QUANTUM UNCERTAINTY, COHERENCE AND QUANTUM SPEED LIMIT

By

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. **"**Quantum Speed Limit For Mixed States Using Experimentally Realizable Metric", Debasis Mondal, Arun Kumar Pati, *Physics Letter A*, **2016,** *380*, 1395-1400.

2. "Quantum Coherence Sets The Quantum Speed Limit", Debasis Mondal, Chandan Datta, Sk Sazim, *Physics Letter A*, **2016**, *380*, 689-695.

3. "Non-Local Advantage of Quantum Coherence", Debasis Mondal, Tanumoy Pramanik, Arun Kumar Pati, *Physical Revew A*, **2017,** *95*, 010301(R).

4. "Tighter Uncertainty and Reverse Uncertainty Relations", Debasis Mondal, Shrobona Bagchi, Arun Kumar Pati, *Physics Revew A*, **2017,** *95*, 052117.

Other Publications

1. "Generalized Fubini-Study Metric and Fisher Information Metric", Debasis Mondal, arXiv:1503.04146

2. "Steerability of Quantum Coherence in Accelerated Frame", Debasis Mondal, Chiranjib Mukhopadhyay, arXiv:1510.07556.

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Dedicated to my Maa! You are wonderful!

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Contents

Synopsis

In quantum mechanics, a fundamental goal is how to influence a system and control its evolution so as to achieve faster and controlled evolution. Quantum mechanics imposes a fundamental limit to the speed of quantum evolution, conventionally known as the quantum speed limit (QSL). With the advent of quantum information and communication theory, it has been established as an important notion for developing the ultra-speed quantum computer and communication channel, identification of precision bounds in quantum metrology, the formulation of computational limits of physical systems and the development of optimal control algorithms. The first major result in this direction was put forward by Mandelstam and Tamm, whose sole motivation was to give a new perspective to the energy-time uncertainty relation.

There have been rigorous attempts to achieve more and more tighter bounds and to generalize them for mixed states. In the proposed thesis, we address three basic and fundamental questions: (i) what is the ultimate limit of quantum speed? (ii) Can we measure this speed of quantum evolution in the interferometry by measuring a physically realizable quantity? And the third question we ask: (iii) Can we relate various properties of states and evolution operators such as quantum correlations, quantum coherence, dimension *etc.* with QSL?

To address the issues raised in the previous section, we proved a new quantum speed limit for mixed states. Then, we showed that the QSL can be measured in the interference experiment by measuring the visibility between the initial and the final states for unitary evolutions as well as completely positive trace preserving evolutions. We also showed that the bound based on the interference visibility is tighter than the most of the existing bounds in the literature. To address the third issue, we showed an intriguing connection between the QSL and the observable measure of quantum coherence for unitary evolutions. An important implication of our results is that quantum coherence plays an important role in setting the QSL and it can be used to control the speed of quantum evolutions.

Moreover, we also focus on the study of QSL in the bipartite scenario, where a part of the system is considered to be the controller of the evolution of the other part. It is well known that quantum correlations affects the evolutions of the total quantum systems. On the other hand, how a part of the system affects the evolution speed limit of the other part using the quantum correlation or non-locality is still an important unanswered question. In this thesis, we show that non-locality plays an important role in setting the QSL of a part of the system. In particular, we studied the effects of non-locality on quantum coherence of a part of a bi-partite system. This, in turn, clarified the role of quantum non-locality on QSL from the intriguing connection between QSL and the quantum coherence.

It is well-known that quantum mechanical uncertainty relations play an important role in setting the QSL for quantum systems. With the advent of quantum information theory, uncertainty relations have been established as important tools for wide range of applications. To name a few, uncertainty relations have been used in formulating quantum mechanics (where we can justify the complex structure of the Hilbert space or as a fundamental building block for quantum mechanics and quantum gravity). Further, it has been used in entanglement detection, security analysis of quantum key distribution in quantum cryptography, quantum metrology and quantum speed limit. In most of these fields, particularly, in quantum entanglement detection and quantum metrology or quantum speed limit, where a small fluctuation in an unknown parameter of the state of the system is needed to detect, state-independent relations are not very useful. Again, the existing certainty relations are all in terms of entropy and the bounds are always state independent. Thus, a focus on the study of the state dependent and tighter uncertainty and the reverse uncertainty relations based on the variance is a need of the hour. In the proposed thesis, we address this issue.

Another striking feature of the most of the existing tighter uncertainty bounds is that such bounds depend on arbitrary orthogonal state to the state of the system. It has been shown that an optimization of over the orthogonal states, which maximizes the lower bound, will saturate the inequality. For higher dimensional systems, finding such an orthonormal state, is difficult. Therefore, a focus on to derive a tighter yet optimization-free uncertainty as well as reverse uncertainty relations is needed for the shake of further technological developments and explorations. Here, we try to fulfil this aim and report a few tighter as well as optimization independent uncertainty and reverse uncertainty bounds both in the sum and the product form.

We highlight here the main results obtained in the proposed thesis.

- ➢ We derive a tighter yet experimentally measurable quantum speed limit for the first time. We show that the QSL can be measured by measuring the visibility of the interference pattern due to interference between the initial and the final state.
- ➢ We show for the first time that the speed of quantum evolution may be controlled and manipulated by tuning the observable measure of quantum coherence or asymmetry of the state with respect to the evolution operator.
- ➢ We investigate how to control and manipulate the quantum speed of evolution of a part of a bipartite system by measuring the other part of the system using quantum correlation.
- ➢ We derive a tighter yet optimization free state dependent uncertainty and reverse uncertainty relations for product as well as sum of variances of two incompatible observables. Our results may have the potential to set the stage for addressing another important issue in quantum metrology, i.e., to set the upper bound of error in measurement and the upper bound of the time of quantum evolutions.

The content of the proposed thesis is divided into three parts. First part contains two chapters—Chapter 2 and Chapter 3. In the second chapter, we derive an experimentally realisable quantum speed limit. We introduce a new metric for the non-degenerate density operator evolving along unitary orbit and show that this is experimentally realizable operation dependent metric on the quantum state space. Using this metric, we obtain the geometric uncertainty relation that leads to a new quantum speed limit. Furthermore, we argue that this gives a tighter bound for the evolution time compared to any other bound. We also obtain a Margolus-Levitin and Chau bound for mixed states using this metric.

Here, we also propose how to measure this new distance and speed limit in quantum interferometry. We show that the speed of quantum evolution can be inferred by measuring the visibility and the phase shift of the interference pattern, when a density operator undergoes unitary evolution. We also show that the bound is tighter than various existing bounds available in the literature. Finally, we generalize the idea and derive a lower bound to the time of quantum evolution for any completely positive trace preserving map using this metric. The third chapter is dedicated to focus on how various properties of states and evolution operators can control the evolution speed. We cast observable measure of quantum coherence or asymmetry as a resource to control the quantum speed limit (QSL) for unitary evolutions. For non-unitary evolutions, QSL depends on that of the state of the system and the environment together. We show that the product of the time bound and the coherence (asymmetry) or the quantum part of the uncertainty behaves in a geometric way under partial elimination and classical mixing of states. These relations give a new insight to the quantum speed limit. We also show that our bound is experimentally measurable and is tighter than various existing bounds in the literature.

In the second part of the thesis, we study the steerability of quantum state from the perspective of quantum coherence and observable measure of quantum coherence, i.e., instead of considering uncertainty relations, we considered another property of quantum states, quantum coherence, to study the condition for the single system description of a part of a bipartite state. Here, we derive the complementarity relations between coherences measured on mutually unbiased bases using various coherence measures such as the l_1-norm, the relative entropy and the skew information. Using these relations, we formulate steering inequalities to check whether one of its subsystems has a single system description from the perspective of quantum coherence. Our results show that not all steerable states are eligible to provide the non-local advantage on quantum coherence. Thus, any arbitrary steerable state cannot provide non-local advantage on the quantum speed limit of a subsystem of the bipartite system. Note that by the word non-local advantage, we mean that the advantage, which cannot be achieved by a single system and exclude advantage due to local operation and classical communication.

The third part contains the optimization free yet tighter state dependent uncertainty and reverse uncertainty relations for the product as well as the sum of variances of two incompatible observables. In the proposed thesis, we show that the uncertainty relation of the product form is most of the times stronger than the Schrödinger uncertainty relation. On the other hand, uncertainty relations for the sum of variances are also shown to be tight enough considering the advantage that the bounds do not need an optimization. Here we also report the reverse uncertainty relations. Both the entropic uncertainty and the certainty relations are known to exist in the literature. These relations based on entropy are state independent. Here, we derive the state-dependent reverse uncertainty relations in terms of variances both in the sum form and the product form for the first time.

It was an well established notion that quantum mechanics sets the lower limit to the time of quantum evolutions. In contrast to that belief, it is now expected that our state dependent certainty relations may be useful also in setting an upper limit to the time of quantum evolutions (reverse bound to the QSL) and in quantum metrology.

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"With great power comes great electricity bill!"

Chapter 1

Introduction

"*Quantum mechanics was, and continues to be, revolutionary, primarily because it demands the introduction of radically new concepts to better describe the world. In addition we have argued that conceptual quantum revolutions in turn enable technological quantum revolutions.*" ——Alain Aspect

Overview

In recent years, various attempts are being made in the laboratory to implement quantum gates, which are the basic building blocks of a quantum computer. Performance of a quantum computer is determined by how fast these logic gates drive the initial state to a final state. An efficient quantum gate should transform the input state into the desired state as fast as possible. Thus, the natural question that arises is: can a quantum state evolve arbitrarily fast? It turns out that quantum mechanics limits the evolution speed of any quantum system, conventionally known as the quantum speed limit (QSL).

Extensive amount of work has already been done on the subject "minimum time required to reach a target state" since the appearance of first major result by Mandelstam and Tamm [1]. However, the notion of quantum speed or speed of transportation of quantum state was first introduced by Anandan-Aharonov using the Fubini-Study metric [2] and subsequently, the same notion was defined in Ref. [3] using the Riemannian metric [4]. It was found that the speed of a quantum state on the projective Hilbert space is proportional to the fluctuation in the Hamiltonian of the system. Using the concept of Fubini-Study metric on the projective Hilbert space, a geometric meaning is given to the probabilities of a two-state system [5]. Furthermore, it was shown that the quantum speed is directly related to the super current in the Josephson junction [6]. In the last two decades, there have been various attempts made in understanding the geometric aspects of quantum evolution for pure as well for mixed states [7–56]. The quantum speed limit (QSL) for the driven [50] and the non-Markovian [56] quantum systems is introduced using the notion of the Bures metric [57]. Very recently, QSL for physical processes was defined by Taddei *et al.* in Ref. [47] using the Bures metric and in the case of open quantum system, the same was introduced by Campo *et al.* in Ref. [48] using the notion of relative purity [46]. In an interesting twist, it has been shown that QSL for multipartite system is bounded by the generalized geometric measure of entanglement [8].

There have been rigorous attempts to achieve more and more tighter bounds and to generalize them for mixed states. In this thesis, we address a few basic and fundamental questions: (i) What is the ultimate limit of quantum speed? (ii) Can we measure the speed of quantum evolutions in the interferometry by measuring a physically realizable quantity? Suppose, there are two different quantum devices G_1 and G_2 performing the same task, i.e., transform the initial state ρ to the same final or the desired state ρ' (see Fig. (1.1)). Therefore, both the devices must correspond to the same CPTP map with different Kraus representations. Now, we ask the question: (iii) Is it possible to distinguish such devices? We observe that it is indeed possible. One way to distinguish such devices is to measure experimentally various properties such as QSL and characterise the dynamics of quantum evolutions performed by both of these gates.

With the advent of quantum information and communication theory, it has been established as an important notion for developing the ultra-speed quantum computer and communication channel, identification of precision bounds in quantum metrology, formulation of computational limits of physical systems, development of quantum optimal control algorithms. Still, most of the bounds in the literature are either not measurable in the experiments or not tight enough. As a result, these bounds cannot be effectively

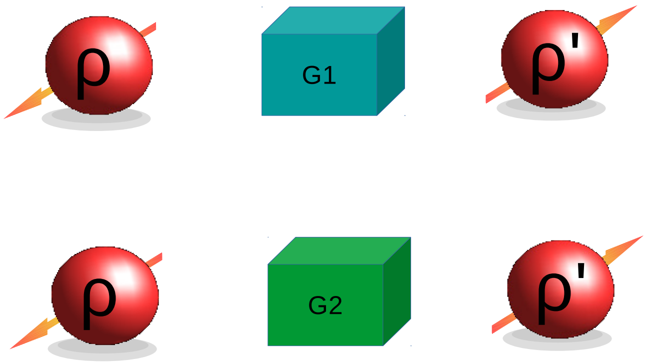

FIGURE 1.1: Two devices G_1 and G_2 represent two different operations performing the same task, i.e., transforming the same initial state ρ to the same desired state ρ'. It is expected that they will take different time to perform the job.

used in the experiments on quantum metrology, quantum communication and specially in the Unruh effect detection *etc.*, where a small fluctuation in a parameter is needed to detect. Therefore, a study to investigate the connection between the speed limit of quantum evolutions and various experimentally measurable quantities is the need of the hour.

In quantum thermodynamics, theoretical understanding is rapidly developing to infer the role of various information theoretic resources in devising the most efficient quantum thermodynamic engines. However, such engines may not always be the most time consuming. To perform a particular job, it may happen that the most efficient engine may take infinite time. Therefore, such engines and thermodynamic tools are not desired. We need the efficient and at the same time faster quantum thermodynamic engines. In this regard, a proper theoretical understanding is necessary how to control and manipulate the speed and efficiency of such engines by using various resources.

Another aspect of studying the QSL is to try to understand how to control and manipulate the speed of quantum evolutions. It will be much more interesting, if one can relate various properties of the states or operations or resources of quantum information theory such as coherence, asymmetry and quantum correlations, non-locality, for example, with the quantum speed limit. Although, these understandings may help us to control and manipulate the speed of quantum communication, quantum algorithm, thermodynamic engines, apart from the particular cases like the Josephson Junction [6] and multipartite scenario [8, 58], there has been little advancement in this direction. Therefore, the question naturally arises: (iv) can we relate such quantities with QSL?

Recently, quantum coherence has taken the center stage in research, specially in quantum biology [59–62] and quantum thermodynamics [63–67] in the last few years. In quantum information theory, it is a general consensus or expectation that it can be projected as a resource [68–71]. This has been the main motivation to quantify and measure coherence [68, 69, 72]. Moreover, it is a crucial resource in the interference phenomenon. Various quantities, such as the visibility and various phases in the interferometry are under scanner and the investigation is on to probe various quantum properties or phenomena, such as the Unruh effect [73–76], quantum speed limit [77], quantum correlation [78] using such quantities in quantum interferometry [79–83]. A proper study of quantum coherence may provide further insight to the development of new techniques to probe such quantum processes in the interferometry and interestingly enough, we show in this thesis that quantum coherence plays an important role in setting the QSL.

Moreover, we also focus on the study of QSL in the bipartite scenario, where a part of the system is considered to be the controller of the evolution of the other part. It is well known that quantum correlation affects the evolution of the total quantum system. In particular, the role of entanglement in dynamical evolutions was studied by Maccone *et al.* in Ref. [28, 29]. On the other hand, how a part of the system affects the evolution speed of the other part using the quantum correlation or non-locality is still an important unanswered question. In this thesis, we study the role of non-locality in gaining the quantum speed limit beyond what could have been achieved by a single system only. In particular, we studied the effects of non-locality on quantum coherence of a part of a bi-partite system. This, in turn, clarified the role of quantum non-locality on QSL from the intriguing connection between QSL and quantum coherence.

This work in the direction of quantum non-locality and QSL in the bi-partite scenario should not be viewed as if it has the only purpose to serve is to show the effect of non-local nature of quantum correlation on QSL. It has fundamental implications from the resource theoretic point of view. We have shown a direct connection between the steering kind of non-locality and the quantum coherence. Both of the properties in quantum information theory have separate resource theories [68–71, 84, 85].

Although, we have observed an advantage in QSL or quantum coherence due to the non-local nature of quantum correlation, we are yet to uncover how to use it in practical scenario. We expect it to have huge implications in understanding the nature apart from technological applications in various developing fields of quantum information theory, specially in quantum communications and quantum thermodynamics. In quantum thermodynamics, this may help us to understand how to control a thermodynamic engine non-locally and use quantum correlations in our favour to construct a faster yet efficient engine.

It has been clarified above that quantum mechanics sets the lower limit to the time of quantum evolutions. This intrigues one to further ask, whether there exists any upper bound to the time of quantum evolutions. The lower bound to the time of quantum evolutions may be regarded as the energy-time uncertainty relations [1]. From the derivation of the Mandelstam-Tamm bound, it is clear that the quantum speed bounds are nothing but the new manifestations of the quantum uncertainty relations and these relations, in particular, variance based state-dependent uncertainty relations play an important role in the derivations of these limits [1, 77, 86, 87]. Naturally, it is expected that to set an upper bound to the time of quantum evolutions, one needs to derive such reverse uncertainty relations.

Quantum mechanics has many distinguishing features from the classical mechanics in the microscopic world. One of those distinguished features is the existence of incompatible observables. As a result of this incompatibility, we have the uncertainty principle and the uncertainty relations. One might ask: To what extent, are the observables in quantum mechanics incompatible? In other word, to what maximal extent is the joint sharp preparation of the state in the eigenbases of incompatible observables impossible?

There have been numerous attempts to address the question by certainty relations. It is noted that there exists entropic uncertainty and certainty relations (or in other words, reverse uncertainty relations) in the literature, which are state independent [88–93]. Therefore, we, here, try to derive state-dependent variance based reverse uncertainty relations.

With the recent developments of quantum information theory, uncertainty relations have also been established as important tools for a wide range of other applications. To name a few, uncertainty relations have been used in formulating quantum mechanics [94] (where we can justify the complex structure of the Hilbert space [95] or as a fundamental building block for quantum mechanics and quantum gravity [96]). Further, it has been used in entanglement detection [97, 98], security analysis of quantum key distribution in quantum cryptography (e.g. see [99]) and quantum metrology and quantum speed limit (QSL) etc. In most of these fields, particularly, in quantum entanglement detection.

Owing to the seminal works by Heisenberg [100], Robertson [101] and Schrödinger [102], lower bounds were shown to exist for the product of variances of two arbitrary observables. Recently, Maccone and Pati have shown stronger uncertainty relations for all incompatible observables [103]. One striking feature of these stronger uncertainty bounds is that they depend on arbitrary orthogonal state to the state of the system [103–107]. It has been shown that an optimization of over the orthogonal states, which maximizes the lower bound, will saturate the inequality. For higher dimensional systems, finding such an orthonormal state, is difficult. Therefore, a focus on to derive an uncertainty relation independent of any optimization and yet tight is needed for the sake of further technological developments and explorations, particularly in quantum metrology. Here, we try to fulfil this aim and report a few tighter as well as optimization independent uncertainty and reverse uncertainty bounds both in the sum and the product form. It was an well established notion that quantum mechanics sets the lower limit to the time of quantum evolutions. In contrast to that belief, it is now expected that our state dependent certainty relations may be useful also in setting an upper limit to the time of quantum evolutions (reverse bound to the QSL) and in quantum metrology.

Uncertainty relations in terms of variances of incompatible observables are generally expressed in two forms—product form and sum form. Although, both of these kinds of uncertainty relations express a limitation in the possible preparations of the system by giving a lower bound to the product or sum of the variances of two observables, product form cannot capture the concept of incompatibility of observables properly because it may become trivial even when observables do not commute. In this sense, uncertainty relations in terms of the sum of variances capture the concept of incompatibility more accurately. It may be noted that earlier uncertainty relations that provide a bound to the sum of the variances comprise a lower bound in terms of the variance of the sum of observables [103, 108], a lower bound based on the entropic uncertainty relations [88–93], sum uncertainty relation for angular momentum observables [109], and also uncertainty for non-Hermitian operators [110, 111].

Preliminary concepts

A complete description of main concepts and results of this thesis requires a brief description of a few preliminary concepts. In this thesis, we deal with the speed of quantum evolutions, which is defined along the evolution path traversed by the state of the quantum system. This distance is defined on the projective Hilbert space. Therefore, before going to introduce the $U(1)$ gauge invariant geodesic distance in the next chapter, here we introduce the idea of projective Hilbert space, quantum state space and quantum evolutions.

Projective Hilbert space

The idea of Hilbert space is generally introduced almost immediately with the introduction of quantum mechanics. It is nothing but a complete metric space of complex vectors $\{|\psi\rangle\}$. One of the most fundamental tenets of quantum mechanics is that the state of a physical system corresponds to a vector in a Hilbert space $\mathcal{H}$, and that the Born rule gives the probability for a system in state $|\psi\rangle$ to be in state $|\phi\rangle$

$$p(\psi, \phi) = \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}. \tag{1.1}$$

This implies $p(c\psi, \phi) = p(\psi, c\phi) = p(\psi, \phi)$ for any $c \in \mathcal{C} \equiv \mathbb{C} - \{0\}$, where $\mathbb{C}$ is the set of complex numbers. Therefore, one can observe that there is no physical way to distinguish the states $|\psi\rangle$ and $c|\psi\rangle$ due to the fact that $p(\psi, c\psi) = p(\psi, \psi) = 1$. This led us to formally define the projective Hilbert space, which is nothing but a space of rays formed by a set of vectors from equivalent classes, i.e., a ray is just a one-dimensional subspace spanned by all the vectors describing the same state. For a state $|\psi\rangle$, the associated ray is defined by

$$R_\psi := \{|\phi\rangle \in \mathcal{H} | \exists c \in \mathcal{C} : |\phi\rangle = c|\psi\rangle\}. \tag{1.2}$$

The equivalence relation on the Hilbert space is formed by identifying all the physically indistinguishable vectors by Born rule, i.e.,

$$\psi \sim \phi \leftrightarrow \psi \in R_\phi. \tag{1.3}$$

Thus, the projective Hilbert space is defined as

$$\mathcal{P}(\mathcal{H}) := (\mathcal{H} - \{0\})/\sim . \tag{1.4}$$

One important thing to note here is that this is not even a vector space and thus, is not a Hilbert space any more. One of the reasons for this is that there is no identity element of addition, i.e., zero vector, which is a requirement for the space to be a vector space. The nice thing about the space is that every element of the space represents a distinct state or ray. In the next chapter, we define a $U(1)$ gauge invariant metric, i.e., the Fubini-Study metric on this space [112]. This is an unique metric on this space but in the following section, we will realise that this space does not cover the whole space of quantum states. It is only a subset of the whole quantum state space and there is no unique $U(1)$ gauge invariant metric on the whole space of quantum states.

Quantum state space

Only the space of pure states form a Hilbert space or projective Hilbert space. One can also define physically equally relevant states, i.e., mixed states by probabilistic mixture

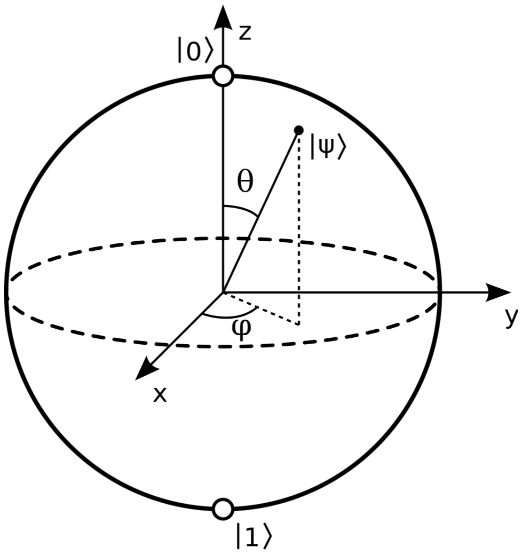

FIGURE 1.2: A 2-dimensional block sphere has been represented by a sphere. All the pure states reside on the boundary of the sphere and only pure states reside there, unlike higher dimensional Bloch-sphere. An arbitrary pure state $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ is placed on the boundary of the sphere as shown above. All the mixed states reside within the boundary.

of pure states. Mathematically, an n-dimensional quantum state can be represented by a positive Hermitian n-dimensional matrix ρ known as density matrix such that

$$\begin{aligned} \mathrm{Tr}\rho &= 1 \\ \mathrm{Tr}\rho^2 &= 1 \qquad \text{for pure states} \\ \mathrm{Tr}\rho^2 &< 1 \qquad \text{for mixed states.} \end{aligned} \tag{1.5}$$

The space of density matrices form a convex set, whose pure points or pure states lie on the boundary of the set. In the quantum state space, there is no unique way to express the quantum mixed states as a probabilistic mixtures of pure states, Unlike the space of states in classical mechanics.

Any density matrix of dimension n can be expressed by n^2-1 number of real independent parameters [112]. For pure states, $2n-2$ number of parameters are sufficient. The space of density matrices $S(\mathcal{H})$ is generally represented by Bloch-sphere. It is important to note that Bloch-sphere in general is not equivalent to n-sphere. This is due to the fact that only pure points reside on the surface of an n-sphere. On the other hand, for any dimension $n \neq 2$, not only the pure states but also the mixed states with $Det(\rho) = 0$

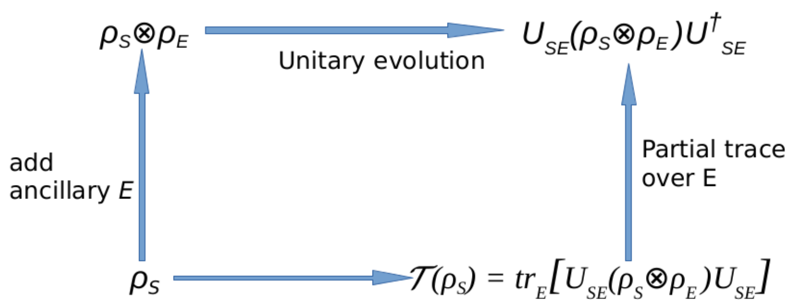

FIGURE 1.3: A Schematic diagram showing the process of a general quantum evolution of a system S with a state $\rho_S \in \mathcal{H}_S$. Any general evolution may be conceived as a joint unitary evolution of system plus environment together such that the system and the environment are in a product state. The final state of the system is found by tracing over the environmental degrees of freedom.

(definition of the boundary of the state space) reside on the boundary of the generalized Bloch-sphere. For $n = 2$, $Det(\rho) = 0$ implies pure states. Therefore, any state, which resides on the boundary of a 2-dimensional Bloch sphere are pure. Thus, 2-dimensional Bloch sphere can equivalently be represented by a 2-sphere (1.2).

Quantum operations

Quantum mechanics dictates the evolution rules of quantum states. Since the state of a quantum system is represented by a positive Hermitian density matrix, we expect that after any allowed operation, the final state should also be a position true class operator. Such evolutions are completely positive trace preserving evolutions and the map, which connects the initial $\rho_S(0)$ and the final state $\rho_S(t)$ is known as completely positive trace preserving map (CPTP) $\mathcal{T} : \rho_S(0) \mapsto \rho_S(t)$. In quantum mechanics, an arbitrary evolution (or CPTP evolution) of a state in general is modelled by adding an ancilla or environment (E) to the system (S) and then evolving them jointly by an unitary evolution U_{SE} and tracing out the ancillary part (1.3). We consider a state $\rho_S \in S(\mathcal{H}_S)$ of a system (S). Any arbitrary evolution of the system can in principle be modelled by an arbitrary joint unitary operator U_{SE} acting on the product state of system (S) and environment (E) (1.3). We consider that the state of the environment is $|0\rangle_E\langle 0|$ and the joint state of the system and environment at time $t = 0$ is $\rho_{SE}(0) = \rho_S \otimes |0\rangle\langle 0| \in S(\mathcal{H}_S \otimes \mathcal{H}_E)$. The final state of the system plus environment together after evolving them by a unitary operator U_{SE} is given by

$$\rho_{SE}(t) = U_{SE}\Big(\rho_S \otimes |0\rangle_E\langle 0|\Big)U_{SE}^\dagger. \tag{1.6}$$

The state of the system at time t can be expressed by a operator-sum representation as given below

$$\begin{aligned}\rho_S(t) &= \mathrm{Tr}_E \rho_{SE} = \sum_i \langle e_i|U_{SE}\rho_S \otimes |0\rangle\langle 0|U_{SE}^\dagger|e_i\rangle \\ &= \sum_i K_i \rho_S(0) K_i^\dagger,\end{aligned} \tag{1.7}$$

where $K_i = \langle e_i|U_{SE}|0\rangle_E$ such that

$$\sum_i K_i^\dagger K_i = I. \tag{1.8}$$

This representation is also known as Kraus-representation [112, 113]. In the next few chapters, we have used this representation to study the speed of arbitrary open quantum evolutions. It is important to note that the Kraus representation is not unique for any CPTP map $\mathcal{T}$. We enumerate below some examples of such quantum channels or maps on the qubit state space to grasp the idea of general quantum evolutions.

a. Bit flip channel

As the name suggests, bit flip channel flips the state. It flips the state $|0\rangle_S$ ($|1\rangle_S$) to $|1\rangle_S$ ($|0\rangle_S$) with probability p and keeps the state intact with probability $(1-p)$. Therefore, one can mathematically express the action of the bit flip channel as

$$\begin{aligned}|0\rangle_S\langle 0| &\rightarrow (1-p)|0\rangle_S\langle 0| + p|1\rangle_S\langle 1| \\ |1\rangle_S\langle 0| &\rightarrow (1-p)|1\rangle_S\langle 0| + p|0\rangle_S\langle 1| \\ |0\rangle_S\langle 1| &\rightarrow (1-p)|0\rangle_S\langle 1| + p|1\rangle_S\langle 0| \\ |1\rangle_S\langle 1| &\rightarrow (1-p)|1\rangle_S\langle 1| + p|0\rangle_S\langle 0|.\end{aligned} \tag{1.9}$$

In matrix representation, the bit flip transformation on a single-qubit state $\rho_S = \begin{pmatrix} \rho^s_{00} & \rho^s_{01} \\ \rho^s_{10} & \rho^s_{11} \end{pmatrix}$ is given as

$$\rho'_S = \mathcal{T}(\rho_S) = \begin{pmatrix} (1-p)\rho^s_{00} + p\rho^s_{11} & (1-p)\rho^s_{01} + p\rho^s_{10} \\ (1-p)\rho^s_{10} + p\rho^s_{01} & (1-p)\rho^s_{11} + p\rho^s_{00} \end{pmatrix}. \tag{1.10}$$

b. Bit-phase flip channel

The action of a bit-phase flip channel on the state of a qubit system can mathematically be shown as

$$\begin{aligned} |0\rangle_S\langle 0| &\rightarrow (1-p)|0\rangle_S\langle 0| + p|1\rangle_S\langle 1| \\ |1\rangle_S\langle 0| &\rightarrow (1-p)|1\rangle_S\langle 0| - p|0\rangle_S\langle 1| \\ |0\rangle_S\langle 1| &\rightarrow (1-p)|0\rangle_S\langle 1| - p|1\rangle_S\langle 0| \\ |1\rangle_S\langle 1| &\rightarrow (1-p)|1\rangle_S\langle 1| + p|0\rangle_S\langle 0|. \end{aligned} \tag{1.11}$$

In the matrix representation, it transforms an arbitrary qubit state $\rho_S = \begin{pmatrix} \rho^s_{00} & \rho^s_{01} \\ \rho^s_{10} & \rho^s_{11} \end{pmatrix}$ as

$$\rho'_S = \mathcal{T}(\rho_S) = \begin{pmatrix} (1-p)\rho^s_{00} + p\rho^s_{11} & (1-p)\rho^s_{01} - p\rho^s_{10} \\ (1-p)\rho^s_{10} - p\rho^s_{01} & (1-p)\rho^s_{11} + p\rho^s_{00} \end{pmatrix}. \tag{1.12}$$

c. Phase damping channel

Phase damping channels keep the diagonal elements of a density matrix unchanged but transform its off-diagonal elements. Phase damping channels act on the qubit states as

$$\begin{aligned} |0\rangle_S|0\rangle_E &\rightarrow \sqrt{1-p}|0\rangle_S|0\rangle_E + \sqrt{p}|0\rangle_S|1\rangle_E \\ |1\rangle_S|0\rangle_E &\rightarrow \sqrt{1-p}|1\rangle_S|0\rangle_E + \sqrt{p}|1\rangle_S|2\rangle_E, \end{aligned} \tag{1.13}$$

where the state of the environment changes with probability p and remains intact with probability $(1-p)$. One can easily show the Kraus representation of the channel as

$$\rho'_S = (1-p)\rho_S + \sum_{i=1}^{2} K_i\rho_S K_i^\dagger, \tag{1.14}$$

where $K_0 = \sqrt{1-p}I$, $K_1 = \sqrt{p}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $K_2 = \sqrt{p}\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$. One can easily show that the Kraus operators not only preserves the normalization condition given in Eq. (1.8) but also satisfy the unital condition,

$$\sum_i K_i K_i^\dagger = I, \tag{1.15}$$

i.e., it always transforms a completely mixed state to a completely mixed state.

d. Amplitude damping channel

The amplitude damping channel is a model for decay of an excited two level atom due to spontaneous emission of photons. An atom, initially in the ground state remains in the ground state under the application of such channel and on the other hand, an atom, initially in the ground state, decays to the ground state with probability p and remains in the excited state with probability $1-p$. Thus, the unitary representation of the channel is given as

$$\begin{aligned} |0\rangle_S|0\rangle_E &\rightarrow |0\rangle_S|0\rangle_E \\ |1\rangle_S|0\rangle_E &\rightarrow \sqrt{1-p}|1\rangle_S|0\rangle_E + \sqrt{p}|0\rangle_S|1\rangle_E. \end{aligned} \tag{1.16}$$

The Kraus representation of the channel is given by

$$\rho'_S = \sum_{i=1}^{2} K_i \rho_S K_i^\dagger, \tag{1.17}$$

where $K_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{smallmatrix}\right)$ and $K_2 = \left(\begin{smallmatrix}0 & \sqrt{p}\\ 0 & 0\end{smallmatrix}\right)$. Thus, the final state turns out to be

$$\rho'_S = \begin{pmatrix} \rho^s_{00} + p\rho^s_{11} & \sqrt{1-p}\rho^s_{01} \\ \sqrt{1-p}\rho^s_{10} & (1-p)\rho^s_{11} \end{pmatrix}. \tag{1.18}$$

Here, p can be considered as the decay rate of the atom. This gives the evolution map of the Bloch vector $\vec{r} = (r_x, r_y, r_z)$ as

$$\vec{r'} \rightarrow (\sqrt{1-p}r_x, \sqrt{1-p}r_y, p + (1-p)r_z). \tag{1.19}$$

As shown above, it is important to note that unlike phase damping channels, amplitude damping channels do not keep the diagonal terms of a density matrix unchanged. Not only that it is not a unital channel as well unlike the phase damping channels.

In the next two chapters, we have studied the quantum speed limits and their tightness for CPTP evolutions.

Outline of the thesis

The content of the thesis is divided into three parts. First part contains two chapters—Chapter 2 and Chapter 3. In the second chapter, we derive an experimentally realisable quantum speed limit. We introduce a new metric for non-degenerate density operator evolving along unitary orbit and show that this is experimentally realizable operation dependent (pseudo-) metric on the quantum state space. Using this metric, we obtain the geometric uncertainty relation that leads to a new quantum speed limit. Furthermore, we argue that this gives a tighter bound for the evolution time compared to any other bound. We also obtain the Margolus-Levitin and the Chau bound for mixed states using this metric.

Here, we also propose how to measure this new distance and speed limit in quantum interferometry. We show that the speed of quantum evolution can be measured by measuring the visibility and the phase shift of the interference pattern due to interference between the initial and the final state. We also show that the bound is tighter than various existing bounds available in the literature. Finally, we generalize the idea and derive a lower bound to the time of quantum evolution for any completely positive trace preserving map using this metric.

The third chapter is dedicated to focus on how various properties of states and evolution operators can control the evolution speed. In particular, we cast the observable measure of quantum coherence or asymmetry as a resource to control the quantum speed limit (QSL) for unitary evolutions. For non-unitary evolutions, QSL depends on that of the state of the system and environment together. We show that the product of the time bound and the coherence (asymmetry) or the quantum part of the uncertainty behaves in a geometric way under partial elimination and classical mixing of states. These relations give a new insight to the quantum speed limit. We also show that our bound is experimentally measurable and is tighter than various existing bounds in the literature.

In the second part of the thesis, we study the steerability of quantum state from the perspective of quantum coherence and observable measure of quantum coherence, i.e., instead of considering uncertainty relations, we consider another property of quantum states, quantum coherence, to study the condition for the single system description of a part of a bipartite state. Here, we derive several complementarity relations between coherences measured on mutually unbiased bases using various coherence measures such as the l_1-norm, the relative entropy and the skew information for a single system. Using these relations, we formulate steering inequalities to check whether one of its subsystems has a single system description from the perspective of quantum coherence. Our results show that not all steerable states are eligible to provide the non-local advantage on quantum coherence or observable measure of quantum coherence of a part of a bipartite system. Thus, any arbitrary steerable state cannot achieve non-local advantage on the quantum speed limit of a part of the bipartite system beyond what may have been achieved by a single system.

The third part contains the optimization free yet tighter state dependent uncertainty and reverse uncertainty relations for the product as well as the sum of variances of two incompatible observables. Here, we show that the uncertainty relation of the product form is most of the times stronger than the Schrödinger uncertainty relation. On the other hand, uncertainty relations for the sum of variances are also shown to be tight enough considering the advantage that the bounds do not need an optimization. Here we also report reverse uncertainty relations. Both the entropic uncertainty and the certainty relations are known to exist in the literature. These relations based on entropy are state independent. Here, we derive the state-dependent reverse uncertainty relations in terms of variances both in the sum form and the product form for the first time.

We highlight here the main results obtained in the thesis. First, we start with the derivation of a tighter yet experimentally measurable quantum speed limit. We show that the QSL can be measured by measuring the visibility of the interference pattern due to interference between the initial and the final state. Second, we show for the first time that the speed of quantum evolution may be controlled and manipulated by tuning the observable measure of quantum coherence or asymmetry of the state with respect to the evolution operator. Third, we investigate how to control and manipulate the quantum speed of evolution of a part of a bipartite system by measuring the other part of the system using quantum correlation. And at last, we derive a tighter yet optimization free state dependent uncertainty and reverse uncertainty relations for product as well as sum of variances of two incompatible observables. Our results may have the potential to set the stage for addressing another important issue in quantum metrology, i.e., to set the upper bound of error in measurement and the upper bound of the time of quantum evolutions.

We conclude the thesis with summary and future directions in the sixth chapter.

Part I

Chapter 2

Quantum speed limit for mixed states using experimentally realizable metric

"Whether you can go back in time is held in the grip of the law of quantum gravity."
——Kip Thorne

Introduction

This chapter is dedicated to the formulation of a connection between the speed of quantum evolutions and the visibility of the interference pattern formed due to the interference between the initial state and the final state. Before going into the details, we briefly discuss the main concepts and various existing results in the literature.

Main concepts and existing literature

Here, we discuss the main concepts behind the idea of quantum speed limit in the existing literature. Although a chronological order would have been a nice choice, here, we have appreciated the creative freedom for the presentation and development of the conceptual framework necessary for the rest of the chapters.

We suppose, a system is evolving from an initial state to a final state. The minimum time the system takes, has fundamental interests in various fields as stated earlier. The first fundamental bound on the time of quantum evolutions was proved by Mandelstam and Tamm [1], whose sole motivation was to derive an alternative energy- time uncertainty relation. Before going into the detail descriptions of QSL, We first discuss about the metric-distance one can define on the projective Hilbert space space. Then we briefly review existing bounds on the time of quantum evolutions.

Fubini-Study metric

Let us discuss about the metric distance defined between pure quantum states. The infinitesimal distance between two pure quantum states is measured by the Fubini-Study (FS) distance [2, 3, 112] which is nothing but a unique $U(1)$ gauge invariant distance. We consider a state of a system to be $|\psi(\theta)\rangle$. Due to an infinitesimal change in θ, the change in the state is given by $|d\psi(\theta)\rangle = d\theta\frac{\partial|\psi(\theta)\rangle}{\partial\theta}$. One can also define another differential form

$$|d\psi_\perp\rangle = |d\psi\rangle - \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}|d\psi\rangle, \tag{2.1}$$

which does not distinguish two collinear vectors, i.e., $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ but $|d\psi\rangle$ does. On the other hand, we want a distance on the projective Hilbert space (1.2.1), where collinear vectors are taken to be the same vector. Therefore, we need $|d\psi_\perp\rangle$.

Now, the angular variation of $|d\psi_\perp\rangle$ is

$$|d\psi_{projec}\rangle = \frac{|d\psi_\perp\rangle}{\sqrt{\langle\psi|\psi\rangle}} = \frac{|d\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}} - \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle^{3/2}}|d\psi\rangle. \tag{2.2}$$

The FS metric [2, 3, 112] on the projective Hilbert space is given by

$$ds^2_{FS} = ds^2 = \langle d\psi_{projec}|d\psi_{projec}\rangle. \tag{2.3}$$

If one considers the state to be normalized, the FS metric due to the infinitesimal change in θ turns out to be

$$ds^2 = \langle d\psi|d\psi\rangle - |\langle\psi|d\psi\rangle|^2. \tag{2.4}$$

Now, suppose a state $|\psi(0)\rangle$ evolves under a unitary operation $U(t) = e^{-i\frac{Ht}{\hbar}}$ to the final state $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, where the Hamiltonian is time independent. The FS metric due to infinitesimal change in the parameter t can easily be shown to be

$$ds^2 = \frac{\Delta H^2}{\hbar^2} dt^2. \tag{2.5}$$

One can then easily define the total distance between the initial state $|\psi(0)\rangle$ and the final state $|\psi(T)\rangle$ at time $t = T$ as

$$s = \int_0^T ds = \frac{\Delta H}{\hbar} T. \tag{2.6}$$

Mandelstam-Tamm bound

We consider that a system with a state $|\psi(0)\rangle$ is evolving to a state $|\psi(t)\rangle$ at time t under a unitary operator $U(t) = e^{-i\frac{Ht}{\hbar}}$. Let us define an operator $A = |\psi(0)\rangle\langle\psi(0)|$, whose expectation value at time t is nothing but $\langle A\rangle = |\langle\psi(0)|\psi(t)\rangle|^2 = \cos^2\theta$ (say). Now, using the Robertson uncertainty relation [101] one obtains

$$\Delta H \Delta A \geq \frac{1}{2}\Big|\langle[A, H]\rangle\Big|. \tag{2.7}$$

Again, we know that $\left|\frac{d\langle A\rangle}{dt}\right| = \hbar\Big|\langle[A, H]\rangle\Big|$ and $\Delta A^2 = \langle A^2\rangle - \langle A\rangle^2 = \cos^2\theta - \cos^4\theta = \frac{\sin^2 2\theta}{4}$. Thus, from the Eq. (2.7), we obtain

$$\begin{aligned} \Delta H\Big|\frac{\sin 2\theta}{2}\Big| &\geq \frac{\hbar}{2}\Big|\frac{d\cos^2\theta}{dt}\Big| \\ &= \frac{\hbar}{2}\Big|\sin 2\theta\frac{d\theta}{dt}\Big| \\ \text{Thus,} \int_0^T \frac{\Delta H}{\hbar} dt = s &\geq |\theta(T)|, \end{aligned} \tag{2.8}$$

where $\frac{s_0(t)}{2} = |\theta(t)| = \cos^{-1}|\langle\psi(0)|\psi(t)\rangle|$ is nothing but the Bures metric between the initial state and the final state. This, in turn, provides the Mandelstam and Tamm bound [1] for the evolution of a state $|\psi(0)\rangle$ to the final state $|\psi(T)\rangle$ at time T as

$$T \geq \hbar\frac{\cos^{-1}|\langle\psi(0)|\psi(t)\rangle|}{\Delta H}, \tag{2.9}$$

where we assumed the Hamiltonian H to be time independent. Here, the inequality $2s \geq s_0$ in Eq. (2.8) states that the geodesic distance along the evolution parameter from the initial state to the final state is no less than the minimum possible geodesic distance between the states. This inequality is important and has been used several times in this thesis.

Margolus-Levitin bound

Another time bound of quantum evolution was provided by Margolus and Levitin [25]. To obtain the bound, we express the state of the system in the eigenbasis of the Hamiltonian H as $|\psi(0)\rangle = \sum_n c_n |E_n\rangle$, where E_n is the eigenvalue of the Hamiltonian in the eigenstate $|E_n\rangle$. Therefore, we have

$$\mathrm{S(t)} = \langle\psi(0)|\psi(t)\rangle = \sum_n |c_n|^2 e^{\frac{iE_n t}{\hbar}}. \tag{2.10}$$

Thus, $\mathrm{Re[S(t)]} = \sum_n |c_n|^2 \cos\frac{E_n t}{\hbar}$ and $\mathrm{Im[S(t)]} = -\sum_n |c_n|^2 \sin\frac{E_n t}{\hbar}$. Using the trigonometric inequality

$$\cos x \geq 1 - \frac{2}{\pi}(x + \sin x) \qquad \text{for all } \; x \geq 0, \tag{2.11}$$

we obtain the Margolus-Levitin bound as

$$t \geq \frac{\hbar}{\langle H\rangle}\big[\frac{\pi}{2} - \mathrm{Im[S(t)]} - \frac{\pi}{2}\mathrm{Re[S(t)]}\big]. \tag{2.12}$$

For orthogonal final state, i.e., $\langle\psi(0)|\psi(t)\rangle = 0$, if the time it takes to evolve a state to its orthogonal state is $t_\perp$, we obtain

$$t_\perp \geq \frac{\pi\hbar}{2\langle H\rangle}. \tag{2.13}$$

Chau Bound

Using the inequality $|\mathrm{Re}(z)| \leq |z|$ and Eq. (2.10), one can easily write

$$|\mathrm{Re}[\mathrm{S(t)}]| = \Big|\sum_n |c_n|^2 \cos\frac{E_n t}{\hbar}\Big| \leq |\mathrm{S(t)}|. \tag{2.14}$$

Now, using a trigonometric inequality $\cos x \geq 1 - A|x|$, where $A \approx 0.725$ as found in [41, 49] by an extensive numerical search, one can easily obtain the Chau bound for pure states. We have

$$1 - A\frac{\langle H\rangle t}{\hbar} \leq \Big|\sum_n |c_n|^2 \cos\frac{E_n t}{\hbar}\Big| \leq |\mathrm{S(t)}|, \tag{2.15}$$

which implies the bound as

$$t \geq \frac{(1 - |\mathrm{S(t)}|)\hbar}{A\langle H\rangle}. \tag{2.16}$$

This bound can further be improved to get a tighter bound as

$$t \geq \frac{\hbar(1 - |\mathrm{S(t)}|)}{AE_{DE}}, \tag{2.17}$$

where E_{ED} is the average absolute deviation from the median (AADM) of the energy as defined by Chau, i.e., $E_{ED} = \sum_n |c_n|^2|E_n - M|$ with M being the median of the E_n's with the distribution p_n.

Interference visibility

The main theme of this chapter is to establish a connection between the speed of quantum evolutions and the visibility of the interference pattern formed due to the interference between the initial state and the final state after quantum evolutions. Therefore, it is necessary to discuss briefly about the interference visibility. The purpose of this section is thus to provide an operationally well defined notion of interference visibility for unitarily evolving mixed quantum states in interferometry following [114].

We consider a Mach-Zehnder interferometer in which a beam of light with state ρ_0 is shined on the beam splitter (B). The state of beams of the interferometer is spanned by a two-dimensional Hilbert space $\tilde{\mathcal{H}} = \{|\tilde{0},\rangle, |\tilde{1}\rangle\}$, which may be considered as wave packets that move in two directions defined by the geometry of the interferometer. In this basis, we may represent the mirrors, beam splitters and the $U(1)$ phase shifters by unitary operators as

$$\begin{aligned} \tilde{U}_M &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tilde{U}_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ \tilde{U}(1) &= \begin{pmatrix} e^{i\xi} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.18}$$

An input pure state of the interferometer $\tilde{\rho} = |\tilde{0}\rangle\langle\tilde{0}|$ (say) transforms into the output state as

$$\begin{aligned} \tilde{\rho}_{out} &= \tilde{U}_B\tilde{U}_M\tilde{U}_1\tilde{U}_B\tilde{\rho}_{in}\tilde{U}_B^\dagger\tilde{U}_1^\dagger\tilde{U}_M^\dagger\tilde{U}_B^\dagger \\ &= \frac{1}{2}\begin{pmatrix} 1+\cos\xi & i\sin\xi \\ -i\sin\xi & 1-\cos\xi \end{pmatrix}. \end{aligned} \tag{2.19}$$

This shows the intensity along $|\tilde{0}\rangle$ as $1+\cos\xi$. Let us now consider that the state of the light beam is made to transform by the interferometer as

$$\rho_0 \to U_i\rho_0U_i^\dagger \tag{2.20}$$

with U_i, which acts only on the internal degrees of freedom of the incident light. We consider that the beam splitters and mirrors keep the state of light beam unaltered. Therefore, we define $U_B = \tilde{U}_B \otimes I_i$ and $U_M = \tilde{U}_M \otimes I_i$, where I_i is the identity matrix acting on the light beam. We define a unitary operator U as

$$U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U_i + \begin{pmatrix} e^{i\xi} & 0 \\ 0 & 0 \end{pmatrix} \otimes I_i, \tag{2.21}$$

which corresponds to the application of U_i along the $|\tilde{1}\rangle$ path and the $U(1)$ phase ξ similarly along $|\tilde{0}\rangle$. Let the incoming state of the beam $\rho_{in} = |\tilde{0}\rangle\langle\tilde{0}|\otimes\rho_0$ be split coherently by a beam splitter and recombined at the second beam splitter after being reflected by two mirrors. A schematic diagram of an interferometer is shown in Fig. (2.1). If the

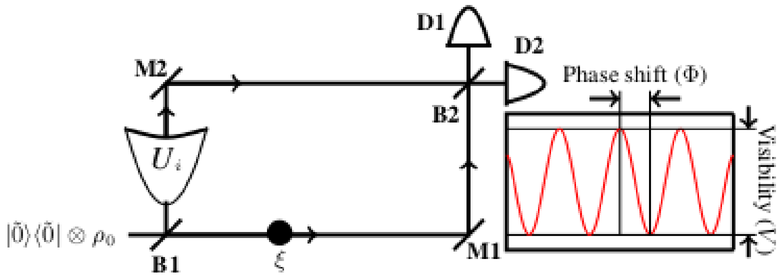

FIGURE 2.1: Schematic diagram of an interferometer. Light beam passes through the 50% beam splitter. ρ_0 part of the beam traverses through the upper arm and $|\tilde{0}\rangle\langle\tilde{0}|$ traverses through the lower arm.

unitary operator U is applied between the first beam splitter and the mirror, the final state turns out to be

$$\rho_{out} = U_B U_M U U_B \rho_{in} U_B^\dagger U^\dagger U_M^\dagger U_B^\dagger. \tag{2.22}$$

Now, using Eq. (2.18), (2.19), (2.20), (2.21) and (2.22), we can easily get the output intensity along $|\tilde{0}\rangle$ as

$$\begin{aligned} I &\propto \mathrm{Tr}(U_i\rho_0 U_i^\dagger + \rho_0 + e^{-i\xi}U_i\rho_0 + e^{i\xi}\rho_0 U_i^\dagger) \\ &\propto 1 + |\mathrm{Tr}(\rho_0 U_i)|\cos[\xi - Arg(\mathrm{Tr}(\rho_0 U_i))]. \end{aligned} \tag{2.23}$$

Therefore, the interference visibility turns out to be $V = |\mathrm{Tr}(\rho_0 U_i)|$. This idea was generalized also for CPTP evolutions [115] and a generalized notion of interference visibility was put forward. We have used this notion of interference visibility later in this chapter.

Here, we start with the introduction of a new operation dependent metric, which can be measured experimentally in the interference of mixed states. We show that using this metric, it is possible to define a new lower limit for the evolution time of any system described by mixed state undergoing unitary evolution. We derive the quantum speed limit using the geometric uncertainty relation based on this new metric. We also obtain a Levitin kind of bound and an improved Chau bound for mixed states using our approach. We show that this bound for the evolution time of a quantum system is tighter than any other existing bounds for unitary evolutions. Most importantly, we propose an experiment to measure this new distance in the interference of mixed states. We argue that the visibility in quantum interference is a direct measure of distance for mixed quantum states. Finally, we generalize the speed limit for the case of completely positive trace preserving evolutions and get a new lower bound for the evolution time using this metric.

The organization of the chapter is as follows. We use an experimentally measurable metric on the quantum state space along the evolution trajectory (2.3) to obtain a new and tighter quantum evolution time bounds for unitary evolutions in section (2.4), followed by examples in section (2.5). In section (2.6), we show that bounds are experimentally measurable. Section (2.8) contains generalization of the metric and the time bounds for completely positive trace preserving(CPTP) maps followed by an example. Then, we conclude in section (2.9).

Metric along unitary orbit

Let $\mathcal{H}$ denotes a finite-dimensional Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of linear operators on $\mathcal{H}$. A density operator ρ is a Hermitian, positive and trace class operator that satisfies $\rho \geq 0$ and $\text{Tr}(\rho) = 1$. Let ρ be a non-degenerate density operator with spectral decomposition $\rho = \sum_k \lambda_k |k\rangle\langle k|$, where λ_k's are the eigenvalues and $\{|k\rangle\}$'s are the eigenstates. We consider a system at time t_1 in a state ρ_1. It evolves under a unitary evolution and at time t_2, the state becomes $\rho_2 = U(t_2, t_1)\rho_1 U^\dagger(t_2, t_1)$. Any two density operators that are connected by a unitary transformation will give a unitary orbit. If $U(N)$ denotes the set of $N \times N$ unitary matrices on $\mathcal{H}^N$, then for a given density operator ρ, the unitary orbit is defined by $\rho' = \{U\rho U^\dagger : U \in U(N)\}$. The most important notion that has resulted from the study of interference of mixed quantum states is the concept of the relative phase between ρ_1 and ρ_2 and the notion of visibility in the interference pattern. The relative phase is defined by [114]

$$\Phi(t_2, t_1) = \text{Arg}\text{Tr}[\rho_1 U(t_2, t_1)] \tag{2.24}$$

and the visibility is defined by

$$V = |\text{Tr}[\rho_1 U(t_2, t_1)]|. \tag{2.25}$$

Note that if $\rho_1 = |\psi_1\rangle\langle\psi_1|$ is a pure state and $|\psi_1\rangle = |\psi(t_1)\rangle \to |\psi_2\rangle = |\psi(t_2)\rangle = U(t_2, t_1)|\psi(t_1)\rangle$, then $|\mathrm{Tr}(\rho_1 U(t_2, t_1))|^2 = |\langle\psi(t_1)|\psi(t_2)\rangle|^2$, which is nothing but the fidelity between two pure states. The quantity $\mathrm{Tr}[\rho_1 U(t_2, t_1)]$ represents the probability amplitude between ρ_1 and ρ_2, which are unitarily connected. Therefore, for the unitary orbit $|\mathrm{Tr}(\rho_1 U(t_2, t_1))|^2$ represents the transition probability between ρ_1 and ρ_2.

All the existing metrics on the quantum state space give rise to the distance between two states independent of the operation. Here, we define a new distance between two unitarily connected states of a quantum system. This distance not only depends on the states but also depends on the operation under which the evolution occurs. Whether a state of a system will evolve to another state depends on the Hamiltonian which in turn fixes the unitary orbit. Let the mixed state traces out an open unitary curve $\Gamma : t \in [t_1, t_2] \to \rho(t)$ in the space of density operators with "end points" ρ_1 and ρ_2. If the unitary orbit connects the state ρ_1 at time t_1 to ρ_2 at time t_2, then the distance between them is defined by

$$D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2})^2 := 4(1 - |\mathrm{Tr}[\rho_1 U(t_2, t_1)]|^2), \tag{2.26}$$

which also depends on the orbit, i.e., $U(t_2, t_1)$. We will show that it is indeed a metric, i.e., it satisfies all the axioms to be a metric.

We know that for any operator A and a unitary operator U, $|\mathrm{Tr}(AU)| \leq \mathrm{Tr}|A|$ with equality for $U = V^\dagger$, where $A = |A|V$ is the polar decomposition of A [113]. Considering $A = \rho = |\rho|$, we get $|\mathrm{Tr}[\rho_1 U(t_2, t_1)]| \leq 1$. This proves the non-negativity, or separation axiom. It can also be shown that $D_U(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$ if and only if there is no evolution along the unitary orbit, i.e., $\rho_1 = \rho_2$ and $U = I$. If there is no evolution along the unitary orbit, then we have $U(t_2, t_1) = I$, i.e., trivial or global cyclic evolution, i.e., $\rho_2 = U(t_2, t_1)\rho_1 U^\dagger(t_2, t_1) = \rho_1$, which in turn implies $D_U(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$. To see the converse, i.e., if $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$, then we have no evolution, consider the purification. We have $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 4(1 - |\langle\Psi_{AB}(t_1)|\Psi_{AB}(t_2)\rangle|^2)$ where $|\Psi_{AB}(t_2)\rangle = U_A(t_2, t_1) \otimes I_B|\Psi_{AB}(t_1)\rangle$ such that $\mathrm{Tr}_B(|\Psi_{AB}(t_1)\rangle\ \langle\Psi_{AB}(t_1)|) = \rho_1$ and $\mathrm{Tr}_B(|\Psi_{AB}(t_2)\rangle\langle\Psi_{AB}(t_2)|) = \rho_2$. In the extended Hilbert space, $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$ implies $|\langle\Psi_{AB}(t_1)|\Psi_{AB}(t_2)\rangle|^2 = 1$ and hence, $\Psi_{AB}(t_1)$ and $\Psi_{AB}(t_2)$ are same up to $U(1)$ phases. To prove the symmetry axiom, we show that the quantity $|\text{Tr}[\rho_1 U(t_2,t_1)]|$ is symmetric with respect to the initial and the final states. In particular, we have

$$|\text{Tr}[\rho_1 U(t_2,t_1)]| = |\text{Tr}[\rho_2 U(t_1,t_2)]| = |\text{Tr}[\rho_2 U(t_2,t_1)]|. \tag{2.27}$$

To see that the new distance satisfies the triangle inequality, consider its purificaton. Let $\rho_A(t_1)$ and $\rho_A(t_2)$ are two unitarily connected mixed states of a quantum system A. If we consider the purification of $\rho_A(t_1)$, then we have $\rho_A(t_1) = \text{Tr}_B[|\Psi_{AB}(t_1)\rangle\langle\Psi_{AB}(t_1)|]$, where $|\Psi_{AB}(t_1)\rangle = (\sqrt{\rho_A(t_1)}V_A \otimes V_B)|\alpha\rangle \in \mathcal{H_A} \otimes \mathcal{H_B}$, V_A, V_B are local unitary operators and $|\alpha\rangle = \sum_i |i^A i^B\rangle$. The evolution of $\rho_A(t_1)$ under $U_A(t_2,t_1)$ is equivalent to the evolution of the pure state $|\Psi_{AB}(t_1)\rangle$ under $U_A(t_2,t_1)\otimes I_B$ in the extended Hilbert space. Thus, in the extended Hilbert space, we have $|\Psi_{AB}(t_1)\rangle \to |\Psi_{AB}(t_2)\rangle = U_A(t_2,t_1)\otimes I_B|\Psi_{AB}(t_1)\rangle$. So, the transition amplitude between two states is given by $\langle\Psi_{AB}(t_1)|\Psi_{AB}(t_2)\rangle = \text{Tr}[\rho_A(t_1)U_A(t_2,t_1)]$. This simply says that the expectation value of a unitary operator $U_A(t_2,t_1)$ in a mixed state is equivalent to the inner product between two pure states in the enlarged Hilbert space. Since, in the extended Hilbert space the purified version of the metric satisfies the triangle inequality, hence the triangle inequality holds also for the mixed states. If ρ_1 and ρ_2 are two pure states, which are unitarily connected then our new metric is the Fubini-Study metric [2, 3, 112] on the projective Hilbert space $\mathbf{CP}(\mathcal{H})$.

Now, imagine that two density operators differ from each other in time by an infinitesimal amount, i.e., $\rho(t_1) = \rho(t) = \sum_k \lambda_k |k\rangle\langle k|$ and $\rho(t_2) = \rho(t+dt) = U(dt)\rho(t)U^\dagger(dt)$. Then, the infinitesimal distance between them is given by

$$dD^2_{U(dt)}(\Gamma_{\rho(t_1)}, \Gamma_{\rho(t_2)}) = 4(1 - |\text{Tr}[\rho(t)U(dt)]|^2). \tag{2.28}$$

If we use the time independent Hamiltonian H for the unitary operator, then keeping terms upto second order, the infinitesimal distance (we drop the subscript) becomes

$$\begin{aligned} dD^2 &= \frac{4}{\hbar^2}[\mathrm{Tr}(\rho(t)H^2) - [\mathrm{Tr}(\rho(t)H)]^2]dt^2 \\ &= \frac{4}{\hbar^2}[\sum_k \lambda_k \langle k|H^2|k\rangle - (\sum_k \lambda_k \langle k|H|k\rangle)^2]dt^2 \\ &= \frac{4}{\hbar^2}[\sum_k \lambda_k \langle \dot{k}|\dot{k}\rangle - (i\sum_k \lambda_k \langle k|\dot{k}\rangle)^2]dt^2, \end{aligned} \tag{2.29}$$

where in the last line we used the fact that $i\hbar|\dot{k}\rangle = H|k\rangle$. Therefore, the total distance travelled during an evolution along the unitary orbit is given by

$$D_{tot} = \frac{2}{\hbar}\int_{t_1}^{t_2} (\Delta H)_\rho \; dt, \tag{2.30}$$

where $(\Delta H)_\rho$ is the uncertainty in the Hamiltonian of the system in the state ρ and is defined as $(\Delta H)^2_\rho = [\mathrm{Tr}(\rho(t)H^2) - [\mathrm{Tr}(\rho(t)H)]^2]$. Thus, it is necessary and sufficient to have non-zero ΔH for quantum system to evolve in time.

Quantum speed limit with new metric

We consider a system A with mixed state $\rho_A(0)$ at time $t = 0$, which evolves to $\rho_A(t_2) = \rho_A(T)$ under a unitary operator $U_A(T)$. We define the Bargmann angle in terms of the purifications of the states $\rho_A(0)$ and $\rho_A(T)$ in the extended Hilbert space $\mathcal{H_A} \otimes \mathcal{H_B}$, i.e., $|\langle\Psi_{AB}(0)|\Psi_{AB}(T)\rangle| = \cos\frac{s_0}{2}$, where $|\Psi_{AB}(T)\rangle = U_A(T,0) \otimes I_B|\Psi_{AB}(0)\rangle$. It has already been shown in the previous section that $|\mathrm{Tr}[\rho_A(0)U_A(T)]| = |\langle\Psi_{AB}(0)|\Psi_{AB}(T)\rangle|$. Therefore, we can define the Bargmann angle between $\rho_A(0)$ and $\rho_A(T)$ as

$$|\mathrm{Tr}[\rho_A(0)U_A(T)]| = \cos\frac{s_0}{2}, \tag{2.31}$$

such that $s_0 \in [0, \pi]$. We know that for pure states

$$\frac{2}{\hbar}\int (\Delta H)_{|\psi_{AB}(0)\rangle} \; dt \geq \cos^{-1}|\langle\psi_{AB}(0)|\psi_{AB}(T)\rangle| \tag{2.32}$$

as was derived in [1, 2]. The inequality in the extended Hilbert space now becomes a property of the state space, i.e., $\frac{2}{\hbar}\int\ (\Delta H)_\rho\ dt \geq s_0$ as shown in Eq. (2.8). This says that the total distance travelled by the density operator $\rho(t)$ as measured by the metric (2.30) is greater than or equal to the shortest distance between $\rho(0)$ and $\rho(T)$ defined by s_0. Using the inequality and the fact that the system Hamiltonian H is time independent, we get the time limit of the evolution as

$$T \geq \frac{\hbar}{\Delta H}\cos^{-1}|\text{Tr}[\rho_A(0)U_A(T)]|. \tag{2.33}$$

This is one of the central results of this chapter with the help of the new metric. This same idea can be extended for the quantum system with time dependent Hamiltonian. The speed limit in this case is given by

$$T \geq \frac{\hbar}{\overline{\Delta H}}\cos^{-1}|\text{Tr}[\rho_A(0)U_A(T)]|, \tag{2.34}$$

where $\overline{\Delta H} = (\frac{1}{T}\int_0^T \Delta H dt)$ is the time averaged energy uncertainty of the quantum system. This may be considered as generalization of the Anandan-Aharonov geometric uncertainty relation for the mixed states. This bound is better and tighter than the bound given in [28, 50] and reduces to the time limit given by Anandan and Aharonov [2] for pure states. There can be some states called intelligent states and some optimal Hamiltonians for which the equality may hold. But in general, it is highly non-trivial to find such intelligent states [24, 26].

To see that Eq. (2.33) indeed gives a tighter bound, consider the following. We suppose that a system in a mixed state ρ_A evolves to ρ'_A under $U_A(t)$. Let S and S' are the sets of purificatons of ρ_A and ρ'_A respectively. In [19, 28, 50], time bound was given in terms of Bures metric [57], i.e., $\min_{|\Psi_{AE}\rangle,|\Phi_{AE}\rangle} 2\cos^{-1}|\langle\Psi_{AE}|\Phi_{AE}\rangle|$ [113], such that $|\Psi_{AE}\rangle \in S$ and $|\Phi_{AE}\rangle \in S'$. But in Eq. (2.33), the time bound is tighter than that given in [19, 28, 50] in the sense that here the bound is in terms of s_0, i.e.,$s_0 = 2\cos^{-1}|\langle\Psi_{AE}|\Phi_{AE}\rangle|$, such that $|\Phi_{AE}\rangle = U_A \otimes I_E|\Psi_{AE}\rangle$ and hence, s_0 is always greater than or equals to the Bures angle [57] defined as $2\cos^{-1}[\text{Tr}\sqrt{\rho_A^{\frac{1}{2}}\rho'_A\rho_A^{\frac{1}{2}}}]$. However, if ρ is pure then the time bound given using our metric and the Bures metric are the same.

We have defined here the quantum speed limit depending on the evolution whereas the speed limits that exist in literature are operation independent. Our result can be experimentally measurable whereas the existing results [19, 28, 50] including the quantum speed limit in [52] cannot be measured directly. This is because we do not know yet how to measure the Bures metric and Uhlmann metric experimentally.

Furthermore, using our formalism, we can derive a Margolus and Levitin kind of time bound [25] for the mixed state. Let us consider the system A with a mixed state $\rho(0)$ at time t=0. Let $\rho(0) = \sum_k \lambda_k |k\rangle\langle k|$ be the spectral decomposition of $\rho(0)$ and it evolves under a unitary operator $U(T)$ to a final state $\rho(T)$. In this case, we have

$$\mathrm{Tr}[\rho(0)U(T)] = \sum_n p_n(\cos\frac{E_nT}{\hbar} + i\sin\frac{E_nT}{\hbar}), \tag{2.35}$$

where we have used $|k\rangle = \sum_n c_n^{(k)}|\psi_n\rangle$, and $|\psi_n\rangle$'s are eigenstates of the Hamiltonian H with $H|\psi_n\rangle = E_n|\psi_n\rangle$, and $p_n = \sum_k \lambda_k |c_n^{(k)}|^2$ with $\sum_n p_n = 1$. Using the inequality $\cos x \geq 1 - \frac{2}{\pi}(x + \sin x)$ for $x \geq 0$, i.e., for positive semi-definite Hamiltonian, we get

$$\mathrm{Re}[\mathrm{Tr}[\rho(0)U(T)]] \geq [1 - \frac{2}{\pi}(\frac{T\langle H\rangle}{\hbar} + \sum_n p_n \sin\frac{E_nT}{\hbar})]. \tag{2.36}$$

Then, from Eq. (2.36), we have

$$T \geq \frac{\pi\hbar}{2\langle H\rangle}[1 - R - \frac{2}{\pi}I], \tag{2.37}$$

where R and I are real and imaginary parts of $\mathrm{Tr}[\rho(0)U(T)]$ and they can be positive as well as negative. Note that when R and I are negative, this can give a tighter bound. This new time bound for mixed states evolving under unitary evolution with non-negative Hamiltonian reduces to $\frac{h}{4\langle H\rangle}$, i.e., the Margolus and Levitin [25] bound in the case of evolution from one pure state to its orthogonal state. Therefore, the time limit of the evolution under unitary operation with Hamiltonian H becomes

$$T \geq \begin{cases} \max\{\frac{s_0\hbar}{2\Delta H}, \frac{\pi\hbar}{2\langle H\rangle}(1 - \frac{2}{\pi}I - R) & \text{if } H \geq 0 \\ \frac{\hbar s_0}{2\Delta H} & \text{otherwise} \end{cases} \tag{2.38}$$

We can also obtain an improved Chau [41, 49] bound for mixed states using our formalism. Using the inequality $|Re(z)| \leq |z|$ and a trigonometric inequality $\cos x \geq 1 - A|x|$, where $A \approx 0.725$ as found in [41], we get $V = |\mathrm{Tr}(\rho U)| \geq |\sum_n p_n \cos(\frac{E_n T}{\hbar})| \geq 1 - \frac{AT}{\hbar} \sum_n p_n |E_n|$. Therefore, the time bound is given by

$$T \geq \frac{(1-V)\hbar}{A < E >}, \tag{2.39}$$

where $< E >$ is the average energy. It can be further modified to get a tighter bound as given by

$$T \geq T_c \equiv \frac{\hbar}{A} \frac{(1-V)}{E_{DE}}, \tag{2.40}$$

where E_{DE} is the average absolute deviation from the median (AADM) of the energy as defined by Chau, i.e., $E_{DE} = \sum_n p_n |E_n - M|$ with M being the median of the E_n's with the distribution p_n. The above bound is more tighter time bound than that given in Eq. (2.33) depending on the distribution formed by the eigenvalues of H for a sufficiently small visibility (V) [41]. Moreover, this new bound is always tighter than the Chau bound [41]. This is because of the fact that $V \leq \mathrm{Tr}\sqrt{\rho^{\frac{1}{2}} \rho' \rho^{\frac{1}{2}}}$.

In the following, we have taken an example in the two dimensional state space and shown that the inequalities in Eq. (2.38) are indeed satisfied by the quantum system.

Example of speed limit for unitary evolution

We consider a general single qubit state

$$\rho(0) = \frac{1}{2}(I + \vec{r}.\vec{\sigma}), \tag{2.41}$$

such that $|r|^2 \leq 1$. The state evolves under a general unitary operator U as

$$\rho(0) \rightarrow \rho(T) = U(T)\rho(0)U^{\dagger}(T), \tag{2.42}$$

where $U = e^{i\frac{a}{\hbar}(\hat{n}.\vec{\sigma}+\alpha I)}$, $a = \omega.T$ and the Hamiltonian

$$H = \omega(\hat{n}.\vec{\sigma} + \alpha I), \tag{2.43}$$

where $\vec{\sigma}= (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $\hat{n}$ is a unit vector. This Hamiltonian H becomes positive semi-definite for $\alpha \geq 1$. It is easy to show, that for $\alpha = 1$,

$$s_0 = 2\cos^{-1}\sqrt{[\cos^2\frac{a}{\hbar} - (\hat{n}.\vec{r})\sin^2\frac{a}{\hbar}]^2 + \cos^2\frac{a}{\hbar}\sin^2\frac{a}{\hbar}(1+\hat{n}.\vec{r})^2}\;, \tag{2.44}$$

where

$$\begin{aligned} \Delta H &= \omega\sqrt{1-(\hat{n}.\vec{r})^2}, \\ R &= [\cos^2\frac{a}{\hbar} - (\hat{n}.\vec{r})\sin^2\frac{a}{\hbar}], \\ I &= \cos\frac{a}{\hbar}\sin\frac{a}{\hbar}(1+\hat{n}.\vec{r}) \\ \text{and} \qquad \langle H\rangle &= \omega(1+\vec{r}.\hat{n}). \end{aligned} \tag{2.45}$$

Using the inequality (2.38), we get

$$T \geq \frac{\hbar}{\omega\sqrt{1-(\hat{n}.\vec{r})^2}}\cos^{-1}\sqrt{\begin{array}{c}[\cos^2\frac{a}{\hbar} - (\hat{n}.\vec{r})\sin^2\frac{a}{\hbar}]^2 + \\ \cos^2\frac{a}{\hbar}\sin^2\frac{a}{\hbar}(1+\hat{n}.\vec{r})^2\end{array}} = f(\beta, T), \tag{2.46}$$

where we define β as $\beta = \hat{n}.\vec{r}$. For $\alpha = 1 = \hbar = \omega$ and $a = \pi/2$, we get that the initial state evolves to $\rho(T) = \frac{1}{2}(I + \vec{r'}.\vec{\sigma})$, where

$$\vec{r'} = (2n_1(\hat{n}.\vec{r}) - r_1, 2n_2(\hat{n}.\vec{r}) - r_2, 2n_3(\hat{n}.\vec{r}) - r_3) \tag{2.47}$$

at time T with the evolution time bound

$$T \geq \max\{\frac{\cos^{-1}(\hat{n}.\vec{r})}{\sqrt{1-(\hat{n}.\vec{r})^2}}, \frac{\pi}{2}\} = \frac{\pi}{2}. \tag{2.48}$$

This shows that the inequality is indeed tight (saturated). For simplicity, we consider a state with parameters $\hat{n} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}})$ and $\vec{r} = (0, 0, \frac{1}{2})$ as an example. Then the state $\rho(0)$ under the unitary evolution $U(T)$ becomes $\rho(T)=\frac{1}{2}(I + \vec{r'}.\vec{\sigma})$, such that

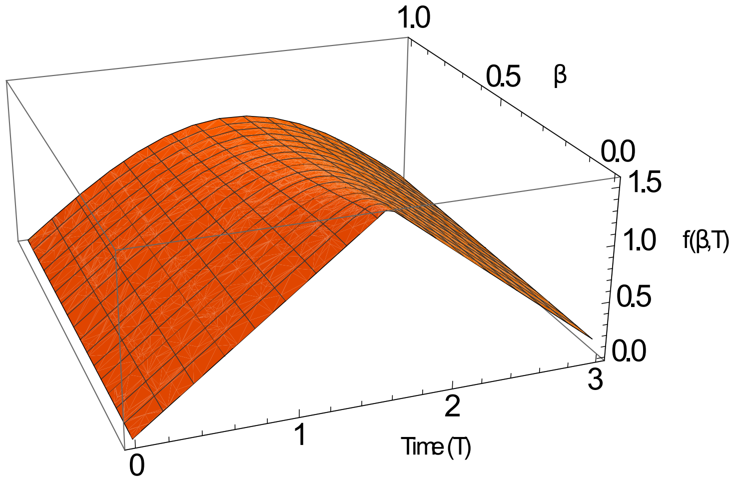

FIGURE 2.2: Here, we plot the quantum evolution time bound $f(\beta, T)$ as given in Eq. (2.46) with β and time (T). From the figure, it is evident that for any qubit state evolving unitarily under arbitrary time independent Hamiltonian (H), evolution time (T) is lower bounded by the function $f(\beta, T)$.

$\vec{r'} = (-\frac{4\sqrt{3}}{15}, \frac{\sqrt{2}}{15}, -\frac{1}{6})$. Therefore, the time bound given by Eq. (2.38) is approximately max[1.09, 0.86], i.e., 1.09 in the units of $\hbar = \omega = 1$. But a previous bound [28, 50] would give approximately 0.31. This shows that our bound is indeed tighter to the earlier ones.

In the sequel, we discuss how the geometric uncertainty relation can be measured experimentally. This is the most important implication of our new approach.

Experimental proposal to measure speed limit

Arguably, the most important phenomenon that lies at the heart of quantum theory is the quantum interference. It has been shown that in the interference of mixed quantum states, the visibility is given by $V = |\mathrm{Tr}(\rho U)|$ and the relative phase shift is given by $\Phi = \mathrm{Arg}[\mathrm{Tr}(\rho U)]$ [114]. In quantum theory both of these play very important roles and they can be measured in experiments [121, 122]. The notion of interference of mixed states has been used to define interference of quantum channels [115]. For pure quantal states, the magnitude of the visibility is the overlap of the states between the upper and lower arms of the interferometer. Therefore, for mixed states one can imagine that $|\mathrm{Tr}(\rho U)|^2$ also represents the overlap between two unitarily connected quantum states.

