaré series for the cusp at ∞ of $\Gamma_0(M)$ of weight k, the above theorem follows.

In some part of the above introduction, we follow the presentation of E. Ghate [20]. In order to see the related results and nice exposition, we refer to E. Ghate [20].

3.2 Preliminaries

3.2.1 *Notations*

We use the following notations:

- K a real quadratic number field;
- D the discriminant of K;
- $\mathcal{O}$ the ring of integers of K;
- $\mathcal{O}^*$ the unit group of $\mathcal{O}$;
- $\mathfrak{d}$ the different of K (the principal ideal $(\sqrt{D})$);
- x' the Galois conjugate over $\mathbb{Q}$ of an element $x \in K$;

$\mathcal{N}(x)$ the norm of x, $\mathcal{N}(x) = xx'$;

$\mathbb{H}$ the upper half plane $\{z \in \mathbb{C} \mid \text{Im } z > 0\}$;

$\mathbb{Z}$ the set of all integers;

k a fixed even integer > 2;

$Cl(K)$ the ideal class group of K.

3.2.2 *The Hilbert modular group and Hilbert modular forms*

For our purpose, we restrict ourself to the real quadratic fields. For more general definitions and results, we refer the books [15] and [82]. Let

$$SL_2(K) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in K,\ ad - bc = 1 \right\}.$$

Then, we have an embedding

$$\begin{array}{ccc} SL_2(K) & \hookrightarrow & SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \hookrightarrow & \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right); \end{array}$$

and $SL_2(K)$ acts on $\mathbb{H} \times \mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right).$$

We also have an action of $SL_2(K)$ on $\mathbb{P}^1(K) = K \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{\alpha}{\beta} = \frac{a\frac{\alpha}{\beta} + b}{c\frac{\alpha}{\beta} + d} = \frac{a\alpha + b\beta}{c\alpha + d\beta}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0. \end{cases}$$

Note that, since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \frac{a}{c}$, the action of $SL_2(K)$ is transitive. We write

$$\Gamma_K := SL_2(\mathcal{O}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathcal{O},\ ad - bc = 1 \right\}$$

which is known as *the Hilbert modular group.* We also define the congruence subgroup $\tilde{\Gamma}_0(N)$ of the Hilbert modular group by

$$\tilde{\Gamma}_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) : c \in N\mathcal{O} \right\}$$

Lemma 3.2.1 *The map*

$$\begin{array}{rcl} \Gamma_K \backslash \mathbb{P}^1(K) & \longrightarrow & Cl(K), \\ (\alpha : \beta) & \mapsto & \alpha\mathcal{O} + \beta\mathcal{O} \end{array}$$

is a bijection.

By Lemma 3.2.1, the number of the cusps of Γ_K is equal to $|Cl(K)|$ (which is called *the class number* of K).

Let $\Gamma \subset SL_2(K)$ be a subgroup which is commensurable with Γ_K (means $\Gamma \cap \Gamma_K$ has finite index in both Γ and Γ_K). Let $(k_1, k_2) \in \mathbb{Z}^2$ be the given integer vector.

Definition 3.2.2 A holomorphic function $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$ is called a *holomorphic Hilbert modular form* of weight (k_1, k_2) for Γ if

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, z_2)\right) = (cz_1 + d)^{k_1}(c'z_2 + d')^{k_2} f(z_1, z_2), \tag{3.1}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If $k_1 = k_2 = k$, then f is said to have *parallel weight* k.

If f is a holomorphic Hilbert modular form for Γ then it is automatically holomorphic at the cusps by the Götzky-Koecher principle, which is as follows. In order to state this principle, we set the following notations. Let $M \subset K$ be a $\mathbb{Z}$-module of rank 2 and let $V \subset \mathcal{O}^*$ be a finite index subgroup such that the group

$$G(M, V) = \left\{ \begin{pmatrix} \epsilon & \mu \\ 0 & \epsilon^{-1} \end{pmatrix} : \mu \in M, \epsilon \in V \right\}$$

is contained in

$$\Gamma_\infty = \left\{ \begin{pmatrix} \epsilon & \nu \\ 0 & \epsilon^{-1} \end{pmatrix} : \nu \in \mathcal{O}, \epsilon \in \mathcal{O}^* \right\}$$

with finite index. Let $M^\vee$ be the dual lattice of M with respect to the trace form on K which can be defined by

$$M^\vee = \{\lambda \in K : tr(\lambda\mu) \in \mathbb{Z},\ for\ all\ \mu \in M\}.$$

The transformation law (3.1) for a special choice of $\gamma \in G(M,V)$ implies that

$$f(z_1 + \mu, z_2 + \mu') = f(z_1, z_2)$$

for all $\mu \in M$. Therefore, f has a convergent Fourier expansion

$$f(z_1, z_2) = \sum_{\nu \in M^\vee} a_\nu e^{2\pi i(\nu z_1 + \nu' z_2)}. \tag{3.2}$$

The Fourier coefficients a_ν are given by

$$a_\nu = \frac{1}{\mathrm{vol}(\mathbb{R}^2/M)} \int_{\mathbb{R}^2/M} f(z_1, z_2) e^{-2\pi i(\nu z_1 + \nu' z_2)} dx_1 dx_2 \tag{3.3}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

An element $\alpha \in K$ with $\alpha > 0$ and $\alpha' > 0$ is called *totally positive* and we denote it by $\alpha \gg 0$.

Theorem 3.2.3 (Götzky-Koecher principle) *Let $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$ be a holomorphic function satisfying $f(\gamma(z_1, z_2)) = (cz_1 + d)^{k_1}(c'z_2 + d')^{k_2} f(z_1, z_2)$, for all $\gamma \in G(M,V)$. Then*

(1) $a_{\epsilon^2\nu} = \epsilon^{k_1}\epsilon'^{k_2} a_\nu$, for all $\nu \in M^\vee$ and $\epsilon \in V$.

(2) $a_\nu \neq 0 \implies \nu = 0$ or $\nu \gg 0$.

By Theorem 3.2.3, a holomorphic Hilbert modular form for the group Γ_K has a Fourier expansion at the cusp ∞ of the form

$$f(z_1, z_2) = a_0 + \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} a_\nu e^{2\pi i(\nu z_1 + \nu' z_2)}.$$

The constant term a_0 is called *the value of f* at the cusp ∞. More generally, if $\kappa \in \mathbb{P}^1(K)$ is a cusp of Γ, then there exists $\rho \in SL_2(K)$ such that $\rho\infty = \kappa$. Then the constant term $a_0 = f(\rho\infty)$. If $(k_1, k_2) \neq (0,0)$, the value of $f(\kappa)$ depends on the choice of ρ (by a non-zero factor).

Definition 3.2.4 A holomorphic Hilbert modular form f is called *a cusp form* if $a_0 = 0$ for all the cusps of Γ.

3.2.3 *Poincaré series for* $\Gamma_0(M)$

In this section we recall the basic facts about the Poincaré series and their Fourier expansion. A more detailed account can be found in Chapter 3 of [27] and Section 3 of [86]. Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index containing $-I$ ($I :=$ the identity matrix) and $\chi : \Gamma \to \{\pm 1\}$ a character such that $\chi(-I) = 1$. We denote by $S_k(\Gamma, \chi)$ the space of cusp forms for Γ of weight k and character χ. A function $f \in S_k(\Gamma, \chi)$ is a holomorphic function in $\mathbb{H}$ satisfying

1. $f \mid A = \chi(A) f$ for all $A \in \Gamma$.

2. f is holomorphic and vanishes at the cusps of Γ.

The second condition means the following: A *cusp* P of Γ is an equivalence class of points of $\mathbb{Q} \cup \{\infty\}$ under the action of Γ. For each cusp P there exists a matrix A_P transforming the cusp P to ∞, that is $A_P^{-1}(\infty) \in P$. The *width* w_P of the cusp P is defined by

$$w_P = [\Gamma_\infty : \Gamma_P], \qquad \Gamma_P = A_P \Gamma A_P^{-1} \cap \Gamma_\infty \tag{3.4}$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$. Since $[SL_2\mathbb{Z} : \Gamma]$ is finite, the index w_P is finite and thus $\Gamma_P = \left\{ \begin{pmatrix} 1 & nw_P \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$. It turns out that the width w_P is independent of the choice of A_P.

Let f be a cusp form in $S_k(\Gamma, \chi)$. Then it follows that the function $f \mid A_P^{-1}$ is periodic with period w_P. Therefore it has a Fourier expansion of the form

$$(f \mid A_P^{-1})(z) = \sum_{n=1}^{\infty} a_n^P(f) e^{2\pi i n z / w_P} \tag{3.5}$$

The complex numbers $a_n^P(f)$ are called the *Fourier coefficients of f at P* and they depend on the choice of A_P. One can verify that a different choice of A_P replaces $a_n^P(f)$ by $\zeta^n a_n^P(f)$, where ζ is a w_Pth root of unity. If $f, g \in S_k(\Gamma, \chi)$, then the *Petersson inner product* of f and g is defined as

$$\langle f, g \rangle = \int_{\mathscr{F}} f(z) \overline{g(z)} y^{k-2} dx dy, \tag{3.6}$$

where $z = x + iy \in \mathbb{H}$ and $\mathscr{F}$ is a fundamental domain for the action of Γ on $\mathbb{H}$. It turns out that the integral converges for $k > 2$ and is independent of the choice of $\mathscr{F}$.

For each $n \geq 1$, the *Poincaré series* for the cusp P is defined by

$$G_n^P(z) = \frac{1}{2} \sum_{A \in \Gamma_P \backslash A_P \Gamma} \chi(A_P^{-1}A) j(A,z)^{-k} e^{2\pi i n Az/w_P} \tag{3.7}$$

where the summation is over the orbits of the left action of Γ_P on $A_P\Gamma$ and $j(A,z) := (cz+d)$, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It can be shown that the above series is convergent for $k > 2$ and independent of the choices of the representatives of A. Also it can be shown that G_n^P is a cusp form in $S_k(\Gamma, \chi)$. Moreover, the Poincaré series G_n^P can also be characterized in terms of the Petersson inner product by the formula

$$\langle f, G_n^P \rangle = \frac{(k-2)!}{(4\pi n)^{k-1}} w_P^k a_n^P(f) \tag{3.8}$$

for all $f \in S_k(\Gamma, \chi)$.

Since G_n^P is a cusp form in $S_k(\Gamma, \chi)$, it has a Fourier expansion of the form (3.5) at each cusp Q of Γ. For simplicity, we take $Q = (\infty)$ and that the width w_∞ is 1. Then we can choose $A_Q = I$ and $\Gamma_Q = \Gamma_\infty$ in (3.4). Thus G_n^P has a Fourier expansion at the cusp ∞

$$G_n^P(z) = \sum_{m=1}^{\infty} g_{nm}^P e^{2\pi i m z}. \tag{3.9}$$

In order to describe the Fourier coefficients, g_{nm}^P, one needs the Bessel function of order $k-1$ which is defined as

$$J_{k-1}(t) = \sum_{r=0}^{\infty} \frac{(-1)^r (t/2)^{2r+k-1}}{r!(r+k-1)!}. \tag{3.10}$$

The following proposition gives the Fourier coefficients, g_{nm}^P, explicitly in terms of the twisted Kloosterman sums and the Bessel functions.

Proposition 3.2.5 *The Poincaré series $G_n^P(z)$ has a Fourier expansion of the form* (3.9) *and the mth Fourier coefficient is given by*

$$g_{nm}^P = \delta_{P\infty}\delta_{nm} + 2\pi(-1)^{k/2} \left(\frac{m w_P}{n}\right)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} H_c^P(n,m) J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{mn}{w_P}}\right) \tag{3.11}$$

where

$$H_c^P(n,m) = \frac{1}{c} \sum_{d (mod\ c)^*} \chi\left(A_P^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) e^{2\pi i c^{-1}(naw_P^{-1}+md)}, \tag{3.12}$$

$J_{k-1}(t)$ is the Bessel function which is defined by (3.10), $\delta_{P\infty}$ and δ_{nm} are the Kronecker delta functions and the summation runs over all $d \pmod c$ such that $(d,c)=1$, $\frac{-d}{c} \in P$ with $A_P^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

The above proposition can be found in §3 of [86].

Let M be a squarefree positive integer and D be the fundamental discriminant of a real quadratic field K such that $D \equiv 1 (\text{mod } 4)$ and dividing M. Let

$$\Gamma = \Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod M \right\}$$

and $\chi : \Gamma_0(M) \to \{\pm 1\}$ be such that

$$\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \epsilon(a) = \epsilon(d) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M).$$

where $\epsilon = \epsilon_D$ is the fundamental character of K with $\epsilon(p) = \left(\frac{D}{p}\right)$ for $p \nmid 2D$. The space $S_k(\Gamma_0(M), \chi)$ is usually denoted by $S(M,k,\epsilon)$. For $x/y,\ x'/y' \in \mathbb{Q} \cup \{\infty\}$ with $(x',y') = (x,y) = 1$, the equation

$$\frac{x'}{y'} = \frac{ax+by}{cx+dy}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$$

can be solved if and only if $(y,M) = (y',M)$. The equivalence classes of $\mathbb{Q} \cup \{\infty\}$ modulo $\Gamma_0(M)$ are thus described by the positive divisions of M. Let D_1 be a division of D. Let the cusp P be given by D_1N, $(N = M/D)$ and write $D_2 = D/D_1$; then $(D_1N, D_2) = 1$, as M is squarefree. Then we can find $p, q \in \mathbb{Z}$ such that $pD_1N + qD_2 = 1$; choose

$$A_P = \begin{pmatrix} D_2 & -p \\ D_1N & q \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{3.13}$$

The cusp P is easily checked to have width D_2. We denote the cusp simply by D_1N; thus for $f \in S(M,k,\epsilon)$ and $D_1 \mid D$ we have the Fourier expansion

$$(f \mid A_{D_1N}^{-1})(z) = \sum_{n=1}^{\infty} a_n^{D_1N}(f) e^{2\pi i n z/D_2} \tag{3.14}$$

The coefficients $a_n^{D_1N}(f)$ are independent of the choice of p and q in (3.13) and given by

$$a_n^{D_1N}(f) = \frac{(4\pi n)^{k-1}}{(k-2)!} D_2^{-k} \langle f, G_n^{D_1N} \rangle \tag{3.15}$$

where $\langle f, G_n^{D_1N} \rangle$ denotes the Petersson inner product of f with $G_n^{D_1N}$ (here $G_n^{D_1N}$ is the nth Poincaré series at the cusp D_1N defined by (3.7)). By Proposition 3.2.5, we have

$$G_n^{D_1N}(z) = \sum_{m=1}^{\infty} g_{nm}^{D_1N} e^{2\pi i m z} \tag{3.16}$$

where

$$g_{nm}^{D_1N} = \delta_{D_1N,M}\delta_{n,m} + 2\pi(-1)^{\frac{k}{2}} \left(\frac{mD_2}{n}\right)^{\frac{k-1}{2}} \sum_{\substack{c=1 \\ D_1N|c}}^{\infty} H_c^{D_1N}(n,m)\, J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{mn}{D_2}}\right),$$

$$H_c^{D_1N}(n,m) = c^{-1} \sum_{\substack{d(mod\ c) \\ (d,c)=1}} \chi\left(A_{D_1N}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) e^{2\pi i c^{-1}(na/D_2+md)}. \tag{3.17}$$

3.3 The function $\omega_m(z_1, z_2)$ and its properties

For an integer $m \geq 0$ and for $z_1, z_2 \in \mathbb{H}$, we define

$$\omega_m(z_1,z_2) = \sideset{}{'}\sum_{\substack{a,\ b\in\mathbb{Z},\ \lambda\in\mathfrak{d}^{-1} \\ \mathcal{N}(\lambda)-ab=\frac{m}{D} \\ N|a}} \frac{1}{(az_1z_2+\lambda z_1+\lambda' z_2+b)^k}, \tag{3.18}$$

where the summation runs over all the tuples (a, b, λ) satisfying the given conditions, and the notation $\sum'$ indicates that, whenever $m = 0$, the triple $(0, 0, 0)$ is omitted.

Theorem 3.3.1 *For an integer $m \geq 0$, the function $\omega_m(z_1, z_2)$ is a Hilbert modular form of weight k with respect to the congruence subgroup $\tilde{\Gamma}_0(N)$ of Γ_K. Moreover, it is a cusp form for all integers $m \geq 1$.*

Proof. Let $m \geq 0$ be a given integer. We first prove that ω_m is a Hilbert modular form of weight k for the congruence subgroup $\tilde{\Gamma}_0(N)$ of Γ_K. In order to prove this we shall prove that ω_m is holomorphic on $\mathbb{H} \times \mathbb{H}$ and it satisfies the modularity condition.

Let $z_1, z_2 \in \mathbb{H}$ be given complex numbers. In the definition of $\omega_m(z_1, z_2)$, the expression $az_1z_2 + \lambda z_1 + \lambda' z_2 + b \neq 0$ for the choice of (a, b, λ) satisfying the conditions. If possible, we assume that $az_1z_2 + \lambda z_1 + \lambda' z_2 + b = 0$ for some tuple (a, b, λ). Then, we get, $z_1 = \dfrac{-\lambda' z_2 - b}{az_2 + \lambda}$. Since the determinant of the matrix $\begin{pmatrix} -\lambda' & -b \\ a & \lambda \end{pmatrix}$ is $-\lambda\lambda' + ab = ab - \mathcal{N}(\lambda) \leq 0$, we get, $z_1 \notin \mathbb{H}$, a contradiction. Hence, $az_1z_2 + \lambda z_1 + \lambda' z_2 + b \neq 0$ for any choice of the tuple (a, b, λ).

Now we shall show that the series ω_m converges absolutely on the compact subsets of $\mathbb{H} \times \mathbb{H}$. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers. Then we have

$$\begin{aligned}
|\omega_m(z_1, z_2)| &\leq \sideset{}{'}\sum_{\substack{a,\ b \in \mathbb{Z},\ \lambda \in \mathfrak{d}^{-1} \\ \mathcal{N}(\lambda) - ab = \frac{m}{D}}} \frac{1}{\left|a\left(z_1 + \frac{\lambda'}{a}\right)\left(z_2 + \frac{\lambda}{a}\right) - \frac{m/D}{a}\right|^k} \\
&\leq \sideset{}{'}\sum_{\substack{a,\ b \in \mathbb{Z},\ \lambda \in \mathfrak{d}^{-1} \\ \mathcal{N}(\lambda) - ab = \frac{m}{D}}} \frac{1}{|a|^k \left(\max\ \left(\left|x_1 + \frac{\lambda'}{a}\right|, |y_1|\right) \max\ \left(\left|x_2 + \frac{\lambda}{a}\right|, |y_2|\right) - \frac{m/D}{a^2}\right)^k}.
\end{aligned}$$

For any real number $R > 0$, let $N(R)$ be the number of elements (a, λ) that occur in the last sum such that

$$R \leq |a| \left(\max\ \left(\left|x_1 + \frac{\lambda'}{a}\right|, |y_1|\right) \max\ \left(\left|x_2 + \frac{\lambda}{a}\right|, |y_2|\right) - \frac{m/D}{a^2}\right) < 2R.$$

The inequality

$$\left|a\left(y_1y_2 - \frac{m/D}{a^2}\right)\right| < 2R$$

implies that $a = O(R)$ where the implicit constants depends on z_1 and z_2. Similarly for fixed a, the inequality

$$\left|a\left(x_1 + \frac{\lambda'}{a}\right)\left(x_2 + \frac{\lambda}{a}\right) - \frac{m/D}{a}\right| < 2R$$

implies that $\lambda = O(R)$. Hence $N(R) = O(R^2)$ and therefore if $z_1, z_2 \in C$, a compact subset of $\mathbb{H} \times \mathbb{H}$, then,

$$|\omega_m(z_1, z_2)| \ll \sum_{n=0}^{\infty} \frac{N(2^n)}{2^{nk}} \ll \sum_{n=0}^{\infty} \frac{1}{2^{n(k-2)}},$$

where the implicit constants depends on z_1 and z_2, which is bounded as $z_1, z_2 \in C$. The last sum converges for all $k > 2$. Therefore, the series ω_m converges absolutely in $\mathbb{C}$ and hence it is holomorphic on $\mathbb{H} \times \mathbb{H}$.

Now, we check the modularity condition as follows. For any complex numbers

$z_1, z_2 \in \mathbb{H}$, we prove that

$$\omega_m\left(\frac{\alpha z_1+\beta}{\gamma z_1+\delta}, \frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\right) = (\gamma z_1+\delta)^k(\gamma' z_2+\delta')^k \omega_m(z_1,z_2). \tag{3.19}$$

For any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, we let

$$\phi_M(z_1,z_2) = \frac{1}{\det M}\frac{d}{dz_1}\left(\frac{1}{z_2 - Mz_1}\right) = \frac{1}{(cz_1z_2 - az_1 + dz_2 - b)^2}$$

where $Mz_1 = \dfrac{az_1+b}{cz_1+d}$. Then, for any $A_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \in GL_2(\mathbb{R})$, one can easily verify that

$$\phi_M(A_1z_1, A_2z_2) = (\gamma_1 z_1 + \delta_1)^2(\gamma_2 z_2 + \delta_2)^2 \phi_{A_2^* M A_1}(z_1,z_2), \tag{3.20}$$

where A_2^* is the adjoint of A_2. Let

$$\mathscr{A} = \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{M}_2(\mathcal{O}) \ : \ N \mid \gamma, \quad M^* = M' \right\}$$

be the set of matrices whose adjoint equal their conjugates over $\mathbb{Q}$, where $M' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$. A typical element M of $\mathscr{A}$ is a matrix which is of the form $\begin{pmatrix} \theta & b\sqrt{D} \\ -a\sqrt{D} & \theta' \end{pmatrix}$ with $a, b \in \mathbb{Z}$, N divides a and $\theta \in \mathcal{O}$. Writing $\theta = -\lambda\sqrt{D}$ for some $\lambda \in \mathfrak{d}^{-1}$, then, we get, $M = \begin{pmatrix} -\lambda\sqrt{D} & b\sqrt{D} \\ -a\sqrt{D} & \lambda'\sqrt{D} \end{pmatrix}$ and hence

$$\phi_M(z_1,z_2) = D^{-1}(az_1z_2 + \lambda z_1 + \lambda' z_2 + b)^{-2}.$$

Thus, with this notation, we see that

$$\omega_m(z_1,z_2) = D^{k/2} \sideset{}{'}\sum_{\substack{M\in\mathscr{A} \\ \det M = -m}} \phi_M(z_1,z_2)^{\frac{k}{2}},$$

where $\sum'$ indicates that, whenever $m = 0$, the zero matrix is omitted in the summation. Now for $A \in \tilde{\Gamma}_0(N)$, we have

$$\begin{aligned} \omega_m(Az_1, A'z_2) &= D^{k/2} \sideset{}{'}\sum_{\substack{M \in \mathscr{A} \\ \det M = -m}} \phi_M(Az_1, A'z_2)^{\frac{k}{2}} \\ &= D^{k/2}(\gamma z_1+\delta)^k(\gamma' z_2+\delta')^k \sideset{}{'}\sum_{\substack{M \in \mathscr{A} \\ \det M = -m}} \phi_{A'^*MA}(z_1, z_2)^{\frac{k}{2}}, \end{aligned}$$

where we have used the equation (3.20). Since A'^*MA belongs to $\mathscr{A}$ with the same determinant as that of M, we observe that the set $\mathscr{A}_m := \{M \in \mathscr{A} : \det M = -m\}$ and $A'^*\mathscr{A}_m A$ are in one to one correspondence. Therefore, we see that ω_m satisfies the modularity condition (3.19). Hence ω_m is automatically holomorphic at the cusps of $\tilde{\Gamma}_0(N)$, by the Götzky-Koecher principle. Therefore, ω_m is a Hilbert modular form for the congruence subgroup $\tilde{\Gamma}_0(N)$ of the Hilbert modular group Γ_K.

Since ω_m for $m > 0$ is a Hilbert modular form for $\tilde{\Gamma}_0(N)$, we have ω_m is invariant with respect to matrices $\begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix}$ where $\varepsilon \in \mathcal{O}^*$, $\mu \in \mathcal{O}$. That is

$$\omega_m(\varepsilon^2 z_1 + \varepsilon\mu, \varepsilon'^2 z_2 + \varepsilon'\mu') = \omega_m(z_1, z_2)$$

Therefore, by the Götzky-Koecher principle, ω_m has a Fourier expansion at the cusp ∞ of the form

$$\omega_m(z_1, z_2) = c_{m0} + \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} c_{m\nu}\, e^{2\pi i(\nu z_1 + \nu' z_2)} \tag{3.21}$$

For any $W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$, we have

$$\begin{aligned} (\omega_m \mid W)(z_1, z_2) &= D^{k/2}(\gamma z_1+\delta)^{-k}(\gamma' z_2+\delta')^{-k} \sideset{}{'}\sum_{\substack{M \in \mathscr{A} \\ \det M = -m}} \phi_M(Wz_1, W'z_2) \\ &= D^{k/2} \sideset{}{'}\sum_{\substack{M \in \mathscr{A} \\ \det M = -m}} \phi_{W'^*MW}(z_1, z_2)^{\frac{k}{2}} \\ &= D^{k/2} \sideset{}{'}\sum_{\substack{M \in \mathscr{B} \\ \det M = -m}} \phi_M(z_1, z_2)^{\frac{k}{2}}, \end{aligned}$$

where

$$\begin{aligned}
\mathscr{B} &= W'^{*}\mathscr{A}W \\
&= W'^{-1}\mathscr{A}W \\
&= W'^{-1}\{M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{M}_2(\mathcal{O}) \mid N \mid \gamma,\ M' = M^*\}W.
\end{aligned}$$

A typical matrix $M \in \mathscr{B}$ has the form

$$M = \begin{pmatrix} \theta & b\sqrt{D} \\ -a\sqrt{D} & \theta' \end{pmatrix}, \qquad \theta \in K, a, b \in \mathbb{Q}.$$

Writing $\theta = \lambda\sqrt{D}$, we obtain

$$(\omega_m \mid W)(z_1, z_2) = \sideset{}{'}\sum_{\substack{(a,\ b,\ \lambda)\in\mathbb{L}, \\ \mathcal{N}(\lambda)-ab=\frac{m}{D}}} \frac{1}{(az_1z_2 + \lambda z_1 + \lambda' z_2 + b)^k},$$

where $\mathbb{L} \subset \mathbb{Q} \times \mathbb{Q} \times K$ is the lattice (i.e. a free $\mathbb{Z}$-module of rank 4) of triples (a, b, λ) for which $W' \begin{pmatrix} \lambda\sqrt{D} & b\sqrt{D} \\ -a\sqrt{D} & -\lambda'\sqrt{D} \end{pmatrix} W^{-1} \in \mathfrak{M}_2(\mathcal{O})$. To show that ω_m is a cusp form, it is enough to show that $c_{m0} = 0$ for the cusp at ∞, because of the similarity between $(\omega_m \mid W)(z_1, z_2)$ and $\omega_m(z_1, z_2)$. It is clear that the method used to find the Fourier expansion of ω_m can be applied to prove that $(\omega_m \mid W)(z_1, z_2)$ has a Fourier series whose constant term vanishes. This proves Theorem 3.3.1.

The Fourier coefficients of ω_m for the cusp at ∞ is computed in the next section. $\square$

3.4 The Fourier coefficients of ω_m

In this section, we shall compute the Fourier coefficients of ω_m explicitly. We follow the method of Zagier [86] to prove the results.

For $m > 0$, write

$$\begin{aligned}
\omega_m(z_1, z_2) &= \sum_{\substack{a\in\mathbb{Z} \\ N|a}} \omega_m^a(z_1, z_2) \\
&= \omega_m^0(z_1, z_2) + 2\sum_{\substack{a=1 \\ N|a}}^{\infty} \omega_m^a(z_1, z_2),
\end{aligned}$$

where

$$\omega_m^a(z_1,z_2) = \sideset{}{'}\sum_{\substack{b\in\mathbb{Z}\\ \lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'-ab=m/D}} (az_1z_2+\lambda z_1+\lambda' z_2+b)^{-k}. \tag{3.22}$$

Observe that ω_m^a satisfies the periodicity property, that is, $\omega_m^a(z_1+\theta, z_2+\theta') = \omega_m^a(z_1,z_2)$, where $\theta\in\mathcal{O}$, and hence each ω_m^a has a Fourier expansion

$$\omega_m^a(z_1,z_2) = \sum_{\nu\in\mathfrak{d}^{-1}} c_{m\nu}^a e^{2\pi i(\nu z_1+\nu' z_2)}. \tag{3.23}$$

Therefore, the Fourier coefficients of ω_m are given by

$$c_{m\nu} = c_{m\nu}^0 + 2\sum_{\substack{a=1\\ N|a}}^{\infty} c_{m\nu}^a. \tag{3.24}$$

We now state the following propositions which were obtained by Zagier in Section 2 of [86].

Proposition 3.4.1 *For $m>0$ and $\nu\in\mathfrak{d}^{-1}$, the Fourier coefficient $c_{m\nu}^0$ defined by equation (3.23) is zero unless $\nu\gg 0$ and $\nu = r\lambda$ with $r\in\mathbb{N}$, $\lambda\in\mathfrak{d}^{-1}$ and $\lambda\lambda' = m/D$, in which case*

$$c_{m\nu}^0 = 2c_k r^{k-1} \text{ with } c_k = \frac{(2\pi i)^k}{(k-1)!}.$$

Proposition 3.4.2 *For $m>0$, $\nu\in\mathfrak{d}^{-1}$ and $a>0$, the Fourier coefficient $c_{m\nu}^a$ defined by equation (3.23) is zero unless $\nu\gg 0$ and is then given by*

$$c_{m\nu}^a = \frac{(2\pi)^{k+1}}{(k-1)!}\frac{D^{\frac{k}{2}-1}}{a}\left(\frac{\mathcal{N}(\nu)}{m}\right)^{\frac{k-1}{2}} G_a(m,\nu)\, J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{m\mathcal{N}(\nu)}{D}}\right).$$

Now, we state the theorem of this section which gives the Fourier coefficients $c_{m\nu}$ defined by (3.21) explicitly:

Theorem 3.4.3 *For $m>0$, the Fourier coefficient $c_{m\nu}$ of $\omega_m(z_1,z_2)$ defined by equation (3.21) is given by*

$$c_{m\nu}=\frac{2(2\pi)^k}{(k-1)!}\left\{(-1)^{\frac{k}{2}}\sum_{\substack{r\in\mathbb{N},\ r|\nu\sqrt{D}\\ \mathcal{N}(\nu\sqrt{D}/r)=-m}} r^{k-1}+2\pi D^{\frac{k}{2}-1}\left(\frac{\mathcal{N}(\nu)}{m}\right)^{\frac{k-1}{2}}\sum_{\substack{a=1\\ N|a}}^{\infty}\frac{1}{a}J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{m\mathcal{N}(\nu)}{D}}\right)G_a(m,\nu)\right\},$$

provided $\nu\gg 0$ and otherwise, $c_{m\nu}=0$, where $G_a(m,\nu)$ is the finite exponential *sum defined by*

$$G_a(m,\nu) = \sum_{\substack{\lambda\in\mathfrak{d}^{-1}/a\mathcal{O}\\ \lambda\lambda'\equiv m/D(mod\ a\mathbb{Z})}} e^{2\pi i(Tr(\nu\lambda))/a}$$

Proof. The proof follows by substituting the Fourier coefficients $c^0_{m\nu}$ and $c^a_{m\nu}$ from Proposition 3.4.1 and Proposition 3.4.2 respectively into (3.24). □

3.5 Finite exponential sums and Kloosterman sums

In this section, we discuss about the finite exponential sums and certain linear sum of Kloosterman sums. We use the results of this section to prove Theorem 3.1.1. Following [86], we define

$$H_b(n,m) = \sum_{\substack{D=D_1D_2\\ D_2|n\\ (b,D_2)=1}} \frac{\psi(D_2)}{D_2}\, H^{D_1}_{bD_1}\left(\frac{n}{D_2}, m\right), \tag{3.25}$$

where $\psi(D_2)$ is the Gauss sum defined by

$$\begin{aligned}\psi(D_2) &= \sum_{x(mod\ D_2)} \left(\frac{x}{D_2}\right) e^{-2\pi i D_1 x/D_2},\\ &= \begin{cases}\left(\frac{D_1}{D_2}\right)\sqrt{D_2} & \text{if } D_1\equiv D_2\equiv 1(\text{mod } 4),\\ -i\left(\frac{D_1}{D_2}\right)\sqrt{D_2} & \text{if } D_1\equiv D_2\equiv 3(\text{mod } 4),\end{cases}\\ &= \left(\frac{-4}{D_2}\right)^{\frac{-1}{2}}\left(\frac{D_1}{D_2}\right)\sqrt{D_2}\end{aligned} \tag{3.26}$$

and

$$H^{D_1}_{bD_1}(n,m) = \frac{1}{bD_1}\left(\frac{bD_1}{D_2}\right)\sum_{\substack{d(\text{mod } bD_1)\\ (d,bD_1)=1}} \left(\frac{-d}{D_1}\right) e^{2\pi i\left(\frac{nD_2^{-1}d^{-1}+md}{bD_1}\right)}. \tag{3.27}$$

In order to prove Theorem 3.1.1, we need an identity relating certain finite exponential sums to Kloosterman sums. More precisely, we need the following proposition:

Proposition 3.5.1 *For $a,m\in\mathbb{Z}$, $\nu\in\mathfrak{d}^{-1}$ and $a>0$, we have*

$$\frac{1}{a\sqrt{D}}\, G_a\left(m,\nu\right) = \sum_{\substack{r|\nu\\ r|a}} H_{a/r}\left(\frac{D\nu\nu'}{r^2}, m\right).$$

where $G_a(m,\nu)$ *is the sum defined as in Theorem 3.4.3, and* $H_b(n,m)$ *is the sum as defined in equation* (3.25).

The above proposition is nothing but proposition proved by D. Zagier in Section 4 of [86].

Let $M \geq 1$ be a squarefree integer. Let D be a fundamental discriminant dividing M, that is, $D \mid M$. Let χ be the fundamental character which is a quadratic character of the associated quadratic field K. Write $N := M/D$.

Proposition 3.5.2 *We have,*

$$\begin{aligned}
\text{(I)}\quad H^{D_1N}_{aD_1N}(n,m) &= \frac{1}{aD_1N}\sum_{\substack{d(mod\ aD_1N)\\(d,aD_1N)=1}} \chi\left(A^{-1}_{D_1N}\begin{pmatrix} a & b\\ aD_1N & d\end{pmatrix}\right) e^{2\pi i\left(\frac{nd^{-1}D_2^{-1}+md}{aD_1N}\right)}\\
&= \frac{1}{aD_1N}\left(\frac{aD_1N}{D_2}\right)\left(\frac{N}{D_2}\right)\sum_{\substack{d(mod\ aD_1N)\\(d,aD_1N)=1}}\left(\frac{-d}{D_1}\right) e^{2\pi i\left(\frac{nD_2^{-1}d^{-1}+md}{aD_1N}\right)};\\
\text{(II)}\quad H_{Na}(n,m) &= \sum_{\substack{D=D_1D_2\\ D_2|n\\ (Na,D_2)=1}}\left(\frac{N}{D_2}\right)\frac{\psi(D_2)}{D_2} H^{D_1N}_{aD_1N}\left(\frac{n}{D_2},m\right).
\end{aligned}$$

Proof. (I) Since $A_{D_1N} = \begin{pmatrix} D_2 & -p\\ D_1N & q\end{pmatrix} \in SL_2(\mathbb{Z})$, we have $pD_1N + qD_2 = 1$. Therefore, we have

$$A^{-1}_{D_1N}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} aq+pc & bq+dp\\ -aD_1N+cD_2 & -bD_1N+dD_2\end{pmatrix} \in \Gamma_0(M)$$

only if $D_2 \mid a$, (since $c = aD_1N$). So a is determined modulo cD_2 by $ad \equiv 1(\mathrm{mod}\ c)$ and $D_2 \mid a$. Hence, we have,

$$\begin{aligned}
\chi\left(A^{-1}_{D_1N}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right) = \epsilon(aq+pc) &= \left(\frac{aq+pc}{D}\right) = \left(\frac{aq+pc}{D_1}\right)\left(\frac{aq+pc}{D_2}\right)\\
&= \left(\frac{aq}{D_1}\right)\left(\frac{pc}{D_2}\right) = \left(\frac{aD_2^{-1}}{D_1}\right)\left(\frac{c(D_1N)^{-1}}{D_2}\right)\\
&= \left(\frac{(dD_2)^{-1}}{D_1}\right)\left(\frac{c(D_1N)^{-1}}{D_2}\right)\\
&= \left(\frac{-d}{D_1}\right)\left(\frac{c}{D_2}\right)\left(\frac{N}{D_2}\right),
\end{aligned}$$

where in the last line we have used the quadratic reciprocity and the fact that $D_1D_2 \equiv 1 \pmod 4$ to get $\left(\frac{D_1}{D_2}\right)\left(\frac{D_2}{D_1}\right) = \left(\frac{-1}{D_1}\right)$.

(II). By the definition of $H_b(n,m)$, we have

$$\begin{aligned}
H_{Na}(n,m) &= \sum_{\substack{D=D_1D_2\\ D_2|n\\ (Na,D_2)=1}} \frac{\psi(D_2)}{D_2}\, H^{D_1}_{NaD_1}\left(\frac{n}{D_2}, m\right)\\
&= \sum_{\substack{D=D_1D_2\\ D_2|n\\ (Na,D_2)=1}} \frac{\psi(D_2)}{D_2}\frac{1}{NaD_1}\left(\frac{NaD_1}{D_2}\right) \sum_{\substack{d \pmod{NaD_1}\\ (d,NaD_1)=1}} \left(\frac{-d}{D_1}\right) e^{2\pi i\left(\frac{\frac{n}{D_2}D_2^{-1}d^{-1}+md}{bD_1}\right)}, \text{ (by (3.27))}\\
&= \sum_{\substack{D=D_1D_2\\ D_2|n\\ (Na,D_2)=1}} \left(\frac{N}{D_2}\right)\frac{\psi(D_2)}{D_2}\, H^{D_1N}_{aD_1N}\left(\frac{n}{D_2}, m\right) \quad \text{(by using part (I)).}
\end{aligned}$$

This proves the proposition. $\square$

Now, we introduce certain linear combinations of Poincaré series: Let $M \geq 1$ be a squarefree integer. Let D be a fundamental discriminant dividing M, that is, $D \mid M$ and n a positive integer. Writing $N := M/D$ as before, we set

$$G_n(z) = \sum_{\substack{D=D_1D_2\\ D_2|n}} \left(\frac{N}{D_2}\right)\psi(D_2)D_2^{-k}G^{D_1N}_{n/D_2}(z) \quad (z \in \mathbb{H}),$$

where the notations are as before. Thus $G_n(z)$ is a linear combination of Poincaré series at certain cusps of $\Gamma_0(M)$. From the Fourier expansion of $G^{D_1N}_n$, we obtain the Fourier expansion of $G_n(z)$:

$$G_n(z) = \sum_{m=1}^{\infty} g_{nm}e^{2\pi imz}$$

with

$$\begin{aligned}
g_{nm} &= \sum_{\substack{D=D_1D_2\\ D_2|n}} \left(\frac{N}{D_2}\right)\psi(D_2)D_2^{-k}g^{D_1N}_{\frac{n}{D_2}m}\\
&= \delta_{nm} + 2\pi(-1)^{\frac{k}{2}}\left(\frac{m}{n}\right)^{\frac{k-1}{2}} \sum_{\substack{D=D_1D_2\\ D_2|n}} \left(\frac{N}{D_2}\right)\frac{\psi(D_2)}{D_2} \sum_{\substack{c=1\\ (c,M)=D_1N}}^{\infty} H^{D_1N}_c\left(\frac{n}{D_2}, m\right)\\
&\qquad \times J_{k-1}\left(\frac{4\pi}{cD_2}\sqrt{mn}\right)
\end{aligned}$$

$$= \quad \delta_{nm} + 2\pi(-1)^{k/2}\left(\frac{m}{n}\right)^{\frac{k-1}{2}} \sum_{b=1}^{\infty} H_{Nb}(n,m) J_{k-1}\left(\frac{4\pi}{NbD}\sqrt{mn}\right). \tag{3.28}$$

In the last line we have used Proposition 3.5.2 to set

$$H_{Nb}(n,m) = \sum_{\substack{D=D_1D_2 \\ D_2|n \\ (b,D_2)=1}} \left(\frac{N}{D_2}\right) \frac{\psi(D_2)}{D_2} H^{D_1N}_{bD_1N}\left(\frac{n}{D_2}, m\right).$$

3.6 The form $\Omega(z_1, z_2; \tau)$

We fix a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ with $D \equiv 1 (\mathrm{mod}\ 4)$, which is squarefree and fix an even integer $k > 2$. We define a function of three variables by

$$\Omega(z_1, z_2; \tau) = \sum_{m=1}^{\infty} m^{k-1} \omega_m(z_1, z_2) e^{2\pi i m \tau} \quad (z_1, z_2, \tau \in \mathbb{H}), \tag{3.29}$$

where $\omega_m(z_1, z_2)$ are the forms defined by (3.18). The series converges absolutely and from the results of Section 3.3 we see that, for fixed $\tau \in \mathbb{H}$, the function $\Omega(z_1, z_2; \tau)$ is a Hilbert cusp form for $\tilde{\Gamma}_0(N)$ of weight k with respect to the variables z_1, z_2. Our goal is to show that, for fixed $z_1, z_2 \in \mathbb{H}$, $\Omega(z_1, z_2; \tau)$ is a cusp form for $\Gamma_0(M)$ of weight k and character χ_D with respect to the variable τ. We do this by proving an identity which expresses Ω as a linear combination of the functions $G_n(\tau)$ constructed in the preceeding section:

Theorem 3.6.1 *For all $z_1, z_2, \tau \in \mathbb{H}$, the identity*

$$\Omega(z_1, z_2; \tau) = \sum_{n=1}^{\infty} n^{k-1} \omega_n^0(z_1, z_2) G_n(\tau) \tag{3.30}$$

holds true.

Proof. We expand both sides by inserting the Fourier expansions of ω_m, ω_n^0 and G_n. By the definition of $\Omega(z_1, z_2; \tau)$, we have the Fourier series of the left-hand side of (3.30) with respect to the variable τ. Its Fourier development with respect to z_1 and z_2 is given by Theorem 3.4.3 which is as follows:

$$\Omega(z_1, z_2; \tau) = \sum_{\substack{m \in \mathbb{Z} \\ m>0}} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} m^{k-1} c_{m\nu} e^{2\pi i m \tau} e^{2\pi i (\nu z_1 + \nu' z_2)} \tag{3.31}$$

with $c_{m\nu}$ as defined in Theorem 3.4.3. We recall that the function $\omega_n^0(z_1, z_2)$ has the Fourier expansion given by (3.23). By Proposition 3.4.1, we have

$$\begin{aligned} \omega_n^0(z_1, z_2) &= 2c_k \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \lambda \gg 0 \\ D\lambda\lambda' = n}} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i (r\lambda z_1 + r\lambda' z_2)} \\ &= 2c_k \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \left(\sum_{r|\nu,\ D\nu\nu' = nr^2} r^{k-1} \right) e^{2\pi i (\nu z_1 + \nu' z_2)}, \end{aligned}$$

where $c_k = \frac{(2\pi i)^k}{(k-1)!}$ and the inner sum is over all natural numbers r such that $\frac{1}{r}\nu \in \mathfrak{d}^{-1}$ and $\mathcal{N}\left(\frac{1}{r}\nu\right) = \frac{n}{D}$ and contains atmost one summand. On the other hand, using the Fourier expansion of $G_n(\tau)$, the right-hand side of (3.30) equals

$$\sum_{m=1}^{\infty} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \left(2c_k \sum_{n=1}^{\infty} n^{k-1} g_{nm} \sum_{\substack{r|\nu \\ D\nu\nu' = r^2 n}} r^{k-1} \right) e^{2\pi i (\nu z_1 + \nu' z_2)} e^{2\pi i m\tau}$$

$$= 2c_k \sum_{m=1}^{\infty} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \left(\sum_{r|\nu} \left(\frac{D\nu\nu'}{r} \right)^{k-1} g_{\frac{D\nu\nu'}{r^2}, m} \right) e^{2\pi i (\nu z_1 + \nu' z_2)} e^{2\pi i m\tau}$$

Comparing this with (3.31), we see that we need to prove that

$$m^{k-1} c_{m\nu} = 2c_k \sum_{r|\nu} \left(\frac{D\nu\nu'}{r} \right)^{k-1} g_{\frac{D\nu\nu'}{r^2}, m}$$

for $m \in \mathbb{Z},\ m > 0,\ \nu \in \mathfrak{d}^{-1}$ and $\nu \gg 0$. Substituting for $c_{m\nu}$ and $g_{\frac{D\nu\nu'}{r^2}, m}$ from Theorem 3.4.3 and (3.28) respectively, we see that the identity to be proved is:

$$m^{k-1} \sum_{\substack{r|\nu\sqrt{D} \\ \frac{D\nu\nu'}{r^2} = m}} r^{k-1} + (-1)^{\frac{k}{2}} 2\pi D^{\frac{k}{2}-1} (m\nu\nu')^{\frac{k-1}{2}} \sum_{\substack{a=1 \\ N|a}}^{\infty} \frac{1}{a} J_{k-1}\left(\frac{4\pi}{a} \sqrt{\frac{m\nu\nu'}{D}} \right) G_a(m, \nu)$$

$$= \sum_{\substack{r|\nu\sqrt{D} \\ \frac{D\nu\nu'}{r^2} = m}} \left(\frac{D\nu\nu'}{r} \right)^{k-1} + (-1)^{\frac{k}{2}} 2\pi (m\nu\nu' D)^{\frac{k-1}{2}} \sum_{r|\nu\sqrt{D}} \sum_{\substack{b=1 \\ N|b}}^{\infty} H_b\left(\frac{D\nu\nu'}{r^2}, m \right) J_{k-1}\left(\frac{4\pi}{br} \sqrt{\frac{m\nu\nu'}{D}} \right).$$

The first terms on the two sides of this identity are equal and comparing the coefficients of $J_{k-1}\left(\frac{4\pi}{a} \sqrt{\frac{m\nu\nu'}{D}} \right)$ on the two sides of the equation, we find that the identity to be proved is

$$\frac{1}{a\sqrt{D}} G_a(m,\nu) = \sum_{\substack{r|\nu \\ r|a}} H_{a/r}\left(\frac{D\nu\nu'}{r^2}, m\right)$$

which is nothing but Proposition 3.5.1. This proves the theorem. □

3.7 The mapping ι_D

In the last section we proved the identity

$$\sum_{m=1}^{\infty} m^{k-1}\omega_m(z_1,z_2)e^{2\pi i m\tau} = \sum_{m=1}^{\infty} m^{k-1}\omega_m^0(z_1,z_2)G_m(\tau)$$

relating the Hilbert modular forms of weight k to the Poincaré series of weight k and character χ_D. These Hilbert modular forms ω_m have been defined by equation (3.18) and its properties have been studied in §3.3 and §3.4. The Poincaré series has been studied in §3.2.3 and a certain variant of Poincaré series (which is $G_m(\tau)$) is studied in §3.5. By using the above identity, we deduce the two statements asserting that some infinite series defines a cusp form. First observe that $G_m(\tau)$ is a cusp form in $S_k(M,\chi_D)$. Therefore, we have,

1. *for each point $(z_1,z_2) \in \mathbb{H}\times\mathbb{H}$, the series*

   $$\sum_{m=1}^{\infty} m^{k-1}\omega_m(z_1,z_2)e^{2\pi i m\tau}$$

   considered as a function of τ, defines a cusp form in $S_k(M,\chi_D)$.

   On the other hand, since we know that the ω_m are Hilbert cusp forms for $\tilde{\Gamma}_0(N)$, we have,

2. *for each point $\tau \in \mathbb{H}$, the series*

   $$\sum_{m=1}^{\infty} m^{k-1}\omega_m^0(z_1,z_2)G_m(\tau)$$

   considered as a function of (z_1,z_2), defines a Hilbert cusp form of weight k for the congruence subgroup $\tilde{\Gamma}_0(N)$ of the Hilbert modular group.

One of the consequence of the above two facts is that the function $\Omega(z_1,z_2;\tau)$ defined by (3.29) is in $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N)) \otimes S_k(M,\chi_D)$. It is well known that the Petersson inner product on $S_k(M,\chi_D)$ is a non-degenerate scalar product and it provides a canonical identification of $S_k(M,\chi_D)$ with its dual $S_k(M,\chi_D)^* = \mathrm{Hom}\,(S_k(M,\chi_D),\mathbb{C})$. Since $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N)) \otimes S_k(M,\chi_D)$ is canonically isomorphic to $S_k(M,\chi_D)\otimes S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$, it turns out that one can identify

$$S_k(M,\chi_D)\otimes S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N)) \simeq \mathrm{Hom}\,(S_k(M,\chi_D), S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$$

and thus we can think of Ω as a linear map

$$\begin{aligned} \Omega : S_k(M,\chi_D) &\longrightarrow S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N)) \\ f &\longmapsto \langle f,\Omega\rangle_\tau = \int_{\mathscr{F}} f(\tau)\overline{\Omega(z_1,z_2;\tau)}y^{k-2}dxdy, \end{aligned}$$

where $\tau = x+iy$ and $\mathscr{F}$ is a fundamental domain for the action of $\Gamma_0(M)$ on $\mathbb{H}$. On the other hand, we have decomposed Ω as a linear sum of Poincaré series and therefore we can compute its Petersson inner product with any cusp form. Let $f \in S_k(M,\chi_D)$ and let $a_n^{D_1N}(f)$ $(n \geq 1, D_1 \mid D, N = M/D)$ be its Fourier coefficients at the various cusps of $\Gamma_0(M)$ as defined by (3.14). By using the definition of G_n as a linear sum of Poincaré series and the defining property (3.15) of Poincaré series, we find that

$$\begin{aligned} n^{k-1}(f,G_n) &= n^{k-1}\sum_{\substack{D=D_1D_2\\ D_2|n}} \left(\frac{N}{D_2}\right)\psi(D_2)D_2^{-k}\langle f, G_{n/D_2}^{D_1N}\rangle \\ &= \frac{(k-2)!}{(4\pi)^{k-1}}\sum_{\substack{D=D_1D_2\\ D_2|n}} \left(\frac{N}{D_2}\right)\psi(D_2)D_2^{k-1}a_{n/D_2}^{D_1N}(f) \end{aligned}$$

and hence

$$\begin{aligned} \langle f,\Omega\rangle_\tau &= \sum_{n=1}^{\infty} n^{k-1}\langle f, G_n\rangle \omega_n^0(z_1,z_2) \\ &= \frac{(k-2)!}{(4\pi)^{k-1}}\sum_{D=D_1D_2} \left(\frac{N}{D_2}\right)\psi(D_2)D_2^{k-1}\sum_{n=1}^{\infty} a_n^{D_1N}(f)\omega_{nD_2}^0(z_1,z_2). \end{aligned}$$

We substitute the Fourier expansion of

$$\omega_n^0(z_1,z_2) = \frac{2(2\pi)^k}{(k-1)!}(-1)^{k/2}\sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ \lambda\gg 0\\ \mathcal{N}(\lambda)=n/D}}\sum_{r=1}^{\infty} r^{k-1}e^{2\pi i(r\lambda z_1 + r\lambda' z_2)}$$

into the above expression which is computed in Proposition 3.4.1 and obtain the following lifting which is given explicitly in terms of Fourier coefficients:

Let $f \in S_k(M, \chi_D)$. Then the Doi-Naganuma lifting of f is defined by

$$\iota_D(f) = \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} c((\nu)\mathfrak{d}) e^{2\pi i(\nu z_1 + \nu' z_2)},$$

where the coefficients $c((\nu)\mathfrak{d})$ is defined by:

$$c((\nu)\mathfrak{d}) = \sum_{r|(\nu)\mathfrak{d}} r^{k-1} \sum_{D_2|(D, \frac{\mathcal{N}(\nu)\mathcal{N}(\mathfrak{d})}{r^2})} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{k-1} a^{D_1 N}_{\frac{\mathcal{N}(\nu)\mathcal{N}(\mathfrak{d})}{r^2 D_2}}(f) \qquad (D_1 \mid D).$$

The first sum is over all positive integers r dividing $(\nu)\mathfrak{d}$, the second sum over all positive integers dividing D and $\mathcal{N}((\nu)\mathfrak{d})/r^2$, $D_1 = D/D_2$ and $\psi(D_2)$ is the Gauss sum defined by (3.26).

Proof of Theorem 3.1.1: It suffices to show that $c((\nu)\mathfrak{d}) = \frac{(k-1)!}{2(2\pi)^k}(-1)^{k/2} c_{m\nu}$. Therefore, we consider

$$c((\nu)\mathfrak{d}) = \sum_{r|(\nu)\mathfrak{d}} r^{k-1} \sum_{D_2|(D, \frac{\mathcal{N}(\nu)\mathcal{N}(\mathfrak{d})}{r^2})} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{k-1} a^{D_1 N}_{\frac{\mathcal{N}(\nu)\mathcal{N}(\mathfrak{d})}{r^2 D_2}}(G_m^\infty).$$

By (3.15), the above equals

$$\begin{aligned}
c((\nu)\mathfrak{d}) &= \sum_{r|(\nu)\sqrt{D}} r^{k-1} \sum_{D_2|(D, \frac{D\nu\nu'}{r^2})} \left(\frac{N}{D_2}\right) \psi(D_2) D_2^{k-1} \frac{(4\pi \frac{D\nu\nu'}{r^2 D_2})^{k-1}}{(k-2)!} D_2^{-k} \langle G_m^\infty, G^{D_1 N}_{\frac{D\nu\nu'}{r^2 D_2}} \rangle \\
&= \sum_{r|(\nu)\sqrt{D}} r^{k-1} \sum_{D_2|(D, \frac{D\nu\nu'}{r^2})} \left(\frac{N}{D_2}\right) \frac{\psi(D_2)}{D_2} \frac{(4\pi \frac{D\nu\nu'}{r^2 D_2})^{k-1}}{(k-2)!} \frac{(k-2)!}{(4\pi m)^{k-1}} g^{D_1 N}_{\frac{D\nu\nu'}{r^2 D_2} m} \\
&= \sum_{r|(\nu)\sqrt{D}} r^{k-1} \sum_{D_2|(D, \frac{D\nu\nu'}{r^2})} \left(\frac{N}{D_2}\right) \frac{\psi(D_2)}{D_2} \left(\frac{D\nu\nu'}{r^2 D_2 m}\right)^{k-1} \Bigg\{ \delta_{D_1 N, M} \delta_{\frac{D\nu\nu'}{r^2 D_2}, m} \\
&+ 2\pi(-1)^{k/2} \left(\frac{m D_2}{\frac{D\nu\nu'}{r^2 D_2}}\right)^{\frac{k-1}{2}} \sum_{\substack{a=1 \\ D_1 N | a}}^{\infty} H_a^{D_1 N}\left(\frac{D\nu\nu'}{r^2 D_2}, m\right) J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{m D\nu\nu'}{r^2 D_2^2}}\right) \Bigg\} \\
&= \sum_{r|(\nu)\sqrt{D}} r^{k-1} \left(\frac{D\nu\nu'}{r^2 m}\right)^{k-1} \delta_{\frac{D\nu\nu'}{r^2}, m} + \sum_{r|(\nu)\sqrt{D}} r^{k-1} \sum_{D_2|(D, \frac{D\nu\nu'}{r^2})} \left(\frac{N}{D_2}\right) \frac{\psi(D_2)}{D_2} \left(\frac{D\nu\nu'}{r^2 D_2 m}\right)^{k-1}
\end{aligned}$$

$$\times\left\{2\pi(-1)^{k/2}\left(\frac{mD_2}{\frac{D\nu\nu'}{r^2D_2}}\right)^{\frac{k-1}{2}}\sum_{\substack{a=1\\(a,M)=D_1N}}^{\infty}H_a^{D_1N}\left(\frac{D\nu\nu'}{r^2D_2},m\right)J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{mD\nu\nu'}{r^2D_2^2}}\right)\right\}$$

$$=\sum_{\substack{r|(\nu)\sqrt{D}\\ \mathcal{N}(\frac{\nu\sqrt{D}}{r})=-m}} r^{k-1}+\sum_{r|(\nu)\sqrt{D}}\left(\frac{D\nu\nu'}{m}\right)^{\frac{k-1}{2}}2\pi(-1)^{k/2}\sum_{D_2|(D,\frac{D\nu\nu'}{r^2})}\left(\frac{N}{D_2}\right)\frac{\psi(D_2)}{D_2}$$

$$\times\sum_{\substack{a=1\\(a,M)=D_1N}}^{\infty}H_a^{D_1N}\left(\frac{D\nu\nu'}{r^2D_2},m\right)J_{k-1}\left(\frac{4\pi}{arD_2}\sqrt{mD\nu\nu'}\right)$$

$$=\sum_{\substack{r|(\nu)\sqrt{D}\\ \mathcal{N}(\frac{\nu\sqrt{D}}{r})=-m}} r^{k-1}+2\pi(-1)^{k/2}\left(\frac{D\nu\nu'}{m}\right)^{\frac{k-1}{2}}\sum_{r|(\nu)\sqrt{D}}\sum_{a=1}^{\infty}\sum_{\substack{D_2|(D,\frac{D\nu\nu'}{r^2})\\(a,D_2)=1}}\left(\frac{N}{D_2}\right)\frac{\psi(D_2)}{D_2}$$

$$\times H_{aD_1N}^{D_1N}\left(\frac{D\nu\nu'}{r^2D_2},m\right)J_{k-1}\left(\frac{4\pi}{aD_1NrD_2}\sqrt{mD\nu\nu'}\right)$$

$$=\sum_{\substack{r|(\nu)\sqrt{D}\\ \mathcal{N}(\frac{\nu\sqrt{D}}{r})=-m}} r^{k-1}+2\pi(-1)^{k/2}\left(\frac{D\nu\nu'}{m}\right)^{\frac{k-1}{2}}\sum_{r|(\nu)\sqrt{D}}\sum_{a=1}^{\infty}H_{aN}\left(\frac{D\nu\nu'}{r^2},m\right)$$

$$\times J_{k-1}\left(\frac{4\pi}{aNr}\sqrt{\frac{m\nu\nu'}{D}}\right),\text{(by Proposition (3.5.2))}$$

$$=\sum_{\substack{r|(\nu)\sqrt{D}\\ \mathcal{N}(\frac{\nu\sqrt{D}}{r})=-m}} r^{k-1}+2\pi(-1)^{\frac{k}{2}}\left(\frac{D\nu\nu'}{m}\right)^{\frac{k-1}{2}}\sum_{\substack{a=1\\N|a}}^{\infty}\frac{1}{\sqrt{D}}\sum_{\substack{r|(\nu)\sqrt{D}\\r|a}}\sqrt{D}\,H_{a/r}\left(\frac{D\nu\nu'}{r^2},m\right)$$

$$\times J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{m\nu\nu'}{D}}\right)$$

$$=\sum_{\substack{r|(\nu)\sqrt{D}\\ \mathcal{N}(\frac{\nu\sqrt{D}}{r})=-m}} r^{k-1}+2\pi(-1)^{k/2}D^{\frac{k}{2}-1}\left(\frac{\mathcal{N}(\nu)}{m}\right)^{\frac{k-1}{2}}\sum_{\substack{a=1\\N|a}}^{\infty}\frac{1}{a}G_a(m,\nu)$$

$$\times J_{k-1}\left(\frac{4\pi}{a}\sqrt{\frac{m\mathcal{N}(\nu)}{D}}\right),\text{(by Proposition (3.5.1))}$$

$$=\frac{(k-1)!}{2(2\pi)^k}(-1)^{k/2}c_{m\nu}.$$

Hence the theorem. □

Theorem 3.7.1 *The map ι_D sends Hecke eigenforms in $S_k(M,\chi_D)$ to Hecke eigenforms in $S_k^{\mathcal{H}}(\tilde{\Gamma}_0(N))$.*

The proof of this theorem follows similar to that given in p. 137 of [82].

