NEXT TO NEXT TO LEADING ORDER CORRECTIONS IN TEV SCALE GRAVITY MODELS

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I, Maguni Mahakhud, hereby declare that the investigations presented in the thesis have been carried out by me. The matter embodied in the thesis is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

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List of Publications arising from the thesis

Journal

1. **"Quark and gluon spin-2 form factors to two loops in QCD" ,**
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2. **"Next-to-Next-to-Leading Order QCD Corrections in Models of TeV-Scale Gravity" ,**
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3. **"Two-Loop QCD Corrections to massive spin-2 resonance$\to 3$ gluons " ,**
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To my parents and my wife Mitali

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Synopsis

The recent discovery of the scalar boson (Higgs boson) by the ATLAS and the CMS collaborations at the Large Hadron Collider (LHC) establishes the Standrad Model as the most successful theory in particle physics which has been tested to a great accuracy.Yet, certain questions are unanswered by SM such as the existence of dark matter,massive neutrinos, the origin of baryon asymmetry in the Universe,the existence of multiple generations of fermions,the hierarchy in fermion masses and mixing,the stability of the Higgs sector under quantum corrctions,the hierarchy problem:why the electroweak scale (1 TeV) is much smaller than the Planck Scale (10^{19} GeV) etc.Over the years, several attempts of Beyond the Standard Model (BSM) has been made to answer these questions, although only with partial success.One such idea envisages a world in more than three space dimensions as a possible solution to some of the above issues.

While such proposals of spacetime more than three spatial dimensions date back to the 1920s,mainly through the work of Kaluza and Klein, in an attempt to unify electromagnetism and gravity, their efforts were failed.Although their initial idea failed, the formalism that they and others developed is still useful now-a-days.Around 1980, string theory proposed again to enlarge the number of space dimensions, this time as a requirement for describing a consistent theory of quantum gravity.The extra dimensions were supposed to be compactified at a scale close to the planck scale, and thus not testable experimentally in the near future.

A different approach was given by Arkani-Hamed, Dimopoulas and Dvali(ADD). They showed that the weakness of gravity could be explained by postulating two or more extra dimensions in which only gravity could propagate. In contrast, the Standard Model (SM) particles are confined to a brane within the extra dimensions.In this model, the effective ($M_{Pl(4+n)}$) and original Planck scale (M_{Pl}) are related by the equation $M_{Pl}^2 \sim R^n M_{Pl(4+n)}^{2+n}$, R being the size of the extra dimension.While keeping the Planck scale constant, the effective Planck Scale can be reduced to the order of a TeV,if the size of the extra dimension is large.(Assuming a toroidal shaped extra dimensions: $R \sim 10^{30/n-19}$m, so for $n \geq 2$,$R < 1$mm).Hence it leads to possible observable consequences in current and future experiments.

A year later, Randall and Sundrum (RS) found a new possibilty using a warped geometry.They postulated a 5−dimensional Anti-de Sitter(AdS) spacetime with a compactification scale of order TeV.The origin of the smallness of the electroweak scale versus the planck scale was explained by the gravitation redshift factor present in the warped AdS metric. In this scenario, gravity is localized on one brane (Planck brane at $y = 0$) in the extra dimension, and th SM particles are located on another(TeV brane at $y = \pi R_c$, $R_c$being the radius of compactification).The scale of new phenomena (Λ_π), in this model,are related to the Planck scale via the equation :$\Lambda_\pi = M_{Pl} e^{k\pi R_c}$. Therefore,$\Lambda_\pi$ can be reduced to $\sim$ 1TeV, if the curvature of this extra dimension (k) is such that $kR_c \sim 12$.

In both these models, only the graviton is allowed to permeate the bulk which leads to different spectrum of spin-2 Kaluza-Klein (KK) modes in 4-dimensions. The spin-2, KK modes couple to the Standard Model (SM) particles through the energy momentum tensor of the SM.

These beyond SM model scenarios could alter the SM predictions by additional virtual KK mode exchanges and real KK mode productions. Dedicated groups in both ATLAS and CMS collaborations are engaged in the analysis for extra dimension searches in various processes like di-lepton, di-photon, mono-jet, mono-photon productions etc. To put stringent bounds on the parameters of these BSM models, control on the theoretical uncertainties is essential. Renormalisation and factorisation scale dependences of a cross section to a particular order in perturbation theory give an estimate of the uncalculated higher order corrections. Presently next-to-leading order (NLO) QCD calculations have been done for di-lepton , di-photon and di-electroweak gauge boson productions via virtual KK modes in addition to the SM contributions. These virtual contributions have been incorporated in the aMC@NLO frame work and results to NLO+PS accuracy are now available for most of the di-final state processes . In all these processes the factorisation scale dependence reduces substantially and in addition the NLO correction is in fact significant. For the above processes, the leading order (LO) is of the order $\mathcal{O}(\alpha_s^0)$, the renormalisation scale dependence starts only at NLO in QCD. To control the renormalisation scale dependence one would have to go to next-to-next-to-leading order (NNLO).

A full NNLO QCD contribution requires the knowledge of graviton-quark-antiquark $G \to q\overline{q}$ and graviton-gluon-gluon $G \to gg$ form factors up to two-loop level in QCD in addition to double real emission and one-loop single real emission scatter processes at the parton level.

In our first work we computed these form factors to two-loop level in QCD by sandwiching the energy momentum tensor of the QCD part of the SM between on-shell gluon and quark states. We have shown that these form factors satisfy Sudakov integro-differential equation with same cusp , collinear and soft anomalous dimensions that contribute to electroweak vector boson and gluon form factors. In addition, they also show the universal behaviour of the infrared poles in ε in accordance with the proposal by Catani

Then we apply these two-loop results to compute next-to-next-to leading order qcd corrctions within soft virtual approximations to TeV scale gravity models. For ADD model we evaluate the contribution to the Drell-Yan cross section, and we present distributions for the di-lepton invariant mass at the LHC with a center-of-mass energy $\sqrt{s_H} = 14$ TeV. We find a large K factor ($K \simeq 1.8$) for large values of invariant mass, which is the region where the ADD graviton contribution dominates the cross section. The increase in the cross section with respect to the previous order result is larger than 10% in the same invariant mass region. We also observe a substantial reduction in the scale uncertainty.For the Randall-Sundrum (RS) model we computed the total single graviton production cross section at the LHC. We find an increase between 10% and 13% with respect to the next-to-leading order prediction, depending on the model parameters. We provide an analytic expression for the NNLO K factor as a function of the lightest RS graviton mass.

Then we computed two loop virtual QCD corrections to the process massive spin $2 \longrightarrow 3$ gluons due to interference of born and two-loop amplitudes. This result constitutes one of the ingredients to full NNLO QCD contribution to production of a massive spin-2 particle along with a jet in the scattering process at the LHC.In particular, this massive spin-2 could be a KK mode of a ADD graviton in large extra dimensional model or a RS KK mode in warped extra dimensional model or a generic massive spin-2.This also provides an opportunity to study the ultraviolet and infrared structures of QCD amplitudes involving tensorial coupling resulting from energy momentum operator.Using dimensional regularization, we find that infrared poles of this amplitude are in agreement with the proposal by Catani confirming the factorization property of QCD amplitudes with tensorial insertion.

List of Publications/Preprints

1. ‡ **Quark and gluon spin−2 form factors to two loops in QCD,**
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A (‡) indicates papers on which this thesis is based.

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Chapter 1

Introduction

Physics is the science of understanding the regularities governing the natural world, finding explanations and making predictions,search for broad, new organising principles which often requires creating and developing novel conceptual frameworks guided by mathematical consistency and tested by experimental evidences.There are four fundamental forces that have been identified for responsible of all natural phenomena.The strong interaction is very strong, but very short ranged.It acts only over ranges of order 10^{-13} centimeter and is responsible for holding the nuclei of atoms together.The electromagnetic force governs atomic level phenomena,binding electrons to atoms and atoms to one another to form molecules and compounds.It is long ranged, but much weaker than the strong force.The weak force is responsible for radioactive decay and neutrino interactions.It has a very short range and as its name indicates,it is very weak. The gravitational force is very weak,but long ranged.Furthermore it is always attractive, and acts between any two pieces of matter in the universe since mass (also energy) is its source.Every force that we experience belongs to one of these four categories, even when the connection is very hard to see.Each type of force acts on a specific property of an object.The specific property refers to the aspect of an object that is necessary for that object to be influenced by the force or to feel the force. For example gravitation force acts on the mass of an object, in case of electromagnetic force the object must have an electric charge,the strong force acts on the color charge of the object and the weak force acts on the weak charge of the object.Although the fundamental forces in our present universe are distinct and have very different characteristics,the current thinking in theoretical Physics is that it was not always so.There is a strong belief that in the very early universe when temperature was very high (Planck Scale$\sim 10^{19}$GeV) compared with today the strong,weak,electromagnetic and gravitation forces were unified into a single force. Only when the temperature dropped did these forces separate from each other, with the gravitation force separating first and then at a still lower temperature the strong force and again at a lower temperature the electromagnetic and weak forces separating to leave us with the four distinct forces that we see in our present universe. The process of the forces separating from each other is called spontaneous symmetry breaking.

At present The Standard Model (SM) of particle physics is considered to be the most successful theory which has been experimentally tested to a great accuracy.It governs all known strong and electroweak interactions.It has been fantastically successful in predicting and explaining data;there are no obvious or overt violations of this theory.After the recent discovery of Higgs boson which is consistent with SM, it gets more strengthened. This doesn't mean it is entirely satisfactory or complete, it has also many dark corners with big question marks.The theory incorporates only three out of four fundamental forces, omitting gravity.It doesn't answer important questions like the origin of dark matter,the hierarchy between electroweak (1 TeV) and Planck scale (10^{19} GeV),massive neutrinos, why there are three generations of quarks and leptons,hierarchy between fermion masses,the stability of Higgs sector under higher order radiative corrections etc. Many different models of physics Beyond the Standard Model (BSM) have been proposed to adress these questions over the years.One such idea envisages a world in more than three space dimensions as a possible solution to some of the above issues.Initially around 1920s Kaluza and Klein proposed the idea of spacetime more than three spatial dimensions in an attempt to unify electromagnetism and gravity.Although their initial ideas were failed, the formalism that they and others developed is still useful now-a-days.Around 1980, string theory proposed again to enlarge the number of space dimensions, for describing a consistent theory of quantum gravity.The extra dimensions were supposed to be compactified at a scale close to the planck scale, and thus not testable experimentally in the near future.

In 1998, Arkani-Hamed, Dimopoulas and Dvali(ADD) proposed a model (ADD)

[1] with large extra dimensions to explain the weakness of gravity by postulating two or more extra dimensions in which only gravity could propagate. In contrast, the Standard Model (SM) particles are confined to a brane within the extra dimensions.In this model, the effective ($M_{Pl(4+n)}$) and original Planck scale (M_{Pl}) are related by the equation $M_{Pl}^2 \sim R^n M_{Pl(4+n)}^{2+n}$, R being the size of the extra dimension.While keeping the Planck scale constant, the effective Planck Scale can be reduced to the order of a TeV,if the size of the extra dimension is large.Hence it leads to possible observable consequences in current and future experiments. A year later, in 1999,Randall and Sundrum (RS) found a new possibilty using a warped geometry [2].They postulated a 5−dimensional Anti-de Sitter(AdS) spacetime with a compactification scale of order TeV.The origin of the smallness of the electroweak scale versus the planck scale was explained by the gravitation redshift factor present in the warped AdS metric.In this scenario, gravity is localized on one brane (Planck brane at $y = 0$) in the extra dimension, and the SM particles are located on another(TeV brane at $y = \pi R_c$, $R_c$being the radius of compactification).The scale of new phenomena (Λ_π), in this model,are related to the Planck scale via the equation :$\Lambda_\pi = M_{Pl} e^{k\pi R_c}$.Therefore,$\Lambda_\pi$ can be reduced to $\sim$ 1TeV, if the curvature of this extra dimension (k) is such that $kR_c \sim 12$.Both of these models are potentially detectable and have different search signatures.In the ADD model there are two ways in which the graviton can be detected: via direct graviton emission in association with a vector boson or via virtual graviton exchange.In the first case,the graviton would be emitted into the extra dimensions and so would escape detection in the detector.Its existence would be deducible by missing transverse energy.The signature would therefore be missing transverse energy and accompanied by a jet(s) or vector boson. The cross section for this process depends on the number of extra dimensions.Virtual graviton exchange would result in, and therefore could be detected by, deviations in dilepton and diboson cross sections and asymmetries expected from those SM processes.For example, in the dilepton channel at Hadron colliders , graviton exchange would not only result in the additional $q\bar{q} \to G \to l^+l^-$ process, but also a gg initiated process $gg \to G \to l^+l^-$, which is not present in the SM.In the ADD model graviton exchange, a broad change in cross section at large invariant mass is expected, due to the summation over closely spaced Kaluza Klein (KK) towers of the graviton.The RS model is also detectable via virtual

graviton exchange and could be detected by an excess in the dilepton,dijet or diboson channel.

Dedicated groups in both ATLAS [3] and CMS [4] collaborations are engaged in the analysis for extra dimension searches in various processes like di-lepton, di-photon, mono-jet, mono-photon productions etc. To put stringent bounds on the parameters of these BSM models, control on the theoretical uncertainties is essential. Renormalisation and factorisation scale dependences of a cross section to a particular order in perturbation theory give an estimate of the uncalculated higher order corrections. Presently next-to-leading order (NLO) QCD calculations have been done for di-lepton [5] , di-photon [6] and di-electroweak gauge boson [7] productions via virtual KK modes in addition to the SM contributions. These virtual contributions have been incorporated in the aMC@NLO frame work and results to NLO+PS [8] accuracy are now available for most of the di-final state processes . In all these processes the factorisation scale dependence reduces substantially and in addition the NLO correction is in fact significant. For the above processes, the leading order (LO) is of the order $\mathcal{O}(\alpha_s^0)$, the renormalisation scale dependence starts only at NLO in QCD. To control the renormalisation scale dependence one would have to go to next-to-next-to-leading order (NNLO).

In this thesis we first study the computation of spin-2 quark gluon form factors to two loop in QCD.This is an important ingredient for full NNLO computation of production of a spin-2 particle.We also study the universal structure of Infrared divergences of these form factors.Next using these form factors we have calculated NNLO QCD corrections in soft-virtual appriximation in TeV Scale gravity models and study their phenomenological impact.Further,we evaluated two loop QCD corrections due to the interference of born and two loop amplitude for the process a massive spin-2 $\rightarrow$ 3gluons.This is a part of the full NNLO QCD corrections to the KK graviton plus one jet processes.

In the next section of this chapter we will briefly discuss the Standard Model and then BSM with ADD and RS Model.In 2nd chapter we discuss some basics of QCD.Third chapter is devoted to the computation of two loop QCD correction to spin-2 form factors.In 4th chapter we have describe the evaluation of NNLO soft-virtual QCD corrections to TeV Scale gravity models. In chapter 5th we study the evaluation of two loop QCD corrections due to interference of born and two loop amplitudes for the process a massive spin-2 $\rightarrow$ 3 gluons.We summarise our results in chapter 6th.Appendix is given in chapter 7th.

1.1 The Standrad Model

The Standard Model (SM) is a local gauge theory with the symmetry group $SU(3)_c \times SU(2)_L \times U(1)_Y$,where $SU(3)_c$ describes the strong interaction and $SU(2)_L \times U(1)_Y$ describes electroweak interactions.Its a non abelian gauge theory that means the symmetry group is non-commutative.Gauge theory is a field theory in which the Lagrangian is invariant under a continuous group of local transformations. Here,gauge alludes to the redundant degrees of freedom in the Lagrangian.The transformations between possible gauges,called gauge transformations form a Lie group.This Lie group is refered to as the symmetry group or gauge group of the theory.Every Lie group is associated with a Lie algebra whose elements are the elements of the group that are infinitesimally close to the identity, and the Lie bracket of the Lie algebra is related to the commutator of two such infinitesimal elements.The Lie group can be build up by means of its infinitesimal generators,a basis for the Lie algebra of the group.For each generator there arises a corresponding vector field, called the gauge field.Gauge fields are included in the Lagrangian to ensure its gauge invariance under the local group transformations.When such a theory is quantized, the quanta of the gauge fields are called gauge bosons. The Standard Model has a total of 12 gauge bosons : the photon,three weak gauge bosons (W^+ ,W^- and Z) and eight gluons.The photon is the force carrier of the electromagnetic interaction,three weak bosons mediate the weak interaction and gluons are the force carrier of the strong interaction.

The Standard Model contains two types of fermionic elementary particles,namely, quarks and leptons.They are the basic building blocks of matter. Quarks have color quantum number, which is responsible for strong interaction. They are the only elementary particles in the SM to experience all four fundamental interactions. Leptons don't have color quantum number.So, they don't experience strong interaction but they experience other three fundamental interactions. There are six types or flavors of quarks: up,down,strange,charm,top and bottom. Up,charm and top quarks (collectively known as up-type quarks) have $+\frac{2}{3}$ electric charge.Down,strange and bottom quarks (collectively known as down-type quarks) have $-\frac{1}{3}$ electric charge. Similarly, there are six types or flavors of leptons : electron,muon and tau and their associated neutrinos. Electron,muon and tau (collectively known as charged leptons) have -1 electric charge and three neutrinos (collectively known as neutral leptons) are electric neutral particles.Again quarks are divided into three families or generations. The first,second and third generation of quarks comprise of up & down, strange & charm and bottom & top quark respectively.Similarly, leptons are divided into three families or generations.The first,second and third generation of leptons comprise of electron & electron neutrino,muon & muon neutrino and tau & tau neutrino. In SM, every particle has an antiparticle with same mass and opposite electric charge as that of the particle.Again all SM fermions are divided into two types: left handed fermions and right handed fermions.The fermions which have left chirality are called left handed fermions and those which have right chirality are called right handed fermions.Left-handed quarks transform as $SU(3)_c$ triplet and $SU(2)_L$ doublet while right handed quarks are $SU(3)_c$ triplet and singlet under $SU(2)_L$.Left-handed leptons transform as $SU(3)_c$ singlet and $SU(2)_L$ doublet while right handed leptons are singlet under both $SU(3)_c$ and $SU(2)_L$.There is no right handed neutrino observed in nature.So they are absent in Standard Model and thus, keeping neutrinos massless. In Standard Model, fermions carry right quantum numbers to cancel anomalies.

All the SM particles are massless when the gauge symmetry is unbroken.The fermion and gauge boson masses are generated through a mechanism called Higgs mechanism, via spontaneous symmetry breaking.To achieve this mechanism,in the SM Lagrangian we add the Higgs scalar potential by writing all renormalisable Higgs self coupling terms.The minimum of this potential has a $O(2)$ symmetry having infinite set of vacua.When one of the vacua is chosen,the symmetry is spontaneously broken.The expectation value of the Higgs field for which the scalar potential attains the minimum, is called vacuum expectation value (vev).Since vacuum is neutral we choose neutral component of Higgs field to get vev.This choice of vacuum breaks $SU(2)_L \times U(1)_Y$, but leaves $U(1)_{em}$ invariant,leaving the photon massless.The color symmetry remains as it is, and thus gluons are massless.Eating the goldstone modes,other three electroweak gauge bosons (W^+ ,W^- and Z) become massive.The vacuum expectation value (vev) should be ~ 246 GeV to agree with the experimental measured masses and couplings of the gauge bosons. The fermion masses are generated after the Higgs gets vev and their masses are proportional to their Yukawa couplings.These Yukawa couplings are free parameters in the SM, and in general non-diagonal which leads to mixing between different quark flavors and similarly for leptons.In the quark and lepton sectors the 3×3 mixing matrices are known as CKM and PMNS matrix respectively.In the SM,the neutrinos are massless due to the absence of right handed neutrinos.

1.2 Beyond Standard Model

Several ideas have been put forth over the years to address the gauge hierarchy problem, one alternate approach is to postulate the existence of extra spatial dimensions wherein only gravity is allowed to propagate. Depending on the geometry of the extra dimension there are different scenarios viz. (i) large extra dimensions (ADD) [1] and (ii) warped extra dimensions models (RS) [2].

1.2.1 ADD Model

In the ADD model there are d flat large extra spatial dimensions of same radii R, compactified on a d-dimensional torus. Due to the larger volume of extra dimension available for gravity, it appears weak in the 4-dimensions where the SM particles and their interactions are restricted to. By Gauss's law the $4+d$ dimensional fundamental Planck scale gets related to 4-dimensional Planck scale (M_P) via the volume factor (R^d) of extra dimension and a large enough volume could result in the fundamental Planck scale of the order of a TeV, there by ensuring the resolution of the hierarchy problem. Propagation of gravity in the large extra dimensions results in a continuous spectrum of Kaluza-Klein (KK) modes in 4-dimensions with small mass splitting of the order of $1/R$.

The interaction of the spin-2, KK modes ($h_{\mu\nu}$) with the SM particles is via the energy momentum tensor $T^{\mu\nu}$ of the SM, which is universally suppressed by a coupling

$\kappa = \sqrt{16\pi}/M_P$,

$$\mathcal{L}_{ADD} = -\frac{\kappa}{2} \sum_{\vec{n}} T^{\mu\nu}(x)\ h^{(\vec{n})}_{\mu\nu}(x). \tag{1.1}$$

Two types of process involving graviton are possible viz. exchange of virtual graviton and production of a real graviton. For processes involving virtual KK mode exchange between SM particles, summation over the high multiplicity KK modes leads to a compensation of the κ suppression. Due to the continuous KK modes spectrum, the summation is replaced by an integral with appropriate density of state $\rho(m_{\vec{n}})$ of KK modes [9]. ADD model being an effective low energy theory, the integral is cutoff at a scale M_S that defines the onset of quantum gravity. The cross section could hence be appreciable at collider energies, giving rise to non-resonant enhancement of the high invariant mass regions of a di-final state production [9–11] or final states involving more particles [12].

Real graviton production leads to missing energy signal and a cross section for the production of a single graviton $d\sigma_{m_{\vec{n}}}$ has to be convoluted with the graviton density of state to get the inclusive cross section. Here too the collective contribution of the KK modes results in observable effects at the collider.

1.2.2 RS Model

The RS model is an alternate extra dimension model with one exponentially warped extra dimension y with radius of compactification r_c, where again only gravity is allowed to propagate. In this model there are two 3-branes; gravity resides on the Planck brane at $y = 0$ and it appear weaker on the TeV brane located at $y = \pi r_c$ due to the exponential warping. A mass scale on the TeV brane $\Lambda_\pi = \overline{M}_P \exp(-k\pi r_c)$ as a result of gravity resides on the Planck brane could be of the order of a TeV, for $kr_c \sim 12$. k is the curvature of the extra dimension. The interaction Lagrangian of the RS KK mode with the SM particles are given by

$$\mathcal{L}_{RS} = -\frac{1}{\Lambda_\pi} \sum_{n=1}^{\infty} T^{\mu\nu}(x)\ h^{(n)}_{\mu\nu}(x). \tag{1.2}$$

The zero mode corresponding to the massless graviton which is M_P suppressed is not included in the sum. As a result of the warped geometry of the extra dimension the characteristic mass spectrum of the KK modes is $M_n = x_n k \exp(-k\pi r_c)$, where x_n are the zeros of the Bessel's function. In the RS case, the resonant production of KK modes would be observed in a pair production of final state SM particles.

Chapter 2

Basics of QCD

Quantum chromodynamics (QCD) is the theory of strong interactions,describing the interaction between quarks and gluons which make up hadrons such as proton,neutron and pion. It is a local non-abelian gauge theory with symmetry group $SU(3)$.The QCD analog of electric charge is a property called color. Gluons are the force carriers of the theory like photons are for the electromagnetic force in quantum electrodynamics.

2.1 Lagrangian of QCD

The dynamics of quarks and gluons are governed by the QCD lagrangian.The full QCD lagrangian is given by

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^a_{\alpha\beta}F^{\alpha\beta}_a}_{\text{pure gauge part}} + \underbrace{\sum_{\text{flavors}} \overline{q}_i \left(i\not{D} - m\right)_{ij} q_j}_{\text{Quark field part}} \quad + \underbrace{\left[-\frac{1}{2\xi}\left(\partial_\mu A_a(x)\right)^2\right]}_{\text{gauge-fixing}}$$
$$+ \underbrace{\left[\partial^\mu \overline{\omega}_a(x)\right]\left[D_\mu(x)\omega_a(x)\right]}_{\text{ghost}}, \tag{2.1}$$

where $F^a_{\alpha\beta}$ is Field Strength Tensor for Spin-1 gluon field A^a_α:

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha - \underbrace{g f^{abc} A^b_\alpha A^c_\beta}_{\text{non-abelian term}}, \tag{2.2}$$

with a, b, c running over the 8 color degrees of freedom of the gluon field. The non-abelian term distinguishes QCD from QED and gives rise to triplet and quartic gluon self interactions which leads to asymptotic freedom and confinement. The gauge fixing term arises due to the fact that classical equation of motion for gluon field A_α^b is not solvable as the operator involves in the equation is not invertible.This is solved by putting constraints on the gauge field through the gauge fixing condition.The unphysical ghost fields are introduced as a technical device to express the modification of the functional integral measure required to compensate for gauge dependence introduced by gauge fixing term.These unphysical ghost fields are necessary to compensate for effects due to the quantum propagation of the unphysical states of the gauge field.

$\alpha_s = \dfrac{g^2}{4\pi}$ is the QCD coupling strength. f^{abc} (where $a, b, c \in \{1, 2, \ldots, 8\}$) are structure constants of $SU(3)$ color group. q_i (where $i \in \{1, 2, 3\}$) is quark fields in triplet color representation. D_α is the covariant derivative

$$(D_\alpha)_{ij} = \partial_\alpha \delta_{ij} + ig\,(t^c A_\alpha^c)_{ij}\,,$$
$$(D_\alpha)_{ab} = \partial_\alpha \delta_{ab} + ig\,(T^c A_\alpha^c)_{ab}\,,$$

where t and T are matrices in the fundamental and adjoint representations of $SU(3)_{\text{color}}$, respectively.

$$\left[t^a, t^b\right] = i f^{abc} t^c, \qquad t^a = \frac{1}{2}\lambda^a,$$
$$\left[T^a, T^b\right] = i f^{abc} T^c, \qquad (T^a)_{bc} = -i f^{abc}.$$

- Normalisation of the t matrices:

$$\text{Tr } t^a t^b = T_R \delta^{ab}, \qquad T_R = \frac{1}{2}.$$

- Color matrices obey the relations:

$$\sum t^a_{ij} t^b_{jk} = C_F \delta_{ik}, \qquad C_F = \frac{N^2 - 1}{2N},$$
$$\text{Tr } T^c T^d = \sum_{a,b} f^{abc} f^{abd} = C_A \delta^{cd}, \qquad C_A = N.$$

Thus for $SU(3)$, $C_F = \frac{4}{3}$ and $C_A = 3$.

The eight $SU(3)$ generators (Gell-Mann matrices) are

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The total anti-symmetric structure constants f^{abc} are given by

$$\left[\lambda^a, \lambda^b\right] = 2if^{abc}\lambda^c, \tag{2.3}$$

with $f^{123} = 1$, $f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$, and all f^{abc} not related to these by permutation are zero.

2.2 Gauge Invariance

QCD Lagrangian is invariant under local gauge transformations, i.e., one can redefine quark fields independently at every point in spacetime,

$$q_i(x) \to q'_i(x) = e^{(it\cdot\theta(x))_{ij}} q_j(x) \equiv \Omega_{ij}(x) q_j(x) \tag{2.4}$$

without changing the physical content. Covariant derivative is so called because it transforms in same way as field itself.

$$D_\alpha q(x) \to D'_\alpha q'(x) \equiv \Omega(x) D_\alpha q(x) \tag{2.5}$$

We can use this to derive transformation property of gluon field A_α

$$D'_\alpha q'(x) = (\partial_\alpha + igt \cdot A'_\alpha)\Omega(x)q(x) \tag{2.6}$$

$$= (\partial_\alpha \Omega(x))q(x) + \Omega(x)\partial_\alpha q(x) + igt \cdot A'_\alpha \Omega(x) q(x), \tag{2.7}$$

where $t \cdot A_\alpha = \sum_a t^a A^a_\alpha$.

Hence,

$$t \cdot A'_\alpha = \Omega(x) t \cdot A_\alpha \Omega^{-1}(x) + \frac{i}{g} (\partial_\alpha \Omega(x)) \Omega^{-1}(x) \tag{2.8}$$

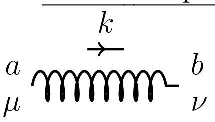
Transformation property of gluon field strength $F_{\alpha\beta}(x)$ is

$$t \cdot F_{\alpha\beta}(x) \to t \cdot F'_{\alpha\beta}(x) = \Omega(x) t \cdot F_{\alpha\beta}(x) \Omega^{-1}(x). \tag{2.9}$$

Contrast this with gauge-invariance of QED field strength. QCD field strength is not gauge-invariant because of self-interaction of gluons. Carriers of the color force are themselves colored unlike the electrically neutral photon.

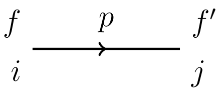
2.3 Feynman Rules

Gluon Propagator



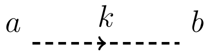
$$= \frac{-i\delta^{ab}}{k^2 + i0} \left(g^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2 + i0} \right)$$

Quark Propagator



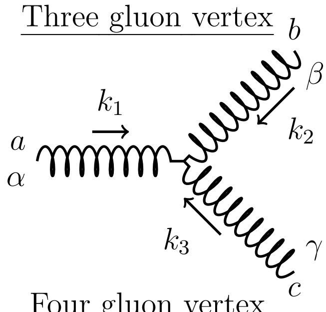
$$= \frac{-i\delta^i_j \delta^f_{f'}}{\not{p} - m_f + i0}$$

Ghost Propagator



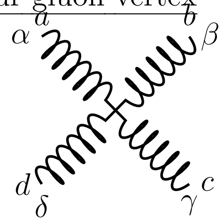
$$= \frac{-i\delta^{ab}}{k^2 + i0}$$

Three gluon vertex



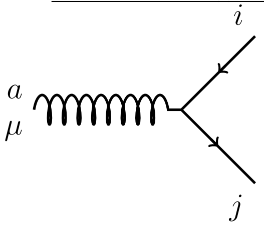
$$= -g f^{abc} \Big[g^{\alpha\beta} (k_1 - k_2)^\gamma + g^{\beta\gamma} (k_2 - k_3)^\alpha + g^{\gamma\alpha} (k_3 - k_1)^\beta \Big]$$

Four gluon vertex



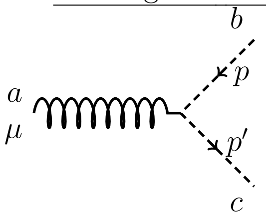
$$= -ig^2 \Big[f^{abe} f^{cde} \left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma} \right) + f^{ace} f^{bde} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\gamma\beta} \right) + f^{ade} f^{bce} \left(g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\gamma} g^{\delta\beta} \right) \Big]$$

Quark-gluon vertex



$$= -ig\gamma^{\mu}\delta_f^{f'}\,(t^a)_i^j$$

Ghost-gluon vertex



$$= -gf^{abc}p'^{\mu}$$

2.4 Running Coupling & Asymptotic Freedom

Consider dimensionless physical obsrvable R which depends on a single large energy scale $Q \gg m$ where m is any mass. Then we can set $m \to 0$ (assuming this limit exists), and dimensional analysis suggests that R should be independent of Q. This is not true in quantum field theory. Calculation of R as a perturbation series in the coupling $\alpha_s = \dfrac{g^2}{4\pi}$ requires renormalisation to remove ultraviolet divergence. This introduces a second mass scale μ; point at which subtractions, that remove divergences, are performed. Then R depends on the ratio $\dfrac{Q}{\mu}$ and is not constant. The renormalised coupling α_s also depends on μ. But μ is arbitrary. Therefore, if we hold bare couplings fixed, R cannot depend on μ. Since R is dimensionless, it can only depend on $\dfrac{Q^2}{\mu^2}$ and the renormalised coupling constant α_s. Hence

$$\mu^2\frac{d}{d\mu^2}R\left(\frac{Q^2}{\mu^2},\alpha_s\right) \equiv \left[\mu^2\frac{\partial}{\partial\mu^2} + \mu^2\frac{\partial\alpha_s}{\partial\mu^2}\frac{\partial}{\partial\alpha_s}\right]R = 0. \tag{2.10}$$

At fixed bare coupling, physical predictions cannot depend on an arbitrary choice of renormalisation scale. Introducing $\tau = \ln\left(\frac{Q^2}{\mu^2}\right)$, $\beta(\alpha_s) = \mu^2\frac{\partial\alpha_s}{\partial\mu^2}$, we have

$$\left[-\frac{\partial}{\partial\tau} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right]R = 0.$$

This renormalisation group equation is solved by defining running coupling $\alpha_s(Q)$:

$$\tau = \int_{\alpha_s}^{\alpha_s(Q)}\frac{dx}{\beta(x)}, \qquad \alpha_s(\mu) \equiv \alpha_s.$$

Then

$$\frac{\partial \alpha_s(Q)}{\partial \tau} = \beta(\alpha_s(Q)), \qquad \frac{\partial \alpha_s(Q)}{\partial \alpha_s} = \frac{\beta(\alpha_s(Q))}{\beta(\alpha_s)},$$

and hence $R\left(\frac{Q^2}{\mu^2}, \alpha_s\right) = R(1, \alpha_s(Q))$. Thus all scale dependence in R comes from running of $\alpha_s(Q)$. Now expanding $\beta(\alpha_s)$ in the powers of α_s:

$$\beta(\alpha_s(Q)) = -\beta_0 \left(\frac{\alpha_S(Q)}{\pi}\right)^2 - \beta_1 \left(\frac{\alpha_s(Q)}{\pi}\right)^3 + \dots \tag{2.11}$$

In perturbative QCD the first coefficient is calculated to be

$$\beta_0 = (33 - 2N_f)/3,$$

where N_f is the number of quark flavors. If we set all β_i beyond β_0 to zero, then by solving the renormalisation group equations we get

$$\alpha_s(Q) = \frac{\alpha_s(\mu)}{1 + (\beta_0/\pi)\alpha_s(\mu)\ln(Q^2/\mu^2)}.$$

When Q increases $\alpha_s(Q)$ decreases. This is called "asymptotic freedom".

2.5 Diverences in QCD, Regularisation and Renormalisation

2.5.1 Ultraviolet Divergences

- In Higher-order Pertubation Theory we encounter Feynman diagrams with closed loops, associated with unconstrained momenta. For every such momentum k^μ, we have to integrate over all values i.e

$$\int \frac{d^4k}{(2\pi)^4}$$

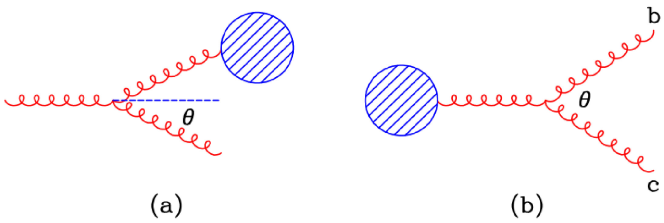
- $\int d^4k\, k^{D-4}$ has **superficial degree of divergence** D:

$$D = \begin{cases} 0 & \Longrightarrow \text{Log divergent} \\ 1 & \Longrightarrow \text{Linearly divergent} \\ 2 & \Longrightarrow \text{Quadratically divergent} \end{cases} \tag{2.12}$$

- If a theory has only a finite set of (classes of) divergent (i.e. cut-off dependent) diagrams, their contributions can be absorbed into redefinitions of the parameters of the theory. This is called **renormalisation**.

2.5.2 Infrared Divergences

Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored. Soft or collinear gluon emission gives infrared divergences in perturbation theory (PT). Light quarks ($m_q \ll \Lambda$) also lead to divergences in the limit $m_q \to 0$ (mass singularities).



If gluon(b) is splitting from incoming line to gluon (a) and gluon (c) in fig (a)

$$p_b^2 = -2E_aE_c(1 - cos\theta) \tag{2.13}$$

Propagator factor $\frac{1}{p_b^2}$ diverges as $E_c \to 0$ (soft singularity) or $\theta \to 0$ (collinear or mass singularity).

If the gluon(a) is splitting from outgoing line to gluon (b) and (c) as in fig.(b),then

$$p_a^2 = -2E_bE_c(1 - cos\theta), \tag{2.14}$$

where E_b and E_c energy of emitted gluons. Propagator diverges when either emitted gluon is soft ($E_b or E_c \to 0$) or when opening angle $\theta \to 0$.

- Similar infrared divergences come in loop diagrams, associated with soft and/or collinear configurations of virtual partons within region of integration of loop momenta.

- Infrared divergences indicate dependence on long-distance aspects of QCD not correctly described by PT. Divergent (or enhanced) propagators imply propagation of partons over long distances. When distance becomes comparable with hadron size $\sim 1fm$, quasi-free partons of perturbative calculation are confined/hadronized non-perturbatively, and apparent divergences disappear.

- We can perform PT calculations, provided we limit ourselves to two classes of observables:
  1. Infrared safe quantities, i.e. those insensitive to soft or collinear divergences. Infrared divergences in PT calculation either cancel between real and virtual contributions or are removed by kinematic factors.
  2. Factorizable quantities, i.e. those in which infrared sensitivity can be absorbed into an overall non-perturbative factor, to be determined experimentally.

- In either case, infrared divergences must be regularized during PT calculation, even though they cancel or factorize in the end.
  1. Gluon mass regularization: introduce finite gluon mass, set to zero at end of calculation. However, as we saw, gluon mass breaks gauge invariance.
  2. Dimensional regularization: analogous to that used for ultraviolet divergences, except we must increase dimension of space-time, $\varepsilon = 2 - \frac{D}{2} < 0$

  Divergences are replaced by powers of $\frac{1}{\varepsilon}$.

2.5.3 Renormalisation

UV divergences are 'dimensional regularised' by reducing number of spacetime dimension $D < 4$:

$$\frac{d^4k}{(2\pi)^4} \longrightarrow \mu^{2\varepsilon} \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\epsilon}}$$

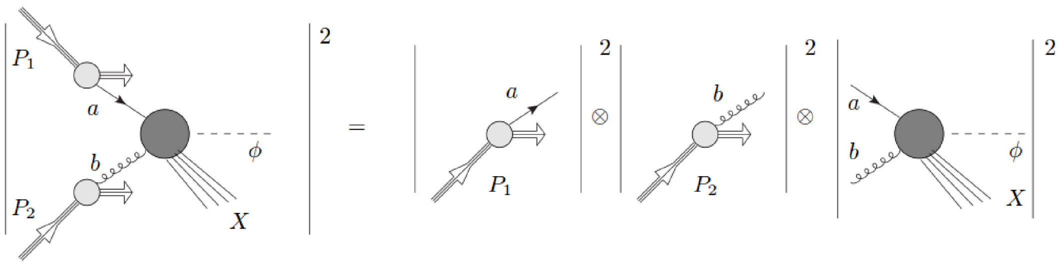
where $\varepsilon = 2 - \frac{D}{2}$. The renormalisation scale μ has been introduced to preserve dimensions of couplings and fields.

Loop integrals of form

$$\int \frac{d^Dk}{(k^2+m^2)^2}$$

lead to poles at $\varepsilon = 0$.

- **minimal subtraction** scheme: subtract poles only, and replace bare coupling by renormalized coupling $\alpha_s(\mu)$.



- **modified minimal subtraction** scheme: In practice poles always appear in the combination

$$\frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E.$$

In this scheme, not just poles but the constant $[\ln(4\pi) - \gamma_E]$ term is also subtracted.

2.6 QCD at hadron Collider

The understanding of the structure of the proton at short distances is one of the key ingredients to be able to predict cross section for processes involving hadrons in the initial state.All processes in hadronic collisions, even those intrinsically of electroweak nature such as the productin of W/Z bosons or photons,are in fact induced by the quark and gluons contained inside the hadron. In the parton model, we can express the hadronic cross section, $\sigma^{P_1 P_2}$ in terms of ultraviolet (UV) renormalised partonic cross sections $\hat{\sigma}_{ab}$, $a, b = q, \overline{q}, g$ and bare parton distribution functions $\hat{f}_c(x_i)$, $i = 1, 2$ and $c = q, \overline{q}, g$ of colliding partons as follows:

$$\sigma^{P_1 P_2}(S, Q^2) = \int dx_1 \int dx_2 \hat{f}_a(x_1) \hat{f}_b(x_2) \hat{\sigma}_{ab}(\hat{s}, Q^2) \tag{2.15}$$

where $S = (P_1 + P_2)^2$ and $\hat{s} = (x_1 P_1 + x_2 P_2)^2$ are the center of mass energies of incoming hadrons and partons respectively. Q is the invariant mass of the final state particle. The sum over a and b is implied for the repeated indices. The bare parton distribution function $\hat{f}_c(x)$ describes the probability of finding a parton of type c which carries a momentum fraction x of the hadron. Since the partonic cross sections are often singular due to collinear kinematics of massless partons we have put "hats" on parton distribution functions $f_c(x_i)$ $(c = a, b$ and $i = 1, 2)$ and on partonic cross sections. The parton distribution functions describe long distance part of the hadronic cross section and hence they are not computable in perturbative QCD.Parton distribution functions are process independent and universal in nature. On the other hand, the bare partonic cross sections $\hat{\sigma}_{ab}$ that describe the short distance part of the reaction can be computed in QCD perturbation theory.

Defining

$$\hat{\Delta}(\hat{\tau}, m^2) \equiv \hat{s}\sigma_{ab}(\hat{s}, m^2), \qquad \hat{\tau} = \frac{m^2}{\hat{s}} \tag{2.16}$$

and using the identity

$$\int d\hat{\tau}\delta(\tau - \hat{\tau}x_1x_2) = \frac{S}{\hat{s}} \qquad \tau = \frac{m^2}{\hat{S}} \tag{2.17}$$

we can express the hadronic cross section, eqn.(2.15) as

$$\sigma^{P_1P_2}(S, m^2) = \frac{1}{S}\int dx_1 \int dx_2 \int d\hat{\tau} \hat{f}_a(x_1)\hat{f}_b(x_2)\hat{\Delta}_{ab}(\hat{\tau}, m^2)\delta(\tau - \hat{\tau}x_1x_2) \tag{2.18}$$

The above expression can be written in "convolution" notation as follows:

$$\sigma^{P_1P_2}(S, m^2) = \frac{1}{S}\hat{f}_a(\tau) \otimes \hat{f}_b(\tau) \otimes \hat{\Delta}_{ab}(\tau, m^2) \tag{2.19}$$

Beyond the leading order in perturbative QCD, the partonic cross sections $\hat{\sigma}_{ab}$ get contributions from subprocesses involving loops with virtual partons as well as from emission of additional real partons. These subprocesses often suffer from various singularities resulting from ultraviolet, soft and collinear regions of loop integrations and also from soft and collinear regions of phase space integrations. These singularities are regularied using dimensional regularisation technique, where the number of space time dimensions is taken to be $n = 4 + \varepsilon$ with ε being a complex number. The singularities in the resulting subprocess partonic cross sections will appear as poles in $1/\varepsilon^k$ where k is an integer. The ultraviolet singularities can be systematically removed by using renormalisation prescription namely $\overline{MS}$. The soft singularities that arise due to massless gluons present in the virtual as well as real emission subprocesses cancel among themselves according to Kinoshita-Lee-Nauenberg (KLN) theorem. The collinear singularities arise when two or more massless partons become collinear to each other. They often do not cancel and are collinear singular beyond leading order.

The colliear singularities present in $\hat{\Delta}(\hat{\tau}, Q^2)$ will go away only if we sum over all the degenerate initial states. This is achieved by the procedure called mass factorisation wherein one redefines the bare parton distribution functions $\hat{f}_c(x)$ at a scale called factorisation scale μ_F in such a way that the collinear singularities in the bare parton cross sections $\hat{\Delta}(\hat{\tau}, m^2)$ are removed. Such a collinear renormalised parton distribution function $f_a(\tau, \mu_F^2)$ is given by,

$$f_a(\tau, \mu_F^2) = \Gamma_{ab}(\tau, \mu_F^2, \varepsilon) \otimes \hat{f}_b(\tau) \tag{2.20}$$

where $\Gamma_{ab}(\tau, \mu_F^2, \varepsilon)$ are renormalisation kernels defined in $\overline{MS}$ scheme. They are expanded in powers of strong coupling constant as

$$\Gamma_{ab}(\tau, \mu_F^2, \varepsilon) = \delta_{ab}\delta(1-\tau) + \frac{\hat{\alpha}_{s,\varepsilon}}{4\pi} S_\varepsilon \frac{(\mu_F^2)^{\frac{\varepsilon}{2}}}{\varepsilon} P_{ab}^{(0)}(\tau) + (\mathcal{O}(\hat{\alpha}_{s,\varepsilon}^2)) \tag{2.21}$$

$P_{ab}^{(0)}(\tau)$ are called Altarelli-Parisi splitting functions. These kernels contain right poles in $1/\varepsilon$ to cancel the collinear singularities in the bare partonic suprocess cross sections. This procedure is called mass factorisation. Note that this introduces a scale μ_F parametrising the arbitrariness inherent in the removal of collinear singularity in the parton cross section through the redefinition of bare parton distribution functions.

After following regularisation,renormalisation and mass factorisation we can express the hadronic cross section to all orders in perturbatin theory as follows:

$$\begin{aligned} \sigma^{P_1 P_2}(S, m^2) &= f_a(\tau, \mu_F^2) \otimes f_b(\tau, \mu_F^2) \otimes \Delta_{ab}(\tau, m^2, \mu_F^2) \\ &= \sum_{a,b=q,\bar{q},g} \int_\tau^1 \frac{dx_1}{x_1} \int_{\tau/x_1}^1 \frac{dx_2}{x_2} f_a(x_1, \mu_F^2)\, f_b(x_2, \mu_F^2)\, \Delta_{ab}\left(\frac{\tau}{x_1\, x_2}, m^2, \mu_F^2\right) \end{aligned}$$

where the finite Δ_{ab} are given by

$$\Delta_{ab}\left(\hat{\tau}, m^2, \mu_F^2\right) = \sum_{i=0}^{\infty} \frac{\alpha_s^i(\mu_R^2)}{4\pi} \Delta_{ab}^{(i)}\left(\hat{\tau}, m^2, \mu_F^2, \mu_R^2\right)$$

The hadronic cross section will be independent of renormalisation (μ_R) and factorisation (μ_F) scales when we include all order results.

2.7 Scale Uncertainties in Hadron Collider

Colliders such as Tevatron at Fermilab and LHC at CERN play important role in high energy Physics phenomenology to confirm SM and constrain various BSM scenarios.

Since these are hadron colliders, the underlying hard processes result from strong interaction force which is governed by Quantum Chromodynamics.Since the process take place at high energies,perturbative approach to understand them has been very successful.Such an approach in perturbative QCD suffers from large uncertainties in the predictions.They are due to renormalisation (μ_R) and factorisation (μ_F) scales resulting from missing higher order quantum corrections.The predictions from Perturbative QCD become stable against these uncertainties provided the observables are computed beyond the leading order in strong coupling constant.It has been observed that such corrections are important to confirm as well as rule out many model predictions in high energy regions.

Chapter 3

Quark and gluon spin-2 form factors to two-loops in QCD

In this chapter we will study quark and gluon spin-2 form factors.In order to calculate full NNLO contributions to processes in large extra dimensional models we require graviton$-$quark$-$antiquark,$G^* \to q\bar{q}$ and graviton$-$gluon$-$gluon,$G^* \to gg$ form factors upto two loop level in QCD in addition to double real emission and one loop single real emission scattering processes at the parton level.G^* denotes the virtual graviton. Here, we take the first step towards the full NNLO computation by evaluating these form factors to two loop level in QCD by sandwiching the energy momentum tensor of the QCD part of the SM between on shell gluon and quark states.We will also discuss the infrared (IR) structure of these form factors using Sudakov's integro$-$differential equation and Catani's proposal on two loop QCD amplitudes.

In the next section of this chapter we will derive the two-loop form factors and then we describe the infrared structure of these form factors. Finally we summarize the chapter and we provide the expanded form factors in powers of ε to desired accuracy in the appendix of this thesis.

3.1 Two loop form factors

We work with the following action that describes the interaction of SM fields with the KK modes of the gravity. To lowest order in κ, the KK modes couple to SM fields through energy momentum tensor of SM. Here, we restrict ourselves to QCD part of the energy momentum tensor:

$$\mathcal{S} = \mathcal{S}_{SM} - \frac{\kappa}{2}\int d^4x \; T^{QCD}_{\mu\nu}(x)\; h^{\mu\nu}(x)\,, \tag{3.1}$$

where $T^{QCD}_{\mu\nu}$ is the energy momentum tensor of QCD [5]:

$$\begin{aligned} T^{QCD}_{\mu\nu} \;=\;& -g_{\mu\nu}\mathcal{L}_{QCD} - F^a_{\mu\rho}F^{a\rho}_{\nu} - \frac{1}{\xi}g_{\mu\nu}\partial^\rho(A^a_\rho\partial^\sigma A^a_\sigma) \\ & +\frac{1}{\xi}(A^a_\nu\partial_\mu(\partial^\sigma A^a_\sigma) + A^a_\mu\partial_\nu(\partial^\sigma A^a_\sigma)) + \frac{i}{4}\Big[\overline{\psi}\gamma_\mu(\overrightarrow{\partial}_\nu - ig_sT^aA^a_\nu)\psi \\ & -\overline{\psi}(\overleftarrow{\partial}_\nu + ig_sT^aA^a_\nu)\gamma_\mu\psi + \overline{\psi}\gamma_\nu(\overrightarrow{\partial}_\mu - ig_sT^aA^a_\mu)\psi \\ & -\overline{\psi}(\overleftarrow{\partial}_\mu + ig_sT^aA^a_\mu)\gamma_\nu\psi\Big] + \partial_\mu\overline{\omega}^a(\partial_\nu\omega^a - g_sf^{abc}A^c_\nu\omega^b) \\ & +\partial_\nu\overline{\omega}^a(\partial_\mu\omega^a - g_sf^{abc}A^c_\mu\omega^b). \end{aligned} \tag{3.2}$$

In the above equation, g_s is the strong coupling constant and ξ is gauge parameter in Lorenz gauge fixing condition. The peculiar feature in the above action is the appearance of the direct coupling of ghost fields $(\omega, \overline{\omega})$ with KK modes [5]. We have kept track of these unphysical contributions along with those coming from gauge fixing term in order to establish the cancellation of their contributions among themselves. We have retained only light flavours in the quark sector.

We compute the relevant form factors by evaluating the truncated matrix elements $\hat{\mathcal{M}}_I$ of $T^{QCD}_{\mu\nu}$ between on-shell gluon ($I = g$) and quark/anti-quark ($I = q, \overline{q}$) states. The symbol $\hat{}$ here and in the following denotes that the quantities are unrenormalised/bare. The $\hat{\mathcal{M}}_I$s in the color space can be expanded as

$$\hat{\mathcal{M}}_I = \hat{\mathcal{M}}^{(0)}_I + \hat{a}_s\left(\frac{Q^2}{\mu^2}\right)^{\frac{\varepsilon}{2}} S_\varepsilon\hat{\mathcal{M}}^{(1)}_I + \hat{a}_s^2\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon} S_\varepsilon^2\hat{\mathcal{M}}^{(2)}_I + \mathcal{O}\left(\hat{a}_s^3\right)\,, I = g, q, \overline{q}\,, \tag{3.3}$$

where the unrenormalised coupling constant $\hat{a}_s = \hat{g}_s^2/16\pi^2$ and the scale μ is introduced to keep g_s dimensionless in dimensional regularisation and the space-time dimension is taken to be $d = 4+\varepsilon$. The scale $Q^2 = -q^2 - i\epsilon$, where q is the momentum transfer. The unrenormalised coupling constant $\hat{a}_s$ is related to the renormalised one $a_s(\mu_R^2)$ by

$$S_\varepsilon\hat{a}_s = Z(\mu_R^2)a_s(\mu_R^2)\left(\frac{\mu^2}{\mu_R^2}\right)^{\frac{\varepsilon}{2}}\,, \qquad S_\varepsilon = \exp\left\{\frac{\varepsilon}{2}\Big[\gamma_E - \ln 4\pi\Big]\right\}\,, \tag{3.4}$$

where the renormalisation constant $Z(\mu_R^2)$ is given by

$$Z(\mu_R^2) = 1 + a_s(\mu_R^2)\frac{2\beta_0}{\varepsilon} + \mathcal{O}\left(a_s^2(\mu_R^2)\right) , \qquad \beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f \tag{3.5}$$

with μ_R-renormalisation scale, $C_A = N, T_F = 1/2$ and n_f the number of active flavours.

Using the $\hat{\mathcal{M}}_I$s, the form factors are defined as

$$\hat{F}_I^{T,(n)} = \frac{\hat{\mathcal{M}}_I^{(0)*} \cdot \hat{\mathcal{M}}_I^{(n)}}{\hat{\mathcal{M}}_I^{(0)*} \cdot \hat{\mathcal{M}}_I^{(0)}} , \tag{3.6}$$

and the symbol $\cdot$ takes care of the color and spin/polarisation sums.

Feynman Amplitudes and Simplification

The Feynman amplitudes that contribute to gluon and quark matrix elements of the energy momentum tensor $T_{\mu\nu}^{QCD}$ at born, one-loop and two-loop levels in QCD are obtained using a computer program QGRAF [13]. We find 12 one-loop and 153 two-loop diagrams that contribute to the matrix element of energy momentum tensor if it is computed between gluon states while 4 one-loop and 54 two-loop diagrams that contribute for quark-antiquark states. We have used a set of in-house FORM [14] routines to convert the QGRAF outputs into a suitable form for further symbolic manipulations. These FORM routines not only replace the symbolic Feynman vertices, propagators by the corresponding Feynman rules but also perform Lorentz contractions, Dirac gamma matrix algebra etc. We have done all our computations in $d = 4 + \varepsilon$ dimensions in order to regulate both ultraviolet (UV) and infrared (IR) divergences. The resulting expressions at this stage contain one and two-loop tensor and scalar integrals. Since the coupling of KK modes with the energy momentum tensor involves quadratic derivatives, we find that the rank of the tensor integrals present in our computation is larger than the rank of integrals appearing in quark and gluon form factors contributing to electroweak vector boson [15–17] and Higgs production cross sections (in the infinite top quark mass limit) [18–21] respectively.

Reduction of tensor integrals

In the past, they were computed using very different methods, that is, different techniques were employed to perform loop integrals. In [15], the method of Feynman parameterisation was used in a judicious way so that after each parametric integration one is left with an integral over the next parameter. In [16], an elegant method, advocated in [22], namely "integration by parts" (IBP) was used. In [17] the integrals were computed using dispersion techniques developed in [23] which uses the Cutkosky rules [24]. In this method, one cuts the Feynman amplitude in all possible ways to obtain the imaginary part and the real part was obtained from the imaginary part via a dispersion relation. In [18], an algorithm [25] which relates l-loop integrals with $n+1$ external legs to $l+1$-loop integrals with n external legs was used to compute two-loop QCD corrections to gluon form factor relevant for Higgs production cross section. It maps the massless two-loop vertex functions onto massless three-loop two-point functions which are relatively easy to compute. In [19], IBP identities were used extensively to compute the gluon form factor. The gluon form factor at two-loop level in $SU(N)$ gauge theory with n_f light flavours was computed in [21] following [23]. All these results were known only to a desired accuracy in ε, say $\mathcal{O}(\varepsilon)$. In [26], using IBP [22] and Lorentz invariance (LI) [27] identities, the authors have shown that the two-loop corrections to electroweak quark and gluon form factors can be expressed in terms of only few master integrals and have also obtained for the first time the closed form solution to one of the master integrals whose result was known only up to few orders in ε.

IBP Identies

The validity of the IBP relations in dimensional regularisation relies on the following two properties: integration is the inverse process of differentiation and surface terms never contribute if the integrals are evaluated in n dimension.Thus the IBP identities follow from the fact that in the dimensional regularization, the integral of the total derivative with respect to any loop momenta vanishes, that is

$$\int \frac{d^dk_1}{(2\pi)^d} \cdots \int \frac{d^dk_L}{(2\pi)^d} \frac{\partial}{\partial k_i} \cdot \left(v_j \frac{1}{\prod_l \mathcal{D}_l^{n_l}} \right) = 0 \tag{3.7}$$

where n_l is an element of $\vec{n} = (n_1, \cdot \cdot \cdot, n_N)$ with $n_l \in Z$, L is the number of loops and $\mathcal{D}_l$s are propagators which depend on the loop and external momenta and also masses. The four vector v_j^μ can be both loop and external momenta. Performing the differentiation on the left hand side and expressing the scalar products of k_i and p_j linearly in terms of $\mathcal{D}_l$'s, one obtains the IBP identities given by

$$\sum_i a_i J(b_{i,1} + n_1, ..., b_{i,N} + n_N) = 0 \tag{3.8}$$

where

$$J(\vec{m}) = J(m_1, \cdots, m_N) = \int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_L}{(2\pi)^d} \frac{1}{\prod_l \mathcal{D}_l^{m_l}} \tag{3.9}$$

with $b_{i,j} \in \{-1, 0, 1\}$ and a_i are polynomial in n_j.

LI identities

The LI identities follow from the fact that the loop integrals are invariant under Lorentz transformations of the external momenta, that is

$$p_i^\mu p_j^\nu \left(\sum_k p_{k[\nu} \frac{\partial}{\partial p_k^{\mu]}} \right) J(\vec{n}) = 0. \tag{3.10}$$

We will closely follow this approach by [26] to achieve our task. Reduction of a large number of one and two-loop tensor integrals that appear in our computation to a few master integrals was achieved by FIRE [28], a Mathematica package, which extensively uses the IBP [22] and LI [27] identities implemented using Laporta algorithm [29]. Note that there are also similar packages namely AIR [30], Reduce [31,32] and most recently LiteRed [33] that can do this reduction. We used LiteRed to cross-check our results obtained using FIRE. At one-loop level, we find that the form factors depend only on one master integral and at two-loop level, there are one one-loop and three two-loop master integrals. These one-loop and two-loop master integrals are now known to all orders in ε and are given in [26]. Below we present our final results for $\hat{F}_I^{T,(n)}$ for $I = g, q; n = 1, 2$ in terms of these master integrals.

For the gluon form factor, we obtain $\hat{F}_g^{T,(0)} = 1$ and

$$\hat{F}_g^{T,(1)} = 2\Bigg[- iA_{2,LO} \Bigg(4C_A \Big(- 68 + 20\ d + 16\ d^2 - 8\ d^3 + \ d^4 \Big) + n_f \Big(- 16 + 32\ d - 15\ d^2 + 2\ d^3 \Big) \Bigg) \Bigg] \Big/ \Big[(-4 + \ d)(-2 + \ d)(2d^2 - 3d - 8) \Big], \tag{3.11}$$

$$
\begin{aligned}
\hat{F}_g^{T,(2)} = -\Bigg[& 16A_{2,LO}^2\Bigg(- 3360 + 5524\ d - 3607\ d^2 + 1169\ d^3 - 188\ d^4 + 12\ d^5\Bigg) \\
& \times \Bigg\{ 4C_A^2\Big(384 + 3584\ d - 7712\ d^2 + 4260\ d^3 + 128\ d^4 - 939\ d^5 + 371\ d^6 \\
& - 62\ d^7 + 4\ d^8\Big) + C_F\ d\Big(1024 - 3616\ d + 4720\ d^2 - 2926\ d^3 + 941\ d^4 \\
& - 153\ d^5 + 10\ d^6\Big)n_f + 8C_A\Big(192 - 288\ d - 28\ d^2 + 322\ d^3 - 255\ d^4 \\
& + 89\ d^5 - 15\ d^6 + \ d^7\Big)n_f \Bigg\} + A_3(-8 + 3\ d)\Bigg\{ 2C_A^2\Big(1720320 + 60414976\ d \\
& - 195105152\ d^2 + 236351744\ d^3 - 120445352\ d^4 - 11375804\ d^5 \\
& + 54553314\ d^6 - 36985777\ d^7 + 13961672\ d^8 - 3324848\ d^9 + 499154\ d^{10} \\
& - 43447\ d^{11} + 1680\ d^{12}\Big) + 2C_F\ d\Big(10379264 - 36831232\ d + 50367872\ d^2 \\
& - 28580992\ d^3 - 3473320\ d^4 + 16083820\ d^5 - 11518542\ d^6 + 4520247\ d^7 \\
& - 1098971\ d^8 + 165551\ d^9 - 14233\ d^{10} + 536\ d^{11}\Big)n_f + C_A\Big(3440640 \\
& + 10901504\ d - 45510400\ d^2 + 62792448\ d^3 - 46643440\ d^4 + 20064592\ d^5 \\
& - 4109776\ d^6 - 494472\ d^7 + 619031\ d^8 - 203281\ d^9 + 36557\ d^{10} - 3641\ d^{11} \\
& + 158\ d^{12}\Big)n_f \Bigg\} + 2(12 - 7\ d + \ d^2)\Bigg(2A_6(-4 + \ d)^2\ d\Big(- 16 + 30\ d - 17\ d^2 \\
& + 3\ d^3\Big)\Bigg\{ C_A^2(3392 - 3664\ d + 284\ d^2 + 794\ d^3 - 298\ d^4 + 32\ d^5) \\
& + 2C_F(-4 + \ d)^2(176 - 26\ d - 35\ d^2 + 8\ d^3)n_f + C_A(-2880 + 1888\ d \\
& + 172\ d^2 - 368\ d^3 + 87\ d^4 - 6\ d^5)n_f \Bigg\} + A_4\Bigg\{ 2C_A^2(-1720320 + 25437184\ d \\
& - 55822976\ d^2 + 41289728\ d^3 + 1696440\ d^4 - 22168812\ d^5 + 16330266\ d^6
\end{aligned}
$$

$$- 6288301\ d^7 + 1498316\ d^8 - 234230\ d^9 + 24945\ d^{10} - 1812\ d^{11} + 72\ d^{12})$$

$$+ 2C_F(-4 + \ d)^2\ d(-479744 + 1418752\ d - 1664968\ d^2 + 989740\ d^3 - 297578\ d^4$$

$$+ 28179\ d^5 + 7786\ d^6 - 2351\ d^7 + 184\ d^8)n_f + C_A(-3440640 + 18589696\ d$$

$$- 42184960\ d^2 + 55760640\ d^3 - 47369168\ d^4 + 26855488\ d^5 - 10323440\ d^6$$

$$+ 2698144\ d^7 - 474715\ d^8 + 54662\ d^9 - 3837\ d^{10} + 130\ d^{11})n_f \Big\}\Big)\Big] \Big/$$

$$\Big[8(-4 + \ d)^3(-3 + \ d)(-2 + \ d)^2(-1 + \ d)\ d(-7 + 2\ d) \times (-5 + 2\ d)(-8 + 3\ d)$$

$$(2d^2 - 3d - 8)\Big], \tag{3.12}$$

where the color factor $C_F = (N^2 - 1)/2N$. For the quark form factor, we obtain $\hat{F}_q^{T,(0)} = 1$ and

$$\hat{F}_q^{T,(1)} = \quad 2\Big[- \frac{i}{2}\ A_{2,LO}\ C_F\ \Big(64 - 34\ d + 5\ d^2\Big)\Big] \Big/ \Big[(d-4)(d-2)\Big], \tag{3.13}$$

$$
\begin{aligned}
\hat{F}_q^{T,(2)} = &-\Bigg[C_F \Bigg(16\ A_{2,LO}^2 \Big(-3360 + 5524\ d - 3607\ d^2 + 1169\ d^3 - 188\ d^4 + 12\ d^5\Big) \\
&\times \Bigg\{C_F\ d \Big(2048 - 5312\ d + 5156\ d^2 - 2432\ d^3 + 619\ d^4 - 84\ d^5 + 5\ d^6\Big) \\
&+ 16\ C_A \Big(192 - 288\ d - 28\ d^2 + 322\ d^3 - 255\ d^4 + 89\ d^5 - 15\ d^6 + d^7\Big) \\
&+ 4 \Big(-4 + d\Big)^2 \Big(48 - 56\ d + 35\ d^2 - 13\ d^3 + 2\ d^4\Big)\ n_f\Bigg\} + A_3 \Big(-8 + 3\ d\Big) \\
&\times \Bigg\{C_A \Big(6881280 - 11370496\ d - 5231104\ d^2 + 24600896\ d^3 - 18172384\ d^4 \\
&- 2105928\ d^5 + 11581460\ d^6 - 8688682\ d^7 + 3558513\ d^8 - 910210\ d^9 \\
&+ 146166\ d^{10} - 13592\ d^{11} + 561\ d^{12}\Big) - 2 \Big(C_F\ d \Big(-13110272 + 59524736\ d \\
&- 119256896\ d^2 + 139910176\ d^3 - 107260248\ d^4 + 56580636\ d^5 - 20992430\ d^6 \\
&+ 5484477\ d^7 - 989746\ d^8 + 117582\ d^9 - 8276\ d^{10} + 261\ d^{11}\Big) - 16 \Big(-4 + d\Big)^2 \\
&\times \Big(13440 - 1936\ d - 63236\ d^2 + 124494\ d^3 - 119835\ d^4 + 68959\ d^5 \\
&- 24789\ d^6 + 5463\ d^7 - 676\ d^8 + 36\ d^9\Big)\ n_f\Big)\Bigg\} + 2 \Big(-12 + 19\ d - 8\ d^2 + d^3\Big) \\
&\times \Bigg(2\ A_6 \Big(-4 + d\Big)^2 d \Big(-16 + 30\ d - 17\ d^2 + 3\ d^3\Big) \Bigg\{2\ C_F \Big(-288 + 192\ d \\
&- 16\ d^2 - 6\ d^3 + d^4\Big) + C_A \Big(416 - 192\ d + 24\ d^2 - 14\ d^3 + 3\ d^4\Big)\Bigg\} \\
&+ A_4 \Bigg\{C_A \Big(6881280 - 23730176\ d + 37309440\ d^2 - 32039680\ d^3 \\
&+ 15239072\ d^4 - 3272584\ d^5 - 345324\ d^6 + 415114\ d^7 - 116931\ d^8 + 17142\ d^9
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&- 1361\ d^{10} + 48\ d^{11}\Big) - 2\ \Big(C_F\ d\ \Big(677888 - 2026112\ d + 2909696\ d^2 \\
&- 2895040\ d^3 + 2126552\ d^4 - 1101532\ d^5 + 384546\ d^6 - 87351\ d^7 + 12286\ d^8 \\
&- 965\ d^9 + 32\ d^{10}\Big) - 4\ \Big(860160 - 1270784\ d + 218048\ d^2 + 766736\ d^3 \\
&- 743952\ d^4 + 330352\ d^5 - 81952\ d^6 + 10967\ d^7 - 533\ d^8 - 36\ d^9 \\
&+ 4\ d^{10}\Big)\ n_f\Big)\Big\}\Big)\Big)\Big]\Big/\Big[16\ \Big(-4+d\Big)^3\ \Big(-3+d\Big)\ \Big(-2+d\Big)^2\ \Big(-1+d\Big)^2\ d \\
&\times\Big(-7+2\ d\Big)\ \Big(-5+2\ d\Big)\ \Big(-8+3\ d\Big)\Big]\,. \qquad (3.15)
\end{aligned}
$$

The exact results for the master integrals A_i $(i = \{2, LO\}, 3, 4, 6)$ can be expressed in terms of Euler Gamma functions and are available in the works on two-loop electroweak form factors [26]. The most difficult crossed two-loop master integral A_6 was solved exactly in [26]. These results are used to present the form factors to order $\mathcal{O}(\varepsilon^4)$ and are given in the appendix. We use them to study the infrared pole structure of these farm factors in the next section. The higher order terms $\mathcal{O}(\varepsilon^i), i > 0$ are also useful to perform ultraviolet renormalisation of the form factors beyond two-loop level.

3.2 Infrared divergence structure

Having obtained these form factors at two-loop level, the next step is to study the infrared pole structure of these factors in order to establish the universal behaviour of these QCD amplitudes. In the past, there have been detailed studies of quark and gluon form factors through Sudakov integro-differential equation [34–38], see also [21,39–42]. Since the KK modes are colour singlet fields, the unrenormalised form factors $\hat{F}_I^T(\hat{a}_s, Q^2, \mu^2, \varepsilon)$ are expected to satisfy similar integro-differential equation that follows from the gauge as well as renormalisation group (RG) invariances. In dimensional regularisation,

$$Q^2 \frac{d}{dQ^2} \ln \hat{F}_I^T \left(\hat{a}_s, Q^2, \mu^2, \varepsilon\right) = \frac{1}{2}\Bigg[K^{T,I}\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \varepsilon\right) + G^{T,I}\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \varepsilon\right)\Bigg], \tag{3.16}$$

where the constants $K^{T,I}$ contain all the poles in ε, and $G^{T,I}$ are finite as ε becomes zero. The RG invariance of $\hat{F}_I^T$ gives

$$\mu_R^2 \frac{d}{d\mu_R^2} K^{T,I}\left(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \varepsilon\right) = -A^{T,I}(a_s(\mu_R^2)),$$

$$\mu_R^2 \frac{d}{d\mu_R^2} G^{T,I}\left(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \varepsilon\right) = A^{T,I}(a_s(\mu_R^2)). \tag{3.17}$$

The quantities $A^{T,I}$ are the cusp anomalous dimensions which are expanded as

$$A^{T,I}(a_s(\mu_R^2)) = \sum_{i=1}^{\infty} a_s^i(\mu_R^2)\ A_i^{T,I}. \tag{3.18}$$

Solving these RG equations, the constants $K^{T,I}$ and $G^{T,I}$ can be obtained in powers of bare coupling constant $\hat{a}_s$. Using these solutions, we obtain,

$$\ln \hat{F}_I^T(\hat{a}_s, Q^2, \mu^2, \varepsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2}{\mu^2}\right)^{i\frac{\varepsilon}{2}} S_\varepsilon^i\ \hat{\mathcal{L}}_{FT}^{I(i)}(\varepsilon), \tag{3.19}$$

where

$$\hat{\mathcal{L}}_{FT}^{I(1)} = \frac{1}{\varepsilon^2}\Bigg(-2A_1^{T,I}\Bigg) + \frac{1}{\varepsilon}\Bigg(G_1^{T,I}(\varepsilon)\Bigg),$$

$$\hat{\mathcal{L}}_{FT}^{I(2)} = \frac{1}{\varepsilon^3}\Bigg(\beta_0 A_1^{T,I}\Bigg) + \frac{1}{\varepsilon^2}\Bigg(-\frac{1}{2}A_2^{T,I} - \beta_0 G_1^{T,I}(\varepsilon)\Bigg) + \frac{1}{2\varepsilon}G_2^{T,I}(\varepsilon). \tag{3.20}$$

The cusp anomalous dimensions $A_i^{T,I}$ can be obtained by comparing eqns.(3.19,3.20) and the results of the form factors, eqns(A.1,A.3,A.4,A.5). We find that they are identical to those obtained in [43], that is, those appearing in gluon and quark form factors, confirming the universality of IR structure of these form factors. The coefficients $G_i^{T,I}(\varepsilon)$ take the following form

$$G_1^{T,I}(\varepsilon) = 2\ B_1^{T,I} + f_1^{T,I} + \sum_{k=1}^{\infty} \varepsilon^k g_1^{\ T,I,k},$$

$$G_2^{T,I}(\varepsilon) = 2\ B_2^{T,I} + f_2^{T,I} - 2\beta_0 g_1^{\ T,I,1} + \sum_{k=1}^{\infty} \varepsilon^k g_2^{\ T,I,k}, \tag{3.21}$$

where again the collinear anomalous dimension $B_i^{T,I}$ and soft anomalous dimension $f_i^{T,I}$ are found to be identical to B_i^I and f_i^I obtained in [21, 44] for quark and gluon form factors. We find that only ${g_i}^{T,I,k}$ are operator dependent.

Another independent check on our computation is done by establishing the connection between these form factors and the very successful proposal by Catani [45] (also see [46]) on one and two-loop QCD amplitudes using the universal factors $\mathbf{I}_I^{(i)}(\varepsilon)$ and $\mathbf{H}_I^{(i)}$, $i = 1, 2$. The all order generalisation of Catani's proposal was obtained by Becher and Neubert [47] and also by Gardi and Magnea [48]. These universal factors capture all the IR poles of n-parton QCD amplitudes up to two-loop level in QCD. Following [45], we proceed by expressing the matrix elements in terms of UV renormalised ones as

$$\hat{\mathcal{M}}_I = \mathbf{M}_I^{(0)} + a_s(\mu_R^2)\mathbf{M}_I^{(1)} + a_s^2(\mu_R^2)\mathbf{M}_I^{(2)} + \mathcal{O}\left(a_s^3(\mu_R^2)\right) \,, \qquad I = g, q, \overline{q}\,. \quad (3.22)$$

Using the universal $\mathbf{I}_I(\varepsilon)$ obtained by Catani, we can write down

$$\begin{aligned} \mathbf{M}_I^{(1)} &= 2\mathbf{I}_I^{(1)}(\varepsilon)\mathbf{M}_I^{(0)}(\varepsilon) + \mathbf{M}_{I,fin}^{(1)}(\varepsilon)\,, \\ \mathbf{M}_I^{(2)} &= 2\mathbf{I}_I^{(1)}(\varepsilon)\mathbf{M}_I^{(1)}(\varepsilon) + 4\mathbf{I}_I^{(2)}(\varepsilon)\mathbf{M}_I^{(0)}(\varepsilon) + \mathbf{M}_{I,fin}^{(2)}(\varepsilon)\,. \end{aligned} \quad (3.23)$$

In terms of these $\mathbf{M}_I^{(i)}$, we find

$$\begin{aligned} \hat{F}_I^{T,(1)} &= 2\mu_R^{\varepsilon}\mathbf{I}_I^{(1)}(\varepsilon) + \hat{F}_{I,fin}^{T,(1)}(\varepsilon)\,, \\ \hat{F}_I^{T,(2)} &= 4\mu_R^{2\varepsilon}\Bigg[\left(\mathbf{I}_I^{(1)}(\varepsilon)\right)^2 + \mathbf{I}_I^{(2)}(\varepsilon) - \frac{\beta_0}{\varepsilon}\left(\mathbf{I}_I^{(1)}(\varepsilon) + \frac{\mu_R^{-\varepsilon}}{2}\hat{F}_{I,fin}^{T,(1)}(\varepsilon)\right) \\ &\quad + \frac{1}{2}\mu_R^{-\varepsilon}\mathbf{I}_I^{(1)}(\varepsilon)\hat{F}_{I,fin}^{T,(1)}(\varepsilon)\Bigg] + \hat{F}_{I,fin}^{T,(2)}(\varepsilon)\,, \end{aligned} \quad (3.24)$$

where

$$\hat{F}_{I,fin}^{T,(i)}(\varepsilon) = \mu_R^{i\varepsilon}\frac{\mathbf{M}_I^{(0)*}\cdot\mathbf{M}_{I,fin}^{(i)}}{\mathbf{M}_I^{(0)*}\cdot\mathbf{M}_I^{(0)}}\,, \qquad i = 1, 2\,. \quad (3.25)$$

The singular universal functions $\mathbf{I}_I^{(i)}$ are given by

$$\begin{aligned}
\mathbf{I}_q^{(1)}(\varepsilon) &= -\frac{e^{-\varepsilon\gamma_E/2}}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}\left(\frac{Q^2}{\mu_R^2}\right)^{\frac{\varepsilon}{2}}\left(4\frac{C_F}{\varepsilon^2}-3\frac{C_F}{\varepsilon}\right), \\
\mathbf{I}_g^{(1)}(\varepsilon) &= -\frac{e^{-\varepsilon\gamma_E/2}}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}\left(\frac{Q^2}{\mu_R^2}\right)^{\frac{\varepsilon}{2}}\left(4\frac{C_A}{\varepsilon^2}-\frac{\beta_0}{\varepsilon}\right), \qquad (3.26)
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_I^{(2)}(\varepsilon) &= -\frac{1}{2}\left(\mathbf{I}_I^{(1)}(\varepsilon)\right)^2+\frac{\beta_0}{\varepsilon}\mathbf{I}_I^{(1)}(\varepsilon) \\
&\quad +\frac{e^{\frac{\varepsilon\gamma_E}{2}}\Gamma(1+\varepsilon)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}\left(-\frac{\beta_0}{\varepsilon}+K\right)\mathbf{I}_I^{(1)}(2\varepsilon)+\mathbf{H}_I^{(2)}\frac{1}{\varepsilon}, \qquad (3.27)
\end{aligned}$$

and

$$K=\left(\frac{67}{18}-\zeta_2\right)C_A-\frac{10}{9}T_F n_f\,. \qquad (3.28)$$

Using our results for $\hat{F}^{T,(i)}$ given in eqns.(A.1,A.3,A.4,A.5) and the results for $\mathbf{I}_I^{(i)}$ given in [45], we obtain $\mathbf{H}_I^{(2)}$:

$$\begin{aligned}
\mathbf{H}_g^{(2)} &= C_A^2\left(-\frac{5}{12}-\frac{11}{24}\zeta_2-\frac{1}{2}\zeta_3\right)+C_A n_f\left(\frac{29}{27}+\frac{1}{12}\zeta_2\right) \\
&\quad +C_F n_f\left(-\frac{1}{2}\right)+n_f^2\left(-\frac{5}{27}\right), \qquad (3.29) \\
\mathbf{H}_q^{(2)} &= C_F^2\left(\frac{3}{8}-3\zeta_2+6\zeta_3\right)+C_A C_F\left(-\frac{245}{216}+\frac{23}{8}\zeta_2-\frac{13}{2}\zeta_3\right) \\
&\quad +C_F n_f\left(\frac{25}{108}-\frac{1}{4}\zeta_2\right). \qquad (3.30)
\end{aligned}$$

The single pole coefficients thus obtained agree with the color diagonal part of eqn.(12) of [47] (see also eqn.(4.21) of [21] for quark and gluon form factors and [49–52] for four parton amplitudes). This serves as a check on our computation and also establishes the proposal by Catani on IR universality of QCD amplitudes with $T_{\mu\nu}$ insertion.

3.3 Conclusions

We study an important ingredient to the full NNLO QCD correction to graviton mediated hadronic scattering processes namely the gluon and quark form factors of energy momentum tensor of the QCD part of the SM up to two-loop level in QCD. We have used dimensional regularisation to obtain these form factors in $SU(N)$ gauge theory with n_f light flavours. Both exact as well as expanded results in ε are presented. The higher order terms in ε of these form factors are important for the ultraviolet renormalisation of these amplitudes at three-loop level. We have shown that these form factors satisfy Sudakov integro-differential equation with same cusp A_I, collinear B^I and soft f^I anomalous dimensions that contribute to electroweak vector boson and gluon form factors. In addition, they also show the universal behaviour of the infrared poles in ε in accordance with the proposal by Catani.

Spin-2 resonance production has been widely studied in the context of the Higgs [53] and BSM models [54]. This two-loop results would further reduce the theoretical uncertainties and hence improve the predictions in disentangling the various postulates. We further apply these two-loop results to the TeV scale gravity models [55] in the next chapter.

Chapter 4

Next to Next to Leading Order soft plus virtual QCD corrections in Models of TeV Scale Gravity

In this chapter we study the NNLO QCD corrections to the graviton production at the LHC for the ADD and RS models, within the soft-virtual approximation. Although a full NNLO calculation requires the evaluation of the double real radiation, real emission from one-loop corrections and the pure virtual two-loop amplitudes, the dominant terms are given by the soft and virtual contributions, which can be obtained in a simpler way. This fact is a general feature of the production of a large invariant mass system in hadronic collisions. Since parton distributions grow fast for small fractions of the hadron momentum, the partonic center-of-mass energy tends to be close to the system invariant mass, and the remaining energy only allows for the emission of soft particles. For this reason, the soft-virtual (SV) approximation is expected to be accurate for a large number of processes.

In the next section of this chapter we present the partonic cross sections for di-lepton production in the ADD model and for single graviton production in the RS model. In section 2 we analyse the phenomenological results for the LHC. Finally, in section 3 we present our conclusions.

4.1 Partonic Cross Section

Using the results of the spin-2 form factor, the complete two-loop corrections for single graviton production and di-lepton production mediated by a graviton can be obtained, for both gluon-gluon and quark-antiquark partonic subprocesses. These contributions include the interference between the two-loop and the tree-level amplitudes and the square of the one-loop amplitudes. These results, computed within the dimensional regularization scheme, are of course divergent in the limit $n \to 4$, being n the space-time dimension. To obtain a finite and physically meaningful quantity we have to add the corresponding real corrections, which cancel the infrared divergences.

On the other hand, in Ref. [56] a universal formula is derived for the NNLO inclusive cross section of any colourless final state process within the soft-virtual approximation. This formula depends on the particular process only through an infrared regulated part of the one and two-loop corrections, which can be obtained from the full virtual result [56]. In this way we can obtain the NNLO corrections to single graviton production and gravity mediated di-lepton production within the soft-virtual approximation. We have also obtained the NNLO-SV result by summing explicitly the soft contributions of Refs. [39–42, 57, 58], arriving to the same results.

We provide here the final results, including the previous orders contributions. For the sake of brevity, we refer to Ref. [59, 60] for the SM contribution to the di-lepton production cross section. We remark that, as it was already noticed in Ref. [61], the interference between SM and gravity contribution to the di-lepton production invariant mass distribution identically vanishes.

We begin with the ADD model. The graviton contribution to the di-lepton invariant mass (Q) distribution at the parton level can be cast in the following way:

$$\frac{d\hat{\sigma}}{dQ^2} = \mathcal{F}_{\text{ADD}}\, z\, \Delta_{ab}(z)\,, \tag{4.1}$$

where $z = Q^2/s$, being s the partonic center-of-mass energy, and a, b denote the type of massless partons ($a, b = g, q, \bar{q}$, with n_f different flavours of light quarks). The constant $\mathcal{F}_{\text{ADD}}$ takes the following form:

$$\mathcal{F}_{\text{ADD}} = \frac{\kappa^4 Q^4}{640\pi^2} \left|\mathcal{D}(Q^2)\right|^2 , \tag{4.2}$$

where the function $\mathcal{D}(Q^2)$ can be expressed as [9]

$$\mathcal{D}(Q^2) = 16\pi \left(\frac{Q^{d-2}}{\kappa^2 M_S^{d+2}}\right) I\left(\frac{M_S}{Q}\right) . \tag{4.3}$$

The integral I is regulated by an ultraviolet cutoff, presumably of the order of M_S [9, 10]. This sets the limit on the applicability of the effective theory (for the di-lepton production this consistency would imply $Q < M_S$). The summation over the non-resonant KK modes yields

$$I(\omega) = -\sum_{k=1}^{d/2-1} \frac{1}{2k}\omega^{2k} - \frac{1}{2}\log(\omega^2 - 1) , \qquad d = \text{even} , \tag{4.4}$$

$$I(\omega) = -\sum_{k=1}^{(d-1)/2} \frac{1}{2k-1}\omega^{2k-1} + \frac{1}{2}\log\left(\frac{\omega+1}{\omega-1}\right) , \qquad d = \text{odd} . \tag{4.5}$$

On the other hand, for the RS model we have for the single graviton production cross section the following expression:

$$\hat{\sigma} = \mathcal{F}_{\rm RS}\, z\, \Delta_{ab}(z) , \tag{4.6}$$

where the constant $\mathcal{F}_{\rm RS}$ takes the following form:

$$\mathcal{F}_{\rm RS} = \frac{1}{\Lambda_\pi^2} . \tag{4.7}$$

Notice that in this case we have $z = M_1^2/s$.

The coefficient function $\Delta_{ab}(z)$, which is independent of the model considered, has a perturbative expansion in terms of powers of the QCD renormalized coupling $\alpha_{\rm S}$:

$$\Delta_{ab}(z) = \sum_{i=0}^{\infty} \left(\frac{\alpha_{\rm S}}{2\pi}\right)^i \Delta_{ab}^{(i)}(z) . \tag{4.8}$$

Soft plus virtual contributions Computation

The production cross section of a heavy particle, namely, a KK graviton or a pair of leptons at the hadron colliders can be computed using

$$\begin{aligned} \sigma^I(s, q^2) &= \sum_{ab} \int dx_1 dx_2 f_a(x_1, \mu_F^2) f_b(x_2, \mu_F^2) \\ &\quad \times \hat{\sigma}_{ab}^I(\hat{s}, q^2, \mu_F^2), \end{aligned} \tag{4.9}$$

where $\hat{s} = x_1 x_2 s$, s is the hadronic center of mass energy, and $\hat{\sigma}^I_{ab}$ is the partonic cross section with initial state partons a and b. $I = g$ for a KK graviton production with $q^2 = m_G^2$ and $I = q$ for DY production with invariant mass of the dileptons being q^2. μ_F is the factorization scale. The threshold contribution at the partonic level, denoted by $\Delta_I^{\rm SV}(z, q^2, \mu_R^2, \mu_F^2)$, normalized by Born cross section $\hat{\sigma}^{I,(0)}_{ab}$ times the Wilson coefficient $C_W^I(\mu_R^2)$, is given by

$$\Delta_I^{\rm SV}(z, q^2, \mu_R^2, \mu_F^2) = \mathcal{C} \exp(\Psi^I(z, q^2, \mu_R^2, \mu_F^2, \epsilon))|_{\epsilon=0} \,, \tag{4.10}$$

where μ_R is the renormalization scale, the dimensionless variable $z = q^2/\hat{s}$, and $\Psi^I(z, q^2, \mu_R^2, \mu_F^2, \epsilon)$ is a finite distribution. The symbol $\mathcal{C}$ implies convolution with the following expansion

$$\mathcal{C} e^{f(z)} = \delta(1-z) + \frac{1}{1!} f(z) + \frac{1}{2!} f(z) \otimes f(z) + \cdots \,. \tag{4.11}$$

Here $\otimes$ means Mellin convolution and $f(z)$ is a distribution of the kind $\delta(1-z)$ and $\mathcal{D}_i$. In $d = 4 + \epsilon$ dimensions,

$$\begin{aligned}
\Psi^I(z, q^2, \mu_R^2, \mu_F^2, \epsilon) = &\Big\{ \ln \Big[Z^I(\hat{a}_s, \mu_R^2, \mu^2, \epsilon) \Big]^2 \\
&+ \ln \Big| \hat{F}^I(\hat{a}_s, Q^2, \mu^2, \epsilon) \Big|^2 \Big\} \delta(1-z) \\
&+ 2\Phi^I(\hat{a}_s, q^2, \mu^2, z, \epsilon) - 2\mathcal{C} \ln \Gamma_{II}(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon) \,,
\end{aligned}$$

where μ is the scale introduced to define the dimensionless coupling constant $\hat{a}_s = \hat{g}_s^2/16\pi^2$ in dimensional regularization, $Q^2 = -q^2$, $Z^I(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)$ is the overall operator renormalization constant, which satisfies

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z^I(\hat{a}_s, \mu_R^2, \mu^2, \epsilon) = \sum_{i=1}^{\infty} a_s^i(\mu_R^2) \gamma_{i-1}^I \,,$$

where $a_s(\mu_R^2)$ is the renormalized coupling constant that is related to $\hat{a}_s$ through strong coupling constant renormalization $Z(a_s(\mu_R^2))$, that is $\hat{a}_s = (\mu/\mu_R)^\epsilon Z(\mu_R^2) S_\epsilon^{-1} a_s(\mu_R^2)$, $S_\epsilon = \exp\left[(\gamma_E - \ln 4\pi)\epsilon/2\right]$. Because of the gauge and renormalization group invariance, the bare form factors $\hat{F}^I(\hat{a}_s, Q^2, \mu^2, \epsilon)$ satisfy the following differential equation [63] :

$$Q^2 \frac{d}{dQ^2} \ln \hat{F}^I = \frac{1}{2} \Big[K^I(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon) + G^I(\hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon) \Big] \,,$$

where K^I contains all the poles in ϵ and G^I contains the terms finite in ϵ. Renormalization group invariance of $\hat{F}^I(\hat{a}_s, Q^2, \mu^2, \epsilon)$ gives

$$\mu_R^2 \frac{d}{d\mu_R^2} K^I = -\mu_R^2 \frac{d}{d\mu_R^2} G^I = -\sum_{i=1}^{\infty} a_s^i(\mu_R^2) A_i^I \,.$$

A_i^I's are the cusp anomalous dimensions. Expanding the μ_R^2 independent part of the solution of the RG equation for G^I, $G^I(a_s(Q^2), 1, \epsilon) = \sum_{i=1}^{\infty} a_s^i(Q^2) G_i^I(\epsilon)$, one finds that G_i^I can be decomposed in terms of collinear B_i^I and soft f_i^I anomalous dimensions as follows [64]:

$$G_i^I(\epsilon) = 2(B_i^I - \gamma_i^I) + f_i^I + C_i^I + \sum_{k=1}^{\infty} \epsilon^k g_i^{I,k} \,, \tag{4.12}$$

where $C_1^I = 0$, $C_2^I = -2\beta_0 g_1^{I,1}$, $C_3^I = -2\beta_1 g_1^{I,1} - 2\beta_0(g_2^{I,1} + 2\beta_0 g_1^{I,2})$, $C_4^I = -2\beta_2 g_1^{I,1} - 2\beta_1(g_2^{I,1} + 4\beta_0 g_1^{I,2}) - 2\beta_0(g_3^{I,1} + 2\beta_0 g_2^{I,2} + 4\beta_0^2 g_1^{I,3})$ and β_i are the coefficients of the β function of strong coupling constant $a_s(\mu_R^2)$, $\mu_R^2 da_s(\mu_R^2)/d\mu_R^2 = \epsilon a_s(\mu_R^2)/2 - \sum_{i=0}^{\infty} \beta_i a_s^{i+2}(\mu_R^2)$. The coefficients $g_i^{I,k}$ can be obtained from the form factors [65].

The mass factorization kernel $\Gamma(z, \mu_F^2, \epsilon)$ removes the collinear singularities which arise due to massless partons and it satisfies the following RG equation :

$$\mu_F^2 \frac{d}{d\mu_F^2} \Gamma(z, \mu_F^2, \epsilon) = \frac{1}{2} P\left(z, \mu_F^2\right) \otimes \Gamma\left(z, \mu_F^2, \epsilon\right) \,. \tag{4.13}$$

$P(z, \mu_F^2)$ are Altarelli-Parisi splitting functions. In perturbative QCD, $P(z, \mu_F^2) = \sum_{i=1}^{\infty} a_s^i(\mu_F^2) P^{(i-1)}(z)$. We find that only diagonal elements of the kernel, $\Gamma_{II}(\hat{a}_s, \mu_F^2, \mu^2, z, \epsilon)$ contribute to threshold corrections because they contain $\delta(1-z)$ and $\mathcal{D}_0$ at every order perturbation theory while the nondiagornal ones are regular functions in z, that is, $P_{II}^{(i)}(z) = 2\left[B_{i+1}^I \delta(1-z) + A_{i+1}^I \mathcal{D}_0\right] + P_{reg,II}^{(i)}(z)$.

The finiteness of $\Delta_I^{\rm SV}$ demands that the soft distribution function $\Phi^I(\hat{a}_s, q^2, \mu^2, z, \epsilon)$ also satisfies Sudakov-type differential equations [42, 57], namely,

$$q^2 \frac{d}{dq^2} \Phi^I = \frac{1}{2}\left[\overline{K}^I(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \epsilon) + \overline{G}^I(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon)\right] .$$

$\overline{K}^I$ and $\overline{G}^I$ take the forms similar to those of K^I and G^I of the form factors in such a way that Ψ^I is finite as $\epsilon \to 0$. The solution to the above equation is found to be

$$\Phi^I = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \left(\frac{i\epsilon}{1-z}\right) \hat{\phi}^{I,(i)}(\epsilon) \tag{4.14}$$

where $\hat{\phi}^{I,(i)}(\epsilon) = [\overline{K}^{I,(i)}(\epsilon) + \overline{G}^{I,(i)}(\epsilon)]/i\epsilon$ and $\mu_R^2 d\overline{K}^I/d\mu_R^2 = -\delta(1-z)\mu_R^2 dK^I/d\mu_R^2$. This implies that $\overline{K}^{I,(i)}(\epsilon)$ can be written in terms of A_i^I and β_i. We define $\overline{\mathcal{G}}_i^I(\epsilon)$ through

$$\sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q_z^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \overline{G}^{I,(i)}(\epsilon) = \sum_{i=1}^{\infty} a_s^i(q_z^2)\overline{\mathcal{G}}_i^I(\epsilon) \tag{4.15}$$

where $q_z^2 = q^2(1-z)^2$. Using the fact that $\Delta_I^{\rm SV}$ is finite as $\epsilon \to 0$, we can express $\overline{\mathcal{G}}_i^I(\epsilon)$ in the form similar to that of $G_i^I(\epsilon)$ by setting $\gamma_i^I = 0, B_i^I = 0$ and replacing $f_i^I \to -f_i^I$ and $g_i^{I,j} \to \overline{\mathcal{G}}_i^{I,j}$. The unknown constants $\overline{\mathcal{G}}_i^{I,j}$ can be extracted from the soft part of the partonic reactions. Since Φ^I results from the soft radiations, the constants $\overline{\mathcal{G}}_i^I(\epsilon)$ are found to be maximally non-abelian [42, 57] satisfying

$$\overline{\mathcal{G}}_i^q(\epsilon) = \frac{C_F}{C_A}\overline{\mathcal{G}}_i^g(\epsilon) \tag{4.16}$$

with $C_A = N$, $C_F = (N^2-1)/2N$, N is the number of colors. Equation (4.16) implies that the entire soft distribution function for the DY production can be obtained from that of Higgs boson production. Substituting Z^I, the solutions for both $\hat{F}^I$ and Φ^I, and Γ_{II} in Eq. (4.10), we obtain $\Delta_I^{\rm SV}$ in powers of $a_s(\mu_R^2)$ as

$$\begin{aligned} \Delta_I^{\rm SV}(z) &= \sum_{i=0}^{\infty} a_s^i(\mu_R^2)\Delta_{I,i}^{\rm SV}(z,\mu_R^2)\,, \qquad \text{where} \\ \Delta_{I,i}^{\rm SV} &= \Delta_{I,i}^{\rm SV}(\mu_R^2)|_\delta \delta(1-z) + \sum_{j=0}^{2i-1}\Delta_{I,i}^{\rm SV}(\mu_R^2)|_{\mathcal{D}_j}\mathcal{D}_j\,. \end{aligned} \tag{4.17}$$

We have set $\mu_R^2 = \mu_F^2 = q^2$ and their dependence can be retrieved using the appropriate renormalization group equation. $\Delta_{I,i}^{\rm SV}(Q^2)$ are finite and they depend on the anomalous dimensions A_i^I, B_i^I, f_i^I and γ_i^I, the β functions coefficients β_i and ϵ expansion coefficients of $G^I(\epsilon)$, $g_j^{I,i}$'s and of the corresponding $\overline{\mathcal{G}}^I(\epsilon)$, $\overline{\mathcal{G}}_j^{I,i}$'s. Up to the two-loop level, all these terms are known to sufficient accuracy to obtain $\Delta_{I,1}^{\rm SV}$ and $\Delta_{I,2}^{\rm SV}$ exactly.

4.1.1 Results

At LO we only have nonzero contributions from $ab = gg$ and $ab = q\bar{q}$ (always equal to $ab = \bar{q}q$), which take the following form:

$$\Delta^{(0)}_{q\bar{q}} = \frac{\pi}{8N_c}\delta(1-z)\,, \tag{4.18}$$

$$\Delta^{(0)}_{gg} = \frac{\pi}{2(N_c^2-1)}\delta(1-z)\,. \tag{4.19}$$

Here N_c stands for the number of quark colors ($N_c = 3$). The NLO contributions, which have been calculated in Ref. [61], can be written in the following way:

$$\begin{aligned}\Delta^{(1)}_{q\bar{q}} &= \left(\frac{\pi}{8N_c}\right) C_F \Bigg[\Big(-10+4\zeta_2\Big)\delta(1-z) + 4\,\mathcal{D}_0 \ln\left(\frac{Q^2}{\mu_F^2}\right) + 8\,\mathcal{D}_1 \\ &+3\delta(1-z)\ln\left(\frac{Q^2}{\mu_F^2}\right) - 2(1+z)\ln\left(\frac{Q^2(1-z)^2}{\mu_F^2 z}\right) - 4\frac{\ln(z)}{1-z} + \frac{8}{3z} - \frac{8z^2}{3}\Bigg]\,,\end{aligned} \tag{4.20}$$

$$\Delta^{(1)}_{q(\bar{q})g} = \left(\frac{\pi}{8N_c}\right)\left[(-\frac{7}{2}+\frac{4}{z}+z+z^2)\ln\left(\frac{Q^2(1-z)^2}{\mu_F^2 z}\right) + \frac{9}{4} - \frac{3}{z} + \frac{9}{2}z - \frac{7}{4}z^2\right] \tag{4.21}$$

$$\begin{aligned}\Delta^{(1)}_{gg} &= \left(\frac{\pi}{2(N_c^2-1)}\right) C_A \Bigg[\Big(-\frac{203}{18}+4\zeta_2\Big)\delta(1-z) + 4\,\mathcal{D}_0 \ln\left(\frac{Q^2}{\mu_F^2}\right) + 8\,\mathcal{D}_1 \\ &+\frac{11}{3}\delta(1-z)\ln\left(\frac{Q^2}{\mu_F^2}\right) + 4(-2+\frac{1}{z}+z-z^2)\ln\left(\frac{Q^2(1-z)^2}{\mu_F^2 z}\right) - 4\frac{\ln(z)}{(1-z)} \\ &-1-\frac{11}{3z}+z+\frac{11z^2}{3}\Bigg] + \left(\frac{\pi}{2(N_c^2-1)}\right) n_f \left[\left(\frac{35}{18}-\frac{2}{3}\ln\left(\frac{Q^2}{\mu_F^2}\right)\right)\delta(1-z)\right].\end{aligned} \tag{4.22}$$

Here μ_F and μ_R stand for the factorization and renormalization scales, and the $SU(N_c)$ Casimir operators are $C_F = \frac{N_c^2-1}{2N_c}$ and $C_A = N_c$. We have also defined the distributions $\mathcal{D}_i$ as

$$\mathcal{D}_i = \left(\frac{\ln^i(1-z)}{1-z}\right)_+\,, \tag{4.23}$$

where the + symbol indicates the usual plus-prescription:

$$\int_0^1 dz\, f_+(z)\, g(z) = \int_0^1 dz\, f(z)\,[g(z)-g(1)]\,. \tag{4.24}$$

The Riemann zeta function is denoted by $\zeta_i \equiv \zeta(i)$.

We present below the NNLO results in the soft-virtual approximation. Within this approximation we have only contributions to the gluon-gluon and quark-antiquark subprocesses, since the terms proportional to $\delta(1-z)$ and $\mathcal{D}_i$ (which are the ones we obtain within the SV approximation) are absent in other channels. The result for the quark-antiquark subprocess is the following:

$$
\begin{aligned}
\Delta_{q\bar{q}}^{(2)SV} = & (\frac{\pi}{8N_c})C_F^2\Big\{\Big[\frac{2293}{48} - \frac{35}{2}\zeta_2 - 31\zeta_3 + \zeta_4 + (-\frac{117}{4} + 6\zeta_2 + 44\zeta_3)\ln(\frac{Q^2}{\mu_F^2}) \\
& + (\frac{9}{2} - 8\zeta_2)\ln^2(\frac{Q^2}{\mu_F^2})\Big]\delta(1-z) + 64\zeta_3\mathcal{D}_0 - (80+32\zeta_2)\mathcal{D}_1 + 32\mathcal{D}_3 \\
& - 8(\mathcal{D}_0(2\zeta_2+5) - 3(\mathcal{D}_1 + 2\mathcal{D}_2))\ln(\frac{Q^2}{\mu_F^2}) + 4(3\mathcal{D}_0 + 4\mathcal{D}_1)\ln^2(\frac{Q^2}{\mu_F^2})\Big\} \\
& + (\frac{\pi}{8N_c})C_A C_F\Big\{\Big[-\frac{5941}{144} + \frac{82}{9}\zeta_2 + 23\zeta_3 - \frac{3}{2}\zeta_4 + (\frac{22}{3}\zeta_2 - 6\zeta_3 + \frac{17}{12})\ln(\frac{\mu_R^2}{\mu_F^2}) \\
& + \frac{11}{4}\ln^2(\frac{\mu_R^2}{\mu_F^2}) + \frac{1}{4}\ln(\frac{\mu_R^2}{Q^2})(24\zeta_3 - 11\ln(\frac{\mu_R^2}{Q^2}) - 79)\Big]\delta(1-z) \\
& + (-\frac{404}{27} + \frac{44}{3}\zeta_2 + 14\zeta_3)\mathcal{D}_0 + (\frac{268}{9} - 8\zeta_2)\mathcal{D}_1 - \frac{44}{3}\mathcal{D}_2 - \frac{44}{3}\mathcal{D}_1\ln(\frac{Q^2}{\mu_R^2}) \\
& + \mathcal{D}_0\ln(\frac{Q^2}{\mu_F^2})[\frac{11}{3}\ln(\frac{\mu_R^4}{\mu_F^2 Q^2}) - 4\zeta_2 + \frac{134}{9}]\Big\} + (\frac{\pi}{8N_c})C_F n_f\Big\{\Big[\frac{461}{72} - \frac{16}{9}\zeta_2 + 2\zeta_3 \\
& + \frac{1}{2}\ln(\frac{\mu_R^2}{Q^2})(\ln(\frac{\mu_R^2}{Q^2}) + 7) - \frac{1}{6}\ln(\frac{\mu_R^2}{\mu_F^2})(8\zeta_2 + 3\ln(\frac{\mu_R^2}{\mu_F^2}) + 1)\Big]\delta(1-z) \\
& + (\frac{56}{27} - \frac{8}{3}\zeta_2)\mathcal{D}_0 - \frac{40}{9}\mathcal{D}_1 + \frac{8}{3}\mathcal{D}_2 + \frac{1}{9}[-2\mathcal{D}_0\ln(\frac{\mu_R^2}{\mu_F^2})(3\ln(\frac{\mu_R^2}{\mu_F^2}) + 10) \\
& + 6\mathcal{D}_0\ln^2(\frac{\mu_R^2}{Q^2}) + 4(5\mathcal{D}_0 - 6\mathcal{D}_1)\ln(\frac{\mu_R^2}{Q^2})]\Big\}\,. \qquad (4.25)
\end{aligned}
$$

On the other hand, the gluon-gluon contribution to the NNLO-SV partonic cross section takes the following form:

$$
\begin{aligned}
\Delta_{gg}^{(2)SV} = & (\frac{\pi}{2(N_c^2-1)})C_A^2\Big\{\Big[\frac{7801}{1296}-\frac{56}{9}\zeta_2-\frac{22}{3}\zeta_3-\frac{1}{2}\zeta_4-\ln(\frac{Q^2}{\mu_R^2})(\frac{121}{18}\ln(\frac{Q^2}{\mu_F^2}) \\
& +\frac{22\zeta_2}{3}-\frac{2233}{108})+\ln(\frac{Q^2}{\mu_F^2})((\frac{121}{12}-8\zeta_2)\ln(\frac{Q^2}{\mu_F^2})+\frac{44\zeta_2}{3}+38\zeta_3-\frac{1945}{54})\Big]\delta(1-z) \\
& +(-\frac{404}{27}+\frac{44}{3}\zeta_2+78\zeta_3)\mathcal{D}_0+(-\frac{544}{9}-40\zeta_2)\mathcal{D}_1-\frac{44}{3}\mathcal{D}_2+32\mathcal{D}_3 \\
& +\ln(\frac{Q^2}{\mu_F^2})[(\frac{55\mathcal{D}_0}{3}+16\mathcal{D}_1)\ln(\frac{Q^2}{\mu_F^2})-\mathcal{D}_0(20\zeta_2+\frac{272}{9})+\frac{88\mathcal{D}_1}{3}+48\mathcal{D}_2] \\
& -\ln(\frac{Q^2}{\mu_R^2})(\frac{22}{3}\mathcal{D}_0\ln(\frac{Q^2}{\mu_F^2})+\frac{44\mathcal{D}_1}{3})\Big\} \\
& +(\frac{\pi}{2(N_c^2-1)})C_A n_f\Big\{\Big[-\frac{2983}{648}-\frac{47}{18}\zeta_2+\frac{16}{3}\zeta_3+\ln(\frac{Q^2}{\mu_R^2})(\frac{22}{9}\ln(\frac{Q^2}{\mu_F^2}) \\
& +\frac{4\zeta_2}{3}-\frac{791}{108})-\ln(\frac{Q^2}{\mu_F^2})(\frac{11}{3}\ln(\frac{Q^2}{\mu_F^2})+\frac{8\zeta_2}{3}-\frac{719}{54})\Big]\delta(1-z) \\
& +(\frac{56}{27}-\frac{8}{3}\zeta_2)\mathcal{D}_0+\frac{100}{9}\mathcal{D}_1+\frac{8}{3}\mathcal{D}_2+\frac{4}{3}\ln(\frac{Q^2}{\mu_R^2})[\mathcal{D}_0\ln(\frac{Q^2}{\mu_F^2})+2\mathcal{D}_1] \\
& -\frac{2}{9}\ln(\frac{Q^2}{\mu_F^2})[15\mathcal{D}_0\ln(\frac{Q^2}{\mu_F^2})-25\mathcal{D}_0+24\mathcal{D}_1]\Big\} \\
& +(\frac{\pi}{2(N_c^2-1)})n_f^2\Big\{\frac{1225}{1296}+\frac{2}{3}\zeta_2+\frac{1}{27}\ln(\frac{\mu_F^2}{Q^2})[9\ln(\frac{\mu_F^2}{Q^2})-6\ln(\frac{\mu_R^2}{Q^2})+35] \\
& -\frac{35}{54}\ln(\frac{\mu_R^2}{Q^2})\Big\}\delta(1-z)+(\frac{\pi}{2(N_c^2-1)})C_F n_f[\frac{61}{12}-4\zeta_3+\ln(\frac{\mu_F^2}{Q^2})]\delta(1-z)\,.
\end{aligned}
\tag{4.26}
$$

These expressions are obtained by keeping only the most divergent terms of the real contributions when $z \to 1$, or equivalently, by keeping only the $\delta(1-z)$ and $\mathcal{D}_i$ distributions in the final result. However, the soft limit can be defined in a more natural way by working in Mellin (or N-moment) space, where instead of distributions in z the dominant contributions are given by continuous functions of the variable N. In fact, it was shown that large subleading terms arise when one attempts to formulate the soft-gluon resummation in z-space, and then all-order resummation cannot be systematically defined in z-space [62]. Also, in Refs. [56, 58] it was shown that the soft-virtual approximation yields better results at NLO and NNLO for Higgs boson production and the Drell-Yan process if defined in N-space.

We will therefore work within the N-space formulation, in which we take the Mellin transform of the coefficient function $\Delta_{ab}(z)$ and drop all those terms that vanish when $N \to \infty$, which is the Mellin space analogous of $z \to 1$.

4.2 Phenomenological Results

4.2.1 ADD Model

In this section we provide the phenomenological results for the LHC, for a center-of-mass energy $\sqrt{s_H} = 14$ TeV. Taking into account the bounds on M_S for different extra dimensions d obtained by ATLAS [66] and CMS [67] collaborations, we choose for our present analysis the following values: $M_S = 3.7$ TeV ($d = 2$), 3.8 TeV ($d = 3$), 3.2 TeV ($d = 4$), 2.9 TeV ($d = 5$) and 2.7 TeV ($d = 6$). We remark that for the SM contribution to the di-lepton production cross section at NNLO we always use the exact result. On the other hand, for the soft-virtual approximation, used only in the NNLO graviton contributions, we always use the Mellin space definition.

To obtain the hadronic cross section we need to convolute the partonic result with the parton distribution functions (PDFs) in the following way:

$$\frac{d\sigma}{dQ^2}(s_H, Q^2) = \sum_{a,b} \int_0^1 dx_1 dx_2 f_{a/h_1}(x_1, \mu_F^2) f_{b/h_2}(x_2, \mu_F^2) \int_0^1 dz\, \delta\left(z - \frac{\tau}{x_1 x_2}\right) \frac{d\hat{\sigma}_{ab}}{dQ^2}(s, Q^2)\,, \tag{4.27}$$

where s_H is the hadronic center-of-mass energy, and $\tau = Q^2/s_H$. In all cases we use the MSTW2008 [68] sets of parton distributions (and QCD coupling) at each corresponding order.

In the first place we want to validate the use of the soft-virtual approximation, checking its accuracy at NLO, where the full result is known. We present the results for $d = 3$ and $M_S = 3.8$ TeV; we obtain similar results with the other sets of parameters.

In Figure 4.1 we show the ratio between the approximation and the full NLO result as a function of the di-lepton invariant mass. We also show the ratio between the previous order (LO) and the NLO cross section. We can observe that the soft-virtual approximation reproduces very accurately the full result, with differences smaller than

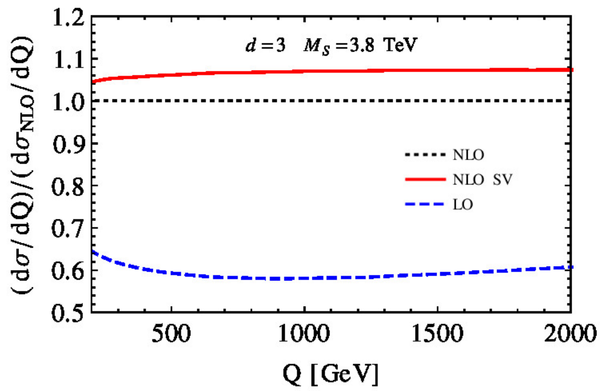


Figure 4.1: Ratio between the NLO-SV approximation and the full NLO result (red solid) compared with the ratio between the LO and the NLO cross sections (blue dashed) as a function of the di-lepton invariant mass.

10%. Using the NLO-SV result clearly improves the accuracy of the prediction, since the previous order fails to reproduce the NLO by a 40%. At NNLO we expect that the SV approximation will be even more accurate, since the size of the corrections is smaller. Comparing with other processes dominated by gluon fusion in which both NNLO-SV and full NNLO have been computed, such as single [19, 20, 60, 69, 70] and double [71, 72] Higgs production, we can expect differences with the exact NNLO result to be smaller than 5%. We recall that the contribution of the gluon-gluon subprocess dominates the graviton production at the LHC in the di-lepton invariant mass region of the current analysis. For instance, at LO it contributes with 73% of the cross section integrated between $Q = 200$ GeV and $Q = 2000$ GeV.

With respect to the theoretical uncertainty, for the total cross section in the range 200 GeV $\leq Q \leq$ 2000 GeV we find a scale variation close to 11% at NLO, while in the case of the NLO-SV this value is about 5%, so that at this order the approximation underestimates the uncertainty by a factor 2.

Once we have checked the validity of the approximation, we continue with the NNLO predictions. We recall that our NNLO results are computed using the exact NLO cross section, and then adding the soft-virtual approximation only for the NNLO gravity corrections. For the SM contributions we use the exact NNLO result. For simplicity, we will denote this computation as NNLO.

In Figure 4.2 we show the di-lepton invariant mass distribution for SM, GR

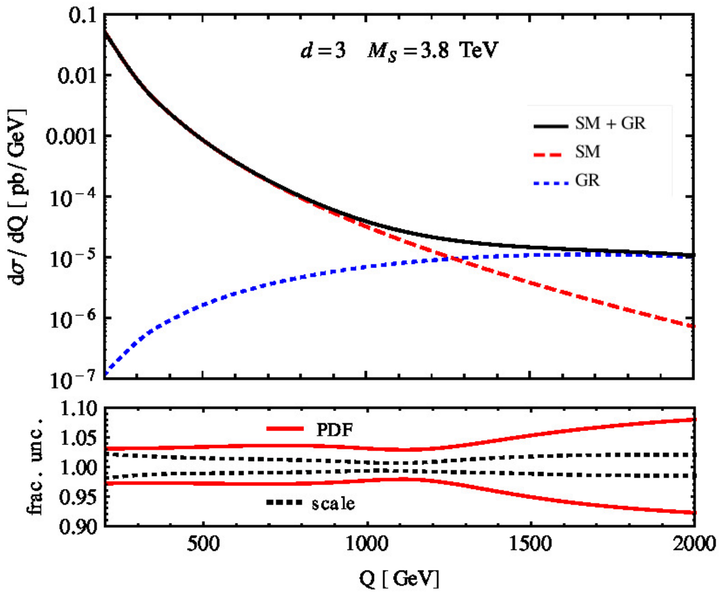


Figure 4.2: Di-lepton invariant mass distribution at the LHC ($\sqrt{s_H} = 14$ TeV) for SM (blue-dotted), gravity (red-dashed) and SM+GR (black-solid) at NNLO. The lower inset gives the fractional scale (black-dotted) and PDF (red-solid) uncertainties.

and SM+GR at NNLO. Deviations from the SM prediction can be observed for $Q \gtrsim 1000$ GeV. For $Q \simeq 1200$ GeV, the SM and gravity contributions are of the same order, while for larger values of invariant mass the graviton mediated processes dominate the cross section.

We have also considered two different sources of theoretical uncertainties: missing higher orders in the QCD perturbative expansion and uncertainties in the determination of the parton flux. To evaluate the size of the former we vary independently the factorization and renormalization scales in the range $0.5\,Q \leq \mu_F, \mu_R \leq 2\,Q$, with the constraint $0.5 \leq \mu_F/\mu_R \leq 2$. With respect to the PDFs uncertainties, we use the 90% C.L. MSTW2008 sets [68]. As we can observe from Figure 4.2 the total scale variation is of $\mathcal{O}(5\%)$ in the whole range of invariant mass. On the other hand, the PDF uncertainty is larger, specially in the gravity dominated invariant mass region, with a total variation close to 15%. This different behaviour for small and large values of invariant mass originates from the larger fractional uncertainty of the gluon-gluon contribution (which dominates the graviton production) compared with the quark-antiquark one (which dominates the SM contribution).

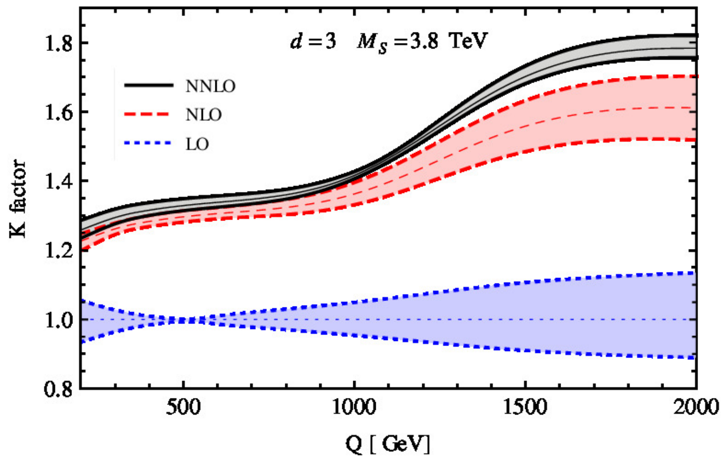


Figure 4.3: K factors as a function of the di-lepton invariant mass. The bands are obtained by varying the factorization and renormalization scales as indicated in the main text. The different curves correspond to the LO (blue-dotted), NLO (red-dashed) and NNLO (black-solid) predictions.

To evaluate the impact of the NNLO corrections we show in Figure 4.3 the corresponding K factor as a function of the di-lepton invariant mass. To normalize we use the LO prediction for $\mu_R = \mu_F = Q$. The bands are obtained by varying the factorization and renormalization scales as indicated before. We also include in the plot the previous order results.

We can observe, both at NLO and NNLO, the transition between the SM and the gravity dominated regions, $Q \lesssim 1000$ GeV and $Q \gtrsim 1000$ GeV respectively. Given that the QCD corrections for the graviton mediated di-lepton production are more sizeable than those of the SM Drell-Yan process, the NNLO K factor goes from $K \simeq 1.3$ to $K \simeq 1.8$ as the value of Q increases. We can also see that there is an overlap between the NLO and NNLO bands for the small invariant mass region, while this does not happen for $Q \gtrsim 1000$ GeV. This might be an effect due to the SV approximation if the underestimation of the uncertainty observed at NLO also holds at NNLO, and we can expect the bands in the gravity dominated region to be larger in the exact NNLO result. However, we also have to consider that an important part of the NNLO scale variation comes from the NLO contribution, for which we use the exact result. At the same time, a small overestimation of the size of the NNLO corrections by the SV approximation (as it was observed at NLO in Figure 4.1) could

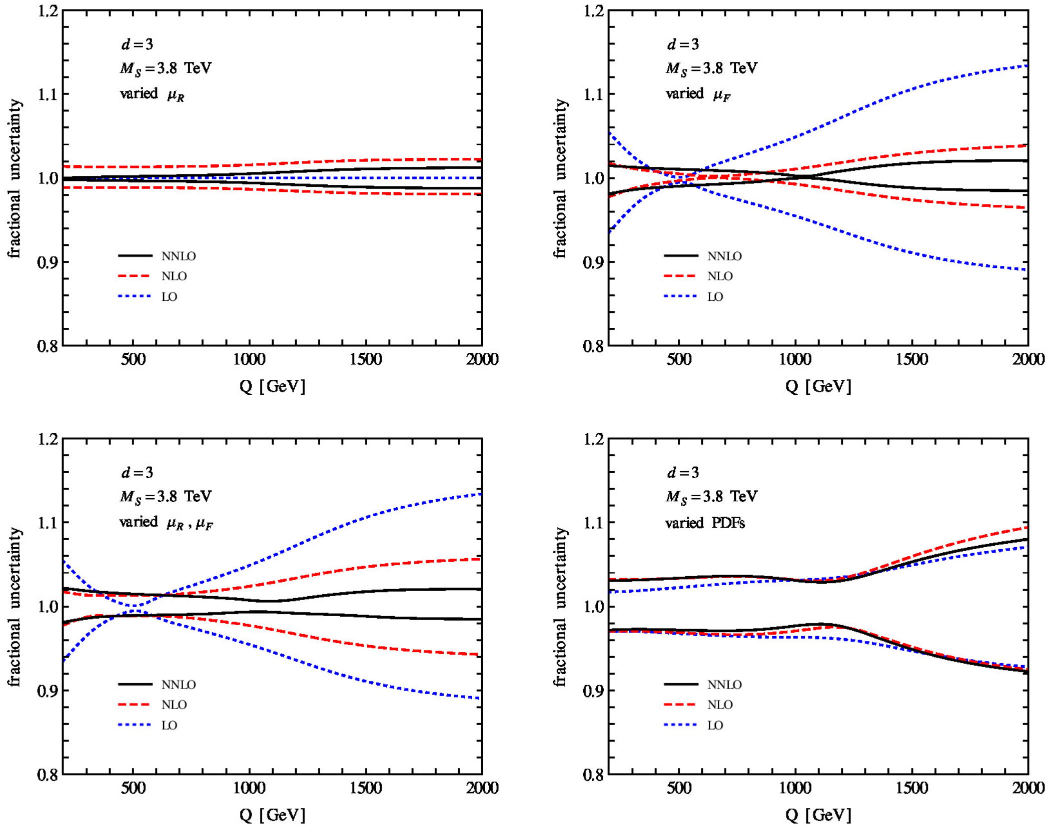


Figure 4.4: Fractional uncertainties of the di-lepton invariant mass distribution coming from μ_R variation (upper-left), μ_F variation (upper-right), μ_R, μ_F variation (down-left) and PDF uncertainties (down right). In all cases we show the LO (blue-dotted), NLO (red-dashed) and NNLO (black-solid) predictions.

be also contributing to this gap between the NLO and NNLO predictions.

In Figure 4.4 we present a more detailed analysis of the the theoretical uncertainties. In the upper-left plot we show the fractional variation of the differential cross section as we vary the renormalization scale in the range $0.5Q \leq \mu_R \leq 2Q$, keeping $\mu_F = Q$. Similarly, in the upper-right plot we vary μ_F keeping μ_R fixed. Finally, in the down-left figure we show the total scale variation, varying simultaneously and independently both scales as indicated before. On the other hand, in the down-right plot we present the fractional variation of the cross section coming from the parton flux determination uncertainties. In all cases we show the LO, NLO and NNLO results.

We can observe that the μ_R dependence starts at NLO, with a total variation going from 3% at $Q = 200$ GeV to 4% at $Q = 2000$ GeV. At NNLO, the uncertainty is substantially reduced for the lower values of invariant mass, with a variation of less than 0.5%, while in the gravity dominated region the reduction is less significant.

As to the μ_F dependence, we can see that there is a zone of minimal variation which tends to move to higher values of invariant mass as we increase the order of the calculation. Aside from that, in the large invariant mass region we can clearly observe how the uncertainty is reduced from LO to NLO and from NLO to NNLO.

The reduction of the uncertainties can be better observed in the total scale variation plot. As mentioned before, we can see that the NNLO total scale uncertainty remains quite constant in the whole range of invariant mass, with a value close to 4%. As we can see from the plot, this result is three times smaller than the previous order uncertainty in the gravity dominated region. On the other hand, for the SM dominated invariant mass region the NLO and NNLO scale variation is of the same order.

Finally, we have the parton flux uncertainties, which as stressed before are the main source of theoretical uncertainties at NNLO. In this case, we can observe that there is no significant difference between the results as we increase the order of the perturbative calculation.

All the analysis described in this section was repeated for each of the model parameter sets, obtaining similar results. In Figure 4.5 we show the di-lepton invariant mass distributions for each of them at NNLO and the corresponding K factors, for $\mu_F = \mu_R = Q$. In all cases we can observe the same transition from the SM- to the GR-dominated region, and the resulting increase in the K factor.

4.2.2 RS Model

We present now the predictions for the single graviton production in the Randall-Sundrum model at the LHC. Taking into account the latest bounds obtained by ATLAS [66] and CMS [67], and the requirement $\Lambda_\pi \lesssim 10$ TeV , we have for each value of $\tilde{k} = k/\overline{M}_P$ a minimum and a maximum value of M_1 allowed. At the same time, precision electroweak data and perturbativity requirements constrain the value of $\tilde{k}$ in the range $0.01 \lesssim \tilde{k} \lesssim 0.1$ (some of these values are already excluded by the experiments). In Table 4.1 we show the values of $\tilde{k}$ we used, and the corresponding

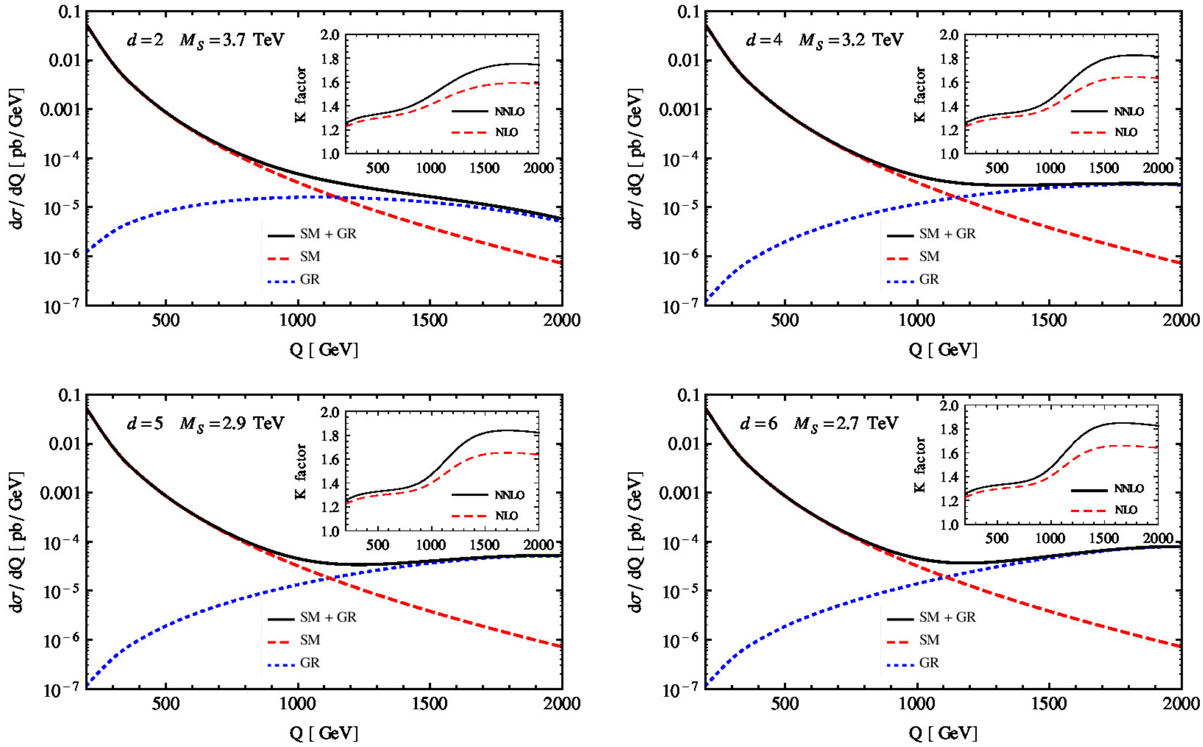


Figure 4.5: Di-lepton invariant mass distribution for SM (blue-dotted), gravity (red-dashed) and SM+GR (black-solid) at NNLO for $M_S = 3.7$ TeV and $d = 2$ (upper-left), $M_S = 3.2$ TeV and $d = 4$ (upper-right), $M_S = 2.9$ TeV and $d = 5$ (down-left) and $M_S = 2.7$ TeV and $d = 6$ (down-right). The inset plots show the corresponding K factors at NLO (red-dashed) and NNLO (black-solid).

minimum and maximum for M_1. These values explore the whole space of allowed parameters.

In Figure 4.6 we show the total cross section as a function of the lightest RS graviton mass for $\tilde{k} = 0.06$ at LO, NLO and NNLO, the latest within the soft-virtual approximation. We can observe the exponential decay as we go to larger values of

$\tilde{k}$	0.04	0.05	0.06	0.07	0.08	0.09	0.1
$M_1^{\min}$[TeV]	1.35	1.55	1.55	1.65	1.7	1.8	1.95
$M_1^{\max}$[TeV]	1.55	1.95	2.3	2.7	3.1	3.45	3.85

Table 4.1: Values of $\tilde{k}$ and M_1 used for the present analysis.

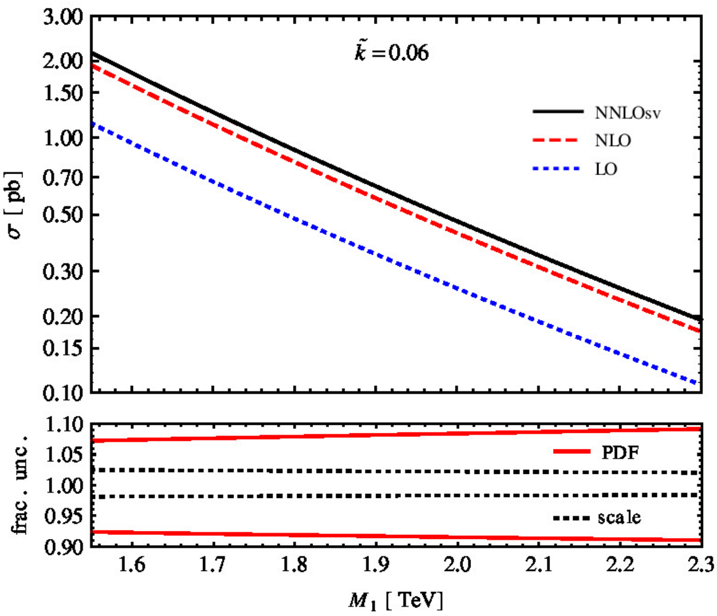


Figure 4.6: Total single graviton production cross section at the LHC ($\sqrt{s_H}$ = 14 TeV) as a function of the lightest RS graviton mass at LO (blue-dotted), NLO (red-dashed) and NNLO (black-solid), the latest within the soft-virtual approximation. The lower inset gives the fractional scale (black-dotted) and PDF (red-solid) uncertainties.

M_1. The lower inset gives the fractional scale and PDF uncertainties. We can observe that the scale variation remains almost constant throughout all the range of masses, with a total uncertainty of less than 5%. Again, we could expect the exact NNLO uncertainty to be larger. The PDF uncertainty is considerably larger, with a variation close to 15% or 20%, depending on the value of M_1.

The NNLO corrections are sizeable. This can be better seen in Figure 4.7, where we show the corresponding K factor, again as a function of M_1. We can observe that the K factor is close to 1.9 for the minimum value of M_1, and goes down to 1.8 as we reach the maximum. This represents an increase close to 15% with respect to the NLO result. We can also notice that the size of the bands, obtained performing the scale variation as indicated before, is considerably smaller at NNLO than in the previous orders.

Given that, for a fixed value of M_1, the size of $\tilde{k}$ only represents an overall normalization, the K factor only depends on M_1. We provide then the following analytic

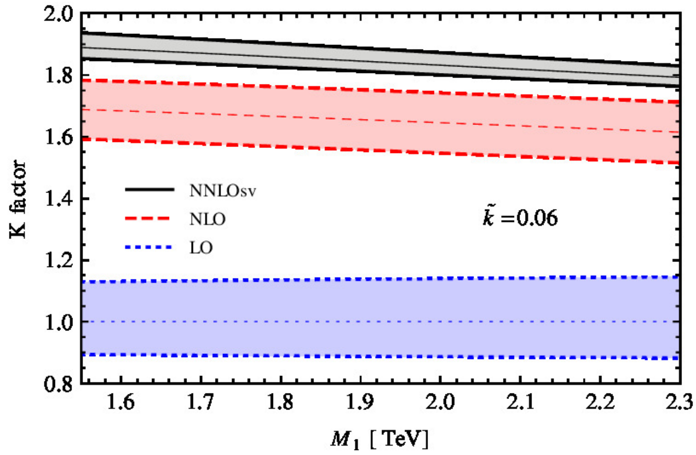


Figure 4.7: K factors as a function of the lightest RS graviton. The bands are obtained by varying the factorization and renormalization scales as indicated in the main text. The different curves correspond to the LO (blue-dotted), NLO (red-dashed) and NNLO (black-solid) predictions, the last within the soft-virtual approximation.

expression that parametrizes the NNLO-SV K factor:

$$K^{SV}_{NNLO} = 2.207 - 0.239 \left(\frac{M_1}{1 \text{ TeV}} \right)^{0.663} . \tag{4.28}$$

This expression is valid for 1.35 TeV $\leq M_1 \leq$ 3.85 TeV, which includes the whole range of allowed values of M_1. The difference between this analytic expression and the exact NNLO-SV result is always smaller than 0.5%. We remark that this expression is valid for any value of $\tilde{k}$ or Λ_π.

4.3 Conclusions

We have calculated the NNLO QCD corrections to the graviton production in models of TeV-scale gravity, working within the soft-virtual approximation, which is known to be very accurate for similar processes. We expect that the differences between our predictions and the exact NNLO result will be smaller than 5%.

For ADD Model, we computed the graviton contribution to the Drell-Yan process, while for the RS model we calculated the single graviton production cross section.

In case of ADD model , with a center-of-mass energy $\sqrt{s_H} = 14$ TeV at LHC, we found a large K factor ($K \simeq 1.8$) for large values of the di-lepton invariant mass. The increment with respect to the previous order result is larger than 10%.

We also observe a significant reduction in the scale uncertainty, with a total variation close to 4%. This value is about three times smaller than the NLO result in the large invariant mass region. On the other hand, for the PDF uncertainty we found a total variation similar to what was found at NLO.

For the RS model we found a similar behaviour with respect to the NNLO QCD corrections. In this case, we also provide a simple analytic parametrization of the NNLO K factor, which only depends on M_1, and is valid for any value of $\tilde{k}$ or Λ_π. Its value goes from 1.92 for $M_1 = 1.35$ TeV to 1.62 for $M_1 = 3.85$ TeV.

Chapter 5

Two Loop QCD corrections to massive spin-2 resonance $\to 3$ gluons

In this chapter we will study the $\mathcal{O}(\alpha_s^3)$ virtual correction in massless QCD to the process $h \to g + g + g$ [73]due to interference of born and two-loop amplitudes, where h is a massive spin-2 particle. This is a part of the full NNLO contributions which also requires square of one loop amplitudes, real emission processes and appropriate mass counter terms. We have assumed a minimal coupling between massive spin-2 field and the fields of the SM. Hence our results are applicable to scattering processes involving a massive spin-2 particle and three gluons such as a massive graviton production with a jet in ADD and RS models or production of a massive spin-2 Higgs like boson along with a jet after appropriate analytical continuation [74] of kinematical variables to the respective physical regions.

Spin-2 field being a rank-2 tensor, we encounter for the first time the two loop amplitudes with higher tensorial integrals resulting from rank-2 derivative couplings of spin-2 fields with the SM ones. In addition, we encounter more than 2000 two loop Feynman amplitudes contributing due to the universal coupling of spin-2 field with all the SM particles. While these increase technical complexities at the intermediate stages of computation, the results confirm the universal infra-red structure of QCD amplitudes. In other words, we find that soft and collinear divergences not only factorise but also agree with the predictions from Catani's work [45] (see also [46]) on two loop QCD amplitudes for multi-leg processes. We also observe that there are no additional UV divergences as the interaction is through energy momentum tensor of the SM which is conserved. Hence, this is also useful to study the field theoretical structure of QCD amplitudes with tensor operator insertions, in particular with the energy momentum tensor of the SM.

In the next section of this chapter, we describe the generic effective Lagrangian that describes coupling of spin-2 fields with those of the SM. Section 2 is devoted to the computational details. Section 3 and Appendix of this thesis contain our final results. In section 4, we conclude with our findings.

5.1 Theory

5.1.1 The effective Lagrangian

We consider the SM with an additional spin-2 field $h^{\mu\nu}$. We assume that the spin-2 field couples minimally with the SM ones through the SM energy momentum tensor $T^{SM}_{\mu\nu}$. Since we are interested only in the QCD effects of the process under study, we restrict ourselves to the QCD part of $T^{SM}_{\mu\nu}$ and hence the action reads [1, 2] as

$$\mathcal{S} = \mathcal{S}_{SM} + \mathcal{S}_h - \frac{\kappa}{2} \int d^4x \; T^{QCD}_{\mu\nu}(x) \; h^{\mu\nu}(x) \,, \tag{5.1}$$

where κ is a dimensionful coupling and $T^{QCD}_{\mu\nu}$ is the energy momentum tensor of QCD given in equation 3.2.

5.1.2 Notation

We consider the decay of a massive spin-2 field into three gluons

$$h(q) \to g(p_1) + g(p_2) + g(p_3). \tag{5.2}$$

The associated Mandelstam variables are defined as

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv (p_2 + p_3)^2, \qquad u \equiv (p_1 + p_3)^2 \tag{5.3}$$

which satisfy

$$s + t + u = M_h^2 \equiv Q^2 \tag{5.4}$$

where M_h is the mass of the spin-2 field. We also define the following dimensionless invariants which appear in harmonic polylogarithms (HPL) [75] and 2dHPL [76, 77] as

$$x \equiv s/Q^2, \qquad y \equiv u/Q^2, \qquad z \equiv t/Q^2 \tag{5.5}$$

satisfying

$$x + y + z = 1. \tag{5.6}$$

5.1.3 Ultraviolet renormalization

We describe here the ultraviolet (UV) renormalization of the matrix elements of the decay of a spin-2 resonance with minimal coupling up to second order in QCD perturbation theory. We regularize the theory in $d = 4 + \epsilon$ dimensions and the dimensionful strong coupling constant in d dimensions is made dimensionless one ($\hat{g}_s$) by introducing the scale μ_0. We expand the unrenormalized amplitude in powers of $\hat{a}_s = \hat{g}_s^2/16\pi^2$ as

$$|\mathcal{M}\rangle = \left(\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon\right)^{\frac{1}{2}} |\hat{\mathcal{M}}^{(0)}\rangle + \left(\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon\right)^{\frac{3}{2}} |\hat{\mathcal{M}}^{(1)}\rangle + \left(\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon\right)^{\frac{5}{2}} |\hat{\mathcal{M}}^{(2)}\rangle + \mathcal{O}(\hat{a}_s^3)\ , \tag{5.7}$$

where $S_\epsilon = \exp[\frac{\epsilon}{2}(\gamma_E - \ln 4\pi)]$ with Euler constant $\gamma_E = 0.5772\ldots$, results from loop integrals beyond leading order. $|\hat{\mathcal{M}}^{(i)}\rangle$ is the unrenormalized color-space vector representing the i^{th} loop amplitude. In $\overline{MS}$ scheme, the renormalized coupling constant $a_s \equiv a_s(\mu_R^2)$ at renormalization scale μ_R is related to unrenormalized coupling constant $\hat{a}_s$ by

$$\begin{aligned}\frac{\hat{a}_s}{\mu_0^\epsilon} S_\epsilon &= \frac{a_s}{\mu_R^\epsilon} Z(\mu_R^2) \\ &= \frac{a_s}{\mu_R^\epsilon}\left[1 + a_s \frac{2\beta_0}{\epsilon} + a_s^2\left(\frac{4\beta_0^2}{\epsilon^2} + \frac{\beta_1}{\epsilon}\right) + \mathcal{O}(a_s^3)\right]\ ,\end{aligned} \tag{5.8}$$

where

$$\beta_0 = \left(\frac{11}{3} C_A - \frac{4}{3} T_F n_f\right), \quad \beta_1 = \left(\frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F n_f - 4 C_F T_F n_f\right) \tag{5.9}$$

with $C_A = N$, $C_F = (N^2-1)/2N$, $T_F = 1/2$ and n_f is the number of active quark flavors. Since, the spin-2 resonance couples to the SM particles through energy momentum tensor (eqn.(5.1)) which is conserved, the coupling constant κ is protected from any UV renormalization. Hence, there will be no additional UV renormalization required other than the strong coupling constant renormalization. Using the eqn.(5.8), we now can express $|\mathcal{M}\rangle$ (eqn.(5.7)) in powers of renormalized a_s with UV finite matrix elements $|\mathcal{M}^{(i)}\rangle$

$$|\mathcal{M}\rangle = (a_s)^{\frac{1}{2}} \left(|\mathcal{M}^{(0)}\rangle + a_s|\mathcal{M}^{(1)}\rangle + a_s^2|\mathcal{M}^{(2)}\rangle + \mathcal{O}(a_s^3) \right) \tag{5.10}$$

where

$$\begin{aligned}
|\mathcal{M}^{(0)}\rangle &= \left(\frac{1}{\mu_R^\epsilon}\right)^{\frac{1}{2}} |\hat{\mathcal{M}}^{(0)}\rangle \,, \\
|\mathcal{M}^{(1)}\rangle &= \left(\frac{1}{\mu_R^\epsilon}\right)^{\frac{3}{2}} \left[|\hat{\mathcal{M}}^{(1)}\rangle + \mu_R^\epsilon \frac{r_1}{2}|\hat{\mathcal{M}}^{(0)}\rangle\right] \,, \\
|\mathcal{M}^{(2)}\rangle &= \left(\frac{1}{\mu_R^\epsilon}\right)^{\frac{5}{2}} \left[|\hat{\mathcal{M}}^{(2)}\rangle + \mu_R^\epsilon \frac{3r_1}{2}|\hat{\mathcal{M}}^{(1)}\rangle + \mu_R^{2\epsilon}\left(\frac{r_2}{2} - \frac{r_1^2}{8}\right)|\hat{\mathcal{M}}^{(0)}\rangle\right]
\end{aligned} \tag{5.11}$$

with

$$r_1 = \frac{2\beta_0}{\epsilon} \,, \qquad r_2 = \left(\frac{4\beta_0^2}{\epsilon^2} + \frac{\beta_1}{\epsilon}\right) . \tag{5.12}$$

5.1.4 Infrared factorization

Beyond leading order, the UV renormalized matrix elements $|\mathcal{M}^{(i)}\rangle$, $i > 0$ contain divergences arising from the infrared sector of massless QCD. They result from soft gluons and collinear massless partons present in the loops. They will cancel against similar divergences coming from real emission contributions in the infrared safe observables order by order in a_s, thanks to KLN theorem [78, 79]. The infrared divergence structure and their factorization property in QCD amplitudes have been well studied for long time. In [45], Catani predicted the infrared divergences of multi-parton QCD amplitudes precisely in dimensional regularization up to two loops excluding two loop single pole in ϵ. In [46], Sterman and Tejeda-Yeomans provided a systematic way of understanding the structure of infrared divergences using factorization properties of the scattering amplitudes along with infrared evolution equations.

They demonstrated the connection of single pole in ϵ to a soft anomalous dimension matrix, later computed in [80, 81]. The structure of single pole term for the electromagnetic and Higgs form factors was first shown in [20, 21]. Using soft collinear effective field theory, Becher and Neubert [47] derived the exact formula for the infrared divergences of scattering amplitudes with an arbitrary number of loops and legs in massless QCD including single pole in dimensional regularization. Using Wilson lines for hard partons and soft and eikonal jet functions in dimensional regularization, Gardi and Magnea also arrived at a similar all order result [48].

According to Catani's prediction, the renormalized amplitudes $|\mathcal{M}^{(i)}\rangle$ for the process (eqn.(5.2)) can be expressed in terms of the universal subtraction operators $\mathbf{I}_g^{(i)}(\epsilon)$ as follows[1]

$$\begin{aligned} |\mathcal{M}^{(1)}\rangle &= 2\,\mathbf{I}_g^{(1)}(\epsilon)\,|\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(1)fin}\rangle \\ |\mathcal{M}^{(2)}\rangle &= 2\,\mathbf{I}_g^{(1)}(\epsilon)\,|\mathcal{M}^{(1)}\rangle + 4\,\mathbf{I}_g^{(2)}(\epsilon)\,|\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(2)fin}\rangle \end{aligned} \tag{5.13}$$

where,

$$\begin{aligned} \mathbf{I}_g^{(1)}(\epsilon) &= \frac{1}{2}\frac{e^{-\frac{\epsilon}{2}\gamma_E}}{\Gamma(1+\frac{\epsilon}{2})}\mathcal{V}_g^{sing}(\epsilon)\left[\Big(-\frac{s}{\mu_R^2}\Big)^{\frac{\epsilon}{2}} + \Big(-\frac{t}{\mu_R^2}\Big)^{\frac{\epsilon}{2}} + \Big(-\frac{u}{\mu_R^2}\Big)^{\frac{\epsilon}{2}}\right] \\ \mathbf{I}_g^{(2)}(\epsilon) &= -\frac{1}{2}\mathbf{I}_g^{(1)}(\epsilon)\Big[\mathbf{I}_g^{(1)}(\epsilon) - \frac{2\beta_0}{\epsilon}\Big] + \frac{e^{\frac{\epsilon}{2}\gamma_E}\,\Gamma(1+\epsilon)}{\Gamma(1+\frac{\epsilon}{2})}\Big[-\frac{\beta_0}{\epsilon} + K\Big]\,\mathbf{I}_g^{(1)}(2\epsilon) \\ &\quad + \mathbf{H}_g^{(2)}(\epsilon) \end{aligned} \tag{5.14}$$

and

$$\mathcal{V}_g^{sing}(\epsilon) = C_A\frac{4}{\epsilon^2} - \frac{\beta_0}{\epsilon}, \qquad K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{10}{9}T_F n_f \tag{5.15}$$

$$\mathbf{H}_g^{(2)}(\epsilon) = \frac{3}{2\epsilon}\left\{C_A^2\Big(-\frac{5}{12} - \frac{11}{24}\zeta_2 - \frac{1}{2}\zeta_3\Big) + C_A n_f\Big(\frac{29}{27} + \frac{1}{12}\zeta_2\Big) - \frac{1}{2}C_F n_f - \frac{5}{27}n_f^2\right\} \tag{5.16}$$

5.2 Calculation of two-loop amplitude

We now describe the computation of $\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle$ & $\langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle$ matrix elements where all the Lorentz and color indices of external particles are summed over. The

[1] The numerical coefficients 2 and 4 with $\mathbf{I}^{(i)}$ come due to the different definition of a_s between ours and Catani.

computation involves large number of Feynman diagrams. We need to perform various algebraic simplifications with Dirac, Lorentz and color indices before the loop integrals are evaluated. Due to tensorial coupling of spin-2 resonance with the SM fields, the loops contain higher rank tensor integrals as compared to the ones normally encountered in the SM processes. We have systematically automated this computation using various symbolic manipulation programs developed in house and few publicly available packages that use FORM [82] and Mathematica. In the following, we describe the method in detail.

5.2.1 Feynman diagrams and simplification

We use QGRAF [13] to generate the Feynman diagrams. We find that there are 4 diagrams at tree level, 108 at one loop and 2362 at two loops, leaving out tadpole and self energy corrections to the external legs. The output of the QGRAF is then converted to the format that is suitable for further symbolic manipulation using FORM and Mathematica. A set of FORM routines is used to perform simplification of the squared matrix elements involving gluon and spin-2 resonance polarization and color sums. We have used Feynman gauge throughout and for the external on-shell gluon legs, physical polarizations are summed using

$$\sum_s \varepsilon^\mu(p_i,s)\varepsilon^{\nu *}(p_i,s) = -g^{\mu\nu} + \frac{p_i^\mu q_i^\nu + q_i^\mu p_i^\nu}{p.q} \tag{5.17}$$

where, p_i is the i^{th}-gluon momentum and q_i is the corresponding reference momentum which is an arbitrary light-like 4-vector. We choose $q_1 = p_2$, $q_2 = p_1$ and $q_3 = p_1$ for simplicity. The spin-2 polarization sum in d dimensions is given by [61]

$$\begin{aligned} B^{\mu\nu;\rho\sigma}(q) &= \left(g^{\mu\rho} - \frac{q^\mu q^\rho}{q.q}\right)\left(g^{\nu\sigma} - \frac{q^\nu q^\sigma}{q.q}\right) + \left(g^{\mu\sigma} - \frac{q^\mu q^\sigma}{q.q}\right)\left(g^{\nu\rho} - \frac{q^\nu q^\rho}{q.q}\right) \\ &\quad - \frac{2}{d-1}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q.q}\right)\left(g^{\rho\sigma} - \frac{q^\rho q^\sigma}{q.q}\right) \end{aligned} \tag{5.18}$$

5.2.2 Reduction of tensor integrals

Beyond leading order, the resulting expressions involve tensorial one and two loop integrals which need to be reduced to a set of scalar integrals using a convenient tensorial reduction procedure. Tensorial reduction is quite straightforward at one loop level but not so at two loop level and beyond. In addition, finding a minimal set of integrals after the tensorial reduction is important to achieve the task with large number of Feynman integrals. A systematic approach to deal with higher rank tensor integrals and large number of scalar integrals is to use Integration by parts (IBP) [22] and Lorentz invariant (LI) [27] identities. While these identities are useful to express the tensorial integrals in terms of a set of master integrals, in practice, the computation becomes tedious due to the appearance of large variety of Feynman integrals involving different set of propagators each requiring a set of IBP and LI identities independently. We have reduced such varieties to a few by shifting the loop momenta suitably using an in-house algorithm which uses FORM. For one-loop diagrams, we can express each Feynman integral to contain terms from one of the following three sets:

$$\{\mathcal{D},\, \mathcal{D}_1,\, \mathcal{D}_{12},\, \mathcal{D}_{123}\}\,, \{\mathcal{D},\, \mathcal{D}_2,\, \mathcal{D}_{23},\, \mathcal{D}_{123}\}\,, \{\mathcal{D},\, \mathcal{D}_3,\, \mathcal{D}_{31},\, \mathcal{D}_{123}\} \tag{5.19}$$

where,

$$\mathcal{D} = k_1^2,\, \mathcal{D}_i = (k_1 - p_i)^2,\, \mathcal{D}_{ij} = (k_1 - p_i - p_j)^2,\, \mathcal{D}_{ijk} = (k_1 - p_i - p_j - p_k)^2 \tag{5.20}$$

In each set in eqn.(5.19), $\mathcal{D}$'s are linearly independent and form a complete basis in the sense that any Lorentz invariant $k_1 \cdot p_i$ can be expressed in terms of $\mathcal{D}$'s. At two loops, there are nine independent Lorentz invariants involving loop momenta k_1 and k_2, namely $\{(k_\alpha \cdot k_\beta), (k_\alpha \cdot p_i)\}, \alpha, \beta = 1, 2;\; i = 1, ..., 3$. After appropriate shifting of loop momenta, we can express each two loop Feynman integral to contain terms belonging to one of the following six sets:

$$\begin{aligned}
&\{\mathcal{D}_0,\, \mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_{1;1},\, \mathcal{D}_{2;1},\, \mathcal{D}_{1;12},\, \mathcal{D}_{2;12},\, \mathcal{D}_{1;123},\, \mathcal{D}_{2;123}\}\\
&\{\mathcal{D}_0,\, \mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_{1;2},\, \mathcal{D}_{2;2},\, \mathcal{D}_{1;23},\, \mathcal{D}_{2;23},\, \mathcal{D}_{1;123},\, \mathcal{D}_{2;123}\}\\
&\{\mathcal{D}_0,\, \mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_{1;3},\, \mathcal{D}_{2;3},\, \mathcal{D}_{1;31},\, \mathcal{D}_{2;31},\, \mathcal{D}_{1;123},\, \mathcal{D}_{2;123}\}\\
&\{\mathcal{D}_0,\, \mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_{1;1},\, \mathcal{D}_{2;1},\, \mathcal{D}_{0;3},\, \mathcal{D}_{1;12},\, \mathcal{D}_{2;12},\, \mathcal{D}_{1;123}\}\\
&\{\mathcal{D}_0,\, \mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_{1;2},\, \mathcal{D}_{2;2},\, \mathcal{D}_{0;1},\, \mathcal{D}_{1;23},\, \mathcal{D}_{2;23},\, \mathcal{D}_{1;123}\}\\
&\{\mathcal{D}_0,\, \mathcal{D}_1,\, \mathcal{D}_2,\, \mathcal{D}_{1;3},\, \mathcal{D}_{2;3},\, \mathcal{D}_{0;2},\, \mathcal{D}_{1;31},\, \mathcal{D}_{2;31},\, \mathcal{D}_{1;123}\}
\end{aligned} \tag{5.21}$$

where,

$$\mathcal{D}_0 = (k_1 - k_2)^2,\ \mathcal{D}_\alpha = k_\alpha^2,\ \mathcal{D}_{\alpha;i} = (k_\alpha - p_i)^2,\ \mathcal{D}_{\alpha;ij} = (k_\alpha - p_i - p_j)^2,$$
$$\mathcal{D}_{0;i} = (k_1 - k_2 - p_i)^2,\ \mathcal{D}_{\alpha;ijk} = (k_\alpha - p_i - p_j - p_k)^2 \tag{5.22}$$

Given the fewer number of sets (eqns.(5.19) & (5.21)), it is easier to use IBP and LI identities using Laporta algorithm [29]. These identities can be generated using publicly available packages such as AIR [30], FIRE [28], REDUZE [31, 32], LiteRed [33, 83] etc. For our computation, we use a Mathematica based package LiteRedV1.51 along with MintV1.1 [84]. This package has the option to exploit symmetry relations within each set and also among different sets.

5.2.3 Master integrals

Using these IBP and LI identities, we reduce all the integrals that appear in our computation to a minimal set of master integrals. For one loop, we get two topologically different master integrals namely box and bubble, see Fig.(5.1) and we find that the master integrals with three propagators are absent. For two loops, we encounter 16 planar and 5 non-planar topologies of master integrals. These master integrals can be related to those that were computed by Gehrmann and Remiddi in their seminal papers [76, 77]. In particular, our set of master integrals does not contain integrals with irreducible numerator, instead we have higher power of propagators. We use IBP and LI identities to express our set of master integrals to those of [76, 77]. All the master integrals are drawn in Fig.(5.1) and Fig.(5.2) up to different permutations of the external momenta p_1, p_2 and p_3. We also observe that some topologies like iXBox1 given in Fig.(5.2) are absent in our final result and find some new topologies namely Glass3S and Kite1 given in Fig.(5.1) which are absent in the [76,77] and those are simply a product of two one loop topologies.

Substituting the master integrals computed by Gehrmann and Remiddi [76,77] in terms of HPLs and 2dHPLs, we obtain the unrenormalized matrix elements $\langle\hat{\mathcal{M}}^{(0)}|\hat{\mathcal{M}}^{(1)}\rangle$ and $\langle\hat{\mathcal{M}}^{(0)}|\hat{\mathcal{M}}^{(2)}\rangle$. We use shuffle algebra to express product of lower weight HPLs as a sum of higher weight HPLs. In the next section we present our results.

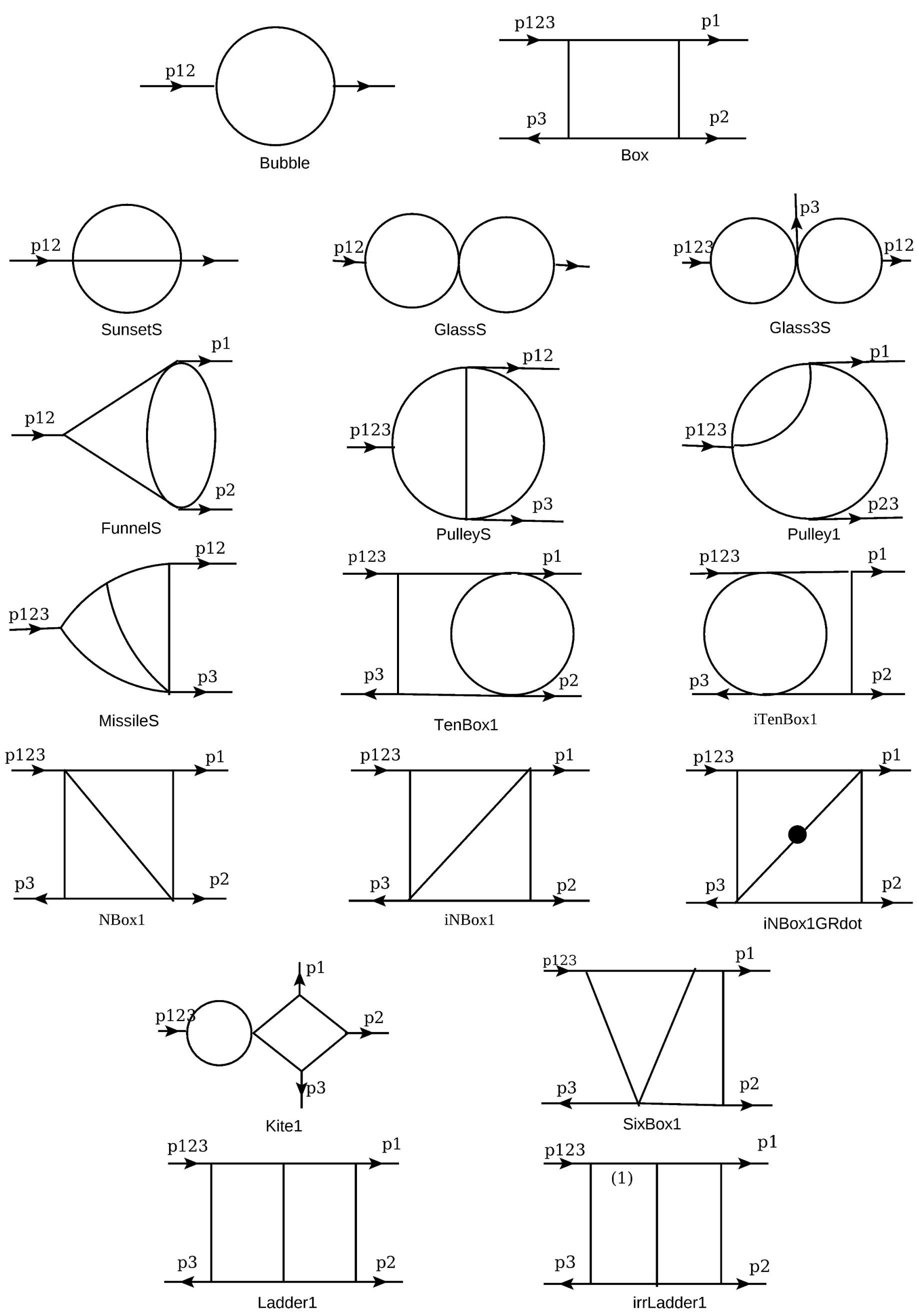


Figure 5.1: Planar topologies of master integrals

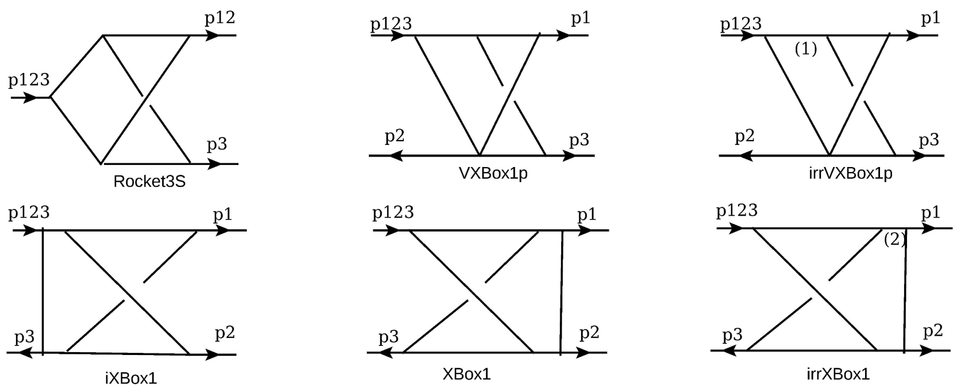


Figure 5.2: Non-Planar topologies of master integrals

5.3 Results

The UV renormalized matrix elements $\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle$ and $\langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle$ are computed using the unrenormalized counter parts through

$$\begin{aligned}\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle &= \left(\frac{1}{\mu_R^\epsilon}\right)^2 \quad \Big[\langle\hat{\mathcal{M}}^{(0)}|\hat{\mathcal{M}}^{(1)}\rangle + \mu_R^\epsilon \frac{r_1}{2}\langle\hat{\mathcal{M}}^{(0)}|\hat{\mathcal{M}}^{(0)}\rangle\Big]\,, \\ \langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle &= \left(\frac{1}{\mu_R^\epsilon}\right)^3 \quad \Big[\langle\hat{\mathcal{M}}^{(0)}|\hat{\mathcal{M}}^{(2)}\rangle + \mu_R^\epsilon \frac{3r_1}{2}\langle\hat{\mathcal{M}}^{(0)}|\hat{\mathcal{M}}^{(1)}\rangle \\ &\qquad + \mu_R^{2\epsilon}\left(\frac{r_2}{2} - \frac{r_1^2}{8}\right)\langle\hat{\mathcal{M}}^{(0)}|\hat{\mathcal{M}}^{(0)}\rangle\Big]. \end{aligned} \tag{5.23}$$

Using eqn.(5.13), we can also express the renormalized matrix elements in terms of $\mathbf{I}_g^{(i)}(\epsilon)$, given by

$$\begin{aligned}\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle &= 2\,\mathbf{I}_g^{(1)}(\epsilon)\,\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle + \langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)fin}\rangle\,, \\ \langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle &= 2\,\mathbf{I}_g^{(1)}(\epsilon)\,\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle + 4\,\mathbf{I}_g^{(2)}(\epsilon)\,\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle + \langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)fin}\rangle\,. \end{aligned} \tag{5.24}$$

Expanding the right hand side of equations (5.23 & 5.24) in powers of ϵ and comparing their coefficients of $\mathcal{O}(\epsilon^0)$, we obtain $\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)fin}\rangle$ and $\langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)fin}\rangle$.

We find that all the poles in ϵ resulting from the soft and collinear partons in eqn.(5.23) are in agreement with those of eqn.(5.24). This serves as an important check on our result. In addition, it establishes the universal structure of infrared poles in QCD amplitudes involving tensorial operator insertion. We also observe that the contributions resulting from the gauge fixing term in eqn.(5.1) cancel exactly with those of ghosts confirming the gauge independence of our result. As we anticipated, the eqn.(5.23) does not require any over all operator renormalization constant due to the conservation of energy momentum tensor and it can be made UV finite through strong coupling constant renormalization (eqn.(5.8)) alone. Below we present our final results

$$\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle \;=\; \mathcal{F}_h\,\mathcal{A}^{(0)}\,, \tag{5.25}$$

$$\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)fin}\rangle \;=\; \mathcal{F}_h\Bigg\{-\frac{\beta_0}{2}\,\mathcal{A}^{(0)}\;\ln\left(-\frac{Q^2}{\mu^2}\right)+\left(\mathcal{A}_1^{(1)}\zeta_2+\mathcal{A}_2^{(1)}\right)\Bigg\}\,, \tag{5.26}$$

$$\begin{aligned}\langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)fin}\rangle \;=\; \mathcal{F}_h\Bigg\{&\frac{3\beta_0^2}{8}\,\mathcal{A}^{(0)}\;\ln^2\left(-\frac{Q^2}{\mu^2}\right)\\ &\langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)fin}\mathcal{F}_h\rangle+\left(-3\;C_A^2\;\zeta_3\;\mathcal{A}^{(0)}+\mathcal{A}_1^{(2)}\zeta_2+\mathcal{A}_2^{(2)}\right)\ln\left(-\frac{Q^2}{\mu^2}\right)\\ &\langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)fin}\mathcal{F}_h\rangle+\left(\mathcal{A}_3^{(2)}\zeta_2^2+\mathcal{A}_4^{(2)}\zeta_3+\mathcal{A}_5^{(2)}\zeta_2+\mathcal{A}_6^{(2)}\right)\Bigg\}\end{aligned} \tag{5.27}$$

where,

$$\begin{aligned}\mathcal{F}_h &= 16\pi^2\kappa^2 N\Big(N^2-1\Big)\,,\\ \mathcal{A}^{(0)} &= \frac{4}{stu}\Big(s^4+2s^3(t+u)+3s^2\left(t^2+u^2\right)+2s\;\left(t^3+u^3\right)+\left(t^2+tu+u^2\right)^2\Big)\,,\\ \mathcal{A}_\alpha^{(1)} &= \mathcal{A}_{\alpha;\mathrm{C_A}}^{(1)}\;C_A+\mathcal{A}_{\alpha;n_f}^{(1)}\;n_f\,,\\ \mathcal{A}_\alpha^{(2)} &= \mathcal{A}_{\alpha;\mathrm{C_A^2}}^{(2)}\;C_A^2+\mathcal{A}_{\alpha;\mathrm{C_A}n_f}^{(2)}\;C_A n_f+\mathcal{A}_{\alpha;\mathrm{C_F}n_f}^{(2)}\;C_F n_f+\mathcal{A}_{\alpha;n_f^2}^{(2)}\;n_f^2\,.\end{aligned} \tag{5.28}$$

The coefficients $\mathcal{A}_{\alpha;\mathcal{C}}^{(i)}$ are given in the appendix except $\mathcal{A}_{6;C_An_f}^{(2)}$, $\mathcal{A}_{6;C_Fn_f}^{(2)}$ & $\mathcal{A}_{6;n_f^2}^{(2)}$ which can be found in the files A6Canf, A6Cfnf and A6nf2 respectively attached with the arXiv 1404.0028.

5.4 Conclusion

The computation of one and two loop QCD results for the process $h \to g + g + g$ is presented. We use dimensional regularization to regulate both UV and IR divergences. Due to the coupling of spin-2 field with the SM ones through rank-2 tensor, we encounter large number of Feynman diagrams with complicated one and two loop Feynman integrals with high powers of loop momenta to scalar ones.The package LiteRed, which uses IBP and LI identities, reduces all such integrals to only few master integrals (MIs).These MIs were already computed by Gehrmann and Remiddi.We find that no overall UV renormalization is required due to the conservation of energy momentum tensor. We also find correct IR structure of the amplitudes which confirms the factorization property of QCD amplitude even with tensorial insertion.

Chapter 6

Summary

As the LHC begins its second run with 13 TeV centre of mass energy we expect some signature of physics Beyond the Standard Model.Since extra dimensional scenarios are possible solutions for hierarchy problem, dedicated groups in both ATLAS and CMS collaborations are engaged in the analysis of extra dimensional searches like di-lepton,di-photon, mono jet,mono photon production etc.To put stringent bounds on the parameters of these BSM models, control on the theoretical uncertainties is essential. Renormalisation and factorisation scale dependences of a cross section to a particular order in perturbation theory give an estimate of the uncalculated higher order corrections.

In this thesis we compute NNLO spin-2 quark and gluon form factors of energy momentum tensor of the QCD part of the SM up to two-loop level in QCD. We have shown that these form factors satisfy Sudakov integro-differential equation with same cusp A_I, collinear B^I and soft f^I anomalous dimensions that contribute to electroweak vector boson and gluon form factors. In addition, they also show the universal behaviour of the infrared poles in ε in accordance with the proposal by Catani.

Using the result of spin-2 quark and gluon form factors, we have calculated the NNLO QCD corrections to the graviton production in models of TeV-scale gravity, working within the soft-virtual approximation. We expect that the differences between our predictions and the exact NNLO result will be smaller than 5%.

We considered the ADD and RS models. For the ADD model, we computed the graviton contribution to the Drell-Yan process, while for the RS model we calculated the single graviton production cross section.

For the ADD model at the LHC, with a center-of-mass energy $\sqrt{s_H} = 14$ TeV, we found a large K factor ($K \simeq 1.8$) for large values of the di-lepton invariant mass. This region is dominated by the graviton contribution, whose QCD corrections are substantially larger than the SM ones. The increment with respect to the previous order result is larger than 10%.

We also observe a substantial reduction in the scale uncertainty, with a total variation close to 4%. This value is about three times smaller than the NLO result in the large invariant mass region. Since at NLO the soft-virtual approximation underestimates the total uncertainty by a factor 2, we can expect the exact NNLO scale variation to be larger. However, given that in our approximation the NLO contribution is treated in an exact way, and given that the NLO is an important contribution to the total NNLO variation, we can also expect the SV approximation to be more accurate at this order with respect to this source of theoretical uncertainty. On the other hand, for the PDF uncertainty we found a total variation similar to what was found at NLO.

For the RS model we found a similar behaviour with respect to the NNLO QCD corrections. In this case, we also provide a simple analytic parametrization of the NNLO K factor, which only depends on M_1, and is valid for any value of $\tilde{k}$ or Λ_π. Its value goes from 1.92 for $M_1 = 1.35$ TeV to 1.62 for $M_1 = 3.85$ TeV.

Further,we computed one and two loop QCD results for the process $h \to g+g+g$. Our result is very general in the sense that it can be used for any scattering process involving production of a massive spin-2 particle that has a universal coupling with the SM fields. We can use it to study the production of a jet with missing energy due to KK graviton escaping the detector or a process with resonant massive spin-2 particle in association with a jet. Since the spin-2 field couples with the SM ones through rank-2 tensor, we not only encounter large number of Feynman diagrams but also the formidable challenge of reducing one and two loop Feynman integrals with high powers of loop momenta to scalar ones. The IBP and LI identities reduce all such integrals to only few master integrals that were already computed by Gehrmann and Remiddi. This computation is the first of the kind involving four point function at two loop level in QCD with tensorial insertion and one massive external state. We find that no overall UV renormalization is required due to the conservation of energy momentum tensor. We also find that our results exhibit the right IR structure confirming the factorization property of QCD amplitude even with tensorial insertion.

Appendix A

A.1 Form factors

We present here the form factors as a series expansion in ε up to $\mathcal{O}(\varepsilon^4)$ for $F_I^{(1)}$ and up to $\mathcal{O}(\varepsilon^2)$) for $F_I^{(2)}$:

$$
\begin{aligned}
\hat{F}_g^{T,(1)} \;=\; & n_f\Bigg[\frac{1}{\varepsilon}\left(-\frac{4}{3}\right)+\left(\frac{35}{18}\right)+\varepsilon\left(-\frac{497}{216}+\frac{1}{6}\zeta_2\right)+\varepsilon^2\left(\frac{6593}{2592}-\frac{7}{18}\zeta_3-\frac{35}{144}\zeta_2\right) \\
& +\varepsilon^3\left(-\frac{84797}{31104}+\frac{245}{432}\zeta_3+\frac{497}{1728}\zeta_2+\frac{47}{480}\zeta_2^2\right)+\varepsilon^4\Bigg(\frac{1072433}{373248}-\frac{31}{120}\zeta_5 \\
& -\frac{3479}{5184}\zeta_3-\frac{6593}{20736}\zeta_2+\frac{7}{144}\zeta_2\zeta_3-\frac{329}{2304}\zeta_2^2\Bigg)\Bigg]+C_A\Bigg[\frac{1}{\varepsilon^2}\left(-8\right)+\frac{1}{\varepsilon}\left(\frac{22}{3}\right) \\
& +\left(-\frac{203}{18}+\zeta_2\right)+\varepsilon\left(\frac{2879}{216}-\frac{7}{3}\zeta_3-\frac{11}{12}\zeta_2\right)+\varepsilon^2\Bigg(-\frac{37307}{2592}+\frac{77}{36}\zeta_3+\frac{203}{144}\zeta_2 \\
& +\frac{47}{80}\zeta_2^2\Bigg)+\varepsilon^3\left(\frac{465143}{31104}-\frac{31}{20}\zeta_5-\frac{1421}{432}\zeta_3-\frac{2879}{1728}\zeta_2+\frac{7}{24}\zeta_2\zeta_3-\frac{517}{960}\zeta_2^2\right) \\
& +\varepsilon^4\Bigg(-\frac{5695811}{373248}+\frac{341}{240}\zeta_5+\frac{20153}{5184}\zeta_3-\frac{49}{144}\zeta_3^2+\frac{37307}{20736}\zeta_2-\frac{77}{288}\zeta_2\zeta_3 \\
& +\frac{9541}{11520}\zeta_2^2+\frac{949}{4480}\zeta_2^3\Bigg)\Bigg] \qquad \text{(A.1)}
\end{aligned}
$$

$$(A.2)$$

$$\begin{aligned}
\hat{F}_g^{T,(2)} &= C_F n_f \Bigg[\frac{1}{\varepsilon}\Bigg(-2\Bigg) + \Bigg(\frac{61}{6} - 8\zeta_3\Bigg) + \varepsilon\Bigg(-\frac{2245}{72} + \frac{59}{3}\zeta_3 + \frac{1}{2}\zeta_2 + \frac{12}{5}\zeta_2^2\Bigg) \\
&+\varepsilon^2\Bigg(\frac{64177}{864} - 14\zeta_5 - \frac{335}{9}\zeta_3 - \frac{83}{24}\zeta_2 + 2\zeta_2\zeta_3 - \frac{179}{30}\zeta_2^2\Bigg) + C_A n_f \Bigg[\frac{1}{\varepsilon^3}\Bigg(8\Bigg) \\
&+\frac{1}{\varepsilon^2}\Bigg(-\frac{40}{3}\Bigg) + \frