CHARACTERIZATION OF CERTAIN ARITHMETICAL FUNCTIONS

By

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STATEMENT BY AUTHOR

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Jay Gopalbhai Mehta

List of Publications arising from the thesis

Journal

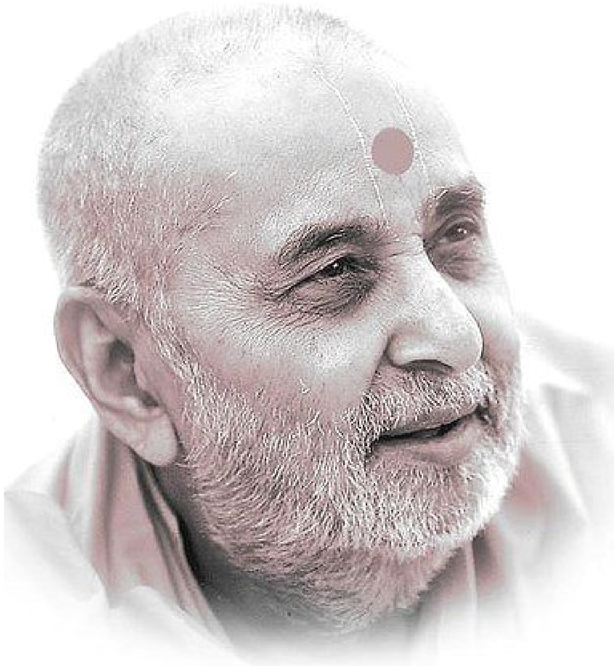
1. "On completely multiplicative complex valued funtions", K. Chakraborty and J. Mehta, *Ann. Univ. Sci. Budapest. Sect. Comput.*, **2012**, *Vol. 38*, 19-24.

2. "On arithmetical functions having constant values in some domain", J. Mehta, *Acta. Math. Hungar.*, **2014**, *Vol. 142, no. 1*, 110-117.

3. "Quasi-uniqueness of Gaussian prime plus one's", Jay Mehta and G. K. Viswanadham, *Int. J. of Number Theory*, **2014**, *Vol. 10, no. 7*, 1783-1790.

Jay Gopalbhai Mehta

Dedicated to my Guru



બીજાના સુખમાં આપણું સુખ.

"In the joy of others, lies our own."

- HDH Pramukh Swami Maharaj

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Abstract

This thesis involves characterization of completely additive and completely multiplicative functions on various domains satisfying some properties. It is divided into three parts of one chapter each.

The first part is on sets characterizing completely additive functions called the sets of uniqueness for functions over non-zero Gaussian integers. In particular, it is proved in Chapter 1 that the set of shifted Gaussian primes in addition to finitely many Gaussian primes is a set of uniqueness for completely additive complex valued functions over the non-zero Gaussian integers.

In the second part, it has been shown that a completely additive function over a finite union of lattices in complex plane, such that the domain is closed under multiplication, having constant values on some family of discs is identically zero.

In the final chapter, we investigate completely multiplicative functions having values that are nearly Gaussian integers, i.e. whose values approach to a Gaussian integer.

CONTENTS

Synopsis

0.1 Introduction

This thesis is about characterizing certain arithmetical functions like (completely) additive and completely multiplicative functions satisfying specific properties over various domains. The thesis can be divided into three parts. The first part comprises of two chapters and the remaining two parts comprises of one chapter each.

The theme of the first part is *set of uniqueness* for (completely) additive functions over the set of non-zero Gaussian integers. The summary of this part is given in Section 0.2 and Section 0.3. The second part of thesis is about completely additive complex valued functions over principal configuration. If such a function assumes constant values in some specific discs of the complex plane, then it is identically zero. A description of contents of this part is given in Section 0.4. The final part of the thesis consists of a chapter on completely multiplicative complex valued functions with nearly Gaussian integer values. The summary of this part is given in Section 0.5.

0.2 Set of quasi-uniqueness of shifted Gaussian primes

A function $f : \mathbb{N} \to \mathbb{R}$ is said to be additive if

$$f(mn) = f(m) + f(n) \tag{1}$$

holds for all $m, n \in \mathbb{N}$ with $(m, n) = 1$ and is said to be completely additive if (1) holds for all $m, n \in \mathbb{N}$. Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the collection of all such additive and completely additive functions respectively.

Clearly, an additive function can be determined by its values at the powers of primes whereas a completely additive function can be determined by its values at primes. Thus, an additive function (respectively a completely additive function) which vanishes at all prime powers (respectively at all primes) vanishes on all of $\mathbb{N}$. It would therefore be interesting to ask that what are the examples of such sets which determine completely additive functions in a similar fashion. Such a set is said to be a *set of uniqueness*. This notion was first introduced by I. Kátai [12]. More precisely,

Definition 0.2.1 A set $A \subset \mathbb{N}$ is a *set of uniqueness* for $\mathcal{A}$ (respectively for $\mathcal{A}^*$) if for every $f \in \mathcal{A}$ (respectively $f \in \mathcal{A}^*$) that vanishes on A necessarily vanishes on all of $\mathbb{N}$.

A more general notion called the *set of quasi-uniqueness*, also given by I. Kátai, is defined as follows:

Definition 0.2.2 We call a set $A \subset \mathbb{N}$ a *set of quasi-uniqueness* for $\mathcal{A}$ (respectively $\mathcal{A}^*$) if there exists a suitable finite set $B \subset \mathbb{N}$ such that $A \cup B$ is a set of uniqueness for $\mathcal{A}$ (respectively $\mathcal{A}^*$).

Let $\mathcal{P}$ denote the set of all rational primes and let $\mathcal{P}+1 := \{p+1 \mid p \in \mathcal{P}\}$. I. Kátai [12] proved that the set $\mathcal{P}+1$ is a set of quasi-uniqueness for completely additive functions (i.e. for $\mathcal{A}^*$) assuming validity of the Riemann-Piltz conjecture. In [13], he again proved the same result but without using any unproven hypothesis.

0.2.1 Extension to Gaussian integers.

The notions of set of uniqueness and set of quasi-uniqueness can be analogously extended for (completely) additive complex valued functions over non-zero Gaussian integers $\mathbb{Z}[i]^*$. Let $\mathcal{A}_{\mathbb{Z}[i]}$ and $\mathcal{A}^*_{\mathbb{Z}[i]}$ denote the family of all additive and completely additive complex valued functions over $\mathbb{Z}[i]^*$ respectively. Clearly $\mathcal{P}[i]$, the set of all Gaussian primes, is a set of uniqueness for $\mathcal{A}^*_{\mathbb{Z}[i]}$. Let $\mathcal{P}[i]+1$ denote the set of Gaussian integers of the form $p+1$ where p is a Gaussian prime. Then, naturally, one would like to ask the following question:

Question 1. Is $\mathcal{P}[i]+1$ a set of quasi-uniqueness for $\mathcal{A}^*_{\mathbb{Z}[i]}$?

An affirmative answer to the above question is obtained by our first result stated below:

Theorem 0.2.3 *There exists a numerical constant K with the property that if a completely additive complex valued function f over $\mathbb{Z}[i]^*$ vanishes at each $p \in \mathcal{P}[i]$ with norm of p not exceeding K and vanishes on $\mathcal{P}[i]+1$, then f necessarily vanishes on all of $\mathbb{Z}[i]^*$.*

The extension of definition of set of uniqueness to completely additive functions over Gaussian integers with some examples and the proof of Theorem 0.2.3 constitutes the first chapter of the first part of this thesis. This is a joint work with G. K. Viswanadham [18].

0.3 Set of uniqueness of shifted Gaussian primes

Not only did Kátai, in [12], proved that the set $\mathcal{P}+1$ is a set of quasi-uniqueness for $\mathcal{A}^*$ but he also conjectured that the set $\mathcal{P}+1$ is, in fact, a set of uniqueness for $\mathcal{A}^*$. In 1974, P. D. T. A. Elliott [6] settled the conjecture completely. More precisely, it follows from Elliott's result that, $\mathcal{P}+1$ is in fact a set of uniqueness for $\mathcal{A}$ and hence for $\mathcal{A}^*$.

This lead us to investigate whether the method of Elliott can be generalized to our setting to conclude that the set of Gaussian primes shifted by 1 is a set of uniqueness for $\mathcal{A}^*_{\mathbb{Z}[i]}$. We obtained an affirmative answer to this by proving the following result:

Theorem 0.3.1 *Every complex valued additive function over $\mathbb{Z}[i]^*$ vanishing on the complement of a finite subset of $\mathcal{P}[i]+1$ in fact vanishes on all of $\mathbb{Z}[i]^*$.*

As a corollary to the above theorem, we have, $\mathcal{P}[i]+1$ is a set of uniqueness for $\mathcal{A}^*_{\mathbb{Z}[i]}$ which is analogous to Kátai's conjecture.

D. Wolke [25] proved that a set $A \subset \mathbb{N}$ is a set of uniqueness for $\mathcal{A}^*$ if and only if every $n \in \mathbb{N}$ can be expressed as a finite product of rational powers of elements of A. As a consequence of this and Elliott's result, one can say that every positive integer can be expressed as a finite product of rational powers of elements of $\mathcal{P}+1$. Such an expression is called *shifted prime factorization* which, unlike prime factorization, may not be unique.

The method of D. Wolke can be extended to Gaussian integers and obtain similar result in this setting. In view of Theorem 0.3.1, we conclude that every non-zero Gaussian integer also has a shifted prime factorization, which again may not be unique. More precisely, we have the following corollary:

Corollary 0.3.2 *Every $\alpha \in \mathbb{Z}[i]^*$ can be written in the following form:*

$$\alpha = \prod_{j=1}^{k} (p_j + 1)^{l_j},$$

where $p_j \in \mathcal{P}[i]$ and $l_j \in \mathbb{Q}$.

The proof of Theorem 0.3.1 and the Corollary 0.3.2 were obtained in [19] co-authored with G. K. Viswanadham. This constitutes the content of another chapter in the first part of the thesis. We recall another well known notion of the set of uniqueness $\mod 1$. We conclude the chapter by a remark that $\mathcal{P} + 1$ is proved to be a set of quasi-uniqueness modulo 1 for $\mathcal{A}^*$, however, it is not yet known whether it is a set of uniqueness $\mod 1$ or not. Furthermore, one may ask whether $\mathcal{P}[i] + 1$ is a set of uniqueness $\mod 1$ or not.

0.4 Arithmetical functions with constant values in some domain

In 1969, I. Kátai proved [14] that any completely additive real valued function on the set of positive integers assuming constant values in some specific intervals of real line is identically zero. In 1991, M. Amer ([1], Theorem 1) generalized Kátai's result for complex valued functions over $\mathbb{Z}[i]^*$. Amer proved that such a function having constant values in some specific discs of the complex plane is identically zero .

This lead to the idea of extending M. Amer's result for completely additive complex valued functions over non-zero lattice points in complex plane as no specific properties of the Gaussian integers were used in his proof except that it is closed under multiplication. However, in general, an arbitrary lattice (say

Λ_1) in $\mathbb{C}$ may not be closed under multiplication. To get through with this, i.e. in order to make our domain closed under multiplication we supplement the lattice Λ_1 with some additional points in such a way that this larger collection of points (which may no longer be a lattice), is still discrete and is closed under multiplication. Such a system of points is called '*principal configuration*' described by Delone (see [4]). These additional points form finitely many lattices $\Lambda_2, \Lambda_3, \ldots \Lambda_h$ having only the origin O in common with the lattice Λ_1 and with each other lattices. Let us denote the principal configuration by Γ. Then we have,

$$\Gamma = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_h.$$

Let Γ be the principal configuration and let $\Lambda_i = \Lambda_i(\omega_i, \omega_i')$ where $\omega_i, \omega_i' \in \mathbb{C}$, $i = 1, 2, \ldots, h$ be the lattices in the principal configuration Γ. We denote the set of all completely additive functions and completely multiplicative complex valued functions over Γ^*, non-zero points of principal configuration, by $\mathcal{A}_\Gamma^*$ and $\mathcal{M}_\Gamma^*$ respectively. Let $S(a, r)(\subseteq \mathbb{C})$ be the closed disc with center a and radius r, i.e.

$$S(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

The following theorem, obtained in [17], generalizes the result of Amer:

Theorem 0.4.1 *Let $f \in \mathcal{A}_\Gamma^*$. Assume that there exists a sequence of complex numbers $z_1, z_2, \ldots$ such that $|z_j| \to \infty$ $(j \to \infty)$ and that*

$$f(\alpha) = A_j \quad \textit{for all } \alpha \in S\big(z_j, (2 + \epsilon)\sqrt{|z_j|}\big),$$

where A_j's are constants and ϵ is some positive constant depending on the principal configuration Γ. Then f vanishes on all of Γ^*.