3.8 The Doi-Naganuma lifting

We now describe the relationship of Theorem 3.1.1 to the analogous construction of K. Doi and H. Naganuma [12] and [64]. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_k(\Gamma_0(M), \left(\frac{D}{\cdot}\right))$ be a cusp form of weight k, where D (dividing M) is the discriminant of real quadratic field $K = \mathbb{Q}(\sqrt{D})$ of class number one. We assume that f is an eigen function of all the Hecke operators T_n, normalized with $a_1 = 1$. Then the associated Dirichlet series

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad (\mathrm{Re}\ s \gg 1)$$

has an Euler product expansion of the form

$$L(f,s) = \prod_{q|N} (1 - a_q q^{-s})^{-1} \prod_{q\nmid N} \left(1 - a_q q^{-s} + \left(\frac{q}{D}\right) q^{k-1-2s}\right)^{-1}$$

(product over all rational primes q and $N = M/D$). Consider the series

$$L(\bar{f},s) = \sum_{n=1}^{\infty} \bar{a}_n n^{-s},$$

where

$$\bar{a}_n = \left(\frac{D}{n}\right) a_n, \quad (n,M) = 1 \tag{3.32}$$

Consider

$$\Phi(s) = L(f,s)L(\bar{f},s) = \prod_{\mathfrak{q}\nmid N} (1 - b(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s} + \mathcal{N}(\mathfrak{q})^{k-1-2s})^{-1} \prod_{\mathfrak{q}|N} (1 - b(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})^{-1}, \tag{3.33}$$

where the product is extended over all prime ideals $\mathfrak{q}$ of $\mathbb{Q}(\sqrt{D})$ and the coefficients are defined by

$$b(\mathfrak{q}) = \begin{cases} a_q & \text{if } q \text{ splits and } (q,M) = 1, \\ a_q^2 + 2q^{k-1} & \text{if } q \text{ inert and } (q,M) = 1. \end{cases} \tag{3.34}$$

Indeed, for those primes q which splits, we know by (3.32) that $a_q = \bar{a}_q$. So the factor $(1 - a_q q^{-s} + q^{k-1-2s})^{-1}$ occurs twice in $L(f,s)L(\bar{f},s)$, and since there are two prime ideals with norm q, it also occurs twice in the product (3.33). For inert primes q, (3.32) implies that $a_q = -\bar{a}_q$. So the corresponding local factor in $L(f,s)L(\bar{f},s)$ is

$$(1 - a_q q^{-s} + q^{k-1-2s})^{-1}(1 + a_q q^{-s} + q^{k-1-2s})^{-1}$$

$$= (1 - a_q^2 q^{-2s} - 2q^{k-1-2s} + q^{2k-2-4s})^{-1}$$

$$= (1 - b(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s} + \mathcal{N}(\mathfrak{q})^{k-1-2s})^{-1}$$

with $\mathfrak{q} = (q)$, $b(\mathfrak{q})$ as in (3.34) and $\mathcal{N}(\mathfrak{q}) = q^2$.

Theorem 3.8.1 *Let $f \in S_k(\Gamma_0(M), \left(\frac{D}{\cdot}\right))$ be a normalized Hecke eigen function, where M is a squarefree integer and $D \equiv 1(mod\ 4)$ is the fundamental discriminant of $K = \mathbb{Q}(\sqrt{D})$ of class number one and $D \mid M$. Let*

$$L(\iota_D(f), s) = \sum_{\mathfrak{m}} c(\mathfrak{m})\mathcal{N}(\mathfrak{m})^{-s}, \qquad Re\ s > \frac{k}{2} + 1$$

be the associated Dirichlet's series to $\iota_D(f)$, where $\iota_D(f)$ is as defined in Theorem 3.1.1 and the summation is over all non-zero integral ideals $\mathfrak{m}$ of K. Then

$$L(\iota_D(f), s) = L(f,s)L(\bar{f},s).$$

Proof. By the definition of $c(\mathfrak{a})$, for primes $\mathfrak{p} \nmid N$, we have,

$$c(\mathfrak{p}) = \begin{cases} a_p & \text{if } p \text{ splits,} \\ a_p^2 + 2p^{k-1} & \text{if } p \text{ is inert.} \end{cases} \tag{3.35}$$

Also, using the Euler product of $L(f,s)L(\bar{f},s)$, we find that $L(\iota_D(f), s)$ and $L(f,s)L(\bar{f},s)$ agree up to finitely many Euler factors, but they satisfy the same functional equation. Hence they are equal. This proves the theorem. □

It is clear from Theorem 3.8.1 that the mapping ι_D follows the Doi-Naganuma description. Thus, the modular form $\Omega(z_1, z_2; \tau)$ in three variables given by (3.29) has an interpretation as the *kernel* (in the sense of integral operators) of the Doi-Naganuma lifting.

Chapter 4

Newforms of half-integral weight and Jacobi forms

4.1 Introduction

Let k, N, α be positive integers, $k \geq 2$, $\alpha \geq 2$, N odd and χ be a Dirichlet character modulo $2^\alpha N$. We denote the the space of cusp forms of weight $k+1/2$ for $\Gamma_0(2^\alpha N)$ with character $\chi_0 := \left(\frac{4\epsilon}{\cdot}\right)\chi$, $\epsilon := \chi(-1)$ by $S_{k+1/2}(2^\alpha N, \chi_0)$ and the space of cusp forms of weight $2k$, level $2^{\alpha-2}N$ with character χ^2 by $S_{2k}(2^{\alpha-2}N, \chi^2)$. Both these spaces are equipped with the Petersson inner product. When the character χ_0 and χ are principal (trivial), we denote these spaces by $S_{k+1/2}(2^\alpha N)$ and $S_{2k}(2^{\alpha-2}N)$.

By the works of G. Shimura [77] and S. Niwa [66], there exist linear operators $\mathcal{S}_{t,\chi}$ indexed by squarefree integers t, $\epsilon(-1)^k t > 0$ which commute with the action of Hecke operators $T(n^2)$, $(n, 2N) = 1$ and map the space $S_{k+1/2}(2^\alpha N, \chi_0)$ into the space $S_{2k}(2^{\alpha-1}N, \chi^2)$. W. Kohnen [34], [35] introduced a canonical subspace $S^+_{k+1/2}(4N, \chi_0)$, called the Kohnen plus space in $S_{k+1/2}(4N, \chi_0)$. He defined the Hecke operators $T^+(n^2)$ for all integers $n \geq 1$, $(n, N) = 1$, which are nothing but the Hecke operators $T(n^2)$ introduced by Shimura, except for $p = 2$ where $T^+(4)$ is the new Hecke operator preserving the plus space. He then defined the modified Shimura lifts $\mathcal{S}^+_{D,\chi_0}$, called Shimura Kohnen lifts, indexed by fundamental discriminants D, $\epsilon(-1)^k D > 0$, which commutes with the action of Hecke operators:

$$f \mid T^+(n^2)\mathcal{S}^+_{D,\chi_0} = f \mid \mathcal{S}^+_{D,\chi_0} T(n),$$

for all $f \in S^+_{k+1/2}(4N, \chi_0)$ and for all $(n, N) = 1$. He proved that the linear operator $\mathcal{S}^+_{D,\chi_0}$ maps the space $S^+_{k+1/2}(4N, \chi_0)$ into the space $S_{2k}(N, \chi_0^2)$.

To study the Hecke theory of half-integral weight forms, one needs the trace identity which is a powerful tool to understand the multiplicity of a Hecke eigenform of half-integral weight which corresponds to an integral weight newform. In this connection, the trace computation has been carried out by S. Niwa [66] as suggested by G. Shimura in [77]. Following the lines of S. Niwa [66], Kohnen [34,], [35] achieved this goal in the plus space. The equality of traces shows that there is a Hecke equivarient isomorphism ψ between the respective half-integral and integral weight spaces. More precisely, the existence of a Hecke equivarient isomorphism

$$\psi : S_{k+1/2}(4N, \chi_0) \longrightarrow S_{2k}(2N)$$

follows from the work of S. Niwa [66] and the existence of an isomorphism

$$\psi^+ : S^+_{k+1/2}(4N, \chi_0) \longrightarrow S_{2k}(N)$$

follows from the work of Kohnen [34, 35]. Both the results are valid under the assumption that N is odd and squarefree and χ_0 is a real even character $\pmod{4N}$. In [80], M. Ueda extended these results and derived the Hecke equivarient isomorphism

$$\psi : S_{k+1/2}(2^\alpha N, \chi_0) \longrightarrow S_{2k}(2^{\alpha-1}N)$$

where $\alpha = 3$ with χ_0 is a real character and $\alpha = 4$ with $\chi_0 = \left(\frac{8}{\cdot}\right)$ is the even quadratic primitive Dirichlet character modulo 8. Using this, several authors starting with Kohnen studied the Hecke theory for half-integral weight forms.

Kohnen [34, 35] initiated the study of the theory of newforms for the plus space $S^+_{k+1/2}(4N, \chi_0)$ along the lines of Atkin-Lehner [2], where N is odd and squarefree and $\chi_0^2 = 1$. Using the trace identities proved by Niwa[66], M. Manickam, B. Ramakrishnan and T. C. Vasudevan [51] set up the theory of newforms for the full space $S_{k+1/2}(4N, \chi_0)$ where N is odd and squarefree and $\chi_0^2 = 1$. Recently, the work of M. Manickam, J. Meher and B. Ramakrishnan [53] shows the absence of newforms in $S_{k+1/2}(16N)$, that is $S^{new}_{k+1/2}(16N) = \{0\}$. This motivated us to look into the other cases of half integral weight spaces where there is no nonzero Hecke eigenforms. In this chapter, we present the theory of newforms for certain higher level Hecke eigenforms of half integral weight. Using this, we also obtain similar results in the case of Hecke eigenforms of both holomorphic and skew-holomorphic Jacobi forms of integral weight. The contents of this chapter is published in [42].

We first consider the space $S_{k+1/2}(4N, \chi_0)$ and set up the theory of newforms, where χ is a primitive Dirichlet character modulo N(N odd and squarefree) such that χ^2 is also a primitive Dirichlet character modulo N. Next, by using the Eichler-Zagier isomorphism $\mathcal{Z}_1$ studied in [49], we derive the theory of newforms for Jacobi forms and for skew-holomorphic Jacobi forms of weight $k+1$, index 1, level N, character χ, where χ is a primitive Dirichlet character modulo N such that χ^2 is also a primitive Dirichlet character modulo N and $\epsilon(-1)^k$ is negative, (or) positive according as they are holomorphic or skew-holomorphic Jacobi forms respectively. The theory of newforms of Jacobi cusp forms of index m and squarefree level with real character has already been set up by M. Manickam and B. Ramakrishnan in [49].

In the recent work [53], the theory of newforms for the space of cusp forms of weight $k+1/2$, for $\Gamma_0(2^\alpha N)$, ($\alpha = 3$ or $4, N$ odd and squarefree) with real character has been set up and they noticed that the space of newforms $S^{new}_{k+1/2}(16N)$ becomes trivial and on the other hand, the space of newforms $S^{new}_{k+1/2}(16N, \left(\frac{8}{\cdot}\right))$ for $\Gamma_0(16N)$ with the even quadratic primitive character modulo 8 is isomorphic to the space $S^{new}_{2k}(8N)$ of level $8N$ under a certain linear combination of Shimura maps. Hence, it is natural to look into the other cases, where this phenomenon occurs. In the case $\alpha = 5$, we consider the spaces $S_{k+1/2}(32N)$ and $S_{k+1/2}(32N, \left(\frac{8}{\cdot}\right))$. It is to be noted that they are isomorphic under the W-operator $W(32)$. Using the dimension formulas and by explicit decomposition of each old class of Hecke eigenforms, we deduce the fact that $S^{new}_{k+1/2}(32N) = \{0\}$. Next, we consider the action of Shintani map $\mathcal{S}^*_D$ indexed by a fundamental discriminant $D \equiv 1 (\mathrm{mod}\, 4)$ on each normalized newform $F \in S^{new}_{2k}(2^{\alpha-2}N)$ for $\alpha \geq 6$. It is important to observe that in the cases where $\alpha = 2, 3, 4, 5$, the trace identity gives the explicit image $F|\mathcal{S}^*_D$. Indeed, equality of traces shows that each normalized integral weight newform F is associated to a non-zero cusp form f of half-integral weight which is unique upto a scalar multiplication and it is eigenform for almost all the Hecke operators such that

$$\frac{F|\mathcal{S}^*_D}{< F, F >} = \lambda_{k,D} \frac{\overline{a_f(|D|)} f}{< f, f >},$$

where $a_f(|D|)$ is the $|D|$-th Fourier coefficient of f and $\lambda_{k,D}$ is an explicit constant depending only on k and D.

In all other cases, that is, when $\alpha \geq 6$, we consider the calculations carried out by Kohnen in [36] and observe that if $(-1)^k m \equiv 1 (\mathrm{mod}\, 4)$, the image of the mth Poincaré series $\wp_{k+1/2, 2^\alpha N; m}$ in $S_{k+1/2}(2^\alpha N)$ under the Dth Shimura map

$\mathcal{S}_D$ for $D \equiv 1 (\mathrm{mod}\, 4)$ and squarefree is equal to a certain period function in $S_{2k}(2^{\alpha-2}N)$. In particular the Shintani map $\mathcal{S}_D^*$ indexed by odd fundamental discriminant D maps $S_{2k}^{new}(2^{\alpha-2}N)$ into the space $S_{k+1/2}^{new}(2^\alpha N)$. In this case, the twisting operator $R_{\left(\frac{8}{\cdot}\right)}$ preserves the space $S_{k+1/2}^{new}(2^\alpha N)$ and the only newform which is in the kernel is the zero function and hence the square of the operator is identity on Hecke eigenforms and newforms. Since this operator is self adjoint and commuting with the action of Hecke operators $T(n^2), (n, 2N) = 1$, a normalized newform $F \in S_{2k}^{new}(2^{\alpha-2}N)$ lifts to two non-zero Hecke eigenforms $f_1, f_2 \in S_{k+1/2}^{new}(2^\alpha N)$ under some non-zero Shintani lifts. Thus, if $\alpha \geq 6$ the space $S_{k+1/2}^{new}(2^\alpha N)$ is non-trivial. This chapter gives the details of the above results.

4.2 Preliminaries

In this section, we recall some basic facts regarding modular forms of half integral weight and Jacobi forms. The theory of half-integral weight forms was first developed by G. Shimura [77]. Let $\mathbb{C}$ be the complex plane and $\mathbb{H}$ be the upper half-plane consisting of complex numbers $\tau \in \mathbb{C}$ with Im $(\tau) > 0$. For complex numbers $z \neq 0$, x, let $z^x = e^{x \log z}$, $\log z = \log |z| + i \arg z$, $-\pi < \arg z \leq \pi$. Let ζ be a fourth root of unity. For integers a, b, let (a, b) denote the greatest common divisor of a and b. If m is an integer, by $\mu (\mathrm{mod}\, m)$ we mean μ varies over all integers which are incongruent modulo m.

Let G denote the four-sheeted covering of $GL_2^+(\mathbb{Q})$ defined as the set of all ordered pairs $(\alpha, \phi(\tau))$, where $\phi(\tau)$ is a holomorphic function on $\mathbb{H}$ such that $\phi^2(\tau) = \zeta^2 (c\tau + d)/\sqrt{\det \alpha}$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$. Then G is a group with multiplication $(\alpha, \phi(\tau))(\beta, \psi(\tau)) = (\alpha\beta, \phi(\beta\tau)\psi(\tau))$. For $\Gamma_0(4)$ and its subgroups, we take the embedding $\Gamma_0(4) \hookrightarrow G$ as $\Gamma_0(4)^* :=$ the collection $\{(\alpha, j(\alpha, \tau))\}$, where

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \quad \text{and} \quad j(\alpha, \tau) = \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{-1/2} (c\tau + d)^{1/2}.$$

Here $\left(\frac{c}{d}\right)$ denotes the generalised quadratic residue symbol and $\left(\frac{-4}{d}\right)^{1/2}$ is equal to 1 or i according as d is 1 or 3 modulo 4. Let $\left(\frac{8}{\cdot}\right)$ be the even quadratic character modulo 8. We use the notation α^* for the image of $\alpha \in \Gamma_0(4)$ in G. Let $k \geq 2$ be a natural number. For a complex valued function f defined on the upper half-plane $\mathbb{H}$ and an element $(\alpha, \phi(\tau)) \in G$, define the stroke operator by $f \mid_{k+1/2} (\alpha, \phi(\tau))(\tau) = \phi(\tau)^{-2k-1} f(\alpha\tau)$. We omit the subscript $k+1/2$ wherever there is no ambiguity.

Let $k \geq 2$, N be natural numbers, N odd. Write $M = 2^\alpha N$, $\alpha \geq 0$. For $\alpha \geq 2$, a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight $k + 1/2$ for $\Gamma_0(M)$ with even character $\chi \pmod M$ if $f|_{k+1/2}(\gamma, j(\gamma,\tau))(\tau) = \chi(d)f(\tau)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$ and if f is holomorphic at the cusps of $\Gamma_0(M)$. Further, if it vanishes at all the cusps, it is called a cusp form. The set of cusp forms $S_{k+1/2}(M, \chi)$ defined as above forms a complex vector space. Also, we denote the space of modular forms by $M_{k+1/2}(M, \chi)$. If χ is the principal character, the spaces of modular forms and cusp forms are respectively denoted by $M_{k+1/2}(M)$, $S_{k+1/2}(M)$. When $\alpha \geq 0$, we denote by $S_k(M, \chi)$ the space of cusp forms of weight k on $\Gamma_0(M)$ with character χ. If χ is the principal character, the space is denoted by $S_k(M)$. The Fourier expansion of a cusp form f at the cusp infinity is denoted as $f(\tau) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi i n\tau}$. For a prime p, the p-th Hecke operator on $S_{k+1/2}(M, \chi)$ is denoted by $T(p^2)$ if $p \nmid M$ and by $U(p^2)$ if $p \mid M$ and on $S_k(M, \chi)$ the Hecke operator is denoted by $T(p)$ if $p \nmid M$ and by $U(p)$ if $p \mid M$. By a Hecke eigenform in $S_{k+1/2}(M, \chi)$, we mean a non-zero function in the space which is a simultaneous eigenform for all the Hecke operators $T(n^2), (n, M) = 1$. By a normalised Hecke eigenform in $S_k(M, \chi)$, we mean a newform in the space whose first Fourier coefficient is one. For any positive integer n, the operators $U(n)$ and $B(n)$ are defined on formal series by

$$U(n) : \sum_{m=1}^{\infty} a(m)e^{2\pi i m\tau} \longmapsto \sum_{m=1}^{\infty} a(mn)e^{2\pi i m\tau},$$

$$B(n) : \sum_{m=1}^{\infty} a(m)e^{2\pi i m\tau} \longmapsto \sum_{m=1}^{\infty} a(m)e^{2\pi i n m\tau}.$$

The Petersson inner product for forms f, g in $S_\lambda(M, \chi)$ is defined by

$$\langle f, g\rangle = \frac{1}{i_M} \int_{\mathcal{F}} f(\tau)\overline{g(\tau)} v^{\lambda-2} du\, dv,$$

where $\mathcal{F}$ is a fundamental domain for the action of $\Gamma_0(M)$ on $\mathbb{H}$, i_M is the index of $\Gamma_0(M)$ in $SL_2(\mathbb{Z})$ and $\tau = u + iv$. Here $\alpha \geq 0$ if $\lambda = k$ and $\alpha \geq 2$ if $\lambda = k + \frac{1}{2}$. For details on modular forms of half integral weight, we refer to [32].

4.2.1 *W-operators and projection operator* $\mathcal{P}_+$

For a prime p with $p^l \parallel M$ ($\alpha \geq 0$ case), we denote the Atkin-Lehner W-operator on $S_k(M, \chi)$ by W_{p^l}. We also have W-operators for half-integral weight forms. For $p = 2$, we define the analogous Atkin-Lehner W-operator $W(2^\alpha)$ on

$S_{k+1/2}(2^\alpha N)$, $\alpha \geq 2$ and N odd, as follows:

$$W(2^\alpha) = \left(\begin{pmatrix} 2^\alpha x & y \\ 2^\alpha N w & 2^\alpha \end{pmatrix}, 2^{\alpha/4} e^{i\pi/4} (Ncw+1)^{1/2} \right)$$

where x, y and w are integers satisfying $y \equiv 1 (\mathrm{mod}\ 2^\alpha)$, $2^\alpha x - Nwy = 1$. Note that the W-operator defined above is independent of the choice of the integers x, y, w with the given condition. The operator $W(2^\alpha)$ maps $S_{k+1/2}(2^\alpha N, \chi)$ into $S_{k+1/2}(2^\alpha N, \left(\frac{2^\alpha}{\cdot}\right)\chi)$ and $W^2(2^\alpha) = I$ on $S_{k+1/2}(2^\alpha N, \left(\frac{2^\alpha}{\cdot}\right)\chi)$, where I denotes the identity operator.

We now define the projection operator $\mathcal{P}_+$ on $S_{k+1/2}(M)$, where $\alpha \geq 3$ (Note that $M = 2^\alpha N$). Let $\xi = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4} \right)$ and $\xi' = \left(\begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, e^{-\pi i/4} \right)$. Then a formal computation shows that ξ (and hence ξ') preserves the space $S_{k+1/2}(M)$ if $\alpha \geq 4$. However, if $\alpha = 3$, we have

$$\xi + \xi' : S_{k+1/2}(M) \to S_{k+1/2}(M).$$

We define

$$\mathcal{P}_+ := \frac{1}{2} \left(\frac{1}{\sqrt{2}} \left(\frac{8}{2k+1} \right) (\xi + \xi') + I \right). \tag{4.1}$$

Then

$$f \mid \mathcal{P}_+(\tau) = \sum_{(-1)^k n \equiv 0,1 \pmod 4} a_n e^{2\pi i n \tau} \in S_{k+1/2}(M),$$

where $f(\tau) = \sum_{n \geq 1} a_n e^{2\pi i n \tau} \in S_{k+1/2}(M)$. Thus, if $\alpha = 3$, we have

$$S^+_{k+1/2}(M) = S_{k+1/2}(M) \mid \mathcal{P}_+.$$

If $\alpha = 3$, it turns out that the projection operater $\mathcal{P}_+$ acts as an injective operator on $\wp_{k+1/2,4M;t}$, the t-th Poincare series in $S_{k+1/2}(M)$ when $(-1)^k t > 0$ and $t \equiv 0, 1 (\mathrm{mod}\, 4)$. For $\alpha \geq 3$, the image of the Poincaré series $\wp_{k+1/2,M;t} \in S_{k+1/2}(M)$ under $\mathcal{P}_+$ is denoted by $P_{k+1/2,M;t}$, the t-th Poincare series on $S^+_{k+1/2}(4M)$ when $(-1)^k t > 0$ and $t \equiv 0, 1 (\mathrm{mod}\, 4)$.

$$\text{i.e.,}\quad P_{k+1/2,M;t}(\tau) := \wp_{k+1/2,M;t} \mid \mathcal{P}_+(\tau).$$

Let us make the following observations which we need later.

A direct computation shows that $\mathcal{P}_+$ maps $S_{k+1/2}(32N, \left(\frac{8}{\cdot}\right))$ into itself. However, if the functions f, $f \mid \mathcal{P}_+$ are in $S_{k+1/2}(16N, \left(\frac{8}{\cdot}\right))$ then $f = 0$. To get this, let $\eta = \frac{1}{2\sqrt{2}}\left(\frac{8}{2k+1}\right)$. Then,

$$f \mid \mathcal{P}_+ = f \mid \mathcal{P}_+ \begin{pmatrix} 1 & 0 \\ 16N & 1 \end{pmatrix}^*$$

$$= \eta f \mid \left(\left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4}\right) + \left(\begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, e^{-\pi i/4}\right)\right) \mid \begin{pmatrix} 1 & 0 \\ 16N & 1 \end{pmatrix}^* + \frac{1}{2}f.$$

The right hand side of above equals

$$\eta f \mid \begin{pmatrix} 1+4N & -N \\ 16N & 1-4N \end{pmatrix}^* \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4}\right) + \frac{1}{2}f$$
$$+\eta f \mid \begin{pmatrix} 1-4N & -N \\ 16N & 1+4N \end{pmatrix}^* \left(\begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, e^{-\pi i/4}\right)$$

and since N is odd and $f \in S_{k+1/2}(16N, \left(\frac{8}{\cdot}\right))$, we see that the above simplifies to

$$-\eta f \mid (\xi + \xi') + 1/2f.$$

From this we get

$$f \mid \xi + \xi' = 0,$$

and hence we have

$$f \mid \mathcal{P}_+ = \frac{1}{2}f,$$

which implies that f is in the plus space. Thus, we get $f \mid \mathcal{P}_+ = f$ which is possible only when $f = 0$.

4.2.2 *Binary quadratic forms and characters*

We let $SL_2(\mathbb{Z})$ act on integral binary quadratic forms $[a,b,c](x,y) = ax^2+bxy+cy^2$ by

$$[a,b,c]\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(x,y) = [a,b,c](\alpha x + \beta y, \gamma x + \delta y).$$

For an integer D with $D \equiv 0,1 \pmod 4$ and a form $Q = [a,b,c]$ whose discriminant $b^2 - 4ac = \Delta$ which is divisible by D with $\Delta/D \equiv 0,1 \pmod 4$. Define the genus character on binary quadratic forms as done in [21] by

$$\chi_D(Q) = \begin{cases} 0 & \text{if } (a,b,c,D) > 1 \\ \left(\frac{D}{r}\right) & \text{if } (a,b,c,D) = 1 \text{ where } Q \text{ represents } r,\ (r,D) = 1. \end{cases}$$

If Q represents both r and s, then $4rs$ may be written as $x^2 - Dzy^2$ for some $x, y, z \in \mathbb{Z}$. Therefore $\left(\frac{D}{r}\right) = \left(\frac{D}{s}\right)$ so that $\chi_D(Q)$ is well-defined. Note that the value $\chi_D(Q)$ depends only on the $SL_2(\mathbb{Z})$-equivalence class of Q.

4.2.3 *Shimura and Shintani liftings*

Let t be a squarefree integer with $\epsilon(-1)^k t > 0$, $\epsilon = \chi(-1)$, χ a Dirichlet character modulo $2^\alpha N$ as before. Then the t-th Shimura map on the space $S_{k+1/2}(2^\alpha N, \chi_0)$ is defined by

$$f \mid \mathcal{S}_{t,\chi}(\tau) = \sum_{n \geq 1} \left(\sum_{d|n,(d,2^\alpha N)=1} \chi(d) \left(\frac{4t}{d}\right) d^{k-1} a_f(|t|n^2/d^2) \right) e^{2\pi i n\tau}. \tag{4.2}$$

Let $S^*_{t,\chi}$ be the adjoint of $S_{t,\chi}$ with respect to the Petersson inner product. If $\alpha = 2$, as done in [50], we put

$$\mathcal{S}_t = \begin{cases} W(4)U(4)W(4)\mathcal{S}_{t,\chi} & \text{if } t \equiv 1 (\text{mod } 4), \\ (W(4)U(4))^2 \mathcal{S}_{t,\chi} & \text{if } t \equiv 2, 3 (\text{mod } 4). \end{cases} \tag{4.3}$$

Here, we summarise the Shintani lifting obtained in ([50], [71]). Let $\alpha = 2$, N be squarefree and χ be a primitive Dirichlet character modulo N such that $\chi^2 \neq 1$. If t is a squarefree integer, $\epsilon(-1)^k t > 0$. Write

$$D_0 = \begin{cases} t & \text{if } t \equiv 1 (\text{mod } 4) \\ 4t & \text{if } t \equiv 2, 3 (\text{mod } 4) \end{cases}; \beta = \begin{cases} 4 & \text{if } t \equiv 1 (\text{mod } 4) \\ 5 & \text{if } t \equiv 2, 3 (\text{mod } 4) \end{cases}; c' = \begin{cases} 1 & \text{if } t \equiv 1 (\text{mod } 4) \\ 2 & \text{if } t \equiv 2, 3 (\text{mod } 4). \end{cases}$$

Then for $F \in S_{2k}(2N, \chi^2)$, we have

$$F \mid \mathcal{S}^*_t(\tau) = (-1)^{[k/2]} 2^{k-\beta+2} N^{-k+1/2} \bar{R}_{\chi,t} \sum_{m \geq 1} r_{k,2N,\chi}(F; \Delta m) e^{2\pi i m\tau},$$

where $\bar{R}_{\chi,t}$ is the complex conjugate of the Gauss sum $R_{\chi,t}$ given by

$$R_{\chi,t} = (N|D_0|)^{-1/2} \left(\frac{4\chi(-1)t}{-1}\right)^{-1/2} \sum_{r (\text{mod } N|D_0|)} \chi(r) \left(\frac{t}{r}\right) e^{2\pi i r/(N|D_0|)}$$

and

$$r_{k,2N,\chi}(F; \Delta m) = \sum \chi(c) \chi_{D_0}(Q) \int_{C_Q} F(z)(az^2 - bz + c)^{k-1} dz. \tag{4.4}$$

In the above, $\chi_{D_0}(Q)$ is as defined in §4.2.2 and the sum is over all $\Gamma_0(2N)$-equivalent quadratic forms $Q = [a, b, c]$ with discriminant $b^2 - 4ac = \Delta m$, $\Delta = 4|t|N^2$ and $a \equiv 0 (\text{mod } 2^{c'} N^2)$; C_Q is the image in $\Gamma_0(2N)\backslash\mathbb{H}$ of the semicircle $a|z|^2 + b\Re(z) + c = 0$ oriented from $(-b-\sqrt{\Delta m})/2a$ to $(-b+\sqrt{\Delta m})/2a$ if $a \neq 0$, or of the vertical line $b\Re(z) + c = 0$ oriented from $-c/b$ to $i\infty$ if $b > 0$ and from $i\infty$ to $-c/b$ if $b < 0, a = 0$. Define

$$\mathcal{S}^*_{t,\chi} = \begin{cases} \mathcal{S}^*_t(2^{-2k+1}W(4)U(4)W(4) - 2^{-k}W(4)) & \text{if } t \equiv 1 (\text{mod } 4), \\ \mathcal{S}^*_t(3 \cdot 2^{-2k} - 2^{-3k+1}W(4)U(4)) & \text{if } t \equiv 2, 3 (\text{mod } 4). \end{cases}$$

Then $\mathcal{S}^*_{t,\chi} : S_{2k}(2N, \chi^2) \to S_{k+1/2}(4N, \chi_0)$ is adjoint to the t-th Shimura lifting $\mathcal{S}_{t,\chi}$ with respect to the Petersson scalar product.

4.2.4 *Shintani lifting and special value of L-function*

Let F be a normalised newform in $S_{2k}(\ell, \chi^2)$, where $\ell = 2^{\alpha-1}N$, $\alpha \geq 2$, N odd and χ be a primitive Dirichlet character modulo ℓ', $\ell' \mid \ell$ and let $D \equiv 1 (\text{mod } 4)$ be a fundamental discriminant with $\epsilon(-1)^k D > 0, (D, \ell) = 1$. Then eqs. (6) and (10) of [49] relate the (D, r)-th Fourier coefficient of the Shintani lift of F (which will be a Jacobi form of weight $k+1$, index 1, level ℓ and character χ) and the special value $L(F, \overline{\chi}\left(\frac{D}{\cdot}\right), k)$. When $D \equiv r^2 (\text{mod } 4)$, combining this with the fact that

$$F|\mathcal{S}^*_{D,r}|\mathcal{Z}_1 = F|\mathcal{S}^*_D,$$

implies that $|D|$-th Fourier coefficient of $F \mid \mathcal{S}^*_D$ is a constant multiple of $L(F, \overline{\chi}\left(\frac{D}{\cdot}\right), k)$. In the above $\mathcal{S}^*_{D,r}$ is the Shintani map which takes cusp forms $F \in S_{2k}(\ell, \chi^2)$ to Jacobi cusp form in $J^{cusp}_{k+1,1}(\ell, \chi)$ defined by eq. (9) in [49] and $\mathcal{Z}_1$ is the Eichler-Zagier map defined by (4.15) in §4.2.5. Indeed, to compute the integral

$$r_{k,\ell,\chi}(F; \Delta|D|) = \sum \chi(c)\chi_D(Q) \int_{C_Q} F(z)(az^2 - bz + c)^{k-1} dz, \tag{4.5}$$

where the sum varies over all $\Gamma_0(\ell)$-equivalent quadratic forms $Q = [a, b, c]$ with discriminant $b^2 - 4ac = D^2\ell^2$, and $a \equiv 0 (\text{mod } \ell/2)$ with $b \equiv 0 \pmod{\ell}$, $(c, \ell/2) = 1$, we select the inequivalent representatives as the set of quadratic forms $[0, D\ell, \mu]$ with $\mu (\text{mod } D\ell)$. This can be done as follows:

Given a quadratic form $Q = [a, b, c]$ with the above conditions, write the associated matrix of Q as $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$. Let $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ be a matrix in $\Gamma_0(\ell)$. In order to get the representative, there should exist a matrix A in $\Gamma_0(\ell)$ such that $A^t[a,b,c]A = [0, D\ell, \mu]$ with $\mu \pmod{D\ell}$. That is

$$\begin{pmatrix} r & t \\ s & u \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} 0 & D\ell/2 \\ D\ell/2 & \mu \end{pmatrix}.$$

The left hand side of the above equals

$$\begin{pmatrix} r^2a+rtb+t^2c & rsa+b/2(st+ru)+tuc \\ rsa+b/2(st+ru)+tuc & s^2a+sub+u^2c \end{pmatrix} := \begin{pmatrix} a^* & b^*/2 \\ b^*/2 & c^* \end{pmatrix}.$$

Therefore, we have $r^2a+rtb+t^2c = 0$ so that $(t/r)^2c + (t/r)b + a = 0$. This quadratic equation has real solutions because the discriminant $b^2-4ac = D^2\ell^2$ (which is perfect square) is positive. The solutions are given by $t/r = (-b \pm |D|\ell)/2c$ so that $t = \frac{-b+|D|\ell}{(-b+|D|\ell,2c)}$ and $r = \frac{2c}{(-b+|D|\ell,2c)}$ or $t = \frac{-b-|D|\ell}{(-b-|D|\ell,2c)}$ and $r = \frac{2c}{(-b-|D|\ell,2c)}$. We easily see that ${b^*}^2 - 4a^*c^* = D^2\ell^2$ and hence $b^*/2 = D\ell/2$.
It is enough to show that μ varies $\pmod{2tM}$. Assuming if $\begin{pmatrix} 0 & D\ell/2 \\ D\ell/2 & \mu_1 \end{pmatrix}$ is equivalent to $\begin{pmatrix} 0 & D\ell/2 \\ D\ell/2 & \mu_2 \end{pmatrix}$,

$$\text{i.e.,} \quad \begin{pmatrix} r & t \\ s & u \end{pmatrix} \begin{pmatrix} 0 & D\ell/2 \\ D\ell/2 & \mu_2 \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} 0 & D\ell/2 \\ D\ell/2 & \mu_1 \end{pmatrix}$$

$$\text{i.e.,} \quad \begin{pmatrix} D\ell tr + \mu_2 t^2 & \frac{D\ell}{2}(st+ru) + \mu_2 tu \\ \frac{D\ell}{2}(st+ru) + \mu_2 tu & D\ell su + \mu_2 u^2 \end{pmatrix} = \begin{pmatrix} 0 & D\ell/2 \\ D\ell/2 & \mu_1 \end{pmatrix}.$$

Therefore, this gives $D\ell tr + \mu_2 t^2 = 0$, which implies that either $t = 0$ or if $t \neq 0$ then $D\ell r + \mu_2 t = 0$ and so $\mu_2 = -D\ell r/t$. Also, we have $\frac{D\ell}{2}(st+ru) + \mu_2 tu = D\ell/2$. Consider the case that $t \neq 0$, then $st + ru - 2(r/t)tu = 1$, which is a contradiction to the fact that $ru - st = 1$. Therefore, $t = 0$ so that $ru = 1$, which implies that $u = \pm 1$. Hence $\mu_1 \equiv \mu_2 \pmod{D\ell}$.
Hence the integral (4.5) becomes a non-zero constant times the special value $L(F, \overline{\chi}\left(\frac{D}{\cdot}\right), k)$.

4.2.5 *Holomorphic Jacobi forms and skew-holomorphic Jacobi forms of index* 1

In this section, we shall give some preliminaries on Jacobi forms and skew-holomorphic Jacobi forms of index 1. For a general theory of Jacobi forms, we refer to the monograph of M. Eichler and D. Zagier [13] and for skew-holomorphic

Jacobi forms, we refer to the works of N. P. Skoruppa [78], [79]. First we consider the holomorphic Jacobi forms. Let $M \geq 1$ be an integer and χ be a Dirichlet character modulo M.

Let $\Gamma^J(M) := \Gamma_0(M) \ltimes (\mathbb{Z} \times \mathbb{Z})$ denote the generalized Jacobi group. For any pair $X = (\gamma, (\lambda, \mu)) \in \Gamma^J(1)$ and any function ϕ on $\mathbb{H} \times \mathbb{C}$, define

$$\phi|_{k,1}X(\tau, z) = e\left(\lambda^2\tau + 2\lambda z - (z + \lambda\tau + \mu)^2 \frac{c}{c\tau + d}\right)(c\tau + d)^{-k} \times \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right), \tag{4.6}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Definition 4.2.1 (Holomorphic Jacobi forms) Let χ be any Dirichlet character modulo M. A function ϕ on $\mathbb{H} \times \mathbb{C}$ is said to be a holomorphic Jacobi form of weight k and index 1 with respect to the Jacobi group $\Gamma^J(M)$ and character χ, if it satisfies the following conditions:

1. $\phi(\tau, z)$ is a holomorphic function,
2. $\phi|_{k,1}X(\tau, z) = \chi(d)\phi(\tau, z)$ for all $X = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma^J(M)$,
3. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the function $\phi|_{k,1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (0,0)\right)(\tau, z)$ has a Fourier expansion of the form
$$\sum_{\substack{n, r \in \mathbb{Q} \\ 4n \geq r^2}} c_\phi(n, r) e(n\tau + rz).$$

If ϕ satisfies the stronger condition $c_\phi(n, r) = 0$ unless $4n > r^2$, then it is called a holomorphic Jacobi cusp form.

We call $c_\phi(n, r)$, the (n, r)-th Fourier coefficient of the holomorphic Jacobi form ϕ.

Remark 4.2.2 Sometimes we write simply Jacobi forms for holomorphic Jacobi forms when there is no confusion.

The set of all holomorphic Jacobi forms as defined above forms a $\mathbb{C}$-vector space and is denoted by $J_{k,1}(M, \chi)$, the space of holomorphic Jacobi forms of weight k, index 1 for $\Gamma_0(M)$ with character χ, where χ is a Dirichlet character modulo

M. We denote the vector subspace of all holomorphic Jacobi cusp forms by $J_{k,1}^{\text{cusp}}(M,\chi)$. If χ is a trivial character, then we write these spaces as $J_{k,1}(M)$ and $J_{k,1}^{\text{cusp}}(M)$ respectively.

Definition 4.2.3 (Petersson inner product) For holomorphic Jacobi cusp forms ϕ and ψ on $\Gamma^J(M)$, we define the Petersson scalar product of them as follows.

$$\langle \phi, \psi \rangle := \frac{1}{[\Gamma^J(1) : \Gamma^J(M)]} \int_{\Gamma^J(M)\backslash \mathbb{H}\times\mathbb{C}} \phi(\tau,z)\overline{\psi(\tau,z)}e^{-4\pi y^2/v}v^{k-3}dudvdxdy, \tag{4.7}$$

where $\tau = u + iv$ and $z = x + iy$.

Poincaré series (holomorphic case): For $n \in \mathbb{Z}, r \in \mathbb{Z}$ with $4n > r^2$ and $k > 3$, we define the (n,r)-th holomorphic Jacobi Poincaré series of exponential type by

$$P_{(n,r)}(\tau,z) := \sum_{X\in\Gamma^J_\infty\backslash\Gamma^J(M)} \overline{\chi}(d)\ e^{(n,r)}|_{k,1}X(\tau,z) \qquad (\tau\in\mathbb{H}, z\in\mathbb{C}), \tag{4.8}$$

where $X = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda,\mu)\right)$, $\Gamma^J_\infty = \left\{\left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, (0,\mu)\right) | m\in\mathbb{Z}, \mu\in\mathbb{Z}\right\}$ and $e^{(n,r)}(\tau,z) = e(n\tau + rz)$.

The Poincaré series on $J_{k,1}^{\text{cusp}}(M,\chi)$ is characterized by using the Petersson scalar product in terms of the Fourier coefficients which we give below. Let

$$\phi(\tau,z) = \sum_{\substack{m,r\in\mathbb{Z}\\4m>r^2}} c_\phi(m,r)e(m\tau+rz)\ \in J_{k,1}^{\text{cusp}}(M,\chi).$$

Then, the (n,r)-th Poincaré series $P_{(n,r)}$ in $J_{k,1}^{\text{cusp}}(M,\chi)$ is uniquely determined by

$$\langle \phi, P_{(n,r)} \rangle = \lambda_{k,D,M} c_\phi(n,r), \tag{4.9}$$

with $D = r^2 - 4n$ and

$$\lambda_{k,D,M} = \frac{\Gamma(k-3/2)}{\pi^{k-3/2}[\Gamma^J(1):\Gamma^J(M)]}|D|^{-k+3/2}. \tag{4.10}$$

Let us now recall the definition of a skew-holomorphic Jacobi form. For any pair $X = (\gamma,(\lambda,\mu)) \in \Gamma^J(1)$ and any function ϕ on $\mathbb{H}\times\mathbb{C}$, define

$$\phi|^*_{k,1}X(\tau,z) = e\left(\lambda^2\tau + 2\lambda z - (z+\lambda\tau+\mu)^2\frac{c}{c\tau+d}\right)|c\tau+d|^{-1}\overline{(c\tau+d)}^{1-k}$$

$$\times\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right), \tag{4.11}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

As before, let $M \geq 1$ be an integer and χ be a Dirichlet character modulo M.

Definition 4.2.4 (Skew-holomorphic Jacobi forms) Let χ be any Dirichlet character modulo M. A function ϕ on $\mathbb{H} \times \mathbb{C}$ is said to be a skew-holomorphic Jacobi form of weight k and index 1 with respect to the Jacobi group $\Gamma^J(M)$ and character χ, if it satisfies the following conditions.

1. $\phi(\tau, z)$ is a smooth function in $\tau \in \mathbb{H}$ and holomorphic in $z \in \mathbb{C}$,
2. $\phi|^*_{k,1}X(\tau, z) = \chi(d)\phi(\tau, z)$ for all $X = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma^J(M)$,
3. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the function $\phi|_{k,1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (0,0)\right)(\tau, z)$ has a Fourier expansion of the form

$$\sum_{\substack{n,r\in\mathbb{Q} \\ 4n\leq r^2}} c_\phi(n,r)e(n\tau + \frac{i}{2}(r^2-4n)\ \mathrm{Im}\ \tau + rz).$$

If ϕ satisfies the stronger condition $c_\phi(n,r) = 0$ unless $4n < r^2$, for all $\gamma \in SL_2(\mathbb{Z})$, then it is called a skew-holomorphic Jacobi cusp form.

We call $c_\phi(n,r)$, the (n,r)-th Fourier coefficient of the skew- holomorphic Jacobi form ϕ.

The set of all skew-holomorphic Jacobi forms as defined above forms a $\mathbb{C}$-vector space and we denote it by $J^*_{k,1}(M,\chi)$. We denote the vector subspace of all skew-holomorphic Jacobi cusp forms by $J^{*,\mathrm{cusp}}_{k,1}(M,\chi)$. If χ is a trivial character, then we write these spaces as $J^*_{k,1}(M)$ and $J^{*,\mathrm{cusp}}_{k,1}(M)$ respectively.

Definition 4.2.5 (Petersson inner product) For skew-holomorphic Jacobi cusp forms ϕ and ψ of weight k and index 1 on $\Gamma^J(M)$, we define the Petersson scalar product of them similar to the holomrphic case as follows.

$$\langle\phi,\psi\rangle := \frac{1}{[\Gamma^J(1):\Gamma^J(M)]}\int_{\Gamma^J(M)\backslash\mathbb{H}\times\mathbb{C}} \phi(\tau,z)\overline{\psi(\tau,z)}e^{-4\pi y^2/v}v^{k-3}dudvdxdy, \tag{4.12}$$

where $\tau = u + iv$ and $z = x + iy$.

Poincaré series (skew-holomorphic case): For $n \in \mathbb{Z}, r \in \mathbb{Z}$ with $4n < r^2$ and $k > 3$, define the (n,r)-th skew-holomorphic Jacobi Poincaré series of exponential type by

$$P^*_{(n,r)}(\tau, z) := \sum_{X \in \Gamma^J_\infty \backslash \Gamma^J(M)} \overline{\chi}(d) e_*^{(n,r)}|^*_{k,1} X(\tau, z) \qquad (\tau \in \mathbb{H}, z \in \mathbb{C}), \tag{4.13}$$

where $X = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right)$, $\Gamma^J_\infty = \left\{ \left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) | m \in \mathbb{Z}, \mu \in \mathbb{Z}^{(g,1)} \right\}$ and $e_*^{(n,r)}(\tau, z) = e(n\tau + \frac{i}{2}(r^2 - 4n) \text{ Im } \tau + rz)$. Using the definition and the absolute convergence of the series $P^*_{(n,r)}$, we get the transformation formula. Moreover it can be shown that $P^*_{(n,r)} \in J^{*,\text{cusp}}_{k,1}(M, \chi)$.

Let $\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4n < r^2}} c_\phi(n,r) \; e(n\tau + \frac{i}{2}(r^2 - 4n) \text{ Im } \tau + rz) \;\; \in J^{*,cusp}_{k,1}(M, \chi)$.

Then, the (n,r)-th Poincaré series $P^*_{(n,r)}$ in $J^{*,\text{cusp}}_{k,1}(M, \chi)$ is uniquely determined by

$$\langle \phi, P^*_{(n,r)} \rangle = \lambda_{k,D,M} c_\phi(n,r), \tag{4.14}$$

where $\lambda_{k,D,M}$ is the same constant as defined by (4.10) in the holomorphic case.

Eichler-Zagier map: Define the map $\mathcal{Z}_1$ on $J^{\text{cusp}}_{k+1,1}(M, \chi)$, (usually known as the Eichler-Zagier map) as follows:

$$\begin{aligned} \mathcal{Z}_1 : J^{\text{cusp}}_{k+1,1}(M, \chi) \;&\longrightarrow\; S^+_{k+1/2}\left(4M, \left(\frac{4\chi(-1)}{\cdot}\right)\chi\right), \\ \sum_{\substack{D<0, r \in \mathbb{Z} \\ D \equiv r^2 (\text{mod } 4)}} a(D,r) \exp\left(\frac{r^2 - D}{4}\tau + rz\right) \;&\longmapsto\; \sum_{\substack{D<0 \\ D \equiv 0,1 (\text{mod } 4)}} a(|D|) \exp(|D|\tau). \end{aligned} \tag{4.15}$$

For $M = 1$, $\mathcal{Z}_1$ is nothing but the map defined by Eichler and Zagier in [13], which is a canonical map from $J^{\text{cusp}}_{k+1,1}(1)$ onto $S^+_{k+1/2}(4)$. When $2 \nmid M$, $\mathcal{Z}_1$ is the map defined in [52] in connection with the Saito-Kurokawa descent. In §4.4, we will prove that the Eichler-Zagier map $\mathcal{Z}_1$ as defined above is an Hecke equivarient isomorphism and preserving the inner product structure.

4.3 Newform theory for $S_{k+1/2}(4N, \chi_0)$

In this section, we consider $\alpha = 2$ and consider the space $S_{k+1/2}(4N, \chi_0)$, where χ is a primitive Dirichlet character modulo N such that χ^2 is also a primitive Dirichlet character modulo N, N an odd and squarefree positive integer and $\chi_0 := \left(\frac{4\chi(-1)}{\cdot}\right)\chi$. The following result is derived by Serre and Stark in [73].

However, we present a proof here which uses the nature of the Gauss sum associated with primitive characters. A similar calculation has been carried out for integral weight forms by H. Iwaniec [27] in Chapter 6, p. 109.

Proposition 4.3.1 *If* $f(\tau) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi i n\tau} \in M_{k+1/2}(4N, \chi_0)$ *such that its Fourier coefficients* $a_f(n)$ *satisfy the condition* $a_f(n) = 0$*, for* $n \geq 1,\ (n, 2N) = 1$*, then* $f = 0$*.*

Proof. Assume that $a_f(n) = 0$, for $n \geq 1,\ (n, 2N) = 1$ and $f \neq 0$. Then, there exists a divisor N_1 of $2N$ such that $a_f(nN_1) \neq 0$ for some $n \geq 1$. We note that for a given integer $u \pmod{4N}, (u, 2N) = 1$, there exists a unique integer $v \pmod{4N}$ such that $uv \equiv -1 \pmod{4N}$ and

$$\left(\begin{pmatrix} 1 & \frac{u}{4N} \\ 0 & 1 \end{pmatrix}, 1\right) \left(\begin{pmatrix} 0 & \frac{-1}{4N} \\ 4N & 0 \end{pmatrix}, \sqrt{4N\tau}\right) \left(\begin{pmatrix} 1 & \frac{-v}{4N} \\ 0 & 1 \end{pmatrix}, 1\right)$$
$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \left(\frac{N}{-v}\right) \left(\frac{-4}{-v}\right)^{-1/2}\right) \begin{pmatrix} u & \frac{-uv-1}{4N} \\ 4N & -v \end{pmatrix}^*,$$

where $\begin{pmatrix} u & \frac{-uv-1}{4N} \\ 4N & -v \end{pmatrix} \in \Gamma_0(4N)$. Since N is squarefree, we write $2N = N_1N_2$ with $(N_1, N_2) = 1$. Let

$$g = \sum_{u \pmod{4N}^*} \left(\frac{u}{N_2}\right) f \mid \left(\begin{pmatrix} 1 & \frac{u}{4N} \\ 0 & 1 \end{pmatrix}, 1\right).$$

Then, we have

$$g \left| \left(\begin{pmatrix} 0 & \frac{-1}{4N} \\ 4N & 0 \end{pmatrix}, \sqrt{4Nz}\right) = f \right| \sum_{v \pmod{4N}^*} \left(\frac{N}{-v}\right) \left(\frac{-4}{-v}\right)^{-1/2} \chi(-v) \left(\frac{-v}{N_2}\right) \begin{pmatrix} 1 & \frac{v}{4N} \\ 0 & 1 \end{pmatrix}^*$$
$$= \sum_{n=0}^{\infty} a_f(n) \left(\frac{-1}{N_2}\right) \sum_{v \pmod{4N}^*} \left(\frac{N}{-v}\right) \left(\frac{-4}{-v}\right)^{-k-1/2} \chi(-v) \left(\frac{v}{N_2}\right) e^{2\pi i n \frac{v}{4N}} e^{2\pi i n\tau}.$$

Since χ is a primitive Dirichlet character mod N, the Gauss sum

$$\sum_{v \pmod{4N}^*} \left(\frac{N}{-v}\right) \left(\frac{-4}{-v}\right)^{-k-1/2} \chi(-v) \left(\frac{v}{N_2}\right) e^{2\pi i n \frac{v}{4N}}$$

vanishes whenever $(n, 2N) > 1$. But by our assumption, we have $a_f(n) = 0$ for all $(n, 2N) = 1$. Thus, we get

$$\begin{aligned} 0 = g &= \sum_{u (\bmod\ 4N)^*} \left(\frac{u}{N_2}\right) f(\tau + \frac{u}{4N}) \\ &= \sum_{n=0}^{\infty} a_f(n) \sum_{u (\bmod\ 4N)^*} \left(\frac{u}{N_2}\right) e^{2\pi i n \frac{u}{4N}} e^{2\pi i n \tau}. \end{aligned}$$

Comparing the n-th Fourier coefficients, we get

$$a_f(n) \sum_{u (\bmod\ 4N)^*} \left(\frac{u}{N_2}\right) e^{2\pi i n \frac{u}{4N}} = 0, \qquad \forall n \geq 1,$$

replacing n by nN_1, we see that

$$a_f(nN_1) \sum_{u (\bmod\ 4N)^*} \left(\frac{u}{N_2}\right) e^{2\pi i n \frac{u}{4N_2}} = 0.$$

In other words,

$$N_1\ a_f(nN_1) \left(\frac{n}{N_2}\right) \left(\frac{-1}{N_2}\right)^{1/2} \sqrt{N_2} = 0 \qquad \forall n \geq 1,$$

a contradiction. Hence $f = 0$. This proves the proposition. □

Corollary 4.3.2 *If f is a Hecke eigenform in $M_{k+1/2}(4N, \chi_0)$. Then there exists a squarefree integer t, $\epsilon(-1)^k t > 0$, $(t, 2N) = 1$ such that $a_f(|t|) \neq 0$.*

Proof. Since $f \in M_{k+1/2}(4N, \chi_0)$ is a Hecke eigenform. Therefore, we have

$$f \mid T(n^2) = \lambda(n) f.$$

Comparing the $|t|$-th Fourier coefficient on both the sides of the above, we get

$$\sum_{d|n} d^{k-1} \chi_0(d) \left(\frac{4t}{d}\right) a_f\left(\frac{|t| n^2}{d^2}\right) = \lambda(n) a_f(|t|).$$

Suppose on the contrary, that $a_f(|t|) = 0$, for every $(t, N) = 1$. Then by the above relation, we deduce that $a_f(|t|n^2) = 0$, for every $(n, N) = 1$. Since, any positive integer m, $(m, N) = 1$ can be uniquely written as $m = tn^2$, with t-squarefree and $(t, N) = (n, N) = 1$. Therefore, we have $a_f(n) = 0$, for every $(n, N) = 1$. Hence, by Proposition 4.3.1, we have $f = 0$, which is a contradiction. This proves the corollary. □

Multiplicity one result: We now derive the multiplicity one result for Hecke eigenforms in $S^+_{k+1/2}(4N,\chi_0)$ and $S^{new}_{k+1/2}(4N,\chi_0)$. The proof is obtained by using the non-vanishing of Shintani lifts on the normalised Hecke eigenform $F \in S_{2k}(2N,\chi^2)$ and the equality of the dimensions of the corresponding spaces of half-integral and integral weight cusp forms. The equality of the required dimension formula is derived by Kojima [38]. In order to prove the non-vanishing of Shintani lifts, we use §4.2.4 to deduce that the $|D|$-th Fourier coefficient of the D-th Shintani lift of F is equal to a constant multiple of $L(F,\bar{\chi}\left(\frac{D}{\cdot}\right),k)$. Then we use the non-vanishing of $L(F,\bar{\chi}\left(\frac{D}{\cdot}\right),k)$ (which follows from Chapter 6 of [63]) to get our result. More precisely, we have the following theorem.

Theorem 4.3.3 *The multiplicity one result holds good for the space $S^+_{k+1/2}(4N,\chi_0)$. Moreover, the space $S^+_{k+1/2}(4N,\chi_0)$ is isomorphic to the space $S_{2k}(N,\chi^2)$ under a certain linear combination of Shimura lifts.*

Proof. In Theorem 2.1 of [38], the following equality of dimensions was obtained.

$$\dim\, S^+_{k+1/2}(4N,\chi_0) = \dim\, S_{2k}(N,\chi^2).$$

Observe that the full space $S_{2k}(N,\chi^2)$ is the space of newforms since χ^2 is a primitive Dirichlet character modulo N. For each normalised Hecke eigenform $F \in S_{2k}(N,\chi^2)$, the result in Chapter 6 of [63] gives a fundamental discriminant D with $(D,N)=1$ such that $L(F,\overline{\chi}\left(\frac{D}{\cdot}\right),k) \neq 0$. Hence by §4.2.4, we have $a_{F|\mathcal{S}^*_{D,\chi}}(|D|) \neq 0$, which shows that $F \mid \mathcal{S}^*_{D,\chi} \neq 0$.

Now let $d = \dim\, S_{2k}(N,\chi^2)$ and let $F_1, F_2, ..., F_d$ be an orthogonal basis of $S_{2k}(N,\chi^2)$ which are normalised Hecke eigenforms. Then, for some choices of odd fundamental discriminants $D_1,\ldots,D_d$, we have d cusp forms $f_i := F_i \mid \mathcal{S}^*_{D_i,\chi}$, $1 \le i \le d$, which are nonzero in $S^+_{k+1/2}(4N,\chi_0)$ and write $f_j(\tau) = \sum_{n\ge1} a_{f_j}(n)e^{2\pi i n\tau}$, $1 \le j \le d$. Suppose that $\alpha_1 f_1 + \ldots + \alpha_d f_d = 0$, for some $\alpha_i \in \mathbb{C}$, $1 \le i \le d$. Then, for any odd fundamental discriminant D with $\epsilon(-1)^k D > 0$, applying the D^{th} Shimura map $\mathcal{S}_{D,\chi}$ on both the sides of above and using Theorem 1.2 of [70], we see that $f_i \mid \mathcal{S}_{D,\chi}$ and F_i have the same eigenvalues for the Hecke operators $T(n)$, $(n,N)=1$. Therefore by using the multiplicity one which is valid in $S_{2k}(N,\chi^2)$ and comparing the first Fourier coefficient of $f_i \mid \mathcal{S}_{D,\chi}$ and F_i, we see that the above equals

$$\alpha_1 a_{f_1}(|D|)F_1 + \alpha_2 a_{f_2}(|D|)F_2 + \ldots + \alpha_d a_{f_d}(|D|)F_d = 0,$$

from which it follows that

$$\alpha_1 a_{f_1}(|D|) = 0 = \alpha_2 a_{f_2}(|D|) = \ldots = \alpha_d a_{f_d}(|D|).$$

Selecting $D = D_i$, $1 \leq i \leq d$, successively, we get $\alpha_i = 0$, since $a_{f_i}(|D_i|) \neq 0$. This proves that the d forms $f_1, \ldots, f_d$ which have been selected as above form a basis of the space $S^+_{k+1/2}(4N, \chi_0)$. Now we define the space of newforms in $S^+_{k+1/2}(4N, \chi_0)$ as

$$S^{+,new}_{k+1/2}(4N, \chi_0) = \bigoplus_{i=1}^{d} S^{+,new}_{k+1/2}(4N, \chi_0; F_i)$$

where

$$S^{+,new}_{k+1/2}(4N, \chi_0; F_i) = \{f \in S^+_{k+1/2}(4N, \chi_0) : f|T(n^2) = a_{F_i}(n)f, \forall n \geq 1, (n, 2N) = 1\}.$$

Note that $F_i \mid \mathcal{S}^*_{D_i,\chi}$ is a non-zero element in $S^{+,new}_{k+1/2}(4N, \chi_0; F_i)$ and hence dimension of $S^{+,new}_{k+1/2}(4N, \chi_0; F_i)$ is atleast one for $1 \leq i \leq d$. But the spaces $S^+_{k+1/2}(4N, \chi_0)$ and $S_{2k}(N, \chi^2)$ have the same dimension, hence each $S^{+,new}_{k+1/2}(4N, \chi_0; F_i)$ is of dimension one. Therefore multiplicity one result holds in the space $S^{+,new}_{k+1/2}(4N, \chi_0)$ and $S^{+,new}_{k+1/2}(4N, \chi_0) = S^+_{k+1/2}(4N, \chi_0)$.

Now we prove that the spaces $S^+_{k+1/2}(4N, \chi_0)$ and $S_{2k}(N, \chi^2)$ are isomorphic under a certain linear combination of Shimura lifts. Since $f_1, \ldots, f_d$ as above forms a basis of $S^+_{k+1/2}(4N, \chi_0)$ which are common eigenforms of Hecke operators $T(n^2)$, $(n, 2N) = 1$. Moreover, every f_j determines a fundamental discriminant D_j, $\epsilon(-1)^k D_j > 0$ such that $a_{f_j}(|D_j|) \neq 0$. Then the complex polynomial

$$P(X_1, \ldots, X_d) = \prod_{1 \leq j \leq d} (a_{f_j}(|D_1|)X_1 + \ldots + a_{f_j}(|D_d|)X_d)$$

is non-zero, hence there exists $(\beta_1, \ldots, \beta_d) \in \mathbb{C}^d$ with $P(\beta_1, \ldots, \beta_d) \neq 0$. Define $\mathcal{S} = \sum_{j=1}^{d} \beta_j \mathcal{S}_{D_j,\chi}$. Then for every $j \in \{1, \ldots, d\}$, $f_j \mid \mathcal{S}$ lies in $S_{2k}(N, \chi^2)$ and is a non-zero eigenform of all the Hecke operators $T(n)$. The fact that $f_j \mid \mathcal{S}$ is non-zero follows by looking at the first Fourier coefficient of $f_j \mid \mathcal{S}$ and $P(\beta_1, \ldots, \beta_d) \neq 0$. If $f_j \mid \mathcal{S} = f_l \mid \mathcal{S}$ then by Theorem 1.2 of [70], we see that $\mathcal{S}$ commutes with the respective Hecke operators, f_j and f_l have the same eigenvalues for all $T(n^2)$, $(n, 2N) = 1$. Hence by the multiplicity one result in $S^+_{k+1/2}(4N, \chi_0)$ as above, we have $j = l$. From this we see that $\mathcal{S}$ is injective and since the dimension of the respective spaces are equal, it is an isomorphism. This proves the theorem. □

We observe that all the arguments as above also hold good on the full space $S_{k+1/2}(4N,\chi_0)$ and moreover, in Theorem 1.1 of [38], the following equality of dimension is derived:

$$\dim\, S_{k+1/2}(4N,\chi_0) = \dim\, S_{2k}(2N,\chi^2).$$

Hence, we have the following.

Theorem 4.3.4 *The multiplicity one result holds good for the space $S^{new}_{k+1/2}(4N,\chi_0)$ and the space $S^{new}_{k+1/2}(4N,\chi_0)$ is isomorphic to the space $S^{new}_{2k}(2N,\chi^2)$ under a certain linear combination of Shimura-Kohnen lifts.*

Proof. We decompose the spaces $S_{k+1/2}(4N,\chi_0)$ and $S_{2k}(2N,\chi^2)$ respectively as follows.

$$\begin{aligned} S_{k+1/2}(4N,\chi_0) &= S^{new}_{k+1/2}(4N,\chi_0) \bigoplus (S^{+}_{k+1/2}(4N,\chi_0) \oplus S^{+}_{k+1/2}(4N,\chi_0)|U(4)), \\ S_{2k}(2N,\chi^2) &= S^{new}_{2k}(2N,\chi^2) \bigoplus (S_{2k}(N,\chi^2) \oplus S_{2k}(N,\chi^2)|U(2)). \end{aligned}$$

By Theorem 4.3.3, we see that the space $S^{+}_{k+1/2}(4N,\chi_0)$ is isomorphic to the space $S_{2k}(N,\chi^2)$ under a certain linear combination of Shimura lifts $\mathcal{S}$ and therefore $S^{+}_{k+1/2}(4N,\chi_0)|U(4)$ is isomorphic to $S_{2k}(N,\chi^2)|U(2)$ under the lift $\mathcal{S}$. Moreover, by Theorem 1.1 of [38], we know that the spaces $S_{k+1/2}(4N,\chi_0)$ and $S_{2k}(2N,\chi^2)$ have the same dimensions. Hence the spaces $S^{new}_{k+1/2}(4N,\chi_0)$ and $S^{new}_{2k}(2N,\chi^2)$ have the same dimension. Therefore, by using the similar arguments as in Theorem 4.3.3, we see that the theorem follows. □

4.4 Newform theory for Jacobi forms of index 1

Let N be an odd and squarefree integer, let χ be a primitive Dirichlet character modulo N such that χ^2 is also a primitive Dirichlet character modulo N and $2 \nmid k$. Let $\chi_0 = \left(\frac{4}{\cdot}\right)\chi$.

Proposition 4.4.1 *Let N be an odd and squarefree integer, $\chi(-1) = 1$ and $2 \nmid k$. The Eichler-Zagier map $\mathcal{Z}_1 : J^{cusp}_{k+1,1}(N,\chi) \longrightarrow S^{+}_{k+1/2}(4N,\chi_0)$ is a Hecke equivarient isomorphism and preserving the inner product structure. i.e.,*

$$\langle \phi \mid \mathcal{Z}_1, \psi \mid \mathcal{Z}_1 \rangle = const\ \langle \phi, \psi \rangle,$$

where $\phi, \psi \in J^{cusp}_{k+1,1}(N,\chi)$.

Proof. In [49], Manickam and Ramakrishnan proved that for $0 > D \equiv r^2 \pmod 4$,

$$\begin{aligned} P_{(D,r)} \mid \mathcal{Z}_1 &= C^{-1} P_{|D|}, \text{ where } C \text{ is some nonzero known constant.} \\ \text{Therefore, } \langle P_{|D_1|}, P_{|D_2|} \rangle &= C \langle P_{|D_1|}, P_{(D_2,r_2)} \mid \mathcal{Z}_1 \rangle \\ &= C \langle P_{|D_1|} \mid \mathcal{Z}_1^*, P_{(D_2,r_2)} \rangle, \end{aligned}$$

where $\mathcal{Z}_1^*$ is the adjoint of $\mathcal{Z}_1$ with respect to Petersson scalar product and $P_{|D|}$ denotes the $|D|$-th Poincaré series in $S^+_{k+1/2}(4N, \chi_0)$. For any $\phi \in J^{\text{cusp}}_{k+1,1}(N, \chi)$, we have

$$\begin{aligned} \langle P_{|D_1|} \mid \mathcal{Z}_1^*, \phi \rangle &= \langle P_{|D_1|}, \phi \mid \mathcal{Z}_1 \rangle \\ &= \frac{\Gamma(k-1/2)}{(4\pi|D_1|)^{k-1/2}} \overline{a_{\phi|\mathcal{Z}_1}(|D_1|)} \\ &= \frac{\Gamma(k-1/2)}{(4\pi|D_1|)^{k-1/2}} \overline{c_\phi(D_1, r_1)} \\ &= \lambda^{-1}_{k+1,D_1,N} \frac{\Gamma(k-1/2)}{(4\pi|D_1|)^{k-1/2}} \overline{\langle \phi, P_{(D_1,r_1)} \rangle} \\ &= \lambda \langle P_{(D_1,r_1)}, \phi \rangle \text{ (say)}, \end{aligned}$$

where $\lambda_{k+1,D_1,N}$ is defined by (4.10). Taking $\phi = P_{(D_2,r_2)}$, we have

$$\begin{aligned} \langle P_{|D_1|}, P_{|D_2|} \rangle &= C \cdot \lambda \langle P_{(D_1,r_1)}, P_{(D_2,r_2)} \rangle. \\ \text{i.e.,} \quad \langle P_{(D_1,r_1)} \mid \mathcal{Z}_1, P_{(D_2,r_2)} \mid \mathcal{Z}_1 \rangle &= C_1 \cdot \lambda \langle P_{(D_1,r_1)}, P_{(D_2,r_2)} \rangle, \end{aligned}$$

where C_1 is some constant. Since the Poincaré series span the space of Jacobi forms, we have

$$\langle \phi \mid \mathcal{Z}_1, \psi \mid \mathcal{Z}_1 \rangle = \text{const. } \langle \phi, \psi \rangle.$$

Hence $\mathcal{Z}_1$ is an onto, injective and inner product structure preserving linear map and it defines an isomorphism. This completes the proof. □

Theorem 4.3.3 gives the multiplicity one result for the space $S^+_{k+1/2}(4N, \chi_0)$. Therefore, using the isomorphism $\mathcal{Z}_1$ (Proposition 4.4.1), we obtain the following multiplicity one theorem in $J^{\text{cusp}}_{k+1,1}(N, \chi)$.

Theorem 4.4.2 *Let N be an odd and squarefree integer, $k \geq 2$ be an odd integer and χ be an even primitive Dirichlet character modulo N such that χ^2 is also a primitive Dirichlet character modulo N. Then the multiplicity one result holds good in the space $J^{cusp}_{k+1,1}(N, \chi)$.*

In a similar way, we obtain the corresponding result in the case of skew-holomorphic Jacobi forms. (For the Eichler-Zagier map for skew-holomorphic Jacobi forms, we refer to [48].)

Theorem 4.4.3 *Let N be an odd and squarefree integer, $k \geq 2$ be an even integer and χ be an odd primitive Dirichlet character modulo N such that χ^2 is also a primitive Dirichlet character modulo N. The spaces $J^{*,cusp}_{k+1,1}(N, \chi_0)$ and $S^{+}_{k+1/2}(4N, \chi_0)$ are Hecke equivarient isomorphic and preserving the inner product structures. Moreover, the space $J^{*,cusp}_{k+1,1}(N, \chi_0)$ has multiplicity one result.*

4.5 Newform theory for $S_{k+1/2}(32N)$

In order to study the newform theory, we first compute the dimensions of the spaces $S_{k+1/2}(\Gamma_0(2^{a+2}N))$ and $S_{2k}(\Gamma_0(2^a N))$, where $a \geq 3$, N is odd and squarefree. Using the dimension formula as given in Proposition 12 of [54], we have

$$\dim S_{2k}(2^a N) \quad = \quad \frac{2k-1}{12} 2^a N \prod_{p|2N} (1 + \frac{1}{p}) - \frac{1}{2} v_\infty(2^a) 2^{\nu(N)}, \tag{4.16}$$

where

$$v_\infty(2^a) = \begin{cases} 2^{\frac{a+1}{2}} & \text{if } a \text{ is odd,} \\ 2^{a/2} + 2^{\frac{a}{2}-1} & \text{if } a \text{ is even,} \end{cases}$$

and $\nu(N)$ is the number of distinct prime factors of N.
The dimension formulas for the case of half-integral weight were first obtained by Cohen and Oesterlé in [7]. However, we use the formula given in Theorem 1.56, p. 16 of [68] to get

$$\begin{aligned} \dim S_{k+1/2}(2^{a+2}N) \quad &= \quad \frac{2k-1}{24} 2^{a+2} N \prod_{p|2N} (1 + \frac{1}{p}) - \frac{\zeta(k,\ 2^{a+2}N,\ 1)}{2} \prod_{p|N} \lambda(r_p, s_p, p) \\ &= \quad \frac{2k-1}{6} 2^a N \prod_{p|2N} (1 + \frac{1}{p}) - \frac{\zeta(k,\ 2^{a+2}N,\ 1)}{2} 2^{\nu(N)}, \end{aligned} \tag{4.17}$$

where

$$\zeta(k,\ 2^{a+2}N,\ 1) = \begin{cases} 2^{\frac{a+3}{2}} & \text{if } a \text{ is odd,} \\ 2^{a/2} + 2^{\frac{a}{2}+1} & \text{if } a \text{ is even.} \end{cases}$$

Equations (4.16) and (4.17) imply the following lemma.

Lemma 4.5.1 *For an integer $a \geq 3$ and for an odd and squarefree positive integer N, we have*

$$dim\ S_{k+1/2}(2^{a+2}N) = 2\ dim\ S_{2k}(2^a N).$$

In particular, for $a = 3$, we have dim $S_{k+1/2}(32N) = 2$ dim $S_{2k}(8N)$.

Now, we state the main theorem of this section.

Theorem 4.5.2 *If $N \geq 1$ is an odd squarefree positive integer, then we have*

$$S^{new}_{k+1/2}(32N) = \{0\} \tag{4.18}$$

and we have the following decomposition of the full space:

$$\begin{aligned} S_{k+1/2}(32N) = & \bigoplus_{rd|N} \Big\{ S^{+,new}_{k+1/2}(4d) \oplus S^{+,new}_{k+1/2}(4d) \mid U(4) \oplus S^{+,new}_{k+1/2}(4d) \mid U(4)\mathcal{P}_+ \\ & \oplus S^{+,new}_{k+1/2}(4d) \mid U(8)B(2) \oplus S^{+,new}_{k+1/2}(4d) \mid B(4) \\ & \oplus S^{+,new}_{k+1/2}(4d) \mid U(4)B(4) \oplus S^{+,new}_{k+1/2}(4d) \mid U(8)W(8)B(4) \\ & \oplus S^{+,new}_{k+1/2}(4d) \mid R_{\left(\frac{8}{\cdot}\right)} \Big\} \mid U(r^2) \\ & \oplus \bigoplus_{rd|N} \Big\{ S^{new}_{k+1/2}(4d) \oplus S^{new}_{k+1/2}(4d)|\mathcal{P}_+ \oplus S^{new}_{k+1/2}(4d)|U(2)B(2) \\ & \oplus S^{new}_{k+1/2}(4d) \mid B(4) \oplus S^{new}_{k+1/2}(4d) \mid U(2)W(32) \\ & \oplus S^{new}_{k+1/2}(4d) \mid R_{\left(\frac{8}{\cdot}\right)} \Big\} \mid U(r^2) \\ & \oplus \bigoplus_{rd|N} \Big\{ S^{new}_{k+1/2}(8d) \oplus S^{new}_{k+1/2}(8d)|W(16) \oplus S^{new}_{k+1/2}(8d)|B(4) \\ & \oplus S^{new}_{k+1/2}(8d) \mid R_{\left(\frac{8}{\cdot}\right)} \Big\} \mid U(r^2) \\ & \oplus \bigoplus_{rd|N} \Big\{ S^{new}_{k+1/2}\left(16d, \left(\frac{8}{\cdot}\right)\right) \mid B(2) \\ & \oplus S^{new}_{k+1/2}\left(16d, \left(\frac{8}{\cdot}\right)\right)|\mathcal{P}_+ W(32) \Big\} |U(r^2). \end{aligned}$$

Before starting the proof of the above theorem, we state two results which we use frequently in the proof. The following theorem was proved as Theorem 4.1 in [53].

Theorem 4.5.3 *For N an odd and squarefree positive integer, we have*

$$S^{new}_{k+1/2}(16N) = \{0\} \tag{4.19}$$

and

$$
\begin{aligned}
S_{k+1/2}(16N) \;=\; & \bigoplus_{rd|N} \Big\{ S^{+,new}_{k+1/2}(4d) \oplus S^{+,new}_{k+1/2}(4d) \mid U(4) \oplus S^{+,new}_{k+1/2}(4d) \mid U(4)\mathcal{P}_+ \\
& \oplus S^{+,new}_{k+1/2}(4d) \mid U(8)B(2) \oplus S^{+,new}_{k+1/2}(4d) \mid B(4) \\
& \oplus S^{+,new}_{k+1/2}(4d) \mid U(4)B(4) \Big\} \mid U(r^2) \\
& \oplus \bigoplus_{rd|N} \Big\{ S^{new}_{k+1/2}(4d) \oplus S^{new}_{k+1/2}(4d)|\mathcal{P}_+ \oplus S^{new}_{k+1/2}(4d)|U(2)B(2) \\
& \oplus S^{new}_{k+1/2}(4d) \mid B(4) \Big\} \mid U(r^2) \\
& \oplus \bigoplus_{rd|N} \Big\{ S^{new}_{k+1/2}(8d) \oplus S^{new}_{k+1/2}(8d) \mid W(16) \Big\} \mid U(r^2),
\end{aligned}
$$

where $W(16)$ is the W-operator corresponding to the prime $p = 2$ in $S_{k+1/2}(16N)$.

We also state Lemma 5.1 of [53], which is also needed for the proof of Theorem 4.5.2.

Lemma 4.5.4 *The operator $U(2)W(8)$ has the following mapping property:*

$$U(2)W(8) : S_{k+1/2}(4N) \longrightarrow S_{k+1/2}(8N).$$

Moreover, if $f \in S_{k+1/2}(4N)$, then $f \mid U(2)W(8) \in S_{k+1/2}(4N)$ if and only if $f \in S^{+}_{k+1/2}(4N)$, where $W(8)$ is the W-operator on $S_{k+1/2}(8N)$.

We also need the following operator and some of its properties in order to prove Theorem 4.5.2.

The operator $R_{(\frac{8}{\cdot})}$: For a formal series $f(\tau) = \sum_{n\geq 1} a_f(n)e^{2\pi i n\tau}$, define

$$f \mid R_{(\frac{8}{\cdot})}(\tau) = \sum_{n\geq 1} \left(\frac{8}{n}\right) a_f(n)e^{2\pi i n\tau}. \tag{4.20}$$

Then the operator $R_{(\frac{8}{\cdot})}$ defines a linear operator on $S_{k+1/2}(64N, \chi_0)$. If χ is the trivial character, then Ueda [81] proved that $R_{(\frac{8}{\cdot})}$ maps the space $S_{k+1/2}(32N)$ into itself.

Now, we prove the following lemma.

Lemma 4.5.5 *If f and $f \mid R_{(\frac{8}{\cdot})} \in S_{k+1/2}(16N)$, then $f = 0$.*

Proof. Let $f, f \mid R_{\left(\frac{8}{\cdot}\right)} \in S_{k+1/2}(16N)$. Then, for some $\lambda \in \mathbb{C}$, we have

$$f \mid R_{\left(\frac{8}{\cdot}\right)} = \lambda \sum_{u \pmod 8} \left(\frac{8}{u}\right) f \mid \left(\begin{pmatrix} 8 & u \\ 0 & 8 \end{pmatrix}, 1 \right)$$

Now, for each $u \pmod 8$, $(u, 8) = 1$, there exists a unique $v \pmod 8$, $(v, 8) = 1$ such that $v(1 + 2uN) \equiv u \pmod 8$ and we have

$$\left(\begin{pmatrix} 8 & u \\ 0 & 8 \end{pmatrix}, 1 \right) \begin{pmatrix} 1 & 0 \\ 16N & 1 \end{pmatrix}^* \left(\begin{pmatrix} 8 & v \\ 0 & 8 \end{pmatrix}, 1 \right)^{-1} = \left(I, \left(\frac{-4}{N}\right) i \right) \begin{pmatrix} 1 + 2Nu & \frac{u - v(1+2Nu)}{8} \\ 16N & 1 - 2Nv \end{pmatrix}^* .$$

Note that we can take $v = u + 2N$. Then, we have

$$f \mid R_{\left(\frac{8}{\cdot}\right)} = f \mid R_{\left(\frac{8}{\cdot}\right)} \begin{pmatrix} 1 & 0 \\ 16N & 1 \end{pmatrix}^* = \pm i\lambda \sum_{v (\mathrm{mod}\, 8)} f \mid \left(\frac{8}{v}\right) \left(\begin{pmatrix} 8 & v + 2N \\ 0 & 8 \end{pmatrix}, 1 \right)$$

By inserting the Fourier expansion of f, we get $f = 0$. This proves the lemma. □

Proof of Theorem 4.5.2: It is enough to show the direct sum in the respective eigensubspaces. First consider the eigenspace generated by a Hecke eigenform $f \in S^{+,new}_{k+1/2}(4d)$, where d is a fixed divisor of N. Suppose there exists scalars $\alpha, \alpha_i, 1 \leq i \leq 7$ with

$$\begin{aligned} \alpha f \mid R_{\left(\frac{8}{\cdot}\right)} = {} & \alpha_1 f \mid U(8)W(8)B(4) \; + \; \alpha_2 f \mid U(4)B(4) \; + \; \alpha_3 f \mid B(4) \\ & + \alpha_4 f \mid U(8)B(2) \; + \; \alpha_5 f \mid U(4)\mathcal{P}_+ \; + \; \alpha_6 f \mid U(4) \; + \; \alpha_7 f. \end{aligned}$$

Applying $U(4)$ on both the sides, we see that the left hand side of the above vanishes identically. Therefore, the above equals

$$\begin{aligned} -\alpha_1 f \mid U(8)W(8)B(4)U(4) = {} & \alpha_2 f \mid U(4)B(4)U(4) \; + \; \alpha_3 f \mid B(4)U(4) \\ & + \alpha_4 f \mid U(8)B(2)U(4) \; + \; \alpha_5 f \mid U(4)\mathcal{P}_+U(4) \; + \; \alpha_6 f \mid U(4)U(4) \\ & + \alpha_7 f \mid U(4). \end{aligned}$$

Since the right hand side of the above belongs to $S_{k+1/2}(4d)$, we see that

$$\alpha_1 f \mid U(8)W(8) \in S_{k+1/2}(4d).$$

Therefore, $\alpha_1 = 0$. Otherwise, Lemma 4.5.4 shows that $f \mid U(4) \in S^+_{k+1/2}(4d)$, but $f \in S^+_{k+1/2}(4d)$, hence by the lemma proved by Kohnen on p. 69 of [35], we get $f=0$. Thus, we have

$$\alpha f \mid R_{\left(\frac{8}{\cdot}\right)} = \alpha_2 f \mid U(4)B(4) \ + \ \alpha_3 f \mid B(4) \ + \ \alpha_4 f \mid U(8)B(2) \ + \ \alpha_5 f \mid U(4)\mathcal{P}_+$$
$$+ \ \alpha_6 f \mid U(4) \ + \ \alpha_7 f.$$

This implies that both αf and $\alpha f \mid R_{\left(\frac{8}{\cdot}\right)}$ are in the space $S_{k+1/2}(16d)$, and hence by Lemma 4.5.5, $\alpha = 0$. By using Theorem 4.5.3, we see that the remaining sums are direct. Therefore, we get $\alpha_i = 0$, for $2 \le i \le 7$. Thus, all the sums in the eigenspace generated by $S^{+,new}_{k+1/2}(4d)$ are direct.
We now show the direct sum property in the eigenspace generated by $S^{new}_{k+1/2}(4d)$. Let $f \in S^{new}_{k+1/2}(4d)$ be a Hecke eigenform. Suppose for scalars $\alpha_i, 1 \le i \le 6$, we have

$$\alpha_1 f + \alpha_2 f \mid \mathcal{P}_+ + \alpha_3 f \mid U(2)B(2) + \alpha_4 f \mid B(4) + \alpha_5 f \mid U(2)W(32) + \alpha_6 f \mid R_{\left(\frac{8}{\cdot}\right)} = 0.$$

Applying $U(4)$, we see that $\alpha_5 f \mid U(2)W(8) \in S_{k+1/2}(4d)$. But $\alpha_5 f \in S_{k+1/2}(4d)$, and hence Lemma 4.5.4 implies that $\alpha_5 f \in S^+_{k+1/2}(4d)$. Thus, $\alpha_5 = 0$. Since $f \in S_{k+1/2}(4d)$, by Lemma 4.5.5, $f \mid R_{\left(\frac{8}{\cdot}\right)}$ can not be a form in $S_{k+1/2}(16d)$, and so $\alpha_6 = 0$. Therefore, we have

$$\alpha_1 f + \alpha_2 f \mid \mathcal{P}_+ + \alpha_3 f \mid U(2)B(2) + \alpha_4 f \mid B(4) = 0.$$

This implies that $\alpha_i = 0, \ 1 \le i \le 4$, by using Theorem 4.5.3.
Next, we consider the eigenspace generated by $S^{new}_{k+1/2}(8d)$. Let $f \in S^{new}_{k+1/2}(8d)$ be a Hecke eigenform. Suppose for some scalars $\alpha_i, 1 \le i \le 4$, we have

$$\alpha_1 f \mid R_{\left(\frac{8}{\cdot}\right)} + \alpha_2 f \mid B(4) + \alpha_3 f \mid W(16) + \alpha_4 f = 0.$$

Applying $U(4)$ on both the sides and argue as above, first we get $\alpha_2 = 0$. Again by using similar arguments as above we get $\alpha_i = 0, \ i = 1, 3, 4$. Hence, all the sums in the eigenspace generated by $S^{new}_{k+1/2}(8d)$ are direct.
Finally, we consider the eigenspace generated by $S^{new}_{k+1/2}(16d, \left(\frac{8}{\cdot}\right))$. Let f be a Hecke eigenform in $S^{new}_{k+1/2}(16d, \left(\frac{8}{\cdot}\right))$ and suppose that for scalars α_1, α_2, we have

$$\alpha_1 f \mid \mathcal{P}_+ W(32) = \alpha_2 f \mid B(2).$$

Applying $W(32)$ on both the sides, we get

$$\alpha_1 f \mid \mathcal{P}_+ \ = \ \alpha_2 f \mid W(16)$$

$$= \lambda\alpha_2 f, \qquad \text{where } \lambda = \pm 1.$$

By §4.2.1, both f and $f \mid \mathcal{P}_+$ can not be in $S_{k+1/2}(16d, \left(\frac{8}{\cdot}\right))$, it leads to $\alpha_1 = \alpha_2 = 0$. Thus, all the sums in the eigenspace generated by $S^{new}_{k+1/2}(16d, \left(\frac{8}{\cdot}\right))$ are direct. This completes the proof for the direct sum decomposition of $S^{old}_{k+1/2}(32N)$.

We will complete the proof by comparing the dimensions of each side of the decomposition. Since the spaces $S^{+,new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(8d)$ and $S^{new}_{k+1/2}(16d, \left(\frac{8}{\cdot}\right))$ are isomorphic (under the Shimura correspondence) to the spaces $S^{new}_{2k}(d)$, $S^{new}_{2k}(2d)$, $S^{new}_{2k}(4d)$ and $S^{new}_{2k}(8d)$ respectively, we see the arguments as above gives

$$\begin{aligned}
\dim S^{old}_{k+1/2}(32N) &= \sum_{rd|N}(8 \dim S^{new}_{2k}(d) + 6 \dim S^{new}_{2k}(2d) + 4 \dim S^{new}_{2k}(4d) \\
&\qquad + 2 \dim S^{new}_{2k}(8d)) \\
&= 2\sum_{rd|N}(4 \dim S^{new}_{2k}(d) + 3 \dim S^{new}_{2k}(2d) + 2 \dim S^{new}_{2k}(4d) \\
&\qquad + \dim S^{new}_{2k}(8d)) \\
&= 2 \dim S_{2k}(8N) = \dim S_{k+1/2}(32N). \quad \text{(by Lemma 4.5.1)}
\end{aligned}$$

This proves the decomposition of $S_{k+1/2}(32N)$, and as a consequence, we have $S^{new}_{k+1/2}(32N) = \{0\}$. □

4.6 Newform theory for $S_{k+1/2}(2^\alpha N)$, $\alpha \geq 6$

In this section, we consider the space $S_{k+1/2}(2^\alpha N)$, where $\alpha \geq 6$ and $N \geq 1$ is odd and squarefree. For integers $k \geq 2$ and integers D, D' with $D, D' \equiv 0, 1 \pmod 4$ and $DD' > 0$, following Kohnen [36], we define

$$f_{k,\tilde{M}}(z; D, D') = \sum_{\substack{a,b,c\in\mathbb{Z} \\ b^2-4ac=DD' \\ \tilde{M}|a}} \chi_D(a,b,c)(az^2+bz+c)^{-k} \qquad (z \in \mathbb{H}), \tag{4.21}$$

where $\tilde{M} = 2^{\alpha-2}N$. The series converges absolutely uniformly on compact sets, and defines a cusp form of weight $2k$ on $\Gamma_0(\tilde{M})$. The series in (4.21) is identically zero for $(-1)^k D < 0$.

Proposition 4.6.1 *The function $f_{k,\tilde{M}}(z; D, (-1)^k m)$ ($\tilde{M} = 2^{\alpha-2}N$) has the Fourier expansion*

$$f_{k,\tilde{M}}(z; D, (-1)^k m) = \sum_{n\geq 1} c_{k,\tilde{M}}(n; D, (-1)^k m) e^{2\pi i n z}, \tag{4.22}$$

with

$$c_{k,\tilde{M}}(n;D,(-1)^k m) = \frac{2(-2\pi)^k}{(k-1)!}(n^2/(|D|m))^{(k-1)/2}\Bigg[(-1)^{[(k+1)/2]}\left(\frac{D}{n/\sqrt{m/|D|}}\right) \cdot \delta\left(\frac{n}{\sqrt{m/|D|}}\right)|D|^{-1/2} + \pi\sqrt{2}(n^2/(|D|m))^{1/4} \cdot \sum_{a\geq 1,\tilde{M}|a} a^{-1/2} S_{a,D,(-1)^k m}(|D|m,n) J_{k-1/2}\left(\frac{\pi n\sqrt{|D|m}}{a}\right)\Bigg], \tag{4.23}$$

where $\delta(x) = 1$ if x is an integer and is zero otherwise, and

$$S_{a,D,(-1)^k m}(|D|m,n) = \sum_{\substack{b \pmod{2a} \\ b^2 \equiv |D|m \pmod{4a}}} \chi_D\left(a,b,\frac{b^2-|D|m}{4a}\right) e_{2a}(nb) \tag{4.24}$$

is a finite exponential sum and $J_{k-1/2}(t)$ is the Bessel function of order $k-1/2$. (A similar exponential sum was considered in Theorem 3.4.3 of Chapter 3 in the case of real quadratic field).

The proof of Proposition 4.6.1 follows exactly by similar arguments used by Kohnen in §2, Proposition 2 of [36].

For $m \in \mathbb{N}$ with $(-1)^k m \equiv 0,1 \pmod 4$, let $P_{k+1/2,2^\alpha N;m}$ be the m-th Poincaré series in $S^+_{k+1/2}(2^\alpha N)$ as defined in §4.2.1 and it is characterised by

$$\langle g, P_{k+1/2,2^\alpha N;m}\rangle = i^{-1}_{2^\alpha N}\frac{\Gamma(k-1/2)}{(4\pi m)^{k-1/2}} a_g(m), \tag{4.25}$$

for every $g(\tau) = \sum_{n=1}^\infty a_g(n)e^{2\pi i n\tau} \in S_{k+1/2}(2^\alpha N)$. Recall, $i_{2^\alpha N}$ is the index of $\Gamma_0(2^\alpha N)$ in $SL_2(\mathbb{Z})$. Let $m \geq 1$, $(-1)^k m \equiv 0,1 \pmod 4$. Then following similar arguments carried out by Kohnen in §2, Proposition 4 of [36], $P_{k+1/2,2^\alpha N;m}$ has the Fourier expansion

$$P_{k+1/2,2^\alpha N;m}(\tau) = \sum_{n\geq 1,(-1)^k n\equiv 0,1 \pmod 4} g_{k,2^\alpha N;m}(n) e^{2\pi i n\tau} \tag{4.26}$$

with

$$g_{k,2^\alpha N;m}(n) = \frac{2}{3}\left[\delta_{m,n} + (-1)^{\left[\frac{k+1}{2}\right]}\pi\sqrt{2}(n/m)^{\frac{k}{2}-\frac{1}{4}}\sum_{\substack{n\geq 1\\ 2^{\alpha-2}N|c}} H_c(n,m) J_{k-\frac{1}{2}}\left(\frac{\pi}{c}\sqrt{mn}\right)\right] \tag{4.27}$$

Here $\delta_{m,n}$ is the Kronecker delta function and

$$H_c(n,m) = (1-(-1)^k i)\left(1+\left(\frac{4}{c}\right)\right)\frac{1}{4c}\sum_{\delta(4c)^*}\left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k+1/2} e^{2\pi i(n\delta+m\delta^{-1})/(4c)} \tag{4.28}$$

is a Kloostermann type sum and $J_{k-1/2}$ is the Bessel function of order $k-1/2$.

We now state Proposition 5 of [36] proved by Kohnen, which is needed for the proof of Theorem 4.6.3. This gives an identity between finite exponential sums and Kloostermann sums.

Proposition 4.6.2 *Define $S_{a,D,(-1)^k m}(|D|m,n)$ by (4.24) and $H_c(m,n)$ by (4.28). Then for all $a \geq 1$, $n \geq 1$ and $m \geq 1$ with $(-1)^k m \equiv 0, 1 \pmod 4$ we have*

$$S_{a,D,(-1)^k m}(|D|m,n) = \sum_{d|(a,n)}\left(\frac{D}{d}\right)(a/d)^{1/2} H_{a/d}(m, n^2|D|/d^2). \tag{4.29}$$

Let

$$\mathcal{S}_{t,\chi} : S_{k+1/2}(2^\alpha N, \chi_0) \longrightarrow S_{2k}(2^{\alpha-1}N, \chi^2)$$

be the Shimura map defined in §4.2.3, indexed by squarefree integers t, $\epsilon(-1)^k t > 0$ which commutes with the action of Hecke operators $T(n^2)$ $(n, 2N) = 1$. In [36], Kohnen defined the modified Shimura lifts

$$\mathcal{S}^+_{D,\chi_0} : S^+_{k+1/2}(4N, \chi_0) \longrightarrow S_{2k}(N, \chi_0^2),$$

(which we call as Shimura-Kohnen lifts), indexed by fundamental discriminants D, $\epsilon(-1)^k D > 0$, which commutes with the action of Hecke operators in the following sense:

$$f \mid T^+(n^2)\mathcal{S}^+_{D,\chi_0} = f \mid \mathcal{S}^+_{D,\chi_0} T(n),$$

for all $f \in S^+_{k+1/2}(4N, \chi_0)$ and for all $(n, N) = 1$.

Let D be a fundamental discriminant with $D \equiv 1 \pmod 4$, $(-1)^k D > 0$. For $f(\tau) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n\tau} \in S_{k+1/2}(2^\alpha N)$, the D-th Shimura map $\mathcal{S}_D$ on $S_{k+1/2}(2^\alpha N)$ is defined by

$$f \mid \mathcal{S}_D(\tau) = \sum_{n\geq 1}\left(\sum_{d|n,(d,2N)=1}\left(\frac{4D}{d}\right) d^{k-1} a_f(|D|n^2/d^2)\right) e^{2\pi i n\tau}. \tag{4.30}$$

(Note that we drop the χ from the notation in the case of trivial character.)

Proposition 4.6.3 *For $\alpha \geq 6$ and for a fundamental discriminant D with $D \equiv 1 \pmod 4$, $(-1)^k D > 0$, the D-th Shimura map*

$$\mathcal{S}_D : S^+_{k+1/2}(2^\alpha N) \longrightarrow S_{2k}(2^{\alpha-2}N)$$

defined by (4.30) is such that

$$P_{k+1/2,2^\alpha N;m}|\mathcal{S}_D = c(k,D,2^\alpha N)\sum_{t|2^{\alpha-2}N}\mu(t)\left(\frac{D}{t}\right)t^{k-1}f_{k,\frac{2^{\alpha-2}N}{t}}(z;D,(-1)^k m), \quad (4.31)$$

where $P_{k+1/2,2^\alpha N;m}$ is the m-th Poincare series in $S^+_{k+1/2}(2^\alpha N)$, $f_{k,\tilde{M}}(z;D,(-1)^k m)$ is the function defined in (4.21) and

$$c(k,D,2^\alpha N) = (-1)^{[(k+1)/2]}D^{k-1/2}\frac{2}{3}\frac{(k-1)!}{2(-2\pi)^k}.$$

Proof. In order to prove the proposition, it is sufficient to compute the explicit Fourier coefficient of the image of $P_{k+1/2,2^\alpha N;m}(\tau)$ under $\mathcal{S}_D$. Write

$$P_{k+1/2,2^\alpha N;m} \mid \mathcal{S}_D(\tau) = \sum_{n\geq 1}\mathcal{C}(n)e^{2\pi i n\tau}. \quad (4.32)$$

We will now compute the coefficients $\mathcal{C}(n)$. By the definition of $\mathcal{S}_D$, we have

$$\begin{aligned}
\mathcal{C}(n) &= \sum_{\substack{d|n\\(d,2N)=1}}\left(\frac{4D}{d}\right)d^{k-1}a_{P_{k+1/2,2^\alpha N;m}}(|D|n^2/d^2)\\
&= \sum_{\substack{d|n\\(d,2N)=1}}\left(\frac{4D}{d}\right)d^{k-1}g_{k,2^\alpha N;m}\left(\frac{|D|n^2}{d^2}\right), \quad \text{(by (4.27))}\\
&= \sum_{\substack{d|n\\(d,2N)=1}}\left(\frac{4D}{d}\right)d^{k-1}\left(\frac{n^2|D|/d^2}{m}\right)^{k-1/2} g_{k,2^\alpha N;\frac{|D|n^2}{d^2}}(m)\\
&= \frac{2}{3}\sum_{\substack{d|n\\(d,2N)=1}}\left(\frac{4D}{d}\right)d^{k-1}\left(\frac{n^2|D|/d^2}{m}\right)^{k-1/2}\Bigg\{\delta_{\frac{n^2|D|}{d^2},m} + (-1)^{\left[\frac{k+1}{2}\right]}\pi\sqrt{2}\\
&\times\left(\frac{m}{n^2|D|/d^2}\right)^{\frac{k}{2}-\frac{1}{4}}\sum_{\substack{c\geq 1\\2^{\alpha-2}N|c}}H_c\left(m,\frac{n^2|D|}{d^2}\right)J_{k-\frac{1}{2}}\left(\frac{\pi}{c}\sqrt{\frac{mn^2|D|}{d^2}}\right)\Bigg\}. \quad \text{(by (4.27))}
\end{aligned}$$

Denoting by $C_{k,D,N} := \frac{2}{3}(-1)^{[(k+1)/2]}D^{k-1/2}$, we have

$$\mathcal{C}(n) = C_{k,D,N}\left(\frac{n^2}{|D|m}\right)^{\frac{k-1}{2}}\left\{(-1)^{\left[\frac{k+1}{2}\right]}\left(\frac{D}{n/\sqrt{m/|D|}}\right)\delta\left(\frac{n}{\sqrt{m/|D|}}\right)|D|^{-1/2}\right.$$
$$\left. +\pi\sqrt{2}\sum_{\substack{d|n\\(d,2N)=1}}\left(\frac{4D}{d}\right)d^{\frac{-1}{2}}\left(\frac{n^2}{|D|m}\right)^{\frac{1}{4}}\sum_{\substack{c\geq 1\\2^{\alpha-2}N|c}}H_c\left(m,\frac{n^2|D|}{d^2}\right)J_{k-\frac{1}{2}}\left(\frac{\pi}{dc}\sqrt{mn^2|D|}\right)\right\}.$$

Replacing c by $2^{\alpha-2}Nc$ and then cd by c, we get

$$\mathcal{C}(n)= C_{k,D,N}\left(\frac{n^2}{|D|m}\right)^{\frac{k-1}{2}}\left\{(-1)^{\left[\frac{k+1}{2}\right]}\left(\frac{D}{n/\sqrt{m/|D|}}\right)\delta\left(\frac{n}{\sqrt{m/|D|}}\right)|D|^{\frac{-1}{2}}+\pi\sqrt{2}\right.$$
$$\left.\times\sum_{c\geq 1}(2^{\alpha-2}Nc)^{\frac{-1}{2}}\sum_{\substack{d|(n,2^{\alpha-2}Nc)\\(d,2N)=1}}\left(\frac{4D}{d}\right)\left(\frac{2^{\alpha-2}Nc}{d}\right)^{\frac{1}{2}}H_{\frac{2^{\alpha-2}Nc}{d}}\left(m,\frac{n^2|D|}{d^2}\right)J_{k-\frac{1}{2}}\left(\frac{\pi}{2^{\alpha-2}Nc}\sqrt{mn^2|D|}\right)\right\}.$$

In order to prove that the (4.32) equals the right hand side of (4.31), we need to compare the n-th Fourier coefficient of (4.31) with $\mathcal{C}(n)$. By using Proposition 4.6.1 in right hand side of (4.31), we see that the first term of both the sides are plainly equal. By comparing the second term of both the sides, we see that the identity we need to prove is that

$$\sum_{\substack{d|(n,2^{\alpha-2}Nc)\\(d,2N)=1}}\left(\frac{4D}{d}\right)\left(\frac{2^{\alpha-2}Nc}{d}\right)^{\frac{1}{2}}H_{\frac{2^{\alpha-2}Nc}{d}}\left(m,\frac{n^2|D|}{d^2}\right) = \sum_{t|2^{\alpha-2}N}\mu(t)\left(\frac{D}{t}\right)S_{\frac{2^{\alpha-2}Nc}{t},D,(-1)^km}\left(|D|m,\frac{n}{t}\right).$$

By inverting, we see that the above identity is equivalent to Proposition 4.6.2. Therefore, by using Proposition 4.6.2, the above theorem follows. $\square$

We now define the space of newforms in $S_{k+1/2}(2^\alpha N)$ as follows:

$$S^{new}_{k+1/2}(2^\alpha N) := \left\{\begin{array}{c} F|\mathcal{S}^*_D : F\in S^{new}_{2k}(2^{\alpha-2}N) \text{ with } F|W_{N'}=\left(\frac{D}{N'}\right)F, \forall N'|2^{\alpha-2}N, \left(\frac{2^{\alpha-2}N}{N'},N'\right)=1, \\ D\equiv 0,1 \pmod 4 \text{ is a fundamental discriminant with } (-1)^kD>0\end{array}\right\}, \quad (4.33)$$

where for each prime l dividing $2^{\alpha-2}N$, W_l denotes the Atkin-Lehner involution on $S^{new}_{2k}(2^{\alpha-2}N)$.

Theorem 4.6.4 *If $\alpha\geq 6$ then the space $S^{new}_{k+1/2}(2^\alpha N)$ is non trivial.*

Proof. By definition of $S^{new}_{k+1/2}(2^\alpha N)$, it is enough to show the existence of fundamental discriminants D for which $F|\mathcal{S}^*_D$ is non-zero, where F is a Hecke eigenform in $S^{new}_{2k}(2^{\alpha-2}N)$. Since, Murty-Murty [63, Chapter 6] proved the non-vanishing of $L(F, \bar{\chi}\left(\frac{D}{\cdot}\right), k)$ for infinitely many fundamental discriminants D. Then by assuming

$$F|W_{N'} = \left(\frac{D}{N'}\right) F, \text{ for every } N'|2^{\alpha-2}N, \left(\frac{2^{\alpha-2}N}{N'}, N'\right) = 1$$

and using the similar arguments as in Corollary 1 of Kohnen [36], we get $a_{F|\mathcal{S}_D^*}(|D|)$ is non-zero. Thus, $F|\mathcal{S}_D^* \neq 0$. Hence the proof follows. $\square$

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