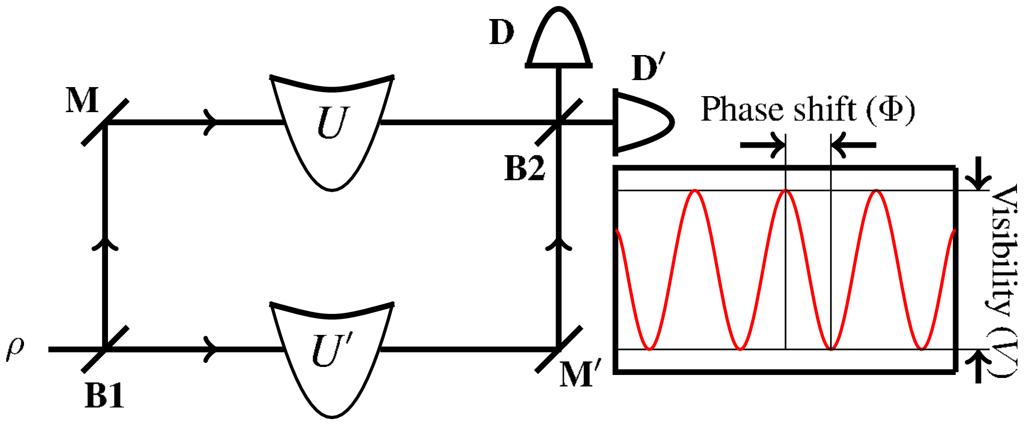

FIGURE 2.3: Mach-Zender interferometer. An incident state ρ is beamed on a 50% beam splitter B1. The state in the upper arm is reflected through M and evolved by a unitary evolution U and the state in the lower arm is evolved by an another unitary evolution U' and then reflected through M′. Beams are combined on an another 50% beam splitter B2 and received by two detectors D and D′ to measure the visibility. By appropriately choosing different unitaries, one can measure the quantum speed and the time limit.

As defined in the previous sections, this visibility can be turned into a distance between ρ and $\rho' = U\rho U^{\dagger}$. In Fig. (2.3), we pass a state ρ of a system through a 50% beam splitter B1. The state in the upper arm is reflected by M and evolved by a unitary evolution operator U and the state in the lower arm is evolved by U' and then reflected through M′. Both the beams in the upper and lower arms are combined on an another 50% beam splitter B2. The beams will interfere with each other. Two detectors are placed in the receiving ends and visibility of the interference pattern is measured by counting the particle numbers received at each ends. To measure the Bargmann angle, we apply $U = U(T)$ in one arm and $U' = I$ in another arm of the interferometer. The visibility $|\mathrm{Tr}[\rho(0)U(T)]| = \cos\frac{s_0}{2}$ will give the Bargmann angle s_0. To measure the quantum speed $v = \frac{2\Delta H}{\hbar}$, one can apply $U = U(t)$ in one arm of the interferometer and one applies $U' = U(t+\tau)$, where τ is very small in another arm of the interferometer. Then, the visibility will be $|\mathrm{Tr}[\rho(t)U(\tau)]|$. Hence, the quantum speed can be measured in terms of this visibility between two infinitesimally unitarily evolved states using the expression

$$V^2 = |\mathrm{Tr}[\rho(t)U(\tau)]|^2 = 1 - \frac{1}{4}v^2\tau^2. \tag{2.49}$$

One can choose τ to be very short time scale with $\tau \ll T$. Once we know the visibility (the value of s_0) then we can verify the speed limit of the evolution for mixed states. Thus, by appropriately changing different unitaries, we can measure the quantum speed and hence the speed limit in quantum interferometry. One can also test Levitin kind of bound using our interferometric set up. Note that R and I of Eq. (2.38) can be calculated from the relative phase Φ of quantum evolution together with the visibility V. The relative phase Φ of mixed state evolution can be measured by determining the shift in the interference pattern in the interferometer [121, 122]. Therefore, with prior knowledge of average of the Hamiltonian, one can test the Levitin bound for the mixed states.

The notion of time bound can be generalized also for the completely positive trace preserving (CPTP) maps. In the next section, we derive the quantum speed limit for CPTP maps.

Speed limit under completely positive trace preserving maps

The metric defined here gives the distance between two states which are related by unitary evolution. Now, consider a system A in a state $\rho_A(0)$ at time $t = 0$, which evolves under CPTP map $\mathcal{E}$ to $\rho_A(T)$ at time $t = T$. The final state $\rho_A(T)$ can be expressed in the following Kraus operator representation form as

$$\rho_A(T) = \mathcal{E}(\rho_A(0)) = \sum_k E_k(T)\rho_A(0)E_k^\dagger(T), \tag{2.50}$$

where $E_k(T)$'s are the Kraus operators with $\sum_k E_k^\dagger(T)E_k(T) = I$. We know that this CPTP evolution can always be represented as a unitary evolution in an extended Hilbert space via the Stinespring dilation. Let us consider, without loss of generality, an initial state $\rho_{AB}(0) = \rho_A(0) \otimes |\nu\rangle_B\langle\nu|$ at time $t = 0$ in the extended Hilbert space. The combined state evolves under $U_{AB}(T)$ to a state $\rho_{AB}(T)$ such that $\rho_A(T) = \mathrm{Tr}_B[\rho_{AB}(T)] = \mathcal{E}(\rho_A(0))$ and $E_k = {}_B\langle k|U_{AB}(T)|\nu\rangle_B$ [112]. Therefore, the time required to evolve the state $\rho_A(0)$ to $\rho_A(T)$ under the CPTP evolution is the same as the time required for the state $\rho_{AB}(0)$ to evolve to the state $\rho_{AB}(T)$ under the unitary evolution $U_{AB}(T)$ in the extended Hilbert space. Following the quantum speed limit for unitary case, we get the time bound to evolve the quantum system from $\rho_A(0)$ to $\rho_A(T)$ as

$$T \geq \frac{\hbar s_0}{2\Delta H_{AB}}, \tag{2.51}$$

where H_{AB} is the time independent Hamiltonian in the extended Hilbert space and s_0 is defined as

$$\cos\frac{s_0}{2} = |\mathrm{Tr}[\rho_{AB}(0)U_{AB}(T)]|. \tag{2.52}$$

Note that the energy uncertainty of the combined system in the extended Hilbert space ΔH_{AB} can be expressed in terms of speed v of evolution of the system and the Bargmann angle s_0 can be expressed in terms of operators acting on the Hilbert space of quantum system. To achieve that, we express probability amplitude $\mathrm{Tr}_{AB}[U_{AB}(T)(\rho_A(0) \otimes |\nu\rangle_B\langle\nu|)]$ in the extended Hilbert space in terms of linear operators acting on the Hilbert space of quantum system as

$$\mathrm{Tr}_{AB}[U_{AB}(T)(\rho_A(0) \otimes |\nu\rangle_B\langle\nu|)] = \mathrm{Tr}_A[\rho_A(0)E_\nu(T)], \tag{2.53}$$

where $E_\nu(T) = {}_B\langle\nu|U_{AB}(T)|\nu\rangle_B$. Here, $|\mathrm{Tr}_A[\rho_A(0)E_\nu(T)]|^2$ is the transition probability between the initial state and the final state of the quantum system under CPTP map. Therefore, we can define the Bragmann angle between $\rho_A(0)$ and $\rho_A(T)$ under the CPTP map as

$$|Tr_A[\rho_A(0)E_\nu(T)]| = \cos\frac{s_0}{2}. \tag{2.54}$$

Similarly, we can define the infinitesimal distance between $\rho_{AB}(0)$ and $\rho_{AB}(dt)$ connected through unitary evolution $U_{AB}(dt)$ with time independent Hamiltonian H_{AB} as

$$\begin{aligned} dD^2_{U_{AB}(dt)} &= 4(1 - |\mathrm{Tr}[\rho_{AB}(t)U_{AB}(dt)]|^2) \\ &= 4(1 - |\mathrm{Tr}[\rho_{AB}(0)U_{AB}(dt)]|^2) \\ &= 4(1 - |\mathrm{Tr}[\rho_A(0)E_\nu(dt)]|^2). \end{aligned} \tag{2.55}$$

Now, keeping terms upto second order, we get the infinitesimal distance as

$$dD^2 = \frac{4}{\hbar^2}[\mathrm{Tr}(\rho_A(0)\tilde{H^2}_A) - [\mathrm{Tr}(\rho_A(0)\tilde{H}_A)]^2]dt^2, \tag{2.56}$$

where $\tilde{H}_A = {}_B\langle\nu|H_{AB}|\nu\rangle_B$ and $\tilde{H^2}_A = {}_B\langle\nu|H^2_{AB}|\nu\rangle_B$. Therefore, the speed of the quantum system is given by

$$v^2 = \frac{4}{\hbar^2}[\mathrm{Tr}(\rho_A(0)\tilde{H^2}_A) - [\mathrm{Tr}(\rho_A(0)\tilde{H}_A)]^2]. \tag{2.57}$$

Note, that this is not a fluctuation in $\tilde{H}_A$. This is because $\tilde{H_A}^2 \neq \tilde{H}^2_A$. Here, $\tilde{H}_A$ can be regarded as an effective Hamiltonian for the subsystem A. Note that the speed can be expressed as

$$v^2 = (\Delta\tilde{H})^2 + \mathrm{Tr}(\rho_A(0)\tilde{H^2}) - \mathrm{Tr}(\rho_A(0)\tilde{H}^2). \tag{2.58}$$

Hence, the time bound for the CPTP evolution from Eq. (2.51) becomes

$$T \geq \frac{2}{v}\cos^{-1}|\mathrm{Tr}_A[\rho_A(0)E_\nu(T)]|, \tag{2.59}$$

Here the interpretation of this limit is different from that of the unitary case. The transition probability in unitary case is symmetric with respect to the initial and the final states. Hence, the time limit can be regarded as the minimum time to evolve the initial state to the final state as well as the final state to the initial state. But the transition probability defined for positive map is not symmetric with respect to the initial and final states of the quantum system. In this case, time limit can only be regarded as the minimum time to evolve the initial state to the final state.

Since we have mapped the time bound to evolve an initial state $\rho_A(0)$ to the final state $\rho_A(T)$ under CPTP evolution with the time bound of corresponding unitary representation $\rho_{AB}(T) = U_{AB}(T)\rho_{AB}(0)\, U^\dagger_{AB}(T)$ of the CPTP map in the extended Hilbert space, this speed limit can be measured in the interference experiment by interfering the two states $\rho_{AB}(0)$ and $\rho_{AB}(T)$ in the extended Hilbert space.

Example of Speed limit under completely positive trace preserving maps

We provide here an example of a general single qubit state $\rho_A(0)=\frac{1}{2}\,(I+\vec{r}.\vec{\sigma})$ at time $t=0$, such that $|r|^2 \leq 1$ evolving under CPTP map $\mathcal{E}$. It evolves to $\rho_A(T)$ at time $t=T$ under completely positive trace preserving (CPTP) map $\mathcal{E}:\rho_A(0)\mapsto\mathcal{E}(\rho_A(0))=\rho_A(T)=\sum_k E_k(T)\rho_A(0)\;E_k^{\dagger}(T)$. This evolution is equivalent to a unitary evolution of $\rho_{AB}(0)=\frac{1}{2}(I+\vec{r}.\vec{\sigma})\otimes|0\rangle\langle 0|\rightarrow\rho_{AB}(T)$ as

$$\rho_{AB}(T)=U_{AB}(T)\rho_{AB}(0)U_{AB}^{\dagger}(T) \tag{2.60}$$

in the extended Hilbert space. The unitary evolution is implemented by a Hamiltonian

$$H=\sum_i \mu_i\sigma_A^i\otimes\sigma_B^i. \tag{2.61}$$

This is a canonical two qubit Hamiltonian up to local unitary operators. With the unitary $U_{AB}(T)=e^{\frac{iT}{\hbar}(\sum_i\mu_i\sigma_A^i\otimes\sigma_B^i)}$, we have the Kraus operators $E_0(T)={}_B\langle 0|U_{AB}(T)|0\rangle_B$ and $E_1(T)={}_B\langle 1|U_{AB}(T)|0\rangle_B$ and it is now easy to show from Eq. (2.59) that the time bound for this CPTP evolution is given by

$$T\geq\frac{\hbar\cos^{-1}K}{\sqrt{\mu_1^2+\mu_2^2+\mu_3^2(1-r_3^2)-2\mu_1\mu_2r_3}}, \tag{2.62}$$

where

$$\begin{aligned}K=[(\cos\theta_1\cos\theta_2\cos\theta_3+r_3\sin\theta_1\sin\theta_2\cos\theta_3)^2 \quad &+ \quad (\sin\theta_1\sin\theta_2 sin\theta_3\\ &+ \quad r_3\cos\theta_1\cos\theta_2\sin\theta_3)^2]^{\frac{1}{2}}\end{aligned} \tag{2.63}$$

and $\theta_1=\frac{\mu_1T}{\hbar}$, $\theta_2=\frac{\mu_2T}{\hbar}$ and $\theta_3=\frac{\mu_3T}{\hbar}$. If we consider $\theta_1=\pi$, $\theta_3=\pi$ then this bound reduces to

$$T\geq\frac{\hbar\theta_2}{\sqrt{\mu_1^2+\mu_2^2+\mu_3^2(1-r_3^2)-2\mu_1\mu_2r_3}}. \tag{2.64}$$

One can also check our speed bound for various CPTP maps and it is indeed respected.

Conclusions

Quantum Interference plays a very important role in testing new ideas in quantum theory. Motivated by interferometric set up for measuring the relative phase and the visibility for the pure state, we have proposed a new and novel measure of distance for the mixed states, which are connected by the unitary orbit. The new metric reduces to the Fubini-Study metric for pure state. Using this metric, we have derived a geometric uncertainty relation for mixed state, which sets a quantum speed limit for arbitrary unitary evolution. In addition, a Levitin kind of bound and an improved Chau bound is derived using our formalism. These new speed limits based on our formalism are tighter than any other existing bounds. Since, the design of the target state is a daunting tusk in quantum control, our formalism will help in deciding which operation can evolve the initial state to the final state much faster. Then, we have proposed an experiment to measure this new distance and quantum speed in the interference of mixed states. The visibility in quantum interference is a direct measure of distance between two mixed states of the quantum system along the unitary orbit. We have shown that by appropriately choosing different unitaries in the upper and lower arm of the interferometer one can measure the quantum speed and the Bargmann angle. This provides us a new way to measure the quantum speed and quantum distance in quantum interferometry. We furthermore, extended the idea of speed limit for the case of density operators undergoing completely positive trace preserving maps. We hope that our proposed metric will lead to direct test of quantum speed limit in quantum interferometry. Our formalism can have implications in quantum metrology [123], precision measurement of the gravitational red shift [124] and gravitationally induced decoherence [125] with mixed states and other areas of quantum information science.

Chapter 3

Quantum coherence sets the quantum speed limit for mixed states

"In relativity, movement is continuous, causally determinate and well defined, while in quantum mechanics it is discontinuous, not causally determinate and not well defined."
——*David Bohm*

Introduction

To build the route to develop the role of quantum coherence and non-locality (quantum steering) in setting the limit to the speed of quantum evolutions, some familiarity with the relevant ideas and notions is necessary. As this chapter is dedicated to establish a connection between quantum coherence and QSL only, in the next subsections, we introduce various measures of quantum coherence and observable measure of quantum coherence (as well as asymmetry) and leave quantum non-locality or in particular, quantum steering for the next chapter.

Quantum coherence and asymmetry

As mentioned earlier, quantum coherence was an already well defined and well established notion in quantum optics and its role in quantum interference was undeniable. It was expected that it might play a crucial role in quantum information theory as well. This was the main motivation to define quantum coherence from the resource theoretic perspective [68, 69, 72]. This eventually led to the development of the resource theory of quantum coherence in quantum information theory [70, 71, 84, 126].

In [68], quantum coherence was viewed as a resource and was argued that like other resources in quantum information theory, quantum coherence also should follow certain properties:

❶ Coherence of an incoherent state must be zero, i.e.,

$$C(\rho) = 0, \text{ where } \rho \in \mathcal{I}, \text{ the set of all the incoherent states,} \tag{3.1}$$

❷ monotonicity under incoherent completely positive and trace preserving maps Ψ_{ICPTP}, i.e.,

$$C(\rho) \geq C(\Phi_{ICPTP}(\rho)), \tag{3.2}$$

❸ monotonicity under selective measurement on average, i.e.,

$$C(\rho) \geq \sum_n p_n C(\rho_n) \text{ for all } \{K_n\} \text{ such that } \sum_n K_n^\dagger K_n = I \text{ and } K_n \mathcal{I} K_n^\dagger \in \mathcal{I}, \tag{3.3}$$

❹ non-increasing under mixing of quantum states, i.e.,

$$\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n) \text{ for any set of states } \{\rho_n\} \text{ with } p_n \geq 0 \text{ and } \sum_n p_n = 1. \tag{3.4}$$

Various measures of quantum coherence like the l_1-norm, the relative entropy of coherence etc. were introduced, which satisfy all of these properties. For a state $\rho \in \mathcal{H}_d$, the l_1-norm measure of quantum coherence in a fixed basis $\{|i\rangle\}$ is given by

$$C^{l_1}(\rho) = \sum_{\substack{i,j \\ i \neq j}} |\rho_{ij}|. \tag{3.5}$$

The relative entropy of coherence is defined as

$$C^E(\rho) = S(\rho_{diag.}) - S(\rho), \tag{3.6}$$

where $S(\sigma)$ is the von-Neumann entropy of the state σ and $\sigma_{diag.}$ is defined as $\sigma_{diag.} = \sum_i \rho_{ii}|i\rangle\langle i|$, when $\sigma = \sum_{i,j} \rho_{ij}|i\rangle\langle j|$.

Observable measure of quantum coherence on the other hand was introduced in Ref. [69]. As evident from the above definitions of measures of quantum coherence, it is a distance, with certain properties, between the state in its eigenbasis and in the basis, in which the quantum state is written. D. Girolami in Ref. [69] defined a measure of quantum coherence of a state with respect to the eigenbasis of an observable instead of the eigenbasis of the state.

$$C_H^S(\rho) = Q(\rho, H) = -\frac{1}{2}\mathrm{Tr}[\sqrt{\rho}, H] \tag{3.7}$$

It was also interpreted as a measure of quantum asymmetry [118–120, 127].

Here, we consider a new notion of Fubini-Study metric for mixed states introduced in [128]. For unitary evolutions, it is nothing but the Wigner-Yanase skew information [129], which only accounts for the quantum part of the uncertainty [71] and a good measure of quantum coherence [69, 116] or asymmetry [118–120, 127], which classifies coherence [130] as a resource. Using this metric, we derive a tighter and experimentally measurable Mandelstam and Tamm kind of QSL for unitary evolutions and later generalize for more general evolutions. This sets a new role for quantum coherence or asymmetry as a resource to control and manipulate the evolution speed.

An important question in the study of quantum speed limit may be how it behaves under classical mixing and partial elimination of states. This is due to the fact that this may help us to properly choose a state or evolution operator to control the speed limit. In this thesis, we try to address this question.

The organization of the chapter is as follows. We introduce the Fubini-Study metric for mixed states along a unitary orbit for our convenience (3.2). Section (3.3) contains a derivation of QSL for unitary evolutions based on the metric defined here. In section (3.4), we propose an experiment to measure our bound. We also study how QSL behaves under classical mixing and partial elimination of states in section (3.5). Section (3.6) deals with a procedure to generalize the QSL defined here. We compare various existing bounds in the literature in section (3.7). In section (3.8), we analyze and compare various existing QSLs with an example of a unitary evolution of a qubit. We also generalize the QSL for general evolutions in section (3.9) and compare existing results with an example for Markovian evolution (3.10). At last, we conclude in section (3.11).

Metric along unitary orbit

Let $\mathcal{H}_A$ denotes the Hilbert space of the system A. Suppose that the system A with a state $\rho(0)$ evolves to $\rho(t)$ under a unitary operator $U = e^{-iHt/\hbar}$. One can define a $U(1)$ gauge invariant distance between the initial and the final state along the evolution parameter t. To derive such a distance along the unitary orbit, we consider the purification of the state in the extended Hilbert space and define the Fubini-Study (FS) metric for pure states. We know that this is the only gauge invariant metric for pure states. We follow the procedure as in [128] to derive a gauge invariant metric for mixed states from this FS metric for pure states. If we consider a purification of the state $\rho(0)$ in the extended Hilbert space by adding an ancillary system B with Hilbert space $\mathcal{H}_B$ as $|\Psi_{AB}(0)\rangle = (\sqrt{\rho(0)}V_A \otimes V_B)|\alpha\rangle \in \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, the state at time t, must be

$$|\Psi_{AB}(t)\rangle = (\sqrt{\rho(t)}V_A \otimes V_B)|\alpha\rangle = (U_A\sqrt{\rho}U_A^\dagger V_A \otimes V_B)|\alpha\rangle, \tag{3.8}$$

where $|\alpha\rangle = \sum_i |i^A i^B\rangle$ and V_A, V_B are unitary operators on the subsystems A and B respectively. The FS metric for a state $|\psi\rangle$ on the projective Hilbert space can be defined as

$$ds^2_{FS} = \langle d\psi_{projec}|d\psi_{projec}\rangle, \tag{3.9}$$

where $|d\psi_{projec}\rangle = \frac{|d\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}} - \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle^{3/2}}|d\psi\rangle$. This is nothing but the angular variation of the perpendicular component of the differential form $|d\psi\rangle$. The angular variation of the perpendicular component of the differential form for the state $|\Psi_{AB}(t)\rangle$ in this case is given by

$$|d\Psi_{AB_{projec}}(t)\rangle = dt(A_\rho - B_\rho)|\alpha\rangle, \tag{3.10}$$

where $A_\rho = (\partial_t\sqrt{\rho(t)}V_A \otimes V_B)$, $B_\rho = |\Psi_{AB}(t)\rangle\langle\Psi_{AB}(t)|\ A_\rho$. Therefore, the FS metric [128] is given by

$$\begin{aligned} ds^2_{FS} &= \langle d\Psi_{AB_{projec}}(t)|d\Psi_{AB_{projec}}(t)\rangle \\ &= dt^2[\langle\alpha|(A^\dagger_\rho A_\rho - A^\dagger_\rho B_\rho - B^\dagger_\rho A_\rho + B^\dagger_\rho B_\rho)|\alpha\rangle] \\ &= \mathrm{Tr}[(\partial_t\sqrt{\rho_t})^\dagger(\partial_t\sqrt{\rho_t})] - |\mathrm{Tr}(\sqrt{\rho_t}\partial_t\sqrt{\rho_t})|^2, \end{aligned} \tag{3.11}$$

where the second term on the last line becomes zero if monotonicity condition is imposed [128].

Now, suppose that the state of the system is evolving unitarily under $U = e^{-iHt/\hbar}$ and at time t, the state $\rho = \rho(t) = U\rho(0)U^\dagger$. We know that square-root of a positive density matrix is unique. If we consider $\rho(0) = \sum_i \lambda_i|i\rangle\langle i|$, then $\rho = \sum_i \lambda_i U|i\rangle\langle i|U^\dagger$ implies

$$\sqrt{\rho} = \sum_i \sqrt{\lambda_i}U|i\rangle\langle i|U^\dagger = U\sqrt{\rho(0)}U^\dagger \tag{3.12}$$

and uniqueness of the positive square-root implies the uniqueness of the relation. One can show this in an another way by considering arbitrary non-hermitian square-root w of the final state ρ and using the relation

$$\rho = ww^\dagger = U\rho(0)U^\dagger = U\sqrt{\rho(0)}\sqrt{\rho(0)}U^\dagger = U\sqrt{\rho(0)}V^\dagger V\sqrt{\rho(0)}U^\dagger, \tag{3.13}$$

where V is arbitrary unitary operator. Thus, one gets the form of these arbitrary non-hermitian square-roots as $w = U\sqrt{\rho(0)}V^\dagger$. Due to uniqueness of the positive square-root of the positive density matrix, Hermiticity condition imposes uniqueness on the arbitrary unitary operators above as $V = U$. Thus, we get $\sqrt{\rho} = U\sqrt{\rho(0)}U^\dagger$, which in turn implies

$$\frac{\partial\sqrt{\rho}}{\partial t} = -\frac{i}{\hbar}[\sqrt{\rho}, H]. \tag{3.14}$$

Using this relation and the Eq. (3.11), we get (dropping the subscript FS)

$$ds^2 = -\frac{dt^2}{\hbar^2}[\mathrm{Tr}[\sqrt{\rho}, H]^2] = 2\frac{dt^2}{\hbar^2}Q(\rho, H). \tag{3.15}$$

The quantity $-[\mathrm{Tr}[\sqrt{\rho}, H]^2] = 2Q(\rho, H)$ in Eq. (3.15) is nothing but the quantum part of the uncertainty as defined in [71] and comes from the total energy uncertainty $(\Delta H)^2$ on the pure states $|\Psi_{AB}\rangle$ in the extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The quantity is also related to the quantum coherence of the state [69]. By integrating the distance, we get the total distance between the initial state $|\Psi_{AB}(0)\rangle$ and the final state $|\Psi_{AB}(T)\rangle$ as

$$s = \int_0^T ds = \frac{1}{\hbar}\sqrt{-\mathrm{Tr}[\sqrt{\rho_1}, H]^2}T, \tag{3.16}$$

where we have considered the Hamiltonian H to be time independent and $\rho(0) = \rho_1$. Here, we see that the distance between the two pure states on the extended Hilbert space can completely be written in terms of the state ρ_1 and the Hamiltonian $H \in S(\mathcal{H}_A)$, the space of all linear operators belongs to the subsystem A and can also be interpreted as a distance between the initial state ρ_1 and the final state $\rho(T) = \rho_2$. Again, we can define the total distance in an another way by considering the Bargmann angle between the initial state and the final state as

$$\begin{aligned} s_0 &= 2\cos^{-1}|\langle\Psi_{AB}(0)|\Psi_{AB}(T)\rangle| \\ &= 2\cos^{-1}\mathrm{Tr}(\sqrt{\rho_1}\sqrt{\rho_2}) \\ &= 2\cos^{-1}A(\rho_1, \rho_2), \end{aligned} \tag{3.17}$$

where the quantity $A(\rho_1, \rho_2) = \mathrm{Tr}(\sqrt{\rho_1}\sqrt{\rho_2})$ is also known as affinity [131] between the states ρ_1 and ρ_2.

Quantum speed limits for unitary evolution

Mandelstam and Tamm in Ref. [1] showed that the total geodesic distance traversed by the system between the pure initial and final states during the evolution is (3.16) is greater than the minimum possible geodesic distance defined by the Bargmann angle between the two states as in (3.17), i.e., $s \geq \frac{s_0}{2}$ as shown in Eq. (2.8). The inequality, in particular, in this case becomes

$$T \geq \frac{\hbar}{\sqrt{2}} \frac{\cos^{-1} A(\rho_1, \rho_2)}{\sqrt{Q(\rho_1, H)}} = \mathcal{T}_l(\rho_1, H, \rho_2). \tag{3.18}$$

This shows that the quantum speed is fundamentally bounded by the observable measure of quantum coherence or asymmetry of the state detected by the evolution Hamiltonian. If an initial state evolves to the same final state under two different evolution operators, the operator, which detects less coherence or asymmetry in the state slows down the evolution. As a result, it takes more time to evolve. We can clarify this fact with a simple example. We consider a system with $|+\rangle$ state. If we evolve the system by unitary operators $U_z = e^{-i\sigma_z t}$ and $U_x = e^{-i\sigma_x t}$, the system will not evolve under U_x with time in the projective Hilbert space. This is due to the fact that the state of the system is incoherent when measured with respect to the evolution operator σ_x, i.e., $[|+\rangle\langle+|, \sigma_x] = 0$. Therefore, quantum coherence or asymmetry of a state with respect to the evolution operator may be considered as a resource to control and manipulate the speed of quantum evolutions. Here it is important to mention that Brody in [54] had also used WY skew information previously to modify the quantum Cramer-Rao bound.

The time bound in Eq. (3.18) can easily be generalized for time dependent Hamiltonian $H(t)$.

Corollary.— *For a time dependent Hamiltonian $H(t)$, the inequality in (3.18) becomes $T \geq \frac{\hbar}{\sqrt{2}} \frac{\cos^{-1} A(\rho_1,\rho_2)}{\overline{\sqrt{Q_T(\rho_1,H(t))}}}$, where $\overline{\sqrt{Q_T(\rho_1, H(t))}} = \frac{1}{T}\int_0^T \sqrt{Q(\rho_1, H(t))}dt$ can be regarded as the time average of the quantum coherence or quantum part of the energy uncertainty.*

One can find other interesting results for time dependant Hamiltonians following other methods given in Ref. [19, 50, 77].

Experimental proposal

Estimation of linear and non-linear functions of density matrices in the interferometry is an important task in quantum information theory and quantum mechanics. D. K. L. Oi *et al.* in [132] gave the first proposal to measure various functions of density matrices in the interferometry directly using the setup in Ref. [114]. Later, the method was used in [133] to measure various overlaps. In [69], a lower bound of the quantum H-coherence, $1/2\sqrt{-\mathrm{Tr}[\rho_1, H]^2}$ was proposed to be measurable using the same procedure. But Here, we show that the quantum H-coherence itself can be measured in the interferometry. We also propose a method to measure the Affinity $A = \mathrm{Tr}(\sqrt{\rho_1}\sqrt{\rho_2})$. For d-dimensional density matrices, $\mathrm{Tr}(\rho_1^n)$ can be measured for $n = 1$ to d by measuring the average of the SWAP operator (V), which in turn gives all the eigenvalues of the state [132] (see FIG. (3.1) also). Using these eigenvalues, we can prepare a state of the form $\sigma_1 = \widetilde{U}\frac{\sqrt{\rho_1}}{\mathrm{Tr}\sqrt{\rho_1}}\widetilde{U}^\dagger$ with arbitrary and unknown unitary $\widetilde{U}$. This is due to the fact that although we know the eigenvalues of the state ρ_1, we don't know its eigenbasis. Now, we put this state σ_1 in one arm and ρ_1 in another arm of the interferometric set up as in Fig. (3.1). This measurement in the interferometry gives the average of the two particle SWAP operator on these two states, which in turn gives the overlap between the two states, i.e., $\mathrm{Tr}(\rho_1 \otimes \sigma_1 V) = \mathrm{Tr}(\rho_1\sigma_1)$. This quantity we get in the measurement is nothing but $\mathrm{Tr}(\rho_1\sigma_1) = \frac{\mathrm{Tr}(\rho_1\widetilde{U}\sqrt{\rho_1}\widetilde{U}^\dagger)}{\mathrm{Tr}(\sqrt{\rho_1})}$. We can calculate the quantity $\frac{\mathrm{Tr}(\rho_1^{3/2})}{\mathrm{Tr}(\sqrt{\rho_1})}$ from the known eigenvalues of the state ρ_1. We can prepare the state $\frac{\sqrt{\rho_1}}{\mathrm{Tr}(\sqrt{\rho_1})}$ from σ_1 by comparing the calculated and the measured results and rotating the polarization axis of the prepared state σ_1 until both the results match. At this point, the prepared state σ_1 and the given state ρ_1 becomes diagonal on the same basis (see [134] for advantage over state tomography). We use this state and similarly prepared another copy of the state to measure $-\mathrm{Tr}[\sigma_1, H]^2$ and the overlap $\mathrm{Tr}(\sigma_1\sigma_2)$ between σ_1 and $\sigma_2 = U\sigma_1 U^\dagger$ ($U = e^{-iHt/\hbar}$) using the method given in [69, 132, 133]. The quantity measured in the experiment $-\mathrm{Tr}[\sigma_1, H]^2$ is nothing but $-\frac{\mathrm{Tr}[\sqrt{\rho_1}, H]^2}{(\mathrm{Tr}\sqrt{\rho_1})^2}$ and similarly the quantity $\mathrm{Tr}(\sigma_1\sigma_2) = \frac{\mathrm{Tr}(\sqrt{\rho_1}\sqrt{\rho_2})}{(\mathrm{Tr}\sqrt{\rho_1})^2}$. The denominator of each of these quantities are known. Therefore, from these measured values, we can easily calculate the quantum coherence $Q(\rho_1, H)$ and the affinity $A(\rho_1, \rho_2)$. This formalism can also be used to measure the Uhlmann fidelity in the experiment.

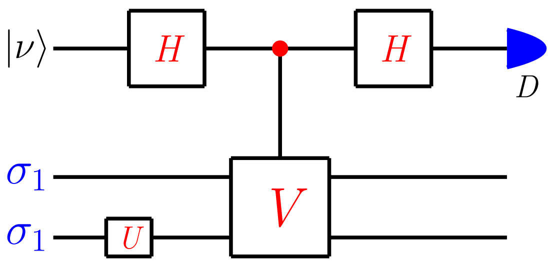

FIGURE 3.1: This network is an experimental configuration to measure $\mathrm{Tr}(\sigma_1\sigma_2)$ and $-\mathrm{Tr}[\sigma_1, H]^2$. a) In both the two lower arms the state σ_1 is fed. The state σ_1 in the lower arm goes through a unitary transformation $U(\tau)$ so that the state changes to σ_2. An ancilla state $|\nu\rangle$ is fed on the upper arm. First this ancilla state undergoes a controlled Hadamard operation (H) followed by a controlled swap operation (V) on the system states and on the ancilla. After that a second Hadamard operation takes place on the ancilla state. The detector (D) measures the probability of getting the ancilla in the same state $|\nu\rangle$. From this probability (P) we can calculate the overlap between the two states by $\mathrm{Tr}(\sigma_1\sigma_2) = 2P - 1$. b) In the second case we measure $-\mathrm{Tr}[\sigma_1, H]^2$. We use the same procedure as described above except here the state σ_1 in the lower arm goes under infinitesimal unitary transformation $U(d\tau)$.

Speed limit under classical mixing and partial elimination of states

We know that the quantum coherence of a state $Q(\rho, H) = -\frac{1}{2}\,\mathrm{Tr}[\sqrt{\rho}, H]^2$ should not increase under classical mixing of states. Therefore, a fundamental question would be to know how the quantum speed limit behaves under classical mixing of states. To answer this question, we consider a state ρ_1, which evolves to ρ_2 under $U = e^{-iHt/\hbar}$. The minimum time required for this evolution will be $\mathcal{T}_l(\rho_1, H, \rho_2)$. If another state σ_1 evolves to σ_2 under the same unitary, the minimum time required similarly be $\mathcal{T}_l(\sigma_1, H, \sigma_2)$. Now, if a system with a state $\gamma_1 = p\rho_1 + (1-p)\sigma_1$ $(0 \leq p \leq 1)$, which is nothing but a state from classical mixing of ρ_1 and σ_1, evolves under the same operator, the final state (say γ_2) becomes $\gamma_2 = U\gamma_1 U^\dagger = p\rho_2 + (1-p)\sigma_2$. The minimum time needed for this evolution must be $\mathcal{T}_l(\gamma_1, H, \gamma_2)$. Now, we can show a nice relation between these time bounds and quantum coherence as

$$\mathcal{U}^H_{\gamma_1\gamma_2} \leq \sqrt{p}\,\mathcal{U}^H_{\rho_1\rho_2} + \sqrt{1-p}\,\mathcal{U}^H_{\sigma_1\sigma_2}, \tag{3.19}$$

where $\mathcal{U}^H_{\chi\eta} = \mathcal{T}_l(\chi, \eta)\sqrt{Q(\chi, H)}$. To prove this inequality, we used a trigonometric inequality of the form, $\cos^{-1}(px + (1-p)y) \leq \sqrt{p}\cos^{-1}x + \sqrt{1-p}\cos^{-1}y$ for $0 \leq x \leq 1$

and $0 \leq y \leq 1$. Quantities of the form of $\mathcal{U}^{H}_{\gamma_1\gamma_2}$ here are nothing but the product of the minimal time of quantum evolution and the quantum part of the uncertainty [71, 135] in the evolution Hamiltonian H. We already know how quantum coherence behaves under classical mixing. This new relation gives an insight to the quantum speed limit and shows how it behaves under classical mixing.

This inequality naturally raises another fundamental question: How does the quantum speed limit of a system behave under discarding part(s) of the system? To answer this question we consider a situation that a system with a state ρ_{ab} evolves to σ_{ab} under the Hamiltonian H_{ab} such that

$$H_{ab} = H_a \otimes I_b + I_a \otimes H_b. \tag{3.20}$$

Corresponding unitary operator U_{ab} is given by

$$U_{ab} = U_a \otimes U_b = e^{-iH_a t} \otimes e^{-iH_b t}. \tag{3.21}$$

Now, if we discard a part of the system b, then the initial state of the system becomes $\rho_a = \mathrm{Tr}_b \rho_{ab}$. The final state of the system is then given by $\sigma_a = \mathrm{Tr}_b \sigma_{ab} = U_a \rho_a U_a^\dagger$. The quantum speed limit before discarding the party must be $\mathcal{T}_l(\rho_{ab}, H_{ab}, \sigma_{ab})$ and that after discarding the party is given by $\mathcal{T}_l(\rho_a, H_a, \sigma_a)$. It can easily be shown that

$$\mathcal{U}^{H_a}_{\rho_a\sigma_a} \leq \mathcal{U}^{H_{ab}}_{\rho_{ab}\sigma_{ab}}. \tag{3.22}$$

To prove it, we use the fact that $A(\rho_a, \sigma_a) \geq A(\rho_{ab}, \sigma_{ab})$. The inequality gives an insight on how the product of the time bound of the evolution and the quantum part of the uncertainty [71, 135] in energy or quantum coherence or asymmetry of the state with respect to the evolution operator behaves if a part of the system is discarded.

Generalization

A more tighter and experimentally realizable time bound can be derived using this bound. To do that let us consider a map Φ, which maps a state to another state, such that

$\Phi : \rho_1 \rightarrow \frac{\rho_1^\alpha}{\mathrm{Tr}(\rho_1^\alpha)}(= \sigma_1)$ is a quantum state. Suppose that the state ρ_1 evolves under a time independent Hamiltonian H to a state ρ_2 after time T. Then, if we consider the final state under this map Φ to be $\sigma_2 = \frac{\rho_2^\alpha}{\mathrm{Tr}(\rho_1^\alpha)}$ (considering the fact that $\mathrm{Tr}\rho_2^\alpha = \mathrm{Tr}\rho_1^\alpha$), the evolution for σ_1 to be σ_2 is governed by the same Hamiltonian H. Therefore, the time bound for the state σ_1 to reach σ_2 under the Hamiltonian H can be written as

$$T \geq \mathcal{T}_l(\sigma_1, H, \sigma_2). \tag{3.23}$$

Now, by mapping the states σ_1 and σ_2 back to the original states ρ_1 and ρ_2 respectively, we get

$$T \geq \max_\alpha \mathcal{T}_l(\rho_1^\alpha, H, \rho_2^\alpha), \tag{3.24}$$

where

$$\mathcal{T}_l(\rho_1^\alpha, H, \rho_2^\alpha) = \frac{\hbar\sqrt{\mathrm{Tr}\rho_1^\alpha}\cos^{-1}|\frac{\mathrm{Tr}(\rho_1^{\alpha/2}\rho_2^{\alpha/2})}{\mathrm{Tr}\rho_1^\alpha}|}{\sqrt{-\mathrm{Tr}[\rho_1^{\alpha/2}, H]^2}}. \tag{3.25}$$

This quantity for $\alpha = 2$, gives not only a tighter bound (than (3.18)) but can also be measured in the interferometry [69, 132]. For $\alpha = 2$, the expression reduces to

$$T \geq \mathcal{T}_l(\rho_1^2, H, \rho_2^2). \tag{3.26}$$

The denominator of the quantity $\mathcal{T}_l(\rho_1^2, H, \rho_2^2)$, is the lower bound of the H coherence, i.e.,

$$-\frac{1}{2}\mathrm{Tr}[\rho_1, H]^2 \leq \sqrt{-\mathrm{Tr}[\sqrt{\rho_1}, H]^2} \tag{3.27}$$

as proved in Ref. [69]. Therefore, the bound given by Eq. (3.26) may become tighter than that given by Eq. (3.18) depending on the purity of the state $\mathrm{Tr}(\rho_1^2)$ and most importantly, can be measured in the experiment. This is due to the fact that the relative purity [132, 133] and the lower bound of the H-coherence [69] both can be measured.

Our results here can also be generalized for more general evolutions, such as dynamical semi-group, quantum channel *et cetera.* (see sec. (3.9) for details).

Comparison with existing bounds

Mandelstam and Tamm's original bound was generalized for mixed states in various ways. Our bound is tighter than that given by the generalization using Uhlmann's fidelity [47, 56, 136]. This is due to the fact that the standard deviation is always lower bounded in the following way [135]

$$\Delta H)_{\rho_1} \geq \frac{\hbar\sqrt{\mathcal{F}_Q}}{2} \geq \sqrt{-\frac{1}{2}\mathrm{Tr}[\sqrt{\rho_1}, H]^2} \tag{3.28}$$

and $F(\rho_1, \rho_2) \geq \mathrm{Tr}(\sqrt{\rho_1}\sqrt{\rho_2})$ (where $F(\rho, \sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$ and $\mathcal{F}_Q$ is the symmetric logarithmic Fisher information as denoted by [47]). Here, it is important to mention that $\mathcal{F}_Q$ becomes time independent for unitary evolutions under time independent Hamiltonians. Therefore, we get

$$\begin{aligned} T &\geq \mathcal{T}_l(\rho_1, H, \rho_2) \geq \frac{\hbar\cos^{-1} F(\rho_1, \rho_2)}{\sqrt{2}(\Delta H)_{\rho_1}} \\ &\geq \frac{\sqrt{2}\cos^{-1} F(\rho_1, \rho_2)}{\sqrt{\mathcal{F}_Q}}. \end{aligned} \tag{3.29}$$

In [48], relative purity $f(t) = \frac{\mathrm{Tr}(\rho_1\rho_t)}{\mathrm{Tr}(\rho_1^2)}$ between two states ρ_1 and ρ_t was considered as a figure of merit to distinguish the two states and a quantum speed limit of evolution under the action of quantum dynamical semi-group was derived. Let us now write the bound given in Eq. (8) in [48], for unitary evolution case as

$$T \geq \frac{4\hbar N}{\pi^2 D}, \tag{3.30}$$

where

$$\begin{aligned} N &= [\cos^{-1}(\frac{Tr(\rho_1\rho_2)}{Tr(\rho_1^2)})]^2 Tr(\rho_1^2) \\ \text{and} \quad D &= \sqrt{-Tr[\rho_1, H]^2}. \end{aligned} \tag{3.31}$$

Then the bound $\mathcal{T}_l(\rho_1^2, H, \rho_2^2)$ given in Eq. (3.24) for $\alpha = 2$ becomes $\frac{\hbar\sqrt{N}}{D}$. One can easily show that

$$T \geq \frac{\hbar\sqrt{N}}{D} \geq \frac{2\hbar\sqrt{N}}{\pi D} \geq \frac{4\hbar N}{\pi^2 D} \tag{3.32}$$

due to the fact that $\frac{4N}{\pi^2} \leq 1$. Thus our bound in Eq. (3.24) is tighter than the bound given in [48] for unitary evolutions (time independent Hamiltonians).

In Ref. [77], quantum speed limits in terms of visibility and the phase shift in the interferometry was derived and was shown to be tighter than the existing bounds then. It is clear that this new bound in Eq. (3.18) or in Eq. (3.24), sometimes may even be tighter than the Mandelstam and Tamm kind of bound for mixed states given in [77] (see case II in sec. (3.8)).

Example of speed limit for unitary evolution

We consider a general single qubit state $\rho(0) = \frac{1}{2}(I + \vec{r}.\vec{\sigma})$, such that $|r|^2 \leq 1$. Let, it evolves under a general unitary operator U, i.e.,

$$\rho(0) \to \rho(T) = U(T)\rho(0)U^\dagger(T), \tag{3.33}$$

where $U = e^{-i\frac{a}{\hbar}(\hat{n}.\vec{\sigma}+\alpha I)}$, $a = \omega.T$ and the time independent Hamiltonian $H = \omega(\hat{n}.\vec{\sigma} + \alpha I)$ ($\vec{\sigma}= (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $\hat{n}$ is a unit vector). This Hamiltonian H becomes positive semi-definite for $\alpha \geq 1$. Therefore, after evolution the state is

$$\rho(T) = \frac{1}{2}(I + \vec{r'}.\vec{\sigma}), \tag{3.34}$$

where $\vec{r'}$ has the elements as given by

$$r'_i = 2n_i(\hat{n}.\vec{r}) \sin^2 \frac{a}{\hbar} + r_i \cos \frac{2a}{\hbar} \qquad (i = 1, 2, 3). \tag{3.35}$$

Using this information we get

$$A\left(\rho, \rho(T)\right) = \frac{1}{2}\Big[\left(\hat{r}.\hat{r}'\right)(1 - \sqrt{m}) + (1 + \sqrt{m})\Big] \tag{3.36}$$

and quantum coherence is

$$Q(\rho, H) = \omega^2 \left(1 - \sqrt{m}\right) |(\hat{r} \times \hat{n})|^2, \tag{3.37}$$

where $\hat{r} = \frac{\vec{r}}{|r|}$, $\hat{r}' = \frac{\vec{r}'}{|r|}$, and $m = 1 - |r|^2$ is a good measure of mixedness of the state upto some factor. The Eq. (3.37) shows that *the maximally coherent states for a fixed mixedness lie on the equatorial plane perpendicular to the direction of the Hamiltonian, under which the state is evolving and the more pure are these states the more coherent they are [137].* Therefore, the time bound for the evolution considering $\hbar = 1 = \omega$ is

$$T \geq \frac{\cos^{-1}\left(\frac{1}{2}\left[(\hat{r}.\hat{r}')\left(1 - \sqrt{m}\right) + \left(1 + \sqrt{m}\right)\right]\right)}{\sqrt{2(1 - \sqrt{m})}|\hat{r} \times \hat{n}|}. \tag{3.38}$$

Case I: Consider the initial state $\rho(0)$ is a maximum coherent state with respect to H (such that $m = 0$ and $|\hat{n} \times \hat{r}| = 1$). Then the initial state will evolve to the final state $\rho(T) = \frac{1}{2}[I + (\hat{r}'.\vec{\sigma})]$ with the quantum time bound of evolution

$$T \geq \frac{\cos^{-1}\left(\frac{(1 + \cos 2a)}{2}\right)}{\sqrt{2}}, \tag{3.39}$$

where $\hat{r}.\hat{r}' = \cos 2a$. For $a = \frac{\pi}{2}$ the quantum time bound is given by $T \geq \frac{\pi}{2\sqrt{2}}$.

Case II: Consider the initial state $\rho(0) = \frac{1}{2}[I + (\vec{r}.\vec{\sigma})]$, such that $\hat{n} \times \hat{r} = \frac{1}{\sqrt{2}} = \hat{n}.\hat{r})$. Then the initial state will evolve to the final state $\rho(T) = \frac{1}{2}[I + (\vec{r}'.\vec{\sigma})]$ with the quantum time bound of evolution

$$T \geq \frac{\cos^{-1}\left(\frac{1}{2}[(\sin^2 \frac{a}{\hbar} + \cos \frac{2a}{\hbar})(1 - \sqrt{m}) + (1 + \sqrt{m})]\right)}{\omega\sqrt{1 - \sqrt{m}}}. \tag{3.40}$$

For $a = \frac{3\pi}{4}$ and $m = 0$, the quantum time bound is given by $T \geq 0.72$ considering ($\hbar = 1 = \omega$), whereas the Mandelstam and Tamm kind of bound given in [77] would be 0.71. Thus our bound is tighter than that given in [77] in this case.

Case III: Consider the example with $\vec{r} = \left(0, 0, \frac{1}{2}\right)$, $\hat{n} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right)$ and $\vec{r}' = \left(-\frac{4\sqrt{3}}{15}, \frac{\sqrt{2}}{15}, -\frac{1}{6}\right)$. For the above evolution under the Hamiltonian H we find the quantum speed limit $T \geq 0.9$ from our bound (3.38). The Mandelstam-Tamm kind of bound derived in [77] would give 1.09. Thus, our bound in this case is slightly weaker than that given in Ref. [77]

Now, the inequality in Eq. (3.19) can also be illustrated with this example. We consider $\rho_1 = \frac{1}{2}(I + \vec{r}_1.\vec{\sigma})$ and $\sigma_1 = \frac{1}{2}(I + \vec{r}_2.\vec{\sigma})$, such that $|r_1| = 1 = |r_2|$, $\hat{r}_1.\hat{n} = \frac{1}{\sqrt{2}}$, $\hat{r}_2.\hat{n} = \frac{\sqrt{3}}{2}$ and $\hat{r}_1.\hat{r}_2 = 0$. The state γ_1 is such that $p = \frac{1}{3}$. Then, under the condition $\omega = 1 = \hbar$, $\mathcal{U}(\rho_1, \rho_2) = 0.43$, $\mathcal{U}(\sigma_1, \sigma_2) = 0.42$ and $\mathcal{U}(\gamma_1, \gamma_2) = 0.34$, which implies the inequality is satisfied.

Quantum speed limit for any general evolution

The effect of environmental noise is inevitable in any information processing device. Hence the study of QSLs in the non-unitary realm is in ultimate demand. For the first time, Taddei et al.[47] and Campo *et al.* [48] extended the MT bound for any physical processes. Later in [77], QSL for arbitrary physical processes was shown to be related to the visibility of the interference pattern. The result of [48] was further improved [138] by Zhang *et al.* to provide a QSL for open systems with an initially mixed state. Other recent studies of QSLs for open quantum systems were made in [139] and [140]. Here, in this section, we are also extending our result for any general *CPTP* evolutions and later, compare our bound with various other QSLs for a Markovian system.

Consider a system with a state ρ_0^S coupled to an environment with a state γ^E, such that the total state of the system and environment together can be written as $\rho_0^{SE} = \rho_0^S \otimes \gamma^E$, initially at time $t = 0$. Suppose that the evolution of the total state is governed by a global unitary operator $U_t = e^{-iH_{SE}t/\hbar}$. The dynamics of the system is given by a one-parameter family of dynamical maps $\rho_t^S \to \mathcal{V}\rho_0^S := e^{\mathcal{L}t}\rho_0^S$ and can also be represented by completely positive trace preserving map. The Fubini-Study distance under such circumstances becomes

$$ds_{FS}^2 = -\frac{dt^2}{\hbar^2}\mathrm{Tr}[\sqrt{\rho_0^S} \otimes \sqrt{\gamma^E}, H_{SE}]^2 = 2\frac{dt^2}{\hbar^2}Q(\rho_0^S, \widetilde{H_S}), \tag{3.41}$$

where $\widetilde{H_S^2} = \mathrm{Tr}_E(H_{SE}^2 I^S \otimes \gamma^E)$ and $\widetilde{H_S} = \mathrm{Tr}_E(H_{SE} I^S \otimes \gamma^E)$[128]. Therefore, the speed limit of the evolution becomes

$$T \geq \frac{\hbar \cos^{-1} A(\rho_0^S \rho_T^S)}{\sqrt{2Q(\rho_0^S, \widetilde{H_S})}}. \tag{3.42}$$

If the typical time scale of the environment is much smaller (larger) than that of the system, the system dynamics can be considered to be Markovian (non-Markovian). Markovian evolutions form a dynamical semi-group $\mathcal{V}$. We consider such a map with time independent generator $\mathcal{L}$, such that (From here onwards we drop the superscript S from the state of the system.)

$$\frac{d\rho_t}{dt} = \mathcal{L}\rho_t, \tag{3.43}$$

where the Lindbland $\mathcal{L}$ takes the form [141–143]

$$\mathcal{L}\rho = \frac{i}{\hbar}[\rho, H] + \frac{1}{2}\sum_{i,j=1}^{n^2-1} c_{ij}\{[A_i, \rho A_j^\dagger] + [A_i\rho, A_j^\dagger]\}. \tag{3.44}$$

Now, a fundamental question will be what is the time bound of quantum evolution in such a situation. To answer this question, one should keep in mind that such an evolution can be written as a reversible unitary evolution of a state in the extended Hilbert space formed by considering an environment with the system. Therefore, The quantum speed bound should not only depend on the coherence of the state of the system but also on the coherence of the environment. In other words, the bound should depend on the coherence dynamics of the system and environment together. One way to get the quantum speed limit is to use the bound for unitary evolution as given in Eq. (3.18). Such a bound will not give tight limit. To get a tighter bound one needs to consider the infinitesimal distance along the parameter of the dynamical map t as given in Eq. (3.11), where we use the fact that $\frac{d\sqrt{\rho_t}}{dt} = \mathcal{L}\sqrt{\rho_t}$. Because, given $\frac{d\rho_t}{dt} = \mathcal{L}\rho_t$, $\frac{d\sqrt{\rho_t}}{dt} = \mathcal{L}\sqrt{\rho_t}$ is always true for positive square-root of the density matrix and this can be shown using the same lines of approach as given for unitary evolution case.

We know that any CPTP evolution is equivalent to a unitary evolution in the extended Hilbert space. Suppose a state $\rho(0)$ is evolving under a CPTP evolution represented by a set of Kraus operators $\{A_i\}$. Thus, we get

$$\rho = \rho(t) = \sum_i A_i \rho(0) A_i^\dagger = Tr_B(U_{AB}\rho(0) \otimes |0\rangle_B\langle 0|U_{AB}^\dagger) \text{ (say)}, \tag{3.45}$$

where U_{AB} is a unitary operator such that

$$A_i =_B \langle i|U_{AB}|0\rangle_B. \tag{3.46}$$

Now, similarly as before, we may write,

$$\begin{aligned} \rho = ww^\dagger &= Tr_B(U_{AB}\sqrt{\rho(0)} \otimes |0\rangle_B\langle 0|U_{AB}^\dagger U_{AB}\sqrt{\rho(0)} \otimes |0\rangle_B\langle 0|U_{AB}^\dagger) \\ &= \sum_{ij} A_i\sqrt{\rho(0)}A_j^\dagger A_j\sqrt{\rho(0)}A_i^\dagger, \end{aligned} \tag{3.47}$$

where we have used the trace preserving condition $\sum_i A_i^\dagger A_i = I$. Uniqueness of positive square-root $\sqrt{\rho}$ implies $\sqrt{\rho} = \sum_i A_i\sqrt{\rho(0)}A_i^\dagger$. Thus, given $\rho = \sum_i A_i\rho(0)A_i^\dagger$, the evolution of the positive square-root of the state must be $\sqrt{\rho} = \sum_i A_i\sqrt{\rho(0)}A_i^\dagger$, although the opposite is not true in general. Any other set of Kraus operators $\{B_i\}$, such that $\sqrt{\rho} = \sum_i B_i\sqrt{\rho(0)}B_i^\dagger$ may give the same final state ρ from the initial state $\rho(0)$ but cannot give rise to the same kind of evolution of state as $\rho = \sum_i A_i\rho(0)A_i^\dagger$. Using this equation, the quantum speed limit of the system under Markovian evolution reduces to

$$T \geq \frac{\cos^{-1} A(\rho_0, \rho_T)}{\overline{\sqrt{2Q_T}}}, \tag{3.48}$$

where $\overline{\sqrt{Q_T}} = \frac{1}{T}\int_0^T \sqrt{Q(\rho_t, \mathcal{L})}dt$ and $2Q(\rho, \mathcal{L}) = \mathrm{Tr}\{(\mathcal{L}\sqrt{\rho})(\mathcal{L}\sqrt{\rho})^\dagger\} - |\mathrm{Tr}(\sqrt{\rho}\mathcal{L}\sqrt{\rho})|^2$. This bound will also hold for non-Markovian dynamics [56].

Example of speed limit for Markovian evolution

Here we study an example for a two level system in a squeezed vacuum channel [144, 145]. Non-unitary part of the Lindbladian in Eq. (3.44) consists of these following operators $A_1 = \sigma$, $A_2 = \sigma^\dagger$ and $A_3 = \frac{\sigma_3}{\sqrt{2}}$, where σ and $\sigma^\dagger$ are the raising and lowering operators for qubit. These two operators describe the transitions between the two levels. With those, we choose c_{ij} as

$$c = \begin{pmatrix} \frac{1}{2T_1}(1-w_{eq}) & -\frac{1}{T_3} & 0 \\ -\frac{1}{T_3} & \frac{1}{2T_1}(1+w_{eq}) & 0 \\ 0 & 0 & \frac{1}{T_2}-\frac{1}{2T_1} \end{pmatrix}, \tag{3.49}$$

where $T_1 = T_w$ represent the decay rate of the atomic inversion into an equilibrium state w_{eq}. T_2 and T_3 are related to T_u and T_v in the following was

$$\begin{aligned} \frac{1}{T_u} &= (\frac{1}{T_2}+\frac{1}{T_3}) \\ \frac{1}{T_v} &= (\frac{1}{T_2}-\frac{1}{T_3}), \end{aligned} \tag{3.50}$$

where T_u and T_v are the decay rates of the atomic dipole. Here, the damping asymmetry between the u and v components is due to the presence of T_3. We describe the first part of the Lindblad in Eq. (3.44) by the Hamiltonian

$$H = \frac{\hbar\Omega}{2}(\sigma+\sigma^\dagger), \tag{3.51}$$

where Ω is the Rabi frequency of the oscillation. Therefore, here, $\mathcal{L}$ describes a two level atom in a laser field subjected to an irreversible de-coherence by its environment. Let the initial state of the atom is given by $\rho_0 = \frac{1}{2}(\mathbb{I}+\vec{r}\cdot\vec{\sigma})$, where $\vec{r} \equiv (r_1, r_2, r_3)$. We can write it in the damping basis as

$$\rho_0 = \sum_i \mathrm{Tr}\{L_i\rho_0\}R_i, \tag{3.52}$$

where L_i and R_i are the left and right eigen-operator respectively with the eigenvalue λ_i. The state after certain time t can be written as

$$\rho_t = e^{\mathcal{L}t}\rho = \sum_i \mathrm{Tr}\{L_i\rho_0\}\Lambda_i R_i = \sum_i \mathrm{Tr}\{R_i\rho_0\}\Lambda_i L_i, \tag{3.53}$$

where $\Lambda_i(t) = e^{\lambda_i t}$. The left eigen-operators for this system can be expressed as

$$\begin{aligned}
L_0 &= \frac{1}{\sqrt{2}} I, \\
L_1 &= \frac{1}{\sqrt{2}} (\sigma^\dagger + \sigma), \\
L_2 &= \frac{1}{\sqrt{2}} (\sigma^\dagger - \sigma) \\
\text{and} \qquad L_3 &= \frac{1}{\sqrt{2}} (-w_{eq} I + \sigma_3).
\end{aligned} \tag{3.54}$$

Similarly the right eigen-operators can be written as

$$\begin{aligned}
R_0 &= \frac{1}{\sqrt{2}} (I + w_{eq}\sigma_3), \\
R_1 &= \frac{1}{\sqrt{2}} (\sigma^\dagger + \sigma), \\
R_2 &= \frac{1}{\sqrt{2}} (\sigma - \sigma^\dagger) \\
\text{and} \qquad R_3 &= \frac{1}{\sqrt{2}} \sigma_3.
\end{aligned} \tag{3.55}$$

One can easily derive the eigenvalues of these operators as

$$\begin{aligned}
\lambda_0 &= 0, \\
\lambda_1 &= -\frac{1}{T_u} = -(\frac{1}{T_2} + \frac{1}{T_3}), \\
\lambda_2 &= -\frac{1}{T_v} = -(\frac{1}{T_2} - \frac{1}{T_3}), \\
\lambda_3 &= -\frac{1}{T_1} = -\frac{1}{T_w}.
\end{aligned} \tag{3.56}$$

Let us denote $\ell_\pm = 1 \pm \sqrt{m}$. The affinity between the initial and the final states for such evolution is given by

$$A(\rho_0, \rho_t) = \frac{1}{2}\Big[\ell_+ - r_3 w_{eq}\big(\Lambda_3(t) - 1\big) + \frac{\ell_-}{|r|^2}\Big(r_1^2 \Lambda_1(t) + r_2^2 \Lambda_2(t) + \Lambda_3(t) r_3^2\Big)\Big] \tag{3.57}$$

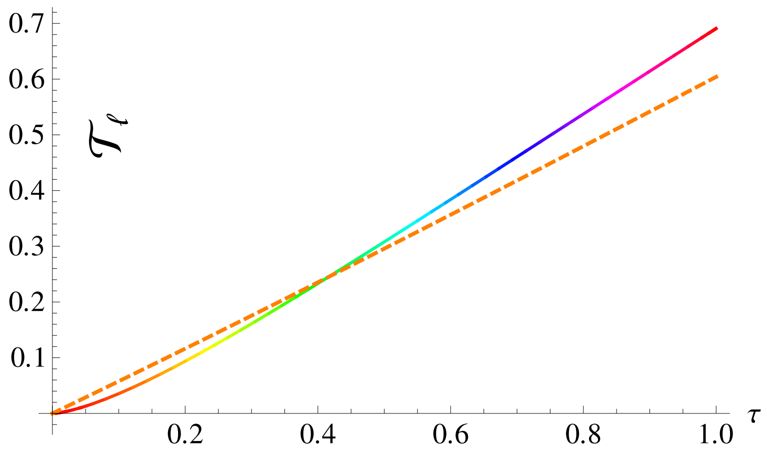

FIGURE 3.2: The bounds $\mathcal{T}_l$ given in Eq. (3.58) (Hue coloured, solid line) and in [48] (Orange coloured, dashed line) have been plotted for $\lambda_1 = -0.9$ with the actual time of evolution T. As seen from the plot, for larger time of interaction, our bound is better than that given in [48].

and the quantum coherence at time t

$$2Q(\rho_t, \mathcal{L}) = \frac{1}{2|r|^2}\Big[\ell_-\Big(r_1^2\lambda_1^2\Lambda_1^2 + r_2^2\lambda_2^2\Lambda_2^2\Big) + \Big(\sqrt{\ell_-}r_3 - w_{eq}|r|\sqrt{\ell_+}\Big)^2 \lambda_3^2\Lambda_3^2\Big] - \frac{1}{4}\Big|\Big[\frac{\ell_-}{|r|^2}\Big(\lambda_1 r_1^2\Lambda_1^2 + \lambda_2 r_2^2\Lambda_2^2\Big) + \lambda_3\Big\{\Big(\frac{r_3\Lambda_3}{|r|}\sqrt{\ell_-}+ \sqrt{\ell_+}\omega_{eq}(1-\Lambda_3)\Big)^2 - \big(r_3\Lambda_3\omega_{eq} + \ell_+\omega_{eq}^2(1-\Lambda_3)\big)\Big\}\Big]\Big|^2,$$

where $m = 1 - |r|^2$. For a simple example, we assume the Lindbladian to be such that $\lambda_3 = 0$ and the initial state to be such that $r_2 = 0 = r_3$, $r_1 = 1$ and $\Lambda_1^T = e^{\lambda_1 T}$ (say). Therefore, the evolution time bound in this case is given by

$$\begin{aligned} T &\geq \frac{2T\cos^{-1}[\frac{1+\Lambda_1^T}{2}]}{\Big|[\Lambda_1^T\sqrt{\frac{1}{4} - \frac{\Lambda_1^T}{2}\sinh(\lambda_1 T)} + \sin^{-1}(\frac{\Lambda_1^T}{\sqrt{2}}) - (\frac{1}{2} + \frac{3\pi}{4})]\Big|} \\ &= \mathcal{T}_l, \end{aligned} \tag{3.58}$$

where the quantity in the denominator $\overline{\sqrt{2Q_T}} = \Big|\frac{1}{2T}[\Lambda_1^T\sqrt{\frac{1}{4} - \frac{\Lambda_1^T}{2}\,\overline{\sinh(\lambda_1 T)}} + \sin^{-1}(\frac{\Lambda_1^T}{\sqrt{2}}) - (\frac{1}{2} + \frac{3\pi}{4})]\Big|$ is nothing but the time average quantum coherence of the state with respect to the evolution operators. In Fig. (3.2), we have plotted bounds $\mathcal{T}_l$ given in Eq. (3.58) and in [48] (see Eq. (9)). As the actual evolution time T increases, the bound given in Eq. (3.58) becomes better and better.

Conclusions

Both, the quantum speed limit and the coherence or asymmetry of a system have been the subjects of great interests. Quantum mechanics limits the speed of evolution, which has adverse effects on the speed of quantum computation and quantum communication protocols. On the other hand, coherence or asymmetry has been projected as a resource in quantum information theory. In this chapter, we define a new role for it as a resource and show that it can be used to control and manipulate the speed of quantum evolution.

A fundamental question is to know how the quantum speed limit behaves under classical mixing and partial elimination of state(s). Therefore, finally answer to this question may help us to choose the state and the evolution operator intelligently for faster evolution. Here, we tried to answer this question for the first time. Our bounds presented here can also be generalized for CPTP evolutions or Markovian processes as well as non-Markovian processes [138, 146].

In a recent paper [69], a protocol was proposed to measure a lower bound of skew information, given by $-\frac{1}{4}\mathrm{Tr}[\rho, H]^2$, experimentally and was argued for, why skew information itself cannot be measured. Not only the skew-information but it was a general consensus that the quantum affinity $A(.,.)$ appears on the numerator of $\mathcal{T}_l(.,.,.)$ also cannot be measured. Here, we show for the first time that both the quantities can indeed be measured (and the formalism can also be used to measure Uhlmann fidelity) in the experiment by recasting them to some other measurable quantities of another properly mapped states. This provides us scopes to apply our theory in a wide range of issues in quantum information theory including quantum metrology, Unruh effect detection, quantum thermodynamics etc. Recently, a number of methods using geometric and Anandan-Aharonov phases have been proposed to detect Unruh effects in analogue gravity models [73–76]. The main issue in such experiments and in general in quantum metrology is to detect a very small fluctuation in some quantity. A potential quantity must be sensitive towards such fluctuation as well as experimentally measurable. By uncovering such a potential quantity $\mathcal{T}_l(.,.,.)$, a formalism to measure Uhlmann fidelity and defining a new role for quantum coherence or asymmetry as a resource, we believe, the present work opens up a wide range of scopes in these directions.

Note: After submitting this work, we noticed another work [147] recently on QSL based on quantum Fisher information. Their work is based on [116] and one of their results resembles our bound for unitary evolutions. For non-unitary evolutions, however, their work is based on the generalization of Quantum Fisher information or Wigner-Yanase skew information. Whereas, our bound is based on the generalization of the Fubini-Study metric for mixed states motivated by [128]. Very recently, another preprint on the arXiv [127], has also appeared on the issues of QSL, coherence and asymmetry. They claimed that coherence is a subset of asymmetry. Their claim does not contradict our work.

Part II

Chapter 4

Non-local advantage of quantum coherence

"Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the old one. I, at any rate, am convinced that He does not throw dice."
——*Albert Einstein*

Introduction

It is evident from the established bounds in the previous two chapters that the limit to the time of quantum evolutions mainly depends on the distance between the initial and the final states and the energy or the energy uncertainty in the system (see Chapter 2). These relations can further be improved and modified and a direct connection between the observable measure of quantum coherence [69, 116] or asymmetry [117–120] and the QSL can be established (see Chapter 3). This is one of the main contributions of this thesis, which leads us to establish the role of quantum non-locality (quantum steering) in setting the speed of quantum evolutions of a part of the system.

In particular, here we address a very fundamental question: Is it possible to gain an advantage in quantum coherence beyond what could have been achieved maximally by a single system with or without non-local influences? We show here that it is indeed possible. This eventually leads us to observe the effects of quantum non-locality (quantum steering) on QSL following the connection between the QSL and the quantum coherence established in the previous chapter.

Steering is a kind of non-local correlation introduced by Schrödinger [102] to reinterpret the EPR-paradox [148]. According to Schrödinger, the presence of entanglement between two subsystems in a bipartite state enables one to control the state of one subsystem by its entangled counter part. Wiseman *et al.* [149] formulated the operational and mathematical definition of quantum steering and showed that steering lies between quantum entanglement and Bell non-locality on the basis of their strength [150]. The notion of the steerability of quantum states is also intimately connected [151] to the idea of remote state preparation [152, 153].

As introduced in Ref. [149], let us consider a hypothetical game to explain the steerability of quantum states. Suppose, Alice prepares two quantum systems, say, A and B in an entangled state ρ_{AB} and sends the system B to Bob. Bob does not trust Alice but agrees with the fact that the system B is quantum. Therefore, Alice's task is to convince Bob that the prepared state is indeed entangled and they share non-local correlation. On the other hand, Bob thinks that Alice may cheat by preparing the system B in a single quantum system, on the basis of possible strategies [154, 155]. Bob agrees with Alice that the prepared state is entangled and they share non-local correlation if and only if the state of Bob cannot be written by local hidden state model (LHS) [149]

$$\rho_A^a = \sum_\lambda \mathcal{P}(\lambda)\, \mathcal{P}(a|A,\lambda)\, \rho_B^Q(\lambda), \tag{4.1}$$

where $\{\mathcal{P}(\lambda), \rho_B^Q\}$ is an ensemble of LHS prepared by Alice and $\mathcal{P}(a|A,\lambda)$ is Alice's stochastic map to convince Bob. Here, we consider λ to be a hidden variable with the constraint $\sum_\lambda \mathcal{P}(\lambda) = 1$ and $\rho_B^Q(\lambda)$ is a quantum state received by Bob. The joint probability distribution on such states, $P(a_{\mathcal{A}_i}, b_{\mathcal{B}_i})$ of obtaining outcome a for the measurement of observables chosen from the set $\{\mathcal{A}_i\}$ by Alice and outcome b for the measurement of observables chosen from the set $\{\mathcal{B}_i\}$ by Bob can be written as

$$P(a_{\mathcal{A}_i}, b_{\mathcal{B}_i}) = \sum_\lambda P(\lambda)\, P(a_{\mathcal{A}_i}|\lambda)\, P_Q(b_i|\lambda), \tag{4.2}$$

where $P_Q(b_i|\lambda)$ is the quantum probability of the measurement outcome b_i due to the measurement of $\mathcal{B}_i$ and $P(a_{\mathcal{A}_i}|\lambda)$ is the probability of the outcome a for the measurement of observables chosen from the set $\{\mathcal{A}_i\}$ by Alice.

Several steering conditions have been derived on the basis of Eq. (4.2) and the existence of single system description of a part of the bi-partite systems [154–156]. It has also been quantified for two-qubit systems [157] In the last few years, several experiments have been performed to demonstrate the steering effect with the increasing measurement settings [154] and with loophole free arrangements [158]. For continuous variable systems, the steerability has also been quantified [159].

Recently, quantum coherence has been established as an important notion, specially in the areas of quantum information theory, quantum biology [59–62, 160] and quantum thermodynamics [63–67]. In quantum information theory, it is expected that it can be used as a resource [69, 70, 161]. This has been the main motivation for recent studies to quantify and develop a number of measures of quantum coherence [69, 72, 161, 162]. Most importantly, operational interpretations of resource theory of quantum coherence have also been put forward [84, 126]. An intriguing connection between quantum coherence and quantum speed limit has been established [77, 130]. However, much work needs to be done to really understand how to control and manipulate coherence so as to use it properly as a resource, particularly, in multipartite scenario.

In this chapter, we, first, study the effects of non-locality on quantum coherence in bi-partite scenario. We derive a set of inequalities for various quantum coherence measures. Violation of any of these inequalities implies that the system has no single system description and it can achieve non-local advantage (the advantage, which cannot be achieved by a single system and LOCC) of quantum coherence. We know that a state is said to be steerable if it does not have a single quantum system description. Thus, the system is steerable as well. Intuitively, for quantum systems, it may seem that all steerable states can achieve the non-local advantage on quantum coherence. But here we show that for mixed states, steerability captured by different steering criteria [154–156] based

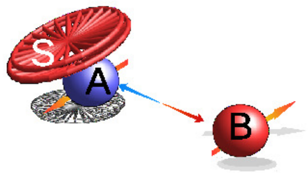

FIGURE 4.1: Coherence of Bob's particle is being steered beyond what could have been achieved by a single system, only by local projective measurements on Alice's particle and classical communications(LOCC).

on uncertainty relations are drastically different from the steerability captured by coherence. In other words, we show that there are steerable states, which cannot achieve the non-local advantage of quantum coherence.

One should note that we do not aim to derive a stronger steering criteria but aim to establish a connection between the steerability and the quantum coherence. This eventually leads us to show the effects of quantum steering on the speed of quantum evolutions.

Complementarity relations

To quantify coherence, we consider the l_1-norm and the relative entropy of coherence as a measure of quantum coherence [161]. We also use the skew information [163], which is an observable measure of quantum coherence [69] and also known as a measure of asymmetry [118–120, 127]. The l_1-norm of coherence of a state ρ is defined as $C^{l_1}(\rho) = \sum_{\substack{i,j \\ i\neq j}} |\rho_{i,j}|$. Now, if a qubit is prepared in either spin up or spin down state along z-direction then the qubit is incoherent, when we calculate the coherence in z-basis (i.e., $C_z^{l_1} = 0$) and is fully coherent in x- and y-basis, i.e., $C_{x\,(y)}^{l_1} = 1$. The l_1-norm of coherence of a general single qubit $\rho = \frac{1}{2}(I + \vec{n}.\vec{\sigma})$ (where $|\vec{n}| \leq 1$ and $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices) in the basis of Pauli matrix σ_i is given by

$$C_i^{l_1}(\rho) = \sqrt{n_j^2 + n_k^2}, \tag{4.3}$$

where $k \neq i \neq j$ and $i, j, k \in \{x, y, z\}$.

Therefore, one may ask, what is the upper bound of $C_{l_1} = C_x^{l_1}(\rho) + C_y^{l_1}(\rho) + C_z^{l_1}(\rho)$ for any general qubit state ρ. Using $C_x^{l_1}C_y^{l_1} + C_x^{l_1}C_z^{l_1} + C_y^{l_1}C_z^{l_1} \leq C_x^2 + C_y^2 + C_z^2 \leq 2$, we find that the above quantity is upper bounded by

$$\sum_{i=x,y,z} C_i^{l_1}(\rho) \leq \sqrt{6}, \tag{4.4}$$

where the equality sign holds for a pure state, which is an equal superposition of all the mutually orthonormal states spanning the state space, i.e.,

$$\rho_{\max}^{\mathcal{C}} = \frac{1}{2}\Big[I + \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z)\Big], \tag{4.5}$$

where I is 2×2 identity matrix. Hence, in the single system description, the quantity $\mathcal{C}_{l_1}$ cannot be larger than $\sqrt{6}$ and the corresponding inequality (4.4) can be thought as a coherence complementarity relation.

Another measure of coherence called the relative entropy of coherence is defined as [161] $C^E(\rho) = S(\rho_D) - S(\rho)$, where $S(\rho)$ is the von-Neumann entropy of the state ρ and ρ_D is the diagonal matrix formed by the diagonal elements of ρ in a fixed basis, i.e., ρ_D is completely decohered state of ρ. This quantity has also been considered as 'wavelike information' in Ref. [164], which satisfies a duality relation. In this case, the sum of coherences of single qubit system in the three mutually unbiased bases for qubit systems is bounded by

$$\sum_{i=x,y,z} C_i^E(\rho) = \sum_{i=x,y,z} \mathcal{H}\left(\frac{1+n_i}{2}\right) - 3\mathcal{H}\left(\frac{1+|\vec{n}|}{2}\right) \leq C_2^m, \tag{4.6}$$

where $\mathcal{H}(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ and $|\vec{n}| = \sqrt{n_x^2 + n_y^2 + n_z^2}$. Using the symmetry, one can easily show that the maximum occurs at $n_x = n_y = n_z = 1/\sqrt{3}$ (i.e., for maximally coherent state given by Eq. (4.5)) and $C_2^m = 2.23$.

Recently, the skew-information [163] has also been considered as an observable measure of coherence of a state [69]. The coherence of a state ρ, captured by an observable $\mathcal{B}$, i.e., the coherence of the state in the basis of eigenvectors of the spin observable σ_i is given by

$$C_i^S = -\frac{1}{2}Tr[\sqrt{\rho}, \sigma_i]^2 = \frac{\left(n_j^2 + n_k^2\right)\left(1 - \sqrt{1 - |\vec{n}|^2}\right)}{|\vec{n}|^2}, \tag{4.7}$$

which is a measure of quantum part of the uncertainty for the measurement of the observable σ_i and hence it does not increase under classical mixing of states [163]. The sum of the coherences measured by skew information in the bases of σ_z, σ_y and σ_x is upper bounded by

$$\sum_{i=x,y,z} C_i^S(\rho) = 2\left(1 - \sqrt{1 - |\vec{n}|^2}\right) \leq 2, \tag{4.8}$$

where the maximum occurs for maximally coherent state $\rho_{\max}^{\mathcal{C}}$ given by Eq. (4.5). The inequalities (4.4), (4.6) and (4.8) are complementarity relations for coherences of a state measured in mutually unbiased bases.

Non-local advantage of quantum coherence

Let us now describe our steering protocol, which we use to observe the effects of steering of the coherence of a part of a bi-partite system. We consider a general two-qubit state of the form of

$$\eta_{AB} = \frac{1}{4}(I^A \otimes I^B + \vec{r}\cdot\sigma^A \otimes I^B + I^A \otimes \vec{s}\cdot\vec{\sigma}^B + \sum_{i,j=x,y,z} t_{ij}\sigma_i^A \otimes \sigma_j^B), \tag{4.9}$$

where $\vec{r} \equiv (r_x, r_y, r_z)$, $\vec{s} \equiv (s_x, s_y, s_z)$, with $|r| \leq 1$, $|s| \leq 1$ and (t_{ij}) is the correlation matrix. Alice may, in principle, perform measurements in arbitrarily chosen bases. For simplicity, we derive the coherence steerability criteria for three measurement settings in the eigenbases of $\{\sigma_x, \sigma_y, \sigma_z\}$. When Alice declares that she performs measurement on the eigenbasis of σ_z and obtains outcome $a \in \{0, 1\}$ with probability $p(\eta_{B|\Pi_z^a}) = Tr[(\Pi_z^a \otimes I_B)\eta_{AB}]$, Bob measures coherence randomly with respect to the eigenbasis of (say) other two of the three Pauli matrices σ_x and σ_y. As Alice's measurement in σ_k basis affects the coherence of Bob's state, the coherence of the conditional state of B, $\eta_{B|\Pi_k^a}$ in the basis of σ_i becomes

$$C_i^{l_1}(\eta_{B|\Pi_k^a}) = \sqrt{\frac{\sum_{j\neq i}\alpha_{jk_a}^2}{\gamma_{k_a}^2}}, \tag{4.10}$$

where $\alpha_{ij_a} = s_i + (-1)^a t_{ji}$, $\gamma_{k_a} = 1 + (-1)^a r_k$ and $i, j, k \in \{x, y, z\}$. Note that the violation of any of the inequalities in Eq. (4.4), (4.6) and (4.8) by the conditional states of Bob implies that the single system description of coherence of the system B does not exist. Thus, the criterion for achieving the non-local advantage on quantum coherence of Bob using the l_1-norm comes out to be

$$\frac{1}{2}\sum_{i,j,a} p(\eta_{B|\Pi^a_{j\neq i}}) C_i^{l_1}(\eta_{B|\Pi^a_{j\neq i}}) > \sqrt{6}, \tag{4.11}$$

where $p(\eta_{B|\Pi^a_j}) = \frac{\gamma_{j_a}}{2}$, $i, j \in \{x, y, z\}$ and $a \in \{0, 1\}$. This inequality forms a volume in 2-qubit state space.

Let us now derive the same criterion following the relative entropy of coherence measure. We can easily show that the eigenvalues of the conditional state of B, $\eta_{B|\Pi^a_i}$ are given by $\lambda^{\pm}_{i_a} = \frac{1}{2} \pm \frac{\sqrt{\sum_j \alpha^2_{ji_a}}}{2\gamma_{i_a}}$. Therefore, the relative entropy of coherence, when Alice measures in Π^a_k, is given by

$$C_i^E(\eta_{B|\Pi^a_k}) = \sum_{p=+,-} \lambda^p_{k_a} \log_2 \lambda^p_{k_a} - \beta^p_{ik_a} \log_2 \beta^p_{ik_a}, \tag{4.12}$$

where the diagonal element $\beta^{\pm}_{ij_a}$ of the conditional state $\eta_{B|\Pi^a_j}$, when expressed in the σ_i^{th} basis is given by $\beta^{\pm}_{ij_a} = \frac{1}{2} \pm \frac{\alpha_{ij_a}}{2\gamma_{j_a}}$. Thus, the criterion for achieving the non-local advantage of quantum coherence becomes (4.6)

$$\frac{1}{2}\sum_{i,j,a} p(\eta_{B|\Pi^a_{j\neq i}}) C_i^E(\eta_{B|\Pi^a_{j\neq i}}) > C_2^m, \tag{4.13}$$

where $i, j \in \{x, y, z\}$ and $a \in \{0, 1\}$. Similarly, we obtain another inequality using the skew information as the observable measure of quantum coherence. The coherence of the conditional state $\eta_{B|\sigma^a_k}$ measured with respect to σ_i in this case is given by

$$C_i^S(\eta_{B|\Pi^a_k}) = \frac{(\sum_{j\neq i} \alpha^2_{jk_a})(1 - \sqrt{1 - (2\lambda^{\pm}_{k_a} - 1)^2})}{\gamma^2_{k_a}(2\lambda^{\pm}_{k_a} - 1)^2}. \tag{4.14}$$

Thus, from Eq. (4.8) we get the coherence steering inequality using the skew-information complementarity relation as

$$\frac{1}{2}\sum_{i,j,a} p(\eta_{B|\Pi^a_{j\neq i}}) C^S_i(\eta_{B|\Pi^a_{j\neq i}}) > 2, \tag{4.15}$$

where $i, j \in \{x, y, z\}$ and $a \in \{0, 1\}$.

It is important to mention here that although the violation of the coherence complementarity relations in Eqs. (4.4), (4.6) and (4.8) implies the steerability of the quantum state and the achievability of the non-local advantage of quantum coherence, its violation highly dependent on the measurement settings (4.5). Therefore, the state of Bob (B) can achieve the non-local advantage of quantum coherence by the help of Alice if at least one of the inequalities in Eqs. (4.11), (4.13) and (4.15) is satisfied but it is not necessary. A better choice of projective measurement bases by Alice may reveal steerability of an apparently unsteerable state with respect to the above inequalities. On the other hand, it is also necessary to show that separable states can never violate the coherence complementarity relations using the present protocol.

Steerability and LHS model

To show that no state with LHS model can violate the coherence complementarity relations, we consider a two-qubit state ρ_{ab}. Suppose, Alice performs a projective measurement in an arbitrary basis Π^a_n, where $a \in \{0, 1\}$ corresponding to two outcomes of the measurement and $n \in \mathcal{Z}^+$ (set of positive integers), each measurement basis associated to an integer. To compare with the coherence complementarity relations, Alice must choose $3\mathcal{Z}^+$ number of measurement bases, making n to run upto $3k$ (say), where $k \in \mathcal{Z}+$. This provides Bob 2^k number of coherence measurement results on a particular Pauli basis. This is due to the fact that for measurements on each basis, Bob can measure coherence randomly only on two of the three mutually unbiased Pauli bases. Bob receives the state $\rho_{B|\Pi^a_n}$, which has a LHS description, i.e., the conditional state of Bob can be expressed by a normalized state as given in Eq. (4.1),

$$\rho_{B|\Pi_n^a} = \frac{\sum_\lambda \mathcal{P}(\lambda)\mathcal{P}(a|\Pi_i^a,\lambda)\rho_B^Q(\lambda)}{\sum_\lambda \mathcal{P}(\lambda)\mathcal{P}(a|\Pi_i^a,\lambda)} \tag{4.16}$$

If the proposed protocol is followed, one can show that the above state in Eq. (4.16) can never violate the coherence complementarity relations. To show that, we start with

$$\begin{aligned}\sum_{a=0,n=1,m=0}^{1,3k,1} p(\rho_{B|\Pi_n^a})C^q_{n\oplus m}(\rho_{B|\Pi_n^a}) &\leq \sum_{a,n,m,\lambda} \mathcal{P}(\lambda)\mathcal{P}(a|\Pi_i^a,\lambda)C^q_{n\oplus m}(\rho_B^Q(\lambda))\\ &= \sum_\lambda \sum_{n=1,m=0}^{3k,1} \mathcal{P}(\lambda)C^q_{n\oplus m}(\rho_B^Q(\lambda)),\end{aligned} \tag{4.17}$$

where we denote $n\oplus m = Mod(n+m,3)+1$ and $q \in \{l_1, E, S\}$, stands for various measures of coherence. In the first inequality, we used the fact that coherence and the observable measure of quantum coherence decreases under classical mixing of states. Here, we also consider $\{C_1^q, C_2^q, C_3^q\} \equiv \{C_x^q, C_y^q, C_z^q\}$. By taking the summation over n and m, one can easily show from the last line of Eq. (4.17) that

$$\begin{aligned}\sum_{a,n,m} p(\rho_{B|\Pi_n^a})C^q_{n\oplus m}(\rho_{B|\Pi_n^a}) &\leq 2^k\sum_\lambda \mathcal{P}(\lambda)\Big(C_x^q(\rho_B^Q(\lambda)) + C_y^q(\rho_B^Q(\lambda)) + C_z^q(\rho_B^Q(\lambda))\Big)\\ &\leq 2^k \sum_\lambda \mathcal{P}(\lambda)\epsilon^q = 2^k\epsilon^q,\end{aligned} \tag{4.18}$$

where $\epsilon^q \in \{\sqrt{6}, 2.23, 2\}$ depending on q. This implies that the coherence complementarity relations can never be violated by any state, which has a LHS description. Mathematically, for any such states

$$\frac{1}{2}\sum_{a=0,n=1,m=0}^{1,3,1} p(\rho_{B|\Pi_n^a})C^q_{n\oplus m}(\rho_{B|\Pi_n^a}) \leq \epsilon^q \tag{4.19}$$

for three measurement settings scenario ($k=1$). Violation to this inequality implies that the state is steerable and Bob can achieve non-local advantage of quantum coherence by Alice.

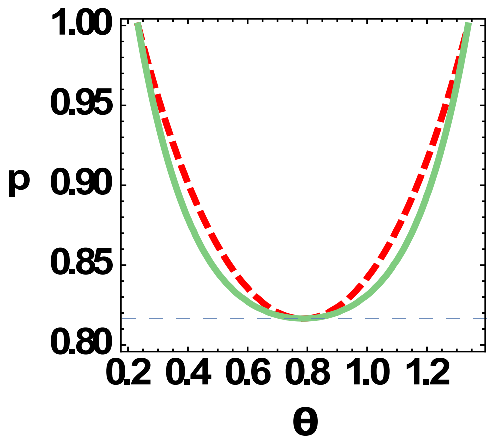

FIGURE 4.2: Filtering operation $F(\theta) = \text{diagonal}\{1/\cos(\theta), 1/\sin\theta\}$ is applied on the Werner state ρ_w. The red coloured dashed line corresponds to the situation, when $F(\theta)$ is applied on Alice and Green solid plot is when it is applied on Bob. The non-local advantage of quantum coherence is not achievable by the resulting state for the ranges of p under the curves. For example, The resulting state is steerable or the state can achieve non-local advantage of quantum coherence from Alice to Bob for $p \geq 0.845$, when $F(\theta \approx 0.5)$ is applied on Bob. The horizontal thin dashed line denotes $p = \sqrt{\frac{2}{3}}$.

Example

Let us now illustrate the coherence steerability condition with an example, say, two qubit Werner state defined by

$$\rho_w = p|\psi^-_{AB}\rangle\langle\psi^-_{AB}| + \frac{(1-p)}{4} I^A \otimes I^B, \tag{4.20}$$

where $|\psi^-_{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and the mixing parameter p is chosen from the range $0 \leq p \leq 1$. For this state, $\vec{r} = 0$, $\vec{s} = 0$, and $t_{xx} = t_{yy} = p$, $t_{zz} = -p$. The state ρ_w is steerable for $p > \frac{1}{2}$, entangled for $p > \frac{1}{3}$ and Bell non-local for $p > \frac{1}{\sqrt{2}}$.

Here, the optimal strategy for Alice to maximize the violation of coherence complementary relation by Bob's conditional state is similar to as stated earlier. With the help of the inequalities (4.11), (4.13) and (4.15) it is easy to show that for the Werner state, the coherence of the state of B is steerable for $p > \sqrt{\frac{2}{3}}$ when one uses the l_1-norm as a measure of coherence, $p > 0.914$ when one uses the relative entropy of coherence as a measure and $p > \frac{2\sqrt{2}}{3}$ for the choice of skew information as a measure of quantum coherence.

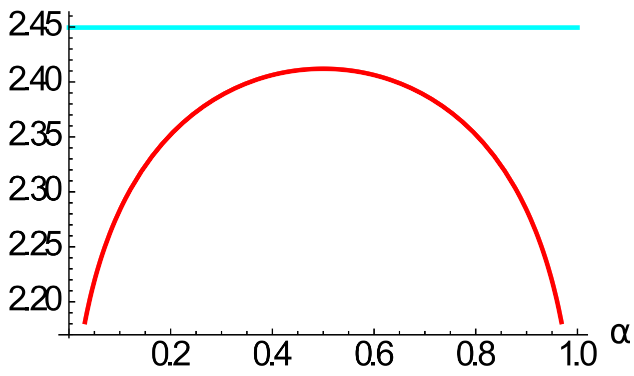

FIGURE 4.3: The state $|\psi_\alpha\rangle_{ab}$ does not show violation of the coherence complementarity relations if Alice performs projective measurements on the Pauli bases. Although, it shows steerability if Alice chooses better measurement settings.

Hence, Alice controls the coherence of Bob's system for $p > \sqrt{\frac{2}{3}}$ whereas Alice controls Bob's state for $p > \frac{1}{2}$. This difference occurs due to participation of noise part ($\frac{I\otimes I}{4}$) in steering the state, whereas, coherence steerability criteria are never influenced by such classical noise. This raises a natural question: Is it possible to increase the range of p to control the coherence of Bob's system using local filtering operations? It has been shown that filtering operations can improve the steerablity [165]. From the Fig. (4.2), it is clear that filtering operation on Bob can increase the range of p to some extent for certain values of θ, for which the resulting state can achieve the non-local advantage of quantum coherence from Alice to Bob. Moreover, any steerable Werner state can be turned into an unsteerable state by local filtering operations [165] (see Fig. (4.2)).

We consider a pure entangled state $|\psi^\alpha_{AB}\rangle = \frac{1}{1+\sqrt{\alpha-\alpha^2}}(\sqrt{\alpha}|++\rangle + \sqrt{1-\alpha}|00\rangle)$. From the Fig. (4.3), it is clearly visible that if Alice performs projective measurements in the Pauli bases, the state of Bob (B) cannot achieve the non-local advantage of quantum coherence. On the other hand, one can construct a set of arbitrary mutually unbiased bases as $|n_z^\pm\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, $|n_x^\pm\rangle = \frac{|n_z^+\rangle \pm |n_z^-\rangle}{\sqrt{2}}$ and $|n_y^\pm\rangle = \frac{|n_z^+\rangle \pm i|n_z^-\rangle}{\sqrt{2}}$. If Alice performs measurements on these bases, for certain values of θ and ϕ, as shown in Fig. (4.4), the coherence complementarity relations are violated.

From the Fig. (4.4), it can also be seen that the states with very low amount of entanglement, cannot easily achieve non-local advantage of quantum coherence but a better measurement settings may reveal its steerability.

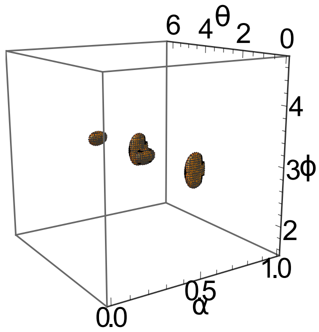

FIGURE 4.4: The state $|\psi_\alpha\rangle_{ab}$ can be turned into a steerable state for α around $\frac{1}{2}$ by performing projective measurements using arbitrary mutually unbiased bases with θ and ϕ, which lie inside the volume. Entanglement of the state is given by linear entropy $S_L = \frac{(1-\alpha)\alpha}{2\left(\sqrt{(1-\alpha)\alpha}+1\right)^2}$ and maximum 0.05 at $\alpha \approx 0.47$

Non-local advantage of quantum speed limits

It is well known that quantum mechanics imposes a fundamental limit to the speed of quantum evolution, conventionally known as quantum speed limit (QSL). Suppose, a quantum system in a state ρ_1 evolves to ρ_2 under a unitary evolution operator U. The minimum time it takes to evolve is of fundamental interest in quantum metrology, quantum computation, quantum algorithm, quantum cryptography and quantum thermodynamics. Here, we apply our observations to understand the effects of steering kind of non-locality on the quantum speed limits (QSL) [1, 77, 86]. Now, the development of quantum technology is at par with the advent of quantum information and computation theory. Various attempts are being made in the laboratory to implement quantum gates, which are basic building blocks of a quantum computer. Performance of a quantum computer is determined by how fast these logic gates drive the initial state to a final state. An efficient quantum gate should transform the input state into the desired state as fast as possible. This naturally presents an important issue, to address how to control and manipulate the speed of quantum evolution so as to achieve faster and efficient quantum gates. Such study may also be useful in quantum thermodynamics and other developing fields of quantum information theory. In quantum thermodynamics, this may help us to understand how to control a thermodynamic engine non-locally and use quantum correlations in our favour to construct faster yet efficient quantum engines.

Therefore, we focus on the study of QSL in the bipartite scenario, where a part of the system is considered to be the controller of the evolution of the other part. It is well known that quantum correlations affects the evolutions of the total quantum systems. On the other hand, how a part of the system affects the evolution speed limit of the other part using the quantum correlation or non-locality is still an important unanswered issue. Here, we show that non-locality plays an important role in setting the QSL of a part of the system. In particular, we studied the effects of non-locality on quantum coherence of a part of a bi-partite system. This, in turn, clarifies the role of quantum non-locality on QSL and the intriguing connection between QSL and quantum coherence.

Let us consider a set of three non-commuting 2-dimensional observables, K_1, K_2 and K_3 for qubit. Then following our result in Eq.(8), one can easily derive a complementarity relation for observable measure of quantum coherence for K_1, K_2 and K_3 in qubit state space. For any qubit state ρ, the relation takes a form

$$\sum_{r=1}^{3} C^S_{K_r}(\rho) \leq m, \tag{4.21}$$

where m is any real number and depends only on the observables. Let us explain this with an example. We consider an arbitrary observable $K = \sum_{i=x,y,z} r_i.\sigma_i$. The skew information or the observable measure of quantum coherence of an arbitrary state $\rho = \frac{1}{2}(I_2 + \sum_{i=x,y,z} n_i.\sigma_i)$, where $\sum_i n_i^2 \leq 1$ with respect to the observable K is given by

$$C^S_K(\rho) = -\frac{1}{2}Tr[\sqrt{\rho}, K]^2 = \frac{\left(1 - \sqrt{1 - |n|^2}\right) |\vec{n} \times \vec{r}|^2}{|n|^2}, \tag{4.22}$$

where $\vec{r} = (r_1, r_2, r_3)$ and $\vec{n} = (n_1, n_2, n_3)$. If we consider $K_1 = \frac{1}{2}I_2 + 2\sigma_x$, $K_2 = \sigma_x + 2\sigma_y$ and $K_3 = I_2 + \sigma_y$, it can be easily shown that the value of the quantity $m = 10$.

Now, we know that the evolution time bound for a state ρ_1 under a time independent Hamiltonian H ($U_H(t) = e^{\frac{-iHt}{\hbar}}$) evolving to $\rho_2 = U_H(T)\rho_1 U_H(T)^\dagger$ at time T is given by (see [77])

$$T \geq T_b(H, \rho_1) = \frac{\hbar}{\sqrt{2}} \frac{\cos^{-1} A(\rho_1, \rho_2)}{\sqrt{C^S_H(\rho_1)}}, \tag{4.23}$$

where $A(\rho_1, \rho_2) = \mathrm{Tr}(\sqrt{\rho_1}\sqrt{\rho_2})$ is the affinity between the initial and the final state. Let us redefine the affinity as $A(\rho_1, \rho_2) = A_H(\rho_1)$.

Now, consider a bi-partite state ρ_{AB} shared between Alice (A) and Bob (B). To verify Alice's control, Bob asks Alice to steer the state of system B in the eigenbasis of K_1, K_2 or K_3. Bob measures the QSL of the conditional state of B, $\rho_{A|\Pi_{K_1}}$ by evolving the state under the unitary evolution governed by K_2 or K_3 in case of the claim by Alice that she controls her state in the eigenbasis of K_1 and so on. Thus, using Eq. (4.21), one can easily show that

$$\frac{1}{2}\sum_{r,s,a} p(\rho_{A|\Pi^a_{K_{s\neq r}}})\left(\frac{\cos^{-1} A_{K_r}(\rho_{A|\Pi^a_{K_{s\neq r}}})}{T_b(K_r, \rho_{A|\Pi^a_{K_{s\neq r}}})}\right)^2 \leq \frac{2m}{\hbar^2}, \tag{4.24}$$

where $r, s \in \{1, 2, 3\}$ and $a \in \{0, 1\}$.

Violation of this inequality, for a set of observables K_1, K_2 and K_3 implies non-local advantage on QSL achievable for the state. In particular, let us consider Werner state $\rho_W = p|\psi^-_{AB}\rangle\langle\psi^-_{AB}| + \frac{(1-p)}{4} I^A \otimes I^B$. For the given set of observables, K_1, K_2 and K_3 if one follows the protocol, one can easily show that the state never achieves the non-local advantage on quantum speed limit although the state achieves the non-local advantage of quantum coherence. On the other hand, if one used $K_1 = \sigma_x$, $K_2 = \sigma_y$ and $K_3 = \sigma_z$, non-local advantage of QSL could have been achieved.

Now, consider the maximally mixed two qubit state $\frac{I^B_2 \otimes I^B_2}{4}$ and the maximally entangled pure two qubit state $|\psi\rangle = |10\rangle + |01\rangle$. For both of the examples, the state of Bob is nothing but the maximally mixed qubit state $\frac{I_2}{2}$. Still, for the second state, we can achieve the non-local advantage of QSL or coherence on Bob by LOCC on Alice's system but this is not possible for the maximally mixed two-qubit state.

Conclusions

In this work, we use various measures of quantum coherence and we derive complementarity relations (4.4), (4.6) and (4.8) between coherences of single quantum system (qubit) measured in the mutually unbiased bases. Using these complementarity relations, we derive conditions (4.11), (4.13) and (4.15), under which the non-local advantage of quantum coherence can be achieved for any general 2-qubit bipartite systems. These conditions also provide a sufficient criteria for state to be steerable. We also show that not all steerable states can achieve the non-local advantage on quantum coherence.

Our results reveal an important connection between quantum non-locality and quantum speed limit. One can show that not all steerable states or for that matter, not even all states, for which non-local advantage on quantum coherence is achievable, can, in principle, achieve non-local advantage on QSL. Only states, which can achieve non-local advantage on observable measure of quantum coherence or asymmetry [118–120, 127] can achieve non-local QSL for those observables.

We also show that our coherence steering criteria are monogamous in the sense that when Alice and Bob share a steerable state, the state shared by Bob and Eve can always be explained by a local hidden state model.

One important application of our results has been uncovered in the detection of Unruh effects as well [166]. It has been shown that the equilibrium state of two accelerating two-level atoms in an extended scalar field cannot achieve non-local advantage of quantum coherence even if they start accelerating with a maximally entangled or steerable state [166]. Instead of detecting a very small Unruh temperature directly or indirectly by measuring a small geometric phase, it is easier to detect a vanishing quantity.

Note: When this article [167] first appeared on arxiv, Fan *et al.* presented a study on the quantum coherence of steered states [168] around the same time. We consider our works to be complementary: though examining a similar topic, our approaches are very different (we consider steering from the existence of a local hidden state model rather than from the perspective of the QSE formalism).

“*So Einstein was wrong when he said, "God does not play dice." Consideration of black holes suggests, not only that God does play dice, but that he sometimes confuses us by throwing them where they can't be seen.*” ——*Stephen Hawking*

Part III

Chapter 5

Tighter uncertainty and reverse uncertainty relations

"Uncertainty is the most stressful feeling" ——*Anonymous*

Introduction

Aim of this thesis has been to explore the connections of QSL with other properties of states and evolution operators. To get further insight into the picture, it is important to focus on the study of quantum uncertainty relations. It is well known that quantum mechanics sets only the lower limit to the time of quantum evolutions [1, 77, 86, 87]. In contrast to this belief, it is now expected that our state dependent reverse uncertainty relations may also be useful in setting an upper time limit of quantum evolutions [169] (reverse bound to the QSL) and in quantum metrology. Thus, the results of our paper are not only of fundamental interest, but can have several applications in diverse areas of quantum physics, quantum information and quantum technology.

The aim of this chapter is two fold. First, we show a set of uncertainty relations in the product as well as the sum forms. The new uncertainty relation in the product form is stronger than the Robertson-Schrödinger uncertainty relation. We also derive an optimization free bound, which is also tighter most of the times than the Robertson-Schrödinger relation. On the other hand, uncertainty relations for the sum of variances are also shown to be tight enough considering the advantage that the bounds do not need an optimization. Second, we prove reverse uncertainty relations for incompatible observables. We derive the state dependent reverse uncertainty relations in terms of variances both in the sum form and the product form. Thus, the uncertainty relation is not the only distinguishing feature but here, we show that reverse uncertainty relation also comes out as an another unique feature of quantum mechanics. If one considers that uncertainty relation quantitatively expresses the impossibility of jointly sharp preparation of incompatible observables, then the reverse uncertainty relation should express the maximum extent to which the joint sharp preparation of incompatible observables is impossible.

This chapter is organised as follows. First, we derive a set of new tighter, optimization free, state-dependent uncertainty relations based on variances of two incompatible observables. We derive uncertainty relations for both the forms,—sum and the product of variances for two observables in the next section (5.2). Second, we then derive reverse uncertainty relations for the sum and the product of variances (5.3) and at last we conclude with the section (5.4).