Similarly, one can also prove the assertion in above theorem for $f \in \mathcal{M}^*_\Gamma$, which does not vanish anywhere. In this f will be the constant function 1 on all of Γ^*. As a corollary, taking $\Gamma = \mathcal{O}_K$, the ring of integers of imaginary quadratic field $K = \mathbb{Q}[\sqrt{d}]$, we have a more general version of Amer's result.

0.5 Arithmetical functions with nearly Gaussian integer values

A function $f : \mathbb{N} \to \mathbb{R}$ is said to be completely multiplicative if

$$f(mn) = f(m)f(n) \qquad (\forall\ m, n \in \mathbb{N}).$$

For any real number z, let $\|z\|$ denote its distance from the nearest integer. A function $f : \mathbb{N} \to \mathbb{R}$ is said to have nearly integer values if $\|f(n)\| \to 0$ as $n \to \infty$.

We say that a real number θ is a Pisot-number if it is an algebraic integer $\theta > 1$, and if all the conjugates $\theta_2, \ldots, \theta_r$, are in the domain $|z| < 1$. It is well known that a Pisot-number satisfies the relation $\|\theta^n\| \to 0$.

I. Kátai and B. Kovács [15] determined the class of completely multiplicative functions with nearly integer values. Such functions satisfying some specific properties have values which are Pisot-numbers.

0.5.1 Extension to functions with nearly Gaussian integer values.

We consider completely multiplicative complex valued functions on the set of positive integers and denote by $\|z\|$ the distance of a complex number z from nearest Gaussian integer. That is,

$$\|z\| = \min_{\gamma\in\mathbb{Z}[i]} |z-\gamma|.$$

Jointly with K. Chakraborty [3] we determined the class of completely multiplicative functions $f:\mathbb{N}\to\mathbb{C}$ with nearly Gaussian integer values, i.e.

$$\|f(n)\| \to 0 \qquad (n\to\infty). \tag{2}$$

The definition of Pisot-number can be evidently extended to generalized Pisot-number with respect to $\mathbb{Z}[i]$ which we called *Gaussian Pisot-number*.

Definition 0.5.1 We shall say that θ is a Gaussian Pisot number if there exists a polynomial $\phi(z)\in\mathbb{Z}[i][z]$ with leading coefficient 1, and $\phi(z)=\prod_{j=1}^{r}(z-\theta_j)$, $\theta_1=\theta$, $|\theta|>1$ and all the conjugates, $\theta_2,\dots,\theta_r$ are in the domain $|z|<1$.

Let f be a completely multiplicative function with values in $\mathbb{Q}(\beta)$, for an algebraic number β. Let $\beta_2,\dots,\beta_r$ be the conjugates of β (with respect to $\mathbb{Z}[i]$). Let $\phi_j(n)$ denote the conjugates of $f(n)$ defined by the substitution $\beta\to\beta_j$. Then clearly, ϕ_j are also completely multiplicative functions. We proved the following result:

Theorem 0.5.2 *Let f be a completely multiplicative complex valued function for which there exists at least one n_0 such that $|f(n_0)|>1$. Let $\mathscr{P}_1$ denote the set of primes p for which $f(p)\neq 0$.*

If (2) holds, then the values $f(p)=\alpha_p$ are Gaussian Pisot numbers and for each $p_1,p_2\in\mathscr{P}_1$ we have $\mathbb{Q}(\alpha_{p_1})=\mathbb{Q}(\alpha_{p_2})$. Let Θ denote one of the values

α_p $(p \in \mathscr{P}_1)$ *and* $\Theta_2, \ldots, \Theta_r$ *be its conjugates* $(i = 2, \ldots, r)$. *Then*

$$\phi_j(n) \to 0 \quad \text{as } n \to \infty, \qquad j = 2, \ldots, r.$$

Conversely, assume that the values $f(p)$ *are zeros or Gaussian Pisot numbers from a given algebraic number field* $\Omega(\Theta)$. *If*

$$\phi_j(p) \to 0 \quad \text{as } p \to \infty, \qquad j = 2, \ldots, r,$$

then (2) *holds.*

The proof of Theorem 0.5.2 is the content of the third and final part of the thesis.

CHAPTER 1

Sets characterizing arithmetical functions

In this chapter, we recall the notion of set of uniqueness *for completely additive arithmetical functions. This notion was first introduced by I. Kátai around 1967. These sets characterizes completely additive functions, for example, the set of rational primes. We defined its analogue for completely additive functions over the set of non-zero Gaussian integers. We prove that the set of shifted Gaussian primes together with finitely many Gaussian primes is a set of uniqueness for completely additive complex valued functions. This is a joint work with G. K. Viswanadham* [18].

1.1 Introduction

Definition 1.1.1 A function $f : \mathbb{N} \to \mathbb{R}$ is said to be additive if

$$f(mn) = f(m) + f(n) \quad (\forall\ m, n \in \mathbb{N} \text{ with } (m, n) = 1),$$

and is said to be completely additive if

$$f(mn) = f(m) + f(n) \quad (\forall\ m, n \in \mathbb{N}).$$

Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of all such additive and completely additive arithmetical functions respectively.

Clearly, an additive function can be determined by its values at the prime powers whereas a completely additive function can be determined by its values at the primes. Thus, an additive function (respectively a completely additive function) over the set of positive integers which vanishes at all prime powers (respectively at all primes) vanishes on all of $\mathbb{N}$. It would therefore be interesting to ask that what are other examples of such sets which determine completely additive functions in a similar fashion.

Let $\mathcal{P}$ denote the set of all rational primes and let

$$\mathcal{P} + 1 := \{p + 1 \mid p \in \mathcal{P}\}.$$

Around 1967, I. Kátai asked whether it is true for completely additive arithmetical functions that $f(\mathcal{P} + 1) = 0$ implies $f(\mathbb{N}) = 0$. More generally, one can ask whether for a subset A of $\mathbb{N}$, it is true that $f(A) = 0$ implies $f(\mathbb{N}) = 0$ for completely additive functions $f : \mathbb{N} \to \mathbb{R}$. If the answer is yes, then we call A, a set of uniqueness for completely additive functions. More precisely,

Definition 1.1.2 A set $A \subset \mathbb{N}$ is a *set of uniqueness* (for completely additive functions) if

$$f(A) = \{0\} \Rightarrow f(\mathbb{N}) = \{0\} \text{ i.e., } f \equiv 0 \quad (\forall\ f \in \mathcal{A}^*).$$

This notion was introduced by I. Kátai [12] in 1968. Clearly, the set of primes $\mathcal{P}$ is an example of set of uniqueness. In some cases, a set itself may not be a set of uniqueness but adding finitely many elements to the set would make it a set of uniqueness. The set $\mathcal{P} \setminus \{p_1, p_2, p_3\}$, where p_1, p_2, p_3 are three distinct primes, is one such example. In [12], Kátai called such a set as set of quasi-uniqueness. This is a more general notion of the set of uniqueness.

Definition 1.1.3 We call a set $A \subset \mathbb{N}$ a *set of quasi-uniqueness* for completely additive functions if there exists a suitable finite set $B \subset \mathbb{N}$ such that $A \cup B$ is a set of uniqueness for completely additive functions.

We state below some more examples of sets of uniqueness for completely additive functions.

Example 1.1.1 Let p_1, p_2, p_3 be three distinct rational primes. The set

$$A = \{p_1^2, p_1^3 p_2^2, p_1^2 p_3\} \cup (\mathcal{P} \setminus \{p_1, p_2, p_3\})$$

is a set of uniqueness.

Example 1.1.2 $A = \{x^2 \mid x \in \mathbb{N}\}$ is clearly a set of uniqueness. In general, one can consider, $A = \{x^k \mid x \in \mathbb{N}\}$ for some $k \in \mathbb{N}$.

Example 1.1.3 $A = \{x^2 + y^2 \mid x, y \in \mathbb{N}\}$ is a set of uniqueness.

Example 1.1.4 If l and k are fixed integers such that $(l, k) = 1$, then the set A containing the prime divisors of k and the arithmetical progression $l + kj$, $(j = 0, 1, \ldots)$ is a set of uniqueness.

Example 1.1.5 The union of the set of primes p in the arithmetical progression $p \equiv -1 \pmod 4$ and of the set of numbers $n^2 + 1$ $(n = 1, 2, \ldots)$ is a set of uniqueness.

In 1968, I. Kátai [12] proved that the set $\mathcal{P}+1 := \{p+1 \mid p \in \mathcal{P}\}$ is a set of quasi-uniqueness assuming validity of the Riemann-Piltz conjecture (GRH). An year later he [13] again proved the same result, by an efficient use of the Bombieri-Vinogradov theorem, without any such assumption.

1.1.1 Extension to Gaussian integers

Let $\mathbb{Z}[i]$ denote the ring of Gaussian integers, i.e.

$$\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}\}.$$

Let $\mathbb{Z}[i]^*$ denote the set of non-zero Gaussian integers. One can extend the notion of additive functions as well as completely additive arithmetical functions to complex valued functions over the Gaussian integers $\mathbb{Z}[i]^*$ as follows:

Definition 1.1.4 A function $f : \mathbb{Z}[i]^* \to \mathbb{C}$ is said to be additive if

$$f(\alpha\beta) = f(\alpha) + f(\beta) \quad (\forall\ \alpha, \beta \in \mathbb{Z}[i]^* \text{ with } (\alpha, \beta) = 1),$$

and is said to be completely additive if

$$f(\alpha\beta) = f(\alpha) + f(\beta) \quad (\forall\ \alpha, \beta \in \mathbb{Z}[i]^*).$$

Let $\mathcal{A}_{\mathbb{Z}[i]}$ and $\mathcal{A}^*_{\mathbb{Z}[i]}$ denote the set of all such additive and completely additive arithmetical functions over $\mathbb{Z}[i]^*$ respectively. As in the case of rational integers, completely additive complex valued function over $\mathbb{Z}[i]^*$ can be determined by its values over the Gaussian primes whereas an additive complex valued function over $\mathbb{Z}[i]^*$ can be determined by its values at the powers of Gaussian primes.

The notions of the set of uniqueness and the set of quasi-uniqueness have evident extension to the additive and completely additive complex valued functions over non-zero Gaussian integers. They are defined as follows:

Definition 1.1.5 A set $A \subset \mathbb{Z}[i]^*$ is said to be a *set of uniqueness with respect to Gaussian integers* if

$$f(A) = \{0\} \Rightarrow f(\mathbb{Z}[i]^*) = \{0\}, \text{ i.e. } f \equiv 0 \quad (\forall\, f \in \mathcal{A}^*_{\mathbb{Z}[i]}).$$

Definition 1.1.6 We call a set $A \subset \mathbb{Z}[i]^*$ a *set of quasi-uniqueness* for $\mathcal{A}^*_{\mathbb{Z}[i]}$ if there exists a suitable finite set $B \subset \mathbb{Z}[i]^*$ such that $A \cup B$ is a set of uniqueness for $\mathcal{A}^*_{\mathbb{Z}[i]}$.

Example 1.1.6 The following are some examples of set of uniqueness with respect to Gaussian integers.

1. The set of all Gaussian primes $\mathcal{P}[i]$.

2. The set $A_k = \{\alpha^k \mid \alpha \in \mathbb{Z}[i]^*\}$, for each $k \in \mathbb{N}$.

3. $A = \{\pi_1^2, \pi_1\pi_2\} \cup (\mathcal{P}[i] \setminus \{\pi_1, \pi_2\})$ for two distinct Gaussian primes π_1, π_2.

4. $A = \{\gamma \in \mathbb{Z}[i]^* \mid \mathrm{N}(\gamma) = p^2,\ p \in \mathcal{P}\}$.

1.2 The main result

Naturally, one would like to ask whether the $\mathcal{P}[i] + 1$ of all Gaussian integers of the form $p + 1$, where p is a Gaussian primes is a set of quasi-uniqueness or not. This is analogous to the result of Kátai. The result proved by J. Mehta and

G. K. Viswanadham [18] answers this question affirmatively. More precisely, we have the following result:

Theorem 1.2.1 *There exists a numerical constant K with the property that if $f : \mathbb{Z}[i]^* \to \mathbb{C}$ is a completely additive function which vanishes on the set $\mathcal{P}[i] + 1$ as well as on all Gaussian primes with norm not exceeding K, then f is identically zero.*
In other words, the set $\mathcal{P}[i] + 1$ is a set of quasi-uniqueness.

Theorem 1.2.1 is same as saying that the set $\mathcal{P}[i]+1$ together with finitely many Gaussian primes with norm not exceeding some constant K is a set of uniqueness for completely additive functions.

1.2.1 Factorization Property

An interesting property of set of uniqueness for completely additive functions over $\mathbb{N}$ is that: every $n \in \mathbb{N}$ can be expressed as a finite product of rational powers of elements of a set of uniqueness A. Also, conversely, if every $n \in \mathbb{N}$ can be expressed as a finite product of rational powers of elements of some set A, then A is a set of uniqueness. It was proved by D. Wolke [25] and independently by Dress and Volkmann [5]. More precisely, we have the following theorem:

Theorem 1.2.2 (Wolke) *A set $A \subset \mathbb{N}$ is a set of uniqueness for completely additive functions if and only if each positive integer n can be written as*

$$n = a_1^{r_1} \dots a_k{}^{r_k}; \quad (a_i \in A,\ r_i \in \mathbb{Q},\ 1 \leq i \leq k,\ k \in \mathbb{N} \cup \{0\}).$$

Along the lines of Wolke's result, one can prove the following theorem which affirms that so is the case for completely additive functions over the set of non-zero Gaussian integers.

Theorem 1.2.3 *A set $A \subseteq \mathbb{Z}[i]^*$ is a set of uniqueness for $\mathcal{A}^*_{\mathbb{Z}[i]}$ if and only if each non-zero Gaussian integer n can be written as a finite product of rational powers of elements of A.*

As a consequence of above theorem and Theorem 1.2.1 we have the following result.

Corollary 1.2.4 *There exists an ineffective constant K such that every non-zero Gaussian integer α can be written in the following form:*

$$\alpha = \prod_{i=1}^{m} {p_i}^{r_i} \prod_{j=1}^{k} (q_j + 1)^{l_j}, \tag{1.1}$$

where $p_i, q_j \in \mathcal{P}[i]$, $r_i, l_j \in \mathbb{Q}$ for $1 \leq j \leq k,\ 1 \leq i \leq m$ and $k, m \in \mathbb{N} \cup \{0\}$ such that $\mathrm{N}(p_i) \leq K$.

In the next couple of sections we give some prerequisites and the proof of the above theorem. In these sections, Section 1.3 and Section 1.4, p and q denotes Gaussian primes unless otherwise specified.

1.3 Prerequisites

For $\alpha = a + ib \in \mathbb{Z}[i]$ $(a, b \in \mathbb{Z})$, let $\overline{\alpha} = a - ib$ denote the conjugate of α. Let $\mathrm{N}(\alpha)$ denote the norm of α defined by

$$\mathrm{N}(\alpha) = a^2 + b^2.$$

For example, $\mathrm{N}(3 - 2i) = 3^2 + 2^2 = 13$. It is easy to see that the norm is multiplicative, i.e. for any $\alpha, \beta \in \mathbb{Z}[i]$,

$$\mathrm{N}(\alpha\beta) = \mathrm{N}(\alpha)\,\mathrm{N}(\beta).$$

The following theorem gives the complete list of invertible elements of $\mathbb{Z}[i]^*$.

Theorem 1.3.1 *The only Gaussian integers which are invertible in $\mathbb{Z}[i]^*$ are ± 1 and $\pm i$.*

These invertible elements are called the units.

Definition 1.3.2 Let α and $\beta \in \mathbb{Z}[i]$. We say that α divides β, we denote by $\alpha|\beta$, if $\beta = \gamma\alpha$ for some $\gamma \in \mathbb{Z}[i]$. In this case, we say that α is a divisor or a factor of β.

Definition 1.3.3 Let α be a Gaussian integer with $\mathrm{N}(\alpha) > 1$. We say that α is a *Gaussian prime* if it has only trivial factors, i.e. ± 1, $\pm i$, $\pm \alpha$, $\pm i\alpha$. If α is not a Gaussian prime, then we say that α is a *composite* number.

The following theorem gives a complete description of primes in $\mathbb{Z}[i]$ in terms of rational primes.

Theorem 1.3.4 *Every prime in $\mathbb{Z}[i]$ is a unit multiple of the following primes:*

1. $1 + i$
2. *π or $\overline{\pi}$ with $\mathrm{N}(\pi) = p$, where p is a rational prime such that $p \equiv 1 \pmod 4$.*
3. *q, where q is a rational prime such that $q \equiv -1 \pmod 4$.*

For any real x, let $\pi_{\mathbb{Q}[i]}(x)$ denote the cardinality of the set of Gaussian primes p with $\mathrm{N}(p) \leq x$. Any rational prime p with $p \equiv 1 \pmod 4$ splits into two Gaussian primes while any rational prime q with $q \equiv -1 \pmod 4$ stays as a Gaussian prime with the norm $\mathrm{N}(q) = q^2$ in the ring of Gaussian integers. As a result, we have

$$\pi_{\mathbb{Q}[i]}(x) = 1 + 2\pi(x, 1, 4) + \pi(\sqrt{x}, -1, 4),$$

where $\pi(x, a, n)$ denotes the number of rational primes p upto x such that $p \equiv a \pmod{n}$. Then by the prime number theorem , we have

$$\pi_{\mathbb{Q}[i]}(x) \sim \frac{x}{\log x}.$$

Clearly, for any completely additive function $f : \mathbb{Z}[i]^* \to \mathbb{C}$ and for any unit $\epsilon \in \mathbb{Z}[i]$, $f(\epsilon) = 0$. So, throughout this chapter, we do not distinguish between the Gaussian primes which differ by unit multiples, that is Gaussian primes that are associates.

Let Φ denote the Euler's phi function for the Gaussian integers, defined by $\Phi(\alpha) = \#\left(\mathbb{Z}[i]/(\alpha)\right)^*$, for $\alpha \in \mathbb{Z}[i]^*$. One can easily see that

$$\Phi(\alpha) = \mathrm{N}(\alpha) \prod_{\substack{p \mid \alpha \\ p \in \mathcal{P}[i]}} \left(1 - \frac{1}{\mathrm{N}(p)}\right).$$

For any real number $x \geq 1$, define

$$\Psi_{\mathbb{Q}[i]}(x) = \sum_{\mathrm{N}(\alpha) \leq x} \Lambda_{\mathbb{Q}[i]}(\alpha),$$

where the function $\Lambda_{\mathbb{Q}[i]}$ is given by

$$\Lambda_{\mathbb{Q}[i]}(\alpha) = \begin{cases} \log \mathrm{N}(\pi) & \text{if } \alpha = \pi^m \text{ for some Gaussian prime } \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi_{\mathbb{Q}[i]}(x, a, q)$ denote the number of Gaussian primes p such that $p \equiv a \pmod q$ and $\mathrm{N}(p) \leq x$, i.e.

$$\pi_{\mathbb{Q}[i]}(x, a, q) = \sum_{\substack{p \in \mathcal{P}[i] \\ \mathrm{N}(p) \leq x \\ p \equiv a \pmod q}} 1.$$

For any non zero Gaussian integer d, clearly

$$\pi_{\mathbb{Q}[i]}(x, d, l) \ll 1 + \frac{x}{\mathrm{N}(d)}.$$

The following result is the Selberg sieve for algebraic number fields given by Rieger in [20]:

Theorem 1.3.5 *Let $M \geq 2$ be any positive integer and let $n_1, n_2, \ldots, n_M$ be distinct ideals of a number field K. Let $z \geq 2$ be any real number. We fix the following notations:*

- $\mathrm{N}(\delta)$- *Norm of the ideal δ.*
- U- *Unit ideal of K.*
- $\mu(\delta)$-*Generalized Möbius function.*

- *$f(\delta)$ is a multiplicative function on the ideals of K such that*

$$1 < f(\delta) \le \infty \quad \text{for } \delta \ne U.$$

- $f_1(t) = \sum_{\delta | t} \mu(\delta) f\left(\frac{t}{\delta}\right)$ *for each ideal t of K.*

- $Z = \sum_{\mathrm{N}(r) \le z} \frac{\mu^2(r)}{f_1(r)}$.

- $S_\delta = \sum_{n;\delta | n} 1$.

- $R_\delta = S_\delta - \frac{M}{f(\delta)}$.

- $\lambda(\delta) = \frac{\mu(\delta)}{Z} \prod_{p|\delta} \left(1 - \frac{1}{f(p)}\right)^{-1} \sum_{\mathrm{N}(r) \le \frac{z}{\mathrm{N}(\delta)}} \frac{\mu^2(r)}{f_1(r)}$.

With the above notations, we have

$$S = \frac{M}{Z} + O\left(\sum_{\mathrm{N}(\delta_1) \le z; \mathrm{N}(\delta_2) \le z} |\lambda(\delta_1)\lambda(\delta_2) R_{[\delta_1,\delta_2]}| \right),$$

where $S = \#\{n : p \nmid n \text{ for } \mathrm{N}(p) \le z\}$.

The proof of the above theorem can be found in [20]. An application of the above theorem yields us the following result:

Theorem 1.3.6 *Let a_i, b_i $(i = 1, \dots, s)$ be a pairs of integers satisfying $(a_i, b_i) = 1$ for $1 \le i \le s$ and define*

$$E = \prod_{i=1}^{s} a_i \prod_{1 \le i < j \le s} (a_i b_j - a_j b_i) \ne 0.$$

For each p, let $\omega(p)$ denote the number of solutions of the congruence

$$(a_1 m + b_1)(a_2 m + b_2) \cdots (a_s m + b_s) \equiv 0 \pmod p$$

and assume that $\omega(p) < \mathrm{N}(p)$ *for all primes* p. *Then for any* $x \geq 2$

$$\begin{aligned}&\#\{m \in \mathbb{Z}[i]^* \text{ with } \mathrm{N}(m) \leq x \mid a_i m + b_i \text{ is prime for } 1 \leq i \leq s\}\\ &\ll_s \frac{x}{\log^s x} \prod_{p \mid E} \left(1 - \frac{1}{\mathrm{N}(p)}\right)^{-(s-\omega(p))}.\end{aligned}$$

Corollary 1.3.7 *Let* $M(x,k)$ *denote the number of pairs of primes* (p,q) *satisfying the conditions* $p + 1 = kq$, $\mathrm{N}(p) \leq x$. *Then*

$$M(x,k) \ll \frac{x}{\Phi(k) \log^2 x}.$$

Proof. Applying Theorem 1.3.6 with $s = 2, a_1 = 1, b_1 = 0, a_2 = k$ and $b_2 = -1$ we get the result.

The following lemma is an analogue to the Siegel- Walfisz theorem for rational primes. It can be obtained in a similar way as in the rational primes case (see [22]).

Lemma 1.3.8 *Let* N *be any positive constant. Then there exists a positive number* $C(N)$ *depending on* N *such that for* $a, q \in \mathbb{Z}[i]$ *with* $\mathrm{N}(q) \leq (\log x)^N$, *we have*

$$\Psi_{\mathbb{Q}[i]}(x,q,a) = \frac{H(q)x}{\Phi(q)} + O(x \exp(-C(N)(\log x)^{\frac{1}{2}})).$$

The proof of above lemma can be found in the Doctoral Thesis of W. Schlackow [22]. We present a sketch of the proof here.

Proof. Let

$$\Psi_{\mathbb{Q}[i]}(x,q,a) = \sum_{\chi(q)}^{(0)} \overline{\chi}(q) \Psi_{\mathbb{Q}[i]}(x,\chi)$$

where the sum is running over all the characters for which $\chi(i) = 1$ and

$$H(q) = \begin{cases} 1 & \text{if } \mathrm{N}(q) = 1 \text{ or } 2 \\ 2 & \text{if } \mathrm{N}(q) = 4 \\ 4 & \text{otherwise.} \end{cases}$$

If χ_0 is the principal character then

$$|\Psi_{\mathbb{Q}[i]}(x, \chi) - \Psi_{\mathbb{Q}[i]}(x)| \ll (\log x) \log(\mathrm{N}(q)).$$

If χ_0 is non principal character then

$$|\Psi_{\mathbb{Q}[i]}(x, \chi)| \ll x \exp(-C(N) \log^{\frac{1}{2}} x).$$

This we get from the identity

$$\Psi_{\mathbb{Q}[i]}(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + O(x \exp(-c \log^{\frac{1}{2}} x))$$

and by Siegel's theorem which says $\beta_1 \leq 1 - c|q|^{\epsilon}$. Using these estimates and the prime number theorem for $\mathbb{Z}[i]$ we get the lemma. □

The lemma given below is an analogue of the Brun-Titchmarsh theorem and can be obtained as an application of the Selberg sieve or the large sieve for algebraic number fields.

Lemma 1.3.9 (Brun-Titchmarsh) *Let $\delta > 0$. Then for $q \in \mathcal{P}[i]$ with $\mathrm{N}(q) \leq x^{1-\delta}$ such that $(a, q) = 1$, we have*

$$\pi_{\mathbb{Q}[i]}(x, q, a) \ll_{\delta} \frac{x}{\Phi(q) \log x}.$$

Proof. The proof is a particular case of Theorem 4 in [10]. □

Let $\mathrm{Li}(x)$ denote the integral $\int_2^x \frac{dt}{\log t}$. The following theorem is the generalization of Bombieri-Vinogradov theorem to the ring of Gaussian integers.

Theorem 1.3.10 (Bombieri-Vinogradov) *For every $A > 0$, there is a $B > 0$ such that*

$$\sum_{\mathrm{N}(q) \leq Q} \left| \pi_{\mathbb{Q}[i]}(x, q, -1) - \frac{4\,\mathrm{Li}(x)}{\Phi(q)} \right| \leq C(A) \frac{x}{(\log Q)^A} \quad (Q \to \infty), \tag{1.2}$$

provided $Q = x^{1/2} \log^B x$.

The proof of the above theorem follows easily from the Corollary of Theorem 3 in [11]. As an immediate consequence of the above theorem, we have the following corollary.

Corollary 1.3.11 *Let $x = Q^2 (\log Q)^B$. Then*

$$\sum_{Q \leq \mathrm{N}(q) \leq 2Q} \left| \pi_{\mathbb{Q}[i]}(x, q, -1) - \frac{4\,\mathrm{Li}(x)}{\Phi(q)} \right| \leq C(A) \frac{x}{(\log Q)^A} \quad (Q \to \infty).$$

1.4 Proof of the main result

In this section, we give the proof of Theorem 1.2.1. The method used here in the proof is similar to that of Kátai in [13].

Let Q_0 be a large constant and $Q_l = 2^l \cdot Q_0$, for $l \geq 1$. Let

$$R_0 = \{z \in \mathbb{C} \mid 1 \leq |z|^2 \leq Q_0\}$$

and

$$R_l = \{z \in \mathbb{C} \mid Q_{l-1} \le |z|^2 \le Q_l\} \ (l \ge 1).$$

Let $\mathscr{S}_l$ be the set of Gaussian primes in the annulus R_l defined by inductively as follows:

Let $\mathscr{S}_0$ be the empty set. Assume that $\mathscr{S}_0, \mathscr{S}_1, \dots, \mathscr{S}_{m-1}$ are defined. Then $\mathscr{S}_m$ is the set of those Gaussian primes $q \in R_m$ for which there exists no k and $p \in \mathcal{P}[i]$ satisfying the following condition:

(A) $p + 1 = kq$, where the prime factors of k in $\mathbb{Z}[i]$ are of norm smaller than Q_{m-1} and do not belong to the set $\bigcup_{i=0}^{m-1} \mathscr{S}_i$.

Let

$$\mathscr{T} \stackrel{\text{def}}{=} \bigcup_{i=0}^{\infty} \mathscr{S}_i.$$

Before proceeding further to prove the theorem, we first prove the following lemma which gives an upper bound for the set $\mathscr{S}_l$.

Lemma 1.4.1 *For sufficiently large Q_0, we have the following estimate:*

$$\mathrm{Card}(\mathscr{S}_l) < \frac{Q_l}{(\log Q_l)^3} \quad (l = 0, 1, 2, \dots). \tag{1.3}$$

Proof. We prove by induction on l. For $l = 0$, since $\mathscr{S}_0$ is empty set, the lemma follows trivially. Suppose the lemma is true for all l with $0 \le l \le m-1$. Now we apply Bombieri-Vinogradov theorem in the form of Corollary 1.3.11 with $Q = Q_{m-1}, A = 4, x = Q_{m-1}^2(\log Q_{m-1})^B$. Then we get a suitably large constant $B = B(A)$ such that

$$\sum_{Q_{m-1} \le \mathrm{N}(q) \le Q_m} \left| \pi_{\mathbb{Q}[i]}(x, q, -1) - \frac{4Li(x)}{\Phi(q)} \right| \le C \frac{x}{(\log Q_m)^4}. \tag{1.4}$$

Hence it follows that

$$\pi_{\mathbb{Q}[i]}(x,q,-1) > \frac{3}{4}\frac{x}{\Phi(q)\log x} \tag{1.5}$$

for all $q \in R_m$ except at most $\frac{Q_m}{(\log Q_m)^3}$ elements of R_m. Now we show that if a prime $q \in R_m$ satisfies the equation (1.5) then $q \notin \mathscr{S}_m$ and thus proving the lemma.

Let $\Pi_{\mathbb{Q}[i]}(x,q)$ denote the number of those $p \in \mathcal{P}[i]$ with $\mathrm{N}(p) \leq x$ for which the condition (**A**) holds. Let N_1 denote the number of Gaussian primes p with $\mathrm{N}(p) \leq x$ for which $p+1 = kq$ and k has at least one Gaussian prime factor in $\mathscr{T}$ with norm less than or equal to $Q_{m-1}^{1-\delta}$ and N_2 denote the number of $p \in \mathcal{P}[i]$, $\mathrm{N}(p) \leq x$ such that $p+1 = kq$, and k has a prime factor with norm greater than $Q_{m-1}^{1-\delta}$. Clearly,

$$\Pi_{\mathbb{Q}[i]}(x,q) \geq \pi_{\mathbb{Q}[i]}(x,q,-1) - N_1 - N_2.$$

Now, we obtain upper bounds for N_1 and N_2, which together with equation (1.5) will give a lower bound for $\Pi_{\mathbb{Q}[i]}(x,q)$.

First we estimate N_2. Suppose if a prime p is contributing to the sum N_2 then we can write $p+1 = jq_1q$. Here

$$\mathrm{N}(j) \leq \frac{x}{\mathrm{N}(q_1q)} \leq xQ_{m-1}^{-2+\delta} \leq x^{2\delta} \quad \text{(say)}. \tag{1.6}$$

By Corollary 1.3.7, we have

$$N_2 \leq \sum_{\mathrm{N}(j)\leq x^{2\delta}} N(x,jq)$$

$$\leq c_1 \frac{x}{\Phi(q)\log^2 x} \sum_{\mathrm{N}(j)\leq x^{2\delta}} \frac{1}{\Phi(j)}$$
$$\leq c_2\delta \frac{x}{\Phi(q)\log x}.$$

Choosing $\delta < \frac{\epsilon}{c_2}$ we get

$$N_2 < \epsilon \frac{x}{\Phi(q)\log x}. \tag{1.7}$$

Now, we estimate N_1. Clearly,

$$N_1 \leq \sum_{\substack{q'\in\mathscr{T} \\ \mathrm{N}(q')\leq Q_{m-1}^{1-\delta}}} \pi_{\mathbb{Q}[i]}(x, qq', -1).$$

By Lemma 1.3.9 and choosing Q_0 large enough that $Q_{m-2} > Q_{m-1}^{1-\delta}$, we have

$$N_1 \leq C(\delta)\frac{x}{\Phi(q)\log x} \sum_{Q_l\leq Q_{m-1}^{1-\delta}} \sum_{q'\in\mathscr{S}_l} \frac{1}{\mathrm{N}(q')}$$
$$\leq C'(\delta)\frac{x}{\Phi(q)\log x} \sum_{l\leq m-2} \frac{\mathrm{Card}(\mathscr{S}_l)}{Q_l}.$$

By the induction hypothesis, we have

$$N_1 \leq C'(\delta)\frac{x}{\Phi(q)\log x} \sum_{l=0}^{\infty} \frac{1}{(\log Q_0 + l)^3} < \epsilon' \frac{x}{\Phi(q)\log x}.$$

First we choose δ suitably small and thereafter choosing a sufficiently large Q_0, we get

$$N_1 + N_2 < \frac{1}{4}\cdot\frac{x}{\Phi(q)\log x}. \tag{1.8}$$

Hence, validity of (1.5) implies that

$$\Pi_{\mathbb{Q}[i]}(x, q) > \frac{3}{4}\frac{x}{\Phi(q)\log x} - \frac{1}{4}\frac{x}{\Phi(q)\log x} > 0.$$

Hence $q \notin \mathscr{S}_m$. This completes the inductive proof of our lemma. $\square$

Now we proceed with the proof of the Theorem 1.2.1. Let $f : \mathbb{Z}[i]^* \to \mathbb{C}$ be any completely additive function satisfying the conditions stated in the theorem with $K \geq Q_0$, where Q_0 is such a large constant as specified in Lemma 1.4.1.

Since, the function f is completely additive function, it suffices to show that f vanishes on all Gaussian primes. First we prove, by induction, that $f(q) = 0$ for all Gaussian primes $q \notin \mathscr{T}$. Since we assume that f is zero on all the Gaussian primes p with $\mathrm{N}(p) \leq K$ and $K \geq Q_0$, we get that $f(q) = 0$ for all $q \in R_0$, i.e. $q \notin \mathscr{S}_0$. Now suppose that

$$f(q) = 0 \text{ for all } q \in R_j,\ q \notin \mathscr{S}_j; \qquad \text{for } j \leq m-1. \tag{1.9}$$

Now, let $q \in R_m$ such that $q \notin \mathscr{S}_m$. Then by the construction of $\mathscr{S}_m$'s there exists a Gaussian prime p and a Gaussian integer such that $p+1 = kq$ with $k = p_1^{\alpha_1} \dots p_r^{\alpha_r},\ p_i \in \mathcal{P}[i],\ \mathrm{N}(p_i) < Q_{m-1},\ p_i \notin \bigcup_{j=0}^{m-1} \mathscr{S}_j;\ (1 \leq i \leq r)$.

Then by the induction hypothesis we have

$$f(k) = f(p_1^{\alpha_1} \dots p_r^{\alpha_r}) = \alpha_1 f(p_1) + \dots + \alpha_r f(p_r) = 0.$$

Hence,

$$f(q) = f(p+1) - f(k) = 0.$$

This proves (1.9) for $j = m$ and hence by induction we get that $f(q) = 0$ for all $q \notin \mathscr{T}$.

Finally we prove that $f(q) = 0$ for all Gaussian primes $q \in \mathscr{T}$. Let $P_{\mathbb{Q}[i]}(y, q)$ denote the number of those Gaussian primes p with $\mathrm{N}(p) \leq y$ for which $p+1 = kq$

and the Gaussian prime factors of k do not belong to $\mathscr{T}$. If

$$P_{\mathbb{Q}[i]}(y, q) > 0$$

for sufficiently large y, then there exits a Gaussian prime p with $N(p) \le y$ for which $p + 1 = kq$ such that Gaussian prime factors of k are not in $\mathscr{T}$. Then by previous case $f(k) = 0$ and hence $f(q) = 0$. Thus it suffices to show that for large y, $P_{\mathbb{Q}[i]}(y, q) > 0$. Clearly,

$$P_{\mathbb{Q}[i]}(y, q) \ge \pi_{\mathbb{Q}[i]}(y, q, -1) - \sum_{\substack{q' \in \mathscr{T} \\ \mathrm{N}(q') \le \frac{y}{\mathrm{N}(q)}}} \pi_{\mathbb{Q}[i]}(y, q'q, -1). \tag{1.10}$$

For sufficiently large y, by Lemma 1.3.8, we have

$$\pi_{\mathbb{Q}[i]}(y, q, -1) > \frac{1}{2} \frac{y}{\mathrm{N}(q) \log y}. \tag{1.11}$$

Now we give an upper bound for the last term on the right hand side of the inequality (1.10) which together with (1.11) gives a lower bound for $P_{\mathbb{Q}[i]}(y, q)$. We have,

$$\begin{aligned} \Sigma' &= \sum_{\substack{q' \in \mathscr{T} \\ \mathrm{N}(q') \le \frac{y}{\mathrm{N}(q)}}} \pi_{\mathbb{Q}[i]}(y, q'q, -1) \\ &\le \frac{Cy}{\mathrm{N}(q) \log y} \sum_{\substack{q' \in \mathscr{T} \\ \mathrm{N}(q') \le y^{\frac{1}{2}}}} \frac{1}{\mathrm{N}(q')} + \frac{c'\pi y}{\mathrm{N}(q)} \sum_{\substack{q' \in \mathscr{T} \\ y^{\frac{1}{2}} \le \mathrm{N}(q') \le \frac{y}{\mathrm{N}(q)}}} \frac{1}{\mathrm{N}(q')} \qquad \text{(Lemma 1.3.9)}. \end{aligned}$$

Now, for sufficiently large Q_0, we have

$$\sum_{\substack{q'\in\mathscr{T}\\ \mathrm{N}(q')\le y^{\frac{1}{2}}}} \frac{1}{\mathrm{N}(q')} < \epsilon.$$

Also by Lemma 1.4.1,

$$\sum_{\substack{q'\in\mathscr{T}\\ y^{\frac{1}{2}}\le \mathrm{N}(q')\le \frac{y}{\mathrm{N}(q)}}} \frac{1}{\mathrm{N}(q')} \le \log y \max_{Q_l\ge y^{\frac{1}{2}}} \sum_{q'\in\mathscr{S}_l} \frac{1}{\mathrm{N}(q')} \le \frac{1}{\log^2 y}.$$

Thus, we have

$$\Sigma' < \frac{1}{4}\frac{y}{\mathrm{N}(q)\log y}.$$

Hence, $P_{\mathbb{Q}[i]}(y,q) > 0$ follows from equations (1.10) and (1.11). This completes the proof of the theorem.

Remark 1.4.2 In Theorem 1.2.1, we proved that the set of Gaussian primes shifted by 1 is a set of quasi-uniqueness. However, there is no significance of the shift 1 in this case. By choosing the numerical constant K, specified in the theorem, suitably large, one can see that the result also holds for any shift $a \in \mathbb{Z}[i]$. Thus, the set of shifted Gaussian primes $\mathcal{P}[i] + a$, where $a \in \mathbb{Z}[i]$ is a set of quasi-uniqueness for completely additive functions over $\mathbb{Z}[i]^*$.

1.5 Set of uniqueness of shifted Gaussian primes

Kátai not only proved that the set $\mathcal{P}+1$ is a set of quasi-uniqueness [12] but he also conjectured that, in fact, it is a set of uniqueness for completely additive functions over $\mathbb{N}$. Later this conjecture was completely settled by P. D. T. A. Elliott [6] who proved a more general result. More precisely, Elliott proved the following:

Theorem 1.5.1 (Elliott) *Let f be an additive function on $\mathbb{N}$ such that $f(p+1) = \text{constant}$ for all sufficiently large primes p. Then $f(2^\nu) = \text{constant}$ for all integers $\nu \geq 1$, and f is zero on all other prime powers.*

The conjecture of Kátai can be proved by taking the constant to be zero in the above theorem. Thus, the set $\mathcal{P}+1$ is a set of uniqueness. Naturally, the question arises that whether the set of Gaussian primes shifted by 1 is a set of uniqueness for $\mathcal{A}^*_{\mathbb{Z}[i]}$ or not. It turns out that it is in fact true. Evidently, the method of Elliott can be extended to the case of Gaussian integers to prove that the set of $\mathcal{P}[i]+1$ is a set of uniqueness for (completely) additive functions over non-zero Gaussian integers. This result is also proved in a joint work with G. K. Viswanadham [19]. More precisely, we proved the following:

Theorem 1.5.2 *Every complex valued additive function on $\mathbb{Z}[i]^*$ that vanishes on the complement of a finite subset of $\mathcal{P}[i]+1$ in fact vanishes on all of $\mathbb{Z}[i]^*$.*

It follows from the above theorem that the set $\mathcal{P}[i]+1$ is, in fact, a set of uniqueness for additive functions and hence for completely additive functions. It is a stronger version of Theorem 1.2.1. It can be proved using sieve techniques in the case of Gaussian integers. The detailed description and proof of Theorem 1.5.2 can be found in the doctoral thesis of G. K. Viswanadham [23].

1.5.1 Shifted-prime factorization

An immediate consequence of Theorem 1.5.1 and Theorem 1.2.2 is a very interesting result that every positive integer n may be expressed in the following form:

$$n = \prod_{i=1}^{m} (p_i + 1)^{r_i},$$

where the p_i's are primes and the exponents r_i's are rationals. K. H. Rosen [21] called such representations as *shifted-prime factorizations.* There are many interesting results, properties and conjectures related to shifted-prime factorizations. One property is that, unlike prime factorization, shifted-prime factorization may not be unique. For example,

$$2 = (3 + 1)^{\frac{1}{2}} = (31 + 1)^{\frac{1}{5}} = (31 + 1)^{\frac{3}{7}} (7 + 1)^{\frac{-8}{21}}.$$

The same holds true in the case of Gaussian integers also. As an application of Theorem 1.5.2 and Theorem 1.2.3 together, we have the following corollary:

Corollary 1.5.3 *Every non-zero Gaussian integer can be expressed as a finite product of rational powers of Gaussian primes shifted by* 1*, i.e. every non-zero Gaussian integer* α *may be written in the following form:*

$$\alpha = \prod_{j=1}^{k} (\pi_j + 1)^{l_j},$$

where $\pi_j \in \mathcal{P}[i]$ *and* $l_j \in \mathbb{Q}$*.*

As in the case of rational primes, the shifted-Gaussian-prime factorization also may not be unique. For example,

$$1 + i = ((-1 + 2i) + 1)^{\frac{1}{2}} = ((-3 + 2i) + 1)^{\frac{1}{3}} = ((-5 - 4i) + 1)^{\frac{1}{5}}.$$

Observe that, $(-1 + 2i)$, $(-3 + 2i)$ and $(-5 - 4i)$ above are Gaussian primes of the second category as described in Theorem 1.3.4.

To conclude this chapter, in the next section, we recall the well-known notion of the set of uniqueness modulo 1 and make some remarks.

1.6 Set of uniqueness modulo 1

The notion of the set of uniqueness modulo 1 was defined and characterized by many mathematicians like Dress and Volkmann and independently by K. H. Indlekofer, by P. D. T. A. Elliott, by P. Hoffman and others. It is defined as follows:

Definition 1.6.1 A set $A \subset \mathbb{N}$ is a *set of uniqueness modulo* 1 if $f(n) \equiv 0 \pmod 1$ for every $n \in A$ implies $f(n) \equiv 0 \pmod 1$ for every $n \in \mathbb{N}$, for all completely additive functions $f : \mathbb{N} \to \mathbb{R}$. In other words,

$$f(A) \subset \mathbb{N} \Rightarrow f(\mathbb{N}) \subset \mathbb{N} \quad (\forall\ f \in \mathcal{A}^*).$$

Clearly, the set of rational primes is an example of set of uniqueness modulo 1.

Example 1.6.1 It was proved by Fehér, Indlekofer and Timofeev that the set

$$E = \{x^2 + y^2 + a\ |x, y \in \mathbb{Z}\}$$

is a set of uniqueness modulo 1 if a is a sum of two squares.

Example 1.6.2 Let p_1, p_2, p_3 be three distinct rational primes. Then the set

$$A = \{p_1^3, p_1^2 p_2, p_1^2 p_3\} \cup (\mathcal{P} \setminus \{p_1, p_2, p_3\})$$

is a set of uniqueness but not a set of uniqueness mod 1.

J. Mehta and G. K. Viswanadham in [18] extended it to the case of Gaussian integers. It is defined as follows:

Definition 1.6.2 A set $A \subset \mathbb{Z}[i]^*$ is a set of uniqueness mod 1 with respect to Gaussian integers if $f(\alpha) \equiv 0 \pmod 1$ for every $\alpha \in A$ implies $f(\gamma) \equiv 0 \pmod 1$ for every $\gamma \in \mathbb{Z}[i]^*$, for all completely additive functions over $\mathbb{Z}[i]^*$. In other words,

$$f(A) \subset \mathbb{Z}[i] \Rightarrow f(\mathbb{Z}[i]^*) \subset \mathbb{Z}[i] \quad (\forall\, f \in \mathcal{A}^*_{\mathbb{Z}[i]}).$$

1.7 Some remarks

Remark 1.7.1 The same argument as in the proof of Theorem 1.2.1 can be used to prove that the set of Gaussian prime plus ones' together with a finite set of Gaussian primes with norm not exceeding some constant K, as in Theorem 1.2.1, is a set of uniqueness mod 1 with respect to Gaussian integers.

Remark 1.7.2 In the case of rational primes, Kátai [13] implicitly proved that there exists finitely many rational primes $q_1, \ldots, q_r$ such that the set $(\mathcal{P} + 1) \cup \{q_1, \ldots, q_r\}$ is a set of uniqueness modulo 1. As mentioned in Remark 1.7.1, here we proved its analogue in the case of $\mathbb{Z}[i]$. However, it is not known whether or not $\mathcal{P} + 1$ is a set of uniqueness modulo 1. It would be very interesting to see if $\mathcal{P}[i] + 1$ is a set of uniqueness modulo 1 or not.

Remark 1.7.3 Many other interesting results related to behaviour of arithmetical functions at shifted primes can be found in the literature, for example Hildebrand [9], Elliott [7, 8], Wirsing [24], etc.

CHAPTER 2

Arithmetical functions having constant values in some domain

In this chapter we consider completely additive complex valued functions over a finite union of lattices, in the complex plane, which is closed under multiplication. We show that such a function having constant values in some family of discs in $\mathbb{C}$ is identically zero function. This work was dedicated to Prof. Imre Kátai on his 75th birthday.

2.1 Introduction

In 1969, I. Kátai [14] proved that any completely additive real valued function on the set of positive integers assuming constant values in some sequence of intervals of real line is identically zero function. In 1991, M. Amer generalized this result for complex valued functions on the set of non-zero Gaussian integers. Amer [1] proved that a completely additive complex valued functions defined on the set of non-zero Gaussian integers having constant values in some sequence of discs of the complex plane is identically zero. The specific property of this sequence of discs is that their centres keep going away and away from origin and their radii also keep increasing.

Naturally, one might like to ask whether the result of Amer can be generalized to completely additive functions over an arbitrary lattice. The aim is to generalize this result for completely additive complex valued functions over non-zero lattice points in complex plane. The proof of Amer uses the 'closure under multiplication' property of $\mathbb{Z}[i]$. However, in general, an arbitrary lattice Λ_1 may not be closed under multiplication. So in order to make our domain closed under multiplication we supplement the lattice Λ_1 with some additional points in such a way that this larger collection of points, which may no longer be a lattice, is still discrete and is closed under multiplication. Such a system of points is called '*principal configuration*' (see [4]). These additional points form finitely many lattices $\Lambda_2, \Lambda_3, \ldots \Lambda_h$ having only the origin O in common with the lattice Λ_1 and with each other lattices. Let Γ denote the principal configuration. Then we have,

$$\Gamma = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_h.$$

Though each lattices Λ_i are closed under addition, Γ may not be closed under addition but it is closed under multiplication. The supplementary points (i.e. the points of $\Lambda_i,\ 2 \leq i \leq h$) are called *auxiliary* or *non-principal.* The principal configuration has many other properties. The geometric picture of above described situation in 2-dimensional case was first given by Klein in 1896 but it can be considered in any dimension.

In [17], we consider completely additive complex valued functions over non-zero points of the principal configuration Γ. We prove that if such a function assumes constant values in a certain sequence of discs then it vanishes on the whole of the domain (see Theorem 2.2.1 of Section 2.2). A similar result can be proved in the same way for completely multiplicative functions (Theorem 2.3.4).

In case the given lattice Λ_1 is already closed under multiplication, we have $h = 1$ and our principal configuration Γ will be the lattice Λ_1 itself. Then, as a particular case, Amer's result follows as the set of non-zero Gaussian integers is a lattice and is closed under multiplication. As a corollary, one can also consider the case of functions over the ring of integers of imaginary quadratic fields.

2.2 The main result

Let Γ be the principal configuration and let $\Lambda_i = \Lambda_i(\omega_i, \omega_i')$ where $\omega_i, \omega_i' \in \mathbb{C}$, $i = 1, 2, \ldots, h$ be the lattices in the principal configuration Γ, i.e. we have

$$\Gamma = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_h.$$

Without loss of generality, we assume that $|\omega_i'| \leq |\omega_i|$ for all $i = 1, 2, \ldots h$. Let Γ^* denote the set of all non-zero points of Γ. Let $\mathcal{A}_\Gamma^*$ and $\mathcal{M}_\Gamma^*$ denote the family of all completely additive functions and completely multiplicative complex valued functions over Γ^* respectively. Let $S(a, r) (\subseteq \mathbb{C})$ be the closed disc with center a and radius r, i.e.

$$S(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Let $\omega' = \omega_j'$ and $\omega = \omega_k$ for some j and k such that

$$|\omega_j'| = \min_{1 \leq i \leq h} \{|\omega_i'|\}; \quad |\omega_k| = \max_{1 \leq i \leq h} \{|\omega_i|\}. \tag{2.1}$$

We have the following result:

Theorem 2.2.1 *Let $f \in \mathcal{A}_\Gamma^*$. Assume that there exists a sequence of complex numbers $z_1, z_2, \ldots$ such that $|z_j| \to \infty$ $(j \to \infty)$. If*

$$f(\alpha) = A_j \quad \text{for all } \alpha \in S\big(z_j, (2+\epsilon)\sqrt{|z_j|}\big),$$

where A_j's are constants and ϵ is a positive constant depending on the principal configuration Γ, then f vanishes on the whole of domain Γ.

2.3 Proof of the main result

In this section we give the proof of Theorem 2.2.1. It follows easily from the course of the following three lemmas. The idea of the proof is similar to the proof of Amer in which case there was only one lattice $\mathbb{Z}[i]$. In our case the domain is a finite union of lattices. In the first lemma we show that f is zero in some disc centred at origin. The following is the lemma:

Lemma 2.3.1 *Let $f \in \mathcal{A}_\Gamma^*$, $z \in \Gamma^*$ with $|z| = M$ such that $M \geq |\omega_j| + |\omega_j'|$. Assume that $f(\alpha) = A$ in the annulus $R = \{\alpha \in \Gamma^* \mid M \leq |\alpha| \leq |\omega_j + \omega_j'|M\}$, where A is some constant. Then f vanishes in the whole disc with radius same as the outer radius of the annulus R, i.e. $f(\alpha) = 0$ for every $\alpha \in \Gamma^*$ with $|\alpha| \leq |\omega_j + \omega_j'|M$.*

Proof. First we prove that the constant A is zero. Let $\lambda = \omega_j + \omega_j' \in \Lambda_j$. Since Γ^* is closed under multiplication, $\lambda z \in \Gamma^*$. We have

$$|\lambda z| = |\lambda||z| = |\omega_j + \omega_j'|M.$$

Therefore, $z, \lambda z \in R$. Since f is completely additive function, we have

$$f(\lambda) = f(\lambda z) - f(z) = 0. \tag{2.2}$$

Let $\lambda^k \in R$ for some $k \in \mathbb{N}$. From the construction of the annulus R it is clear that such k exists. Then $f(\lambda^k) = A$. But

$$f(\lambda^k) = kf(\lambda) = 0.$$

Thus, $A = 0$. Now, it remains to prove that f vanishes in the disc of radius M centred at origin. Let $\alpha \in \Gamma^*$ with $|\alpha| < M$. Choose a suitable positive integer m such that $\alpha\lambda^m \in R$. Then $f(\alpha\lambda^m) = 0$. Consequently, f being completely additive and by (2.2), we have $f(\alpha) = 0$. $\square$

For $r \in \mathbb{R}$, $r > 1$, let

$$[r]_\Gamma = \max_{\substack{\alpha\in\Gamma^* \\ |\alpha|\le r}} |\alpha|,$$

Clearly $[r]_\Gamma \le r$. Let

$$R_N = \{\alpha \in \Gamma^* \mid N \le |\alpha| \le (1+\delta)N\},$$

where $N \in \mathbb{R}$ and δ is an arbitrary number with $0 < \delta < 1$. Then we have the following lemma:

Lemma 2.3.2 *Let $0 < \delta < 1$ be an arbitrary number and let ω, ω' be as in (2.1). There exists a constant N_0 depending on δ and Γ such that if $f \in \mathcal{A}_\Gamma^*$, $N \in \mathbb{R}$ with $N > N_0$ and*

$$f(\alpha) = A \quad \textit{for } \alpha \in R_N,$$

then $f(\alpha)=0$ *for each* $\alpha\in\Gamma^*$ *with* $|\alpha|\leq\left[\frac{\delta N}{2|\omega'|}-|\omega|\right]_\Gamma$.

Proof. Let $\beta\in\Gamma^*$ be such that $|\beta|<N$. Then $\beta\in\Lambda_i^*\subset\Gamma^*$ for some i. Let $E_i=\{\pm\omega_i,\pm\omega_i',0\}$ and $\theta\in E_i$. If

$$\frac{N(1+\delta)}{|\beta|+|\omega|}-\frac{N}{|\beta|-|\omega|}\geq|\omega'| \tag{2.3}$$

holds, then

$$\frac{N(1+\delta)}{|\beta|+|\omega_i|}-\frac{N}{|\beta|-|\omega_i|}\geq|\omega_i'|$$

holds. Then there exists some $\mu\in\Gamma^*$ such that

$$\frac{N}{|\beta|-|\omega_i|}\leq|\mu|\leq\frac{N(1+\delta)}{|\beta|+|\omega_i|}.$$

But then

$$\frac{N}{|\beta+\theta|}\leq|\mu|\leq\frac{N(1+\delta)}{|\beta+\theta|},$$

and so

$$N\leq|(\beta+\theta)\mu|\leq N(1+\delta).$$

This implies $(\beta+\theta)\mu\in R_N$. Since $\theta=0\in E$, we have

$$f(\mu\beta)=A=f\big((\beta+\theta)\mu\big).$$

Since f is completely additive function, we have

$$f(\beta)=f(\beta+\theta).$$

This is true for all $i=1,\ldots,h$. Thus, we have proved that f is constant for all such $\beta\in\Gamma^*$ for which (2.3) holds. Now, we determine for what β (2.3) holds.

Equation (2.3) implies

$$|\omega'||\beta|^2 - N\delta|\beta| + N\delta|\omega| + 2N|\omega| - |\omega'||\omega|^2 \leq 0. \tag{2.4}$$

We shall prove that inequality (2.3) holds for β if

$$|\beta| \in L := \left[|\omega|\left(\frac{4}{\delta}+1\right), \frac{\delta N}{2|\omega'|} - |\omega|\right].$$

As (2.4) is a quadratic in $|\beta|$, it suffices to show that (2.3) holds for the end-points of the interval L.

First let $|\beta| = |\omega|\left(\frac{4}{\delta}+1\right)$. Substituting this value of $|\beta|$ in (2.3), the left hand side of (2.3) will be

$$\frac{\delta^2 N}{|\omega|(4\delta+8)}$$

which is clearly $\geq |\omega'|$ if

$$N \geq |\omega||\omega'|\left(\frac{4}{\delta}+\frac{8}{\delta^2}\right).$$

Now, let $|\beta| = \frac{\delta N}{2|\omega'|} - |\omega|$. Substituting this value of $|\beta|$ in (2.3), the left hand side of (2.3) will be

$$\frac{2\delta^2 N|\omega'| - 8\delta|\omega||\omega'|^2 - 8|\omega||\omega'|^2}{\delta^2 N - 4\delta|\omega||\omega'|}$$

and this is $\geq |\omega'|$ if

$$N \geq |\omega||\omega'|\left(\frac{4}{\delta}+\frac{8}{\delta^2}\right).$$

Hence, we choose the constant N_0 to be $|\omega||\omega'|\left(\frac{4}{\delta}+\frac{8}{\delta^2}\right)$. Thus we have

$$f(\beta) = f(\beta+\theta),$$

whenever $|\beta| \in L$, $\beta \in \Gamma^*$ and $N \geq N_0$. This means that f is constant in the annulus $|\beta| \in L$. Clearly we have

$$\left[\frac{\delta N}{2|\omega'|} - |\omega|\right]_\Gamma \geq |\omega| + |\omega'| \left[|\omega| \left(\frac{4}{\delta} + 1\right)\right]_\Gamma$$

for sufficiently large N, since the left hand side contains N and right hand side is independent of N. Then we can find some $z \in \Gamma^*$ such that

$$(|z|, |\omega_j + \omega'_j||z|) \subseteq (|z|, |\omega| + |\omega'||z|) \subseteq L.$$

Applying Lemma 2.3.1 for $|z|$ in place of M, we get $f(\alpha) = 0$ whenever $|\alpha| \leq |\omega_j + \omega'_j||z|$. But $f(\alpha)$ is constant whenever $|\alpha| \in L$ and hence $f(\alpha) = 0$ for $\alpha \in \Gamma^*$ with $|\alpha| \leq \left[\frac{\delta N}{2|\omega'|} - |\omega|\right]_\Gamma$. $\square$