Tighter uncertainty relations

For any two non-commuting operators A and B, the Robertson-Schrödinger uncertainty relation [102] for the state of the system $|\Psi\rangle$ is given by the following inequality

$$(\Delta A)^2(\Delta B)^2 \geq \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2 + \left|\frac{1}{2i}\langle[A,B]\rangle\right|^2, \tag{5.1}$$

where the averages and variances are taken in the state of the system $|\Psi\rangle$. This relation is a direct consequence of the Cauchy-Schwarz inequality. This uncertainty bound is not optimal. There have been several attempts to tighten the bound [103, 106–108, 170]. Here, we try to tighten the bound further and get another uncertainty relation. We express the two observables A and B in their eigenbasis as $A = \sum_i a_i|a_i\rangle\langle a_i|$ and $B =$

$\sum_i b_i|b_i\rangle\langle b_i|$. Let us define

$$(A - \langle A\rangle) = \bar{A} = \sum_i \tilde{a}_i|a_i\rangle\langle a_i|,$$
$$\text{and} \qquad (B - \langle B\rangle) = \bar{B} = \sum_i \tilde{b}_i|b_i\rangle\langle b_i|. \tag{5.2}$$

We express $|f\rangle = \bar{A}|\Psi\rangle$ and $|g\rangle = \bar{B}|\Psi\rangle$ in the same basis as $|f\rangle = \sum_n \alpha_n|\psi_n\rangle$ and $|g\rangle = \sum_n \beta_n|\psi_n\rangle$, where $\{|\psi_n\rangle\}$ is an arbitrary complete basis. Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\Delta A^2\Delta B^2 &= \langle f|f\rangle\langle g|g\rangle = \sum_{n,m}|\alpha_n|^2|\beta_m|^2 \geq \Big(\sum_n|\alpha_n||\beta_n|\Big)^2 \\ &= \Big(\sum_n|\alpha_n^*\beta_n|\Big)^2 = \Big(\sum_n|\langle\Psi|\bar{A}|\psi_n\rangle\langle\psi_n|\bar{B}|\Psi\rangle|\Big)^2 \\ &= \Big(\sum_n|\langle\Psi|\bar{A}\bar{B}_n^{\psi}|\Psi\rangle|\Big)^2,\end{aligned} \tag{5.3}$$

where $\bar{B}_n^{\psi} = |\psi_n\rangle\langle\psi_n|\bar{B}$, $\alpha_n = \langle\psi_n|\bar{A}|\Psi\rangle$ and $\beta_n = \langle\psi_n|\bar{B}|\Psi\rangle$. We know that

$$\langle\Psi|\bar{A}\bar{B}_n^{\psi}|\Psi\rangle = \frac{1}{2}\big(\langle[\bar{A},\bar{B}_n^{\psi}]\rangle_\Psi + \langle\{\bar{A},\bar{B}_n^{\psi}\}\rangle_\Psi\big). \tag{5.4}$$

One needs to optimize over various complete bases to achieve the tightest bound. Thus, using the relation in Eq. (5.3) and (5.4), the new uncertainty relation can be written as

$$\Delta A^2\Delta B^2 \geq \max_{\{|\psi\rangle\}}\Big(\frac{1}{2}\sum_n\Big|\langle[\bar{A},\bar{B}_n^{\psi}]\rangle_\Psi + \langle\{\bar{A},\bar{B}_n^{\psi}\}\rangle_\Psi\Big|\Big)^2. \tag{5.5}$$

This uncertainty relation is tighter than the Robertson-Schrödinger [101, 102] uncertainty relation given in Eq. (5.1). To prove that let us start with the Eq. (5.3)

$$\begin{aligned}\Big(\sum_n|\langle\Psi|\bar{A}\bar{B}_n^{\psi}|\Psi\rangle|\Big)^2 &\geq \Big|\sum_n\langle\Psi|\bar{A}\bar{B}_n^{\psi}|\Psi\rangle\Big|^2 \\ &= \Big|\sum_n\langle\Psi|\bar{A}\bar{B}|\Psi\rangle\Big|^2,\end{aligned} \tag{5.6}$$

where we have used the fact that $|\sum_i z_i|^2 \leq (\sum_i |z_i|)^2$, $z_i \in \mathbb{C}$ for all i. Here, the last line in Eq. (5.6) is nothing but the bound obtained in Eq. (5.1). Thus, our bound is indeed tighter than the Robertson-Schrödinger uncertainty relation. This uncertainty

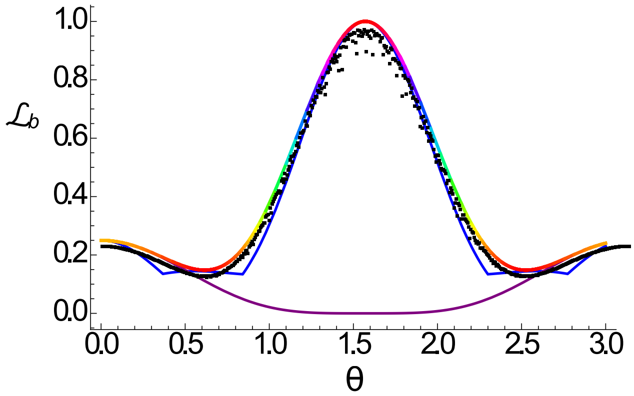

FIGURE 5.1: Here, we plot the lower bound of the product of variances of two incompatible observables, $A = L_x$ and $B = L_y$, two components of the angular momentum for spin 1 particle with a state $|\Psi\rangle = \cos\theta|1\rangle - \sin\theta|0\rangle$, where the state $|1\rangle$ and $|0\rangle$ are the eigenstates of L_z corresponding to eigenvalues 1 and 0 respectively. The blue line shows the lower bound of the product of variances given by (5.8), the purple coloured plot stands for the bound given by Schrödinger uncertainty relation given by Eq. (5.1) and the hue plot denotes the product of two variances. Scattered black points denote the optimized uncertainty bound achieved by Eq. (5.5).

bound in Eq. (5.5) needs an optimization over the sets of complete bases to obtain the tightest bound. One can obtain an optimization-free uncertainty relation using the same Cauchy-Schwarz inequality for real numbers. For that, we consider (say)

$$\begin{aligned} \bar{A}^2 &= \sum_{i,j}(a_i - a_j F_\Psi^{a_j})^2|a_i\rangle\langle a_i| \\ \text{and} \quad \bar{B}^2 &= \sum_{i,j}(b_i - b_j F_\Psi^{b_j})^2|b_i\rangle\langle b_i|, \end{aligned} \tag{5.7}$$

where F_Ψ^x is nothing but the fidelity between the state $|\Psi\rangle$ and $|x\rangle$ ($|x\rangle = |a_i\rangle, |b_i\rangle$), $F(|\Psi\rangle, |x\rangle) = |\langle\Psi|x\rangle|^2$. Using the Cauchy-Schwarz inequality for real numbers, i.e., $\sum_i a_i^2 \sum_j b_j^2 \geq (\sum_i a_i b_i)^2$, we obtain

$$\Delta A^2 \Delta B^2 \geq \left(\sum_i \sqrt{F_\Psi^{a_i}}\sqrt{F_\Psi^{b_i}}|\tilde{a}_i||\tilde{b}_i|\right)^2, \tag{5.8}$$

where $\tilde{a}_i = (a_i - \sum_j F_\Psi^{a_j} a_j)$, $\tilde{b}_i = (b_i - \sum_j F_\Psi^{b_j} b_j)$ and the quantities $\sqrt{F_\Psi^{a_i}}\tilde{a}_i$ and $\sqrt{F_\Psi^{b_i}}\tilde{b}_i$ are arranged such that $\sqrt{F_\Psi^{a_{i+1}}}\tilde{a}_{i+1} \geq \sqrt{F_\Psi^{a_i}}\tilde{a}_i$ and $\sqrt{F_\Psi^{b_{i+1}}}\tilde{b}_{i+1} \geq \sqrt{F_\Psi^{b_i}}\tilde{b}_i$. This is the new uncertainty relation.

Let us now show that the new uncertainty relation in Eq.(5.8) is tighter than the uncertainty relation given by the Robertson-Schrödinger uncertainty relation (5.1). Using these, we define

$$\begin{aligned} |f\rangle &= \bar{A}|\Psi\rangle = \sum_i \langle a_i|\Psi\rangle \tilde{a}_i |a_i\rangle = \Delta A |\Psi_\perp^A\rangle \\ \text{and} \quad |g\rangle &= \bar{B}|\Psi\rangle = \sum_i \langle b_i|\Psi\rangle \tilde{b}_i |b_i\rangle = \Delta B |\Psi_\perp^B\rangle, \end{aligned} \tag{5.9}$$

which imply $\tilde{a}_i = \frac{(\Delta A)\langle a_i|\Psi_\perp^A\rangle}{\langle a_i|\Psi\rangle}$ and $\tilde{b}_i = \frac{(\Delta B)\langle b_i|\Psi_\perp^B\rangle}{\langle b_i|\Psi\rangle}$. The Schrödinger uncertainty relation in Eq. (5.1) is nothing but given by

$$\begin{aligned} \langle f|f\rangle\langle g|g\rangle &= (\Delta A)^2(\Delta B)^2 = \sum_{i,j} F_\Psi^{a_i} \tilde{a}_i^2 F_\Psi^{b_j} \tilde{b}_j^2 \geq |\langle f|g\rangle|^2 \\ &= \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2 + \left|\frac{1}{2i}\langle[A,B]\rangle\right|^2 \\ &= \left|\sum_{i,j} \langle a_i|\Psi\rangle \tilde{a}_i \langle\Psi|b_j\rangle \tilde{b}_j \langle b_j|a_i\rangle\right|^2, \end{aligned} \tag{5.10}$$

which reduces to $\left|\sum_i \langle a_i|\Psi\rangle \tilde{a}_i \langle\Psi|b_i\rangle \tilde{b}_i\right|^2$, when the observables A and B are diagonal in the same basis. Now, using the relation $|\sum_i z_i|^2 \leq (\sum_i |z_i|)^2$, one can show that the quantity

$$\left|\sum_i \langle a_i|\Psi\rangle \tilde{a}_i \langle\Psi|b_i\rangle \tilde{b}_i\right|^2 \leq \left(\sum_i \sqrt{F_\Psi^{a_i}} \sqrt{F_\Psi^{b_i}} |\tilde{a}_i||\tilde{b}_i|\right)^2. \tag{5.11}$$

Therefore, our bound in Eq. (5.8) is tighter than the Schödinger uncertainty bound (5.1) at least when the observables A and B are diagonal in the same basis.

As observed from the Fig. (5.1), the bound given by Eq. (5.5) is one of the tightest bounds reported here but it needs optimization. The bound given by Eq. (5.8) is the only bound, which is tighter than the other bounds most of the time and even surpasses the bound given by Eq. (5.5) but yet does not need any optimization.

However, we know that the product of variances does not fully capture the uncertainty for two incompatible observables, since if the state is an eigenstate of one of the observables, then the product of the uncertainties vanishes. To overcome this shortcoming, the sum of variances was invoked to capture the uncertainty of two incompatible observables. In this regard, stronger uncertainty relations for all incompatible observables

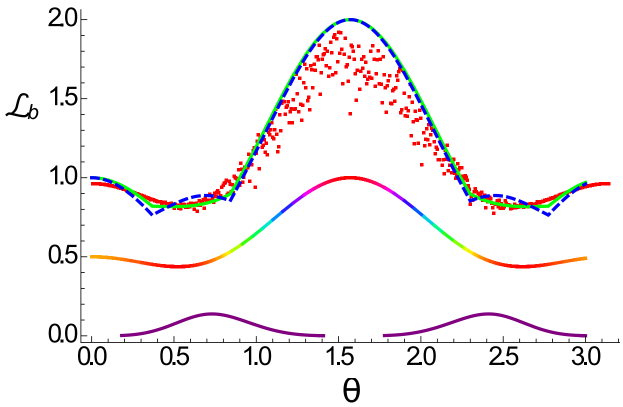

FIGURE 5.2: Here, we plot the lower bound of the sum of variances of two incompatible observables, $A = L_x$ and $B = L_y$, two components of the angular momentum for spin 1 particle with a state $|\Psi\rangle = \cos\theta|1\rangle - \sin\theta|0\rangle$, where the state $|1\rangle$ and $|0\rangle$ are the eigenstates of L_z corresponding to eigenvalues 1 and 0 respectively. Green line shows the lower bound of the sum of variances given by (5.14), blue dashed line is the bound given by (5.13), hue plot denotes the bound given by Eq. (4) in [103] and the purple coloured plot stands for the bound given by Eq. (2) in [108]. Scattered red points are the uncertainty bound achieved by Eq. (3) in [103]. As observed from the plot, the bound given by Eq. (5.15) is one of the tightest bounds in the literature. The bound given by Eq. (3) in [103] is the only bound, which surpasses at only few points.

were proposed in Ref. [103]. But, these uncertainty relations are not always tight and highly dependent on the states perpendicular to the chosen state of the system. Here, we propose new uncertainty relations that perform better than the existing bounds and need no optimization. Similarly, we use the triangle inequality and parallelogram law for real vectors to improve the bound on the sum of variances for two incompatible observables. The triangle inequality provides

$$\Delta A + \Delta B = \sqrt{\sum_i \tilde{a}_i^2 F_\Psi^{a_i}} + \sqrt{\sum_i \tilde{b}_i^2 F_\Psi^{b_i}} \geq \sqrt{\sum_i \left(\tilde{a}_i\sqrt{F_\Psi^{a_i}} + \tilde{b}_i\sqrt{F_\Psi^{b_i}}\right)^2}. \quad (5.12)$$

Thus, the uncertainty relation for sum of variances for two incompatible observables is given by

$$\Delta A^2 + \Delta B^2 \geq \sum_i \left(\tilde{a}_i\sqrt{F_\Psi^{a_i}} + \tilde{b}_i\sqrt{F_\Psi^{b_i}}\right)^2 - 2\Delta A\Delta B = \mathcal{L}_t. \quad (5.13)$$

Now, using the parallelogram law, one can derive another lower bound on the sum of variances of two observables as

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2}\sum_i \left(\tilde{a}_i\sqrt{F_\Psi^{a_i}} + \tilde{b}_i\sqrt{F_\Psi^{b_i}}\right)^2 = \mathcal{L}_p. \quad (5.14)$$

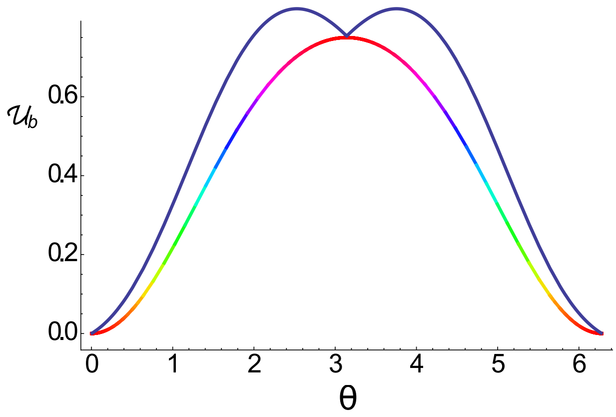

FIGURE 5.3: Here, we plot the upper bound of the product of variances of two incompatible observables, $A = \sigma_x$ and $B = \sigma_z$, two components of the angular momentum for spin $\frac{1}{2}$ particle with a state $\rho = \frac{1}{2}\left(I_2 + \cos\frac{\theta}{2}\sigma_x + \frac{\sqrt{3}}{2}\sin\frac{\theta}{2}\sigma_2 + \frac{1}{2}\sin\frac{\theta}{2}\sigma_z\right)$. Blue coloured line is the upper bound of the product of the two variances given by (5.17) and the hue plot denotes the product of the two variances.

Therefore, we obtain a tighter lower bound on the sum of variances of two observables by writing above two bounds collectively as

$$\Delta A^2 + \Delta B^2 \geq \mathcal{L}_b = \max\{\mathcal{L}_p, \mathcal{L}_t\}. \tag{5.15}$$

Reverse uncertainty relations

Does quantum mechanics restrict upper limit to the product and sum of variances of two incompatible observables? In this section, for the first time, we introduce the reverse bound, i.e., the upper bound to the product and sum of variances of two incompatible observables. To show the reverse uncertainty relation for the product of variances of two observables, we use the reverse Cauchy-Schwarz inequality for positive real numbers [171–174], which states that *for two sets of positive real numbers* $c_1, ..., c_n$ *and* $d_1, ...d_n$*, if* $0 < c \leq c_i \leq C < \infty$*,* $0 < d \leq d_i \leq D < \infty$ *for some constants c, d, C and D for all* $i = 1, ...n$*, then*

$$\sum_{i,j} c_i^2 d_j^2 \leq \frac{(CD + cd)^2}{4cdCD}\left(\sum_i c_i d_i\right)^2. \tag{5.16}$$

Using this inequality, one can easily show that for product of variances of two observables satisfy the relation

$$\Delta A^2 \Delta B^2 \leq \Omega_{ab}^{\Psi} \left(\sum_i \sqrt{F_{\Psi}^{a_i}} \sqrt{F_{\Psi}^{b_i}} |\tilde{a}_i||\tilde{b}_i| \right)^2, \tag{5.17}$$

such that $\Omega_{ab}^{\Psi} = \frac{\left(M_{\Psi}^a M_{\Psi}^b + m_{\Psi}^a m_{\Psi}^b\right)^2}{4 M_{\Psi}^a M_{\Psi}^b m_{\Psi}^a m_{\Psi}^b}$, where

$$\begin{aligned} M_{\Psi}^a &= \max\{\sqrt{F_{\Psi}^{a_i}}|\tilde{a}_i|\}, \\ m_{\Psi}^a &= \min\{\sqrt{F_{\Psi}^{a_i}}|\tilde{a}_i|\}, \\ M_{\Psi}^b &= \max\{\sqrt{F_{\Psi}^{b_i}}|\tilde{b}_i|\}, \text{and} \\ m_{\Psi}^b &= \min\{\sqrt{F_{\Psi}^{b_i}}|\tilde{b}_i|\}. \end{aligned} \tag{5.18}$$

Now, let us derive another reverse uncertainty relation for the sum of variances using the Dunkl-Williams inequality [171]. It is a state dependent upper bound on the sum of variances. The Dunkl-Williams inequality states that *if a, b are non-null vectors in the real or complex inner product space, then*

$$||a - b|| \geq \frac{1}{2}(||a|| + ||b||)||\frac{a}{||a||} - \frac{b}{||b||}||. \tag{5.19}$$

Now, if we take $a = (A - \langle A \rangle)|\Psi\rangle$ and $b = (B - \langle B \rangle)|\Psi\rangle$, then, using the Dunkl Williams inequality we obtain the following equation

$$\Delta A + \Delta B \leq \frac{\sqrt{2}\Delta(A - B)}{\sqrt{1 - \frac{Cov(A,B)}{\Delta A . \Delta B}}}, \tag{5.20}$$

where, $Cov(A, B) = \frac{1}{2}\langle\{A, B\}\rangle - \langle A \rangle\langle B \rangle$ is the *quantum covariance* of the operators A and B in quantum state $|\Psi\rangle$. We know from the Schrödinger's uncertainty relation that $\Delta A^2 \Delta B^2 \geq Cov(A, B)^2 + \frac{1}{4}|\langle[A, B]\rangle|^2$, such that

$$-1 \leq \frac{Cov(A, B)}{\Delta A \Delta B} \leq 1. \tag{5.21}$$

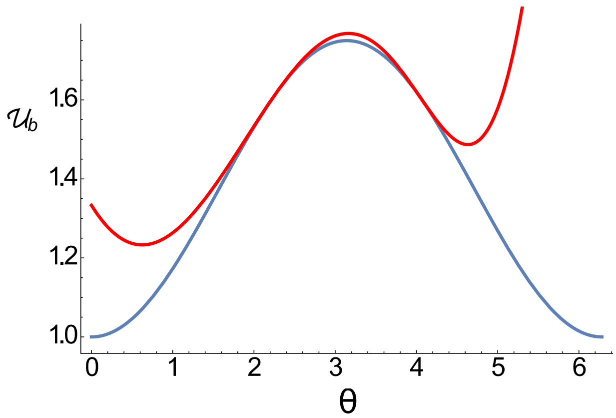

FIGURE 5.4: Here, we plot the upper bound of the sum of variances of two incompatible observables, $A = \sigma_x$ and $B = \sigma_z$, two components of the spin angular momentum for spin $\frac{1}{2}$ particle with a state $\rho = \frac{1}{2}\left(I_2 + \cos\frac{\theta}{2}\sigma_x + \frac{\sqrt{3}}{2}\sin\frac{\theta}{2}\sigma_2 + \frac{1}{2}\sin\frac{\theta}{2}\sigma_z\right)$. Red coloured line is the upper bound of the sum of the two variances given by (5.23) and the blue plot denotes the sum of the two variances.

Thus, the quantity on the denominator in the square root is always positive. On the other hand, one can also show that

$$\sqrt{1 - \frac{Cov(A,B)}{\Delta A.\Delta B}} < \sqrt{2} \tag{5.22}$$

for non-trivial cases. Thus, we have $\Delta A + \Delta B < \Delta(A - B)$ in such cases, though this is a weaker bound than Eq. (5.20). Therefore, by squaring this on the both sides, we obtain an upper bound on the sum of variance as

$$\Delta A^2 + \Delta B^2 \leq \frac{2\Delta(A-B)^2}{\left[1 - \frac{Cov(A,B)}{\Delta A.\Delta B}\right]} - 2\Delta A\Delta B. \tag{5.23}$$

Eq. (5.20) can further be written in terms of the commutator of A and B, by using the Schrödinger's uncertainty relation as

$$\Delta A^2 + \Delta B^2 \leq \frac{2\Delta(A-B)^2}{\left[1 - \sqrt{1 - \frac{|\langle[A,B]\rangle|^2}{4\Delta A^2 \Delta B^2}}\right]} - 2\Delta A\Delta B, \tag{5.24}$$

which is a weaker version of Eq. (5.23). It is evident from the above equation, that the lower bound fixed by the Schrödinger's uncertainty relation, in turn also plays a major role in determining the upper bound on the sum of standard derivations or the sum of variances.

Conclusions

In summary, here we have derived tighter, state-dependent uncertainty relations both in the sum as well as the product form for the variances of two incompatibles observables. We have also introduced state-dependent reversed uncertainty relations based on variances. Significance of the certainty and the uncertainty relations is that for a fixed amount of 'spread' of the distribution of measurement outcomes of one observable, the 'spread' for the other observable is bounded from both sides. Such relations should play important role in quantum metrology, quantum speed limits (QSL) and many other fields of quantum information theory. This is due to the facts that (i) these relations are optimization free (ii) state-dependent (iii) tighter than the most of the existing bounds. Moreover, the reverse uncertainty relations in particular should be useful in setting an upper limit to the time of quantum evolutions [169] and its relation with other properties of states and evolution operators may be explored using these relations.

Chapter 6

Summary and future directions

"Readers probably haven't heard much about it yet, but they will. Quantum technology turns ordinary reality upside down." ——*Michael Chrichton*

In this thesis, we have studied the effects of various quantum information theoretic resources on the speed of quantum evolutions to guage the role of these resources in setting the speed of quantum evolutions. As mentioned earlier, this may help us to understand how to control and manipulate the speed of quantum evolutions, quantum thermodynamic engines, to distinguish quantum gates and in quantum metrology.

Existing speed limits were mainly theoretical constructions based on the geometry of the quantum evolutions in the quantum state space with no experimental realization. Whereas, this thesis contributes enormously to establish a connection between the speed of quantum evolutions and the visibility of the interference pattern formed due to the interference between the initial and the final state.

We also established an intriguing connection between the QSL and the observable measure of quantum coherence or the asymmetry of the state with respect to the evolution operator. In particular, we showed that the bound to the time of quantum evolutions increases with decreasing coherence or asymmetry of the state. Our result had also motivated Marvian *et al.* to show that any measure of QSL is nothing but a measure of quantum asymmetry [130]. We showed how QSL behaves under the partial elimination of quantum states and classical mixing.

Moreover, we showed how the non-local nature of quantum correlation affects the speed of quantum evolutions. It is well known that quantum uncertainty relations play an important role in detecting the steerability of a quantum state by an another entangled party. We, on the other hand, studied the role of quantum coherence and derived the sufficient condition for the steerability of the state. This, in turn, provided us the necessary and sufficient criteria for achieving the non-local advantage of quantum coherence. From the already establish relation between the QSL and quantum coherence, this criteria led to develop the connection between the QSL and non-locality. We derived the necessary and sufficient criteria for achieving the non-local advantage of QSL, which is never possible to achieve by a single system.

Although we know that we can achieve the non-local advantage of QSL beyond what could have been maximally achieved by a single system (the system which has a single system description or LHS model), it is still not clear how to use it in practical situations as a resource. Further study is needed for that purpose. But this thesis genuinely contributes to understanding the effects of non-local nature in QSL and quantum metrology.

In QSL and quantum metrology, apart from the lower bounds in the time of quantum evolutions or the error in measurements, we expect there to exist also the upper bounds. But we are yet to set such bounds and study their properties. Particularly, as we have mentioned before, how to manipulate such a bound and what are the resources that play the role in manipulating them should have important technological implications as well as fundamental implications in understanding the nature.

To address this issue, in the last chapter of this thesis, we derive a few state dependent variance based reverse uncertainty relations in the sum as well as product forms. We first start with the derivation of a few tighter uncertainty relations. One advantage of these new relations as mentioned before is that they are optimization free and at the same time tight enough. These new uncertainty relations show that the incompatibility of two observables depends not on the non-commutativity but on the transition probabilities between the state of the system and the eigenbases of the observables.

Most importantly, unlike previous tighter uncertainty relations, these new uncertainty relations can be reversed. Thus, we provide here reverse uncertainty relations. We believe that such relations may play an important role in setting the upper bound to the time of quantum evolutions and the error in parameters in quantum measurements.

This thesis does not only focus and address the issues in QSL and quantum metrology but also deals a few fundamental issues effectively. First, we have shown a direct connection between the steering kind of non-locality and quantum coherence. It has fundamental implications due to the fact that both of them have separate resource theories [68–71, 84, 85] in quantum information theory. Second, we related the quantum coherence or in particular, quantum asymmetry with QSL.

Apart from the fundamental implications, results from this thesis should have various applications in quantum information theory. First, as mentioned before, it is very important to study and understand the role of various properties of states and observables in controlling the quantum thermodynamic engines. The theoretical development of faster yet efficient thermodynamic engines is necessary. Second, the measurable bounds of QSL should help us to distinguish the quantum gates. This is now inevitable to test this in the experiments. Third, as we have already connected the two resource theories, steering and quantum coherence (or asymmetry as well) in the bi-partite scenario, it is important to study them in the multi-partite scenarios. Forth, apart from the lower time bound, our results on the reverse uncertainty relations may help in setting an upper bound to the time of quantum evolutions. It is well-known that there exists a trade-off relation between the quantum non-locality (Bell) and quantum contextuality. Fifth, One should also focus on to study the connection between the quantum contextuality and quantum steering following our work presented in the chapter 4.

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