Next we prove that if f assumes constant value in some disc $S(a, r)$ as stated in Theorem 2.2.1 then f vanishes on a disc centred at origin and radius depending on a. The lemma is the following:

Lemma 2.3.3 *Let $\epsilon > 0$ be some fixed constant depending on Γ. Then there exists positive numbers N_1 and c, depending on ϵ and Γ such that if $f \in \mathcal{A}^*_\Gamma$, $a \in \mathbb{C}$ with $|a| > N_1$, $r = (2 + \epsilon)\sqrt{|a|}$ and $f(\alpha) = A$ (constant) in the disc $S(a, r)$, then f vanishes at all $\alpha \in \Gamma^*$ with $|\alpha| \leq c\sqrt{|a|}$.*

Proof. Clearly, the disc $S(\alpha, |\omega|)$ contains at least one element of Γ for any $\alpha \in \mathbb{C}$. Let $\beta \in \Gamma^*$ such that $|\beta| \leq \frac{r}{|\omega|}$. Assume that $\beta \in \Lambda^*_i$ for some i. Then there exists some $\mu \in \Gamma^*$ such that

$$\left|\mu - \frac{a}{\beta}\right| \leq |\omega| \leq \frac{r}{|\beta|}. \tag{2.5}$$

Let $E_i = \{\pm\omega_i, \pm\omega_i'\}$ and $\theta \in E_i$ and assume that β is so chosen that the following equation holds:

$$|\omega| + \frac{|a||\theta|}{|\beta||\beta+\theta|} \leq \frac{r}{|\beta+\theta|}. \tag{2.6}$$

Then

$$\begin{aligned}\left|\mu - \frac{a}{\beta+\theta}\right| &\leq \left|\mu - \frac{a}{\beta}\right| + \left|\frac{a}{\beta} - \frac{a}{\beta+\theta}\right| \\ &\leq |\omega| + \frac{|a||\theta|}{|\beta||\beta+\theta|} \leq \frac{r}{|\beta+\theta|}.\end{aligned} \tag{2.7}$$

As a consequence of (2.5) and (2.7), we have $\mu\beta, \mu(\beta+\theta) \in S(a,r)$. By the assumption of the lemma that f assumes constant value in $S(a,r)$, we have

$$f(\mu\beta) = f\big(\mu(\beta+\theta)\big) = A.$$

Since f is completely additive function, we have $f(\beta) = f(\beta+\theta)$ provided that (2.6) holds. This is true for all i, $1 \leq i \leq h$. Thus, we have proved that f is constant for all $\beta \in \Gamma^*$ for which (2.6) holds.

Now, we determine that for what $\beta \in \Gamma^*$, (2.6) holds. By simple computation, we see that (2.6) holds for each $\theta \in E_i$ for each i if

$$|\beta||\omega|(|\beta| + |\omega|) - r|\beta| + |a||\omega| \leq 0.$$

The above inequality holds in the interval $|\beta| \in (x_1, x_2)$ where x_1 and x_2 are the roots of the following quadratic equation:

$$|\omega|x^2 - (r - |\omega|^2)x + |a||\omega| = 0,$$

where $|a|$ is sufficiently large and

$$\epsilon \geq 2(|\omega| - 1).$$

Then we have

$$x_1 = \frac{(r - |\omega|^2) - \sqrt{(r - |\omega|^2)^2 - 4|a||\omega|^2}}{2|\omega|}$$

and

$$x_2 = \frac{(r - |\omega|^2) + \sqrt{(r - |\omega|^2)^2 - 4|a||\omega|^2}}{2|\omega|}.$$

We note that $x_1(|a|) \to \infty$ as $|a| \to \infty$ and that

$$\begin{aligned} \frac{2|\omega|x_1}{r} &= 1 - \sqrt{\left(1 - \frac{|\omega|^2}{r}\right)^2 - \frac{4|a||\omega|^2}{r^2}} - \frac{|\omega|}{r} \\ &= 1 - \sqrt{1 - \left(\frac{2|\omega|}{2+\epsilon}\right)^2} + O\left(\frac{1}{r}\right). \end{aligned}$$

Similarly,

$$\frac{2|\omega|x_2}{r} = 1 + \sqrt{1 - \left(\frac{2|\omega|}{2+\epsilon}\right)^2} + O\left(\frac{1}{r}\right).$$

So, we have

$$\frac{x_2}{x_1} \to \frac{1 + \sqrt{1 - \left(\frac{2|\omega|}{2+\epsilon}\right)^2}}{1 - \sqrt{1 - \left(\frac{2|\omega|}{2+\epsilon}\right)^2}} = b \text{ (say) as } \ |a| \to \infty.$$

Clearly $b > 1$ and so we take $\delta < b - 1$. Then we have

$$\frac{x_2(|a|)}{x_1(|a|)} > 1 + \delta$$

when $|a|$ is sufficiently large enough. Choosing $N = x_1(|a|)$, the conditions of Lemma 2.3.2 are satisfied. Since, $x_1(|a|)$ has the same order as $\sqrt{|a|}$ the lemma follows immediately. $\square$

Now we give the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. The proof follows easily from Lemma 2.3.3. Assume that $f \in \mathcal{A}_\Gamma^*$ and that there exists a sequence $z_1, z_2, \ldots$ of complex numbers such that $|z_j| \to \infty$ and

$$f(\alpha) = A_j \text{ (constant)} \quad \text{ for all } \alpha \in S\big(z_j, (2+\epsilon)\sqrt{|z_j|}\big)$$

for some arbitrary positive constant ϵ. Applying Lemma 2.3.3 here, we get $f(\alpha) = 0$ for each α with $|\alpha| \le c_j\sqrt{|z_j|}$ for some positive constants c_j. But $|z_j| \to \infty$ $(j \to \infty)$. As a consequence, we have $f \equiv 0$. $\square$

A similar assertion can be proved for completely multiplicative complex valued functions over Γ^*. The following is the theorem:

Theorem 2.3.4 *Let $f \in \mathcal{M}_\Gamma^*$, which does not vanish anywhere. Assume that there exists a sequence $z_1, z_2, \ldots$ of complex numbers such that $|z_j| \to \infty$ and that*

$$f(\alpha) = A_j \text{(constant)} \quad \text{ for all } \alpha \in S\big(z_j, (2+\epsilon)\sqrt{|z_j|}\big)$$

for some arbitrary positive constant ϵ depending on Γ. Then $f \equiv 1$.

Corollary 2.3.5 *Taking $\Gamma = \mathbb{Z}[i]$, we have Amer's result. In general, one can take Γ to be the ring of integers of imaginary quadratic field.*

CHAPTER 3

Arithmetical functions with nearly Gaussian integer values

A complex valued function defined over the set of positive integers is said to have nearly Gaussian integer values if its values approach a Gaussian integer. In this chapter we characterize completely multiplicative complex valued functions with nearly Gaussian integer values. This is based on a joint work with K. Chakraborty [3] and was dedicated to Prof. Bui Minh Phong on his 60th birthday.

3.1 Introduction

Definition 3.1.1 A function $f : \mathbb{N} \to \mathbb{R}$ is said to be multiplicative if

$$f(mn) = f(m)f(n) \quad (\forall\ m, n \in \mathbb{N},\ (m, n) = 1)$$

and is said to be completely multiplicative if

$$f(mn) = f(m)f(n) \quad (\forall\ m, n \in \mathbb{N}).$$

Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the family of all multiplicative and completely multiplicative functions respectively. For any real number x, let $\|x\|$ denote its distance from the nearest integer, i.e.

$$\|x\| = \min_{n\in\mathbb{N}} |x-n|.$$

A function $f : \mathbb{N} \to \mathbb{R}$ is said to have nearly integer values if

$$\|f(n)\| \to 0 \quad (n \to \infty).$$

Definition 3.1.2 We say that a real number θ is a Pisot-number if it is an algebraic integer > 1, and if all its Galois conjugates $\theta_2, \ldots, \theta_r$, are in the domain $|z| < 1$.

Example 3.1.1 The *golden ratio*, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$, is a real quadratic integer greater than 1, while the absolute value of its conjugate, $-\phi^{-1} = \frac{1-\sqrt{5}}{2} \approx -0.618$, is less than 1. Its minimal polynomial is $x^2 - x - 1$.

Example 3.1.2 The smallest Pisot number (by Salem in 1944) is the positive root of $x^3 - x - 1$ known as the *plastic constant*.

A well known characteristic property of Pisot numbers is that their powers approach integers. A Pisot-number θ satisfies (see [2]) the following relation:

$$\|\theta^n\| \to 0 \quad (n \to \infty).$$

I. Kátai and B. Kovács [15] determined the class of completely multiplicative real valued functions with nearly integer values. Such functions satisfying some specific properties have values which are Pisot-numbers.

3.1.1 Functions with nearly Gaussian integer values

In this chapter, we consider completely multiplicative functions $f : \mathbb{N} \to \mathbb{C}$. For any complex number z, we denote by $\|z\|$ its distance from the nearest Gaussian integer, i.e.

$$\|z\| = \min_{\gamma \in \mathbb{Z}[i]} |z - \gamma|.$$

K. Chakraborty and J. Mehta [3] determined the class of completely multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ with nearly Gaussian integer values, i.e. functions satisfying the condition

$$\|f(n)\| \to 0 \quad (n \to \infty). \tag{3.1}$$

In the next couple of sections we give the characterization of such functions and prove the main result of [3].

3.2 The main result

The definition of Pisot number can be evidently extended to the case of Gaussian integers. We define a generalized Pisot number as follows:

Definition 3.2.1 We say that a complex number θ is a generalized Pisot number with respect to Gaussian integers if there exists a polynomial $\phi(z) \in \mathbb{Z}[i][z]$ with leading coefficient 1, and $\phi(z) = \prod_{j=1}^{r}(z - \theta_j),\ \theta_1 = \theta,\ |\theta| > 1$ and all the conjugates, $\theta_2, \ldots, \theta_r$ are in the domain $|z| < 1$.

We shall call a generalized Pisot number with respect to $\mathbb{Z}[i]$ as a *Gaussian Pisot number*.

Example 3.2.1 Naturally, any $\theta \in \mathbb{Z}[i]$ is a Gaussian Pisot number since

$$\phi(z) = z - \theta \in \mathbb{Z}[i][z].$$

Let β be an algebraic number and $f : \mathbb{N} \to \mathbb{C}$ be completely multiplicative function with its values $f(n) \in \mathbb{Q}(\beta)$. Let $\beta_2, \ldots, \beta_r$ be the conjugates of β (with respect to $\mathbb{Z}[i]$). Let $\phi_j(n)$ denote the conjugate of $f(n)$ defined by the substitution $\beta \to \beta_j$. Clearly, ϕ_j are completely multiplicative functions for $j = 2, \ldots, r$. Then, we have the following result:

Theorem 3.2.2 *Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function for which there exists at least one n_0 such that $|f(n_0)| > 1$. Let $\mathscr{P}_1$ denote the set of primes p for which $f(p) \neq 0$.*

If (3.1) holds, then the values $f(p) = \alpha_p$ are Gaussian Pisot numbers, for each $p_1, p_2 \in \mathscr{P}_1$ and we have $\mathbb{Q}(\alpha_{p_1}) = \mathbb{Q}(\alpha_{p_2})$. Let Θ denote one of the values α_p $(p \in \mathscr{P}_1)$, $\Theta_2, \ldots, \Theta_r$ its conjugates $(i = 2, \ldots, r)$ and $\phi_2(n), \ldots, \phi_r(n)$ be defined as above. Then

$$\phi_j(n) \to 0 \quad \text{as } n \to \infty \quad (2 \le j \le r). \tag{3.2}$$

Conversely assume that, for any prime p, either f vanishes at p or its value $f(p)$ is a Gaussian Pisot number from a given algebraic number field $\Omega(\Theta)$. If

$$\phi_j(p) \to 0 \quad \text{as } p \to \infty \quad (2 \le j \le r), \tag{3.3}$$

then (3.1) holds.

In the next section we give the proof of Theorem 3.2.2 but first we prove some requisite lemmas.

3.3 Proof of the main result

We first prove the following lemma from which the converse assertion of Theorem 3.2.2 follows immediately.

Lemma 3.3.1 *Let β be an algebraic number and f be a completely multiplicative function with its values $f(n)$ in $\mathbb{Z}[i](\beta)$. Let $\phi_j(n)$ $(2 \leq j \leq r)$ be the conjugates of $f(n)$ as defined above. Assume that $|\phi_j(p)| < 1$ and for $(2 \leq j \leq r)$*

$$\phi_j(p) \to 0 \quad (p \to \infty) \tag{3.4}$$

for all primes p. Then (3.1) holds.

Proof. Clearly, from (3.4) it follows that

$$\phi_j(n) \to 0 \quad (n \to \infty). \tag{3.5}$$

Further, since $\phi_j(n)$ are algebraic for all $j = 2, \ldots, r$, we have

$$f(n) + \phi_2(n) + \cdots + \phi_r(n) = G_n \in \mathbb{Z}[i].$$

Hence, by (3.5) we have

$$\|f(n)\| \to 0 \qquad (n \to \infty). \tag{3.6}$$

□

Now, we proceed to prove the remaining part of Theorem 3.2.2. First we state the following more general theorem due to I. Környei (Theorem 1 in [16]).

Theorem 3.3.2 *Let F be the field of rational numbers of an imaginary quadratic* *field. Let $\alpha_1 \ldots, \alpha_n$ be distinct algebraic numbers, $|\alpha_j| \geq 1$ $(j-1,\ldots,n)$, $p_1(x),\ldots,p_n(x)$ be non zero polynomials with complex coefficients. Then the relation*

$$\lim_{k\to\infty}\left\|\sum_{i=1}^{n} p_i(k)\alpha_i^k\right\| = 0$$

holds if and only if the following assertions are true:

a) *The numbers α_i are algebraic integers.*

b) *The coefficients of $p_i(x)$ are elements of the algebraic extension $F(\alpha_i)$.*

c) *If α_i and α_j are conjugate elements over F, and the corresponding polynomials have the form*

$$p_i(x) = \sum_{u=0}^{t_i} c_u^{(i)} x^u, \quad p_j(x) = \sum_{u=0}^{t_j} c_u^{(j)} x^u,$$

then p_i and p_j have the same degree, $c_u^{(i)}$ and $c_u^{(j)}$ are conjugate elements over F too, and for any such isomorphism τ which is the identical mapping on F and $\tau(\alpha_i) = \alpha_j$, the relations

$$\tau(c_u^{(i)}) = c_u^{(j)} \qquad (u = 0, 1, \ldots, t_i = t_j)$$

hold.

d) *All the conjugates of the α_i-s not occurring in the sum $\sum_{i=1}^{n} p_i(k)\alpha_i^k$ have absolute value less than one.*

e) *The sums*

$$\sum_{i=1}^{n}{}^{*}\, Tr(p_i(k)\alpha_i^k)$$

are algebraic integers in F for every large k ($Tr(\alpha)$ denotes the sum of conjugates of α over F). The asterisk in the sum denotes, that the summation is taken over non-conjugates α_i-s.

This assertion is a generalization of a theorem due to Pisot.

The proof of the above theorem can be found in [16]. As a corollary, we have the following lemma which is a generalization of Lemma 3 in [15].

Lemma 3.3.3 *Let α be an algebraic number with $|\alpha| > 1$, $\lambda \neq 0$ be a complex number and*

$$\|\lambda\alpha^n\| \to 0 \ (n \to \infty). \tag{3.7}$$

Then α is a Gaussian Pisot number and $\lambda \in \mathbb{Q}(\alpha)$.

Proof. The proof follows immediately from assertions a), b) and d) of Theorem 3.3.2. □

Lemma 3.3.4 *Let f be a completely multiplicative function for which (3.1) holds. If $|f(n_0)| > 1$ for atleast one $n_0 \in \mathbb{N}$, then for each values of n either $f(n) = 0$ or $|f(n)| \geq 1$.*

Proof. Assume, on the contrary, $0 < |f(m_0)| < 1$. Let $b = |f(n_0)|$, $a = |f(m_0)|$ and $x_0 = [-3\log a] + 1$. Then for infinitely many k, l pairs of positive integers we have,

$$\frac{-2x_0}{\log a} > k + l\,\frac{\log b}{\log a} > \frac{-x_0}{\log a},$$

since the length of the interval $\left(\frac{-x_0}{\log a}, \frac{-2x_0}{\log a}\right)$ is at least three. For such pairs k, l we have $2^{-2x_0} < a^k b^l < 2^{-x_0}$. Consequently

$$2^{-2x_0} < |f({m_0}^k {n_0}^l)| = |a^k b^l| < 2^{-x_0}.$$

This is a contradiction to the fact that f satisfies (3.1). Hence the lemma. □

Lemma 3.3.5 *Let f be a completely multiplicative function satisfying (3.1). Assume that there exists an $m \in \mathbb{N}$ for which $|f(m)| > 1$. Let $\mathscr{P}_1$ be the set of those primes at which the function f is non-vanishing. Then the values $f(p)$ are Gaussian Pisot numbers for each $p \in \mathscr{P}_1$. Also, for every $p_1, p_2 \in \mathscr{P}$, we have $\mathbb{Q}(\alpha_{p_1}) = \mathbb{Q}(\alpha_{p_2})$, where $\alpha_{p_1} = f(p_1), \alpha_{p_2} = f(p_2)$.*

Proof. Let $f(m) = \alpha$. Since $|\alpha| > 1$ and $\|f(m^k)\| = \|\alpha^k\| \to 0$ $(k \to \infty)$, by Lemma 3.3.3, we say that $f(m)$ is a Gaussian Pisot number.

Now, let n be an arbitrary natural number for which $f(n) \neq 0$. Since $\|f(nm^k)\| = \|f(n)\alpha^k\| \to 0$ $(k \to \infty)$, from Lemma 3.3.3, we deduce that $f(n) \in \mathbb{Q}(\alpha)$. Hence, $\beta = f(n) \in \mathbb{Q}(\alpha)$. Since $\beta \neq 0$, from Lemma 3.3.4 we get that $|\beta| > 1$, and by repeating the above argument for β, we deduce that β is a Gaussian Pisot number and $\alpha \in \mathbb{Q}(\beta)$. Hence, $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. □

Lemma 3.3.6 *Let f be a completely multiplicative function satisfying the relation*

$$\|f(n)\| \leq \varepsilon(n), \tag{3.8}$$

where $\varepsilon(n)$ is a monotonically decreasing function. Then there are following possibilities:

a) f takes values in $\mathbb{Z}[i]$ for every n.

b) For a suitable n, $0 < |f(n)| < 1$. Then $|f(n)| \to 0$ as $n \to \infty$.

c) For a suitable m, $|f(m)| > 1$. Let $\mathscr{P}_1$ denote the whole set of those primes p for which $f(p) \neq 0$. Then there exists a Gaussian Pisot number Θ such that $\mathbb{Q}(f(p)) = \mathbb{Q}(\Theta)$ for each $p \in \mathscr{P}_1$.

Proof. The relation (3.8) involves (3.7). If $0 < |f(n)| < 1$ then from Lemma 3.3.4 we have $|f(m)| \leq 1$ for every m. If $|f(m)| = 1$, then

$$\|f(nm^k)\| = \|f(n)\| \text{ as } k \to \infty.$$

This contradicts (3.1). Consequently, $|f(m)| < 1$ for each $m > 1$. Assume that there exists a subsequence $n_1 < n_2 < \ldots$ such that $f(n_j) \to 1$. Then

$$f(nn_j) \to f(n) \ (j \to \infty)$$

which is a contradiction to (3.1). Consequently, $f(m) \to 0$ as $m \to \infty$. The assertion (c) of the lemma is an immediate consequence of Lemma 3.3.5. □

All the requisite lemmas being stated and proved, now we give the proof of Theorem 3.2.2.

Proof of Theorem 3.2.2. Let f be a completely multiplicative function for which there exists at least one n_0 such that $|f(n_0)| > 1$. Let $\mathscr{P}_1$ denote the set of primes p for which $f(p) \neq 0$. Let us assume that (3.1) holds. Then Lemma 3.3.5 implies that the non-zero values of f are Gaussian Pisot numbers from a given algebraic number field $\Omega(\Theta)$. Hence, by definition $|\phi_j(n)| < 1$ $(2 \leq j \leq r)$. Now, consider the vector

$$\psi(n) = (\phi_2(n), \ldots, \phi_r(n)),$$

and let X denote the set of the limit points of $\psi(n)$ $(n \to \infty)$. Let $(x_2, \ldots, x_r)$ be in X. Since

$$f(n) + \phi_2(n) + \cdots + \phi_r(n) \in \mathbb{Z}[i]$$

and

$$\|f(n)\| \to 0 \quad (n \to \infty),$$

we get that $x_2 + \cdots + x_r$ is a Gaussian integer and $|x_j| \leq 1$. Let m_j be such a sequence for which

$$\psi(m_j) \to (x_2, \ldots, x_r).$$

Then for any $k \in \mathbb{N}$ clearly,

$$\psi({m_j}^k) \to ({x_2}^k, \ldots, {x_r}^k).$$

This implies that ${x_2}^k + \cdots + {x_r}^k \in \mathbb{Z}[i]$ for every $k \in \mathbb{N}$. This can happen only in the case when all x_j's are Gaussian integers. Since $|x_j| \leq 1$ $(2 \leq j \leq r)$, one concludes that $x_j \in \{0, 1, -1, i, -i\}$ for all j. Hence, either $x_j = 0$ or $|x_j| = 1$ for all j.

Now, fix $n \in \mathbb{N}$ such that f does not vanish at n. Then $\phi_j(n) \neq 0$ and $|\phi_j(n)| < 1$. Consequently,

$$\psi(nm_j) \to (\phi_2(n)x_2, \ldots, \phi_r(n)x_r) \in X.$$

For $2 \leq j \leq r$, let $y_j = x_j\phi_j(n)$. If $x_{j_0} \neq 0$ for some j_0, then

$$0 < |y_{j_0}| = |\phi_{j_0}(n)| < 1.$$

But $(y_2, \ldots, y_r)$ is an element of X for which there is a component y_{j_0} such that $|y_{j_0}| < 1$ and $y_{j_0} \neq 0$. This is not possible. As a result, we have

$$\phi_j(n) \to 0 \ \text{ as } n \to \infty, \quad (2 \leq j \leq r).$$

This completes the proof of the theorem. As mentioned earlier, the converse follows from Lemma 3.3.1. $\square$

Theorem 3.3.7 *Let f be a completely multiplicative complex valued function satisfying the condition (3.8). Let us assume that*

$$f(n) \nrightarrow 0$$

and that f takes on at least one value other than Gaussian integer. Then the first assertion in Theorem 3.2.2 holds.

Proof. The proof follows immediately from Lemma 3.3.6 and Theorem 3.2.2. $\square$

Remark 3.3.8 The result can be generalized for completely multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ with values that are nearly integers in imaginary quadratic fields, i.e. $\|f(n)\| \to 0$ $(n \to \infty)$, where

$$\|z\| = \min_{\gamma \in \mathcal{O}_K} |z - \gamma|$$

and $\mathcal{O}_K$ denotes the ring of integers of $K = \mathbb{Q}(\sqrt{d})$, $d < 0$ and d is square free.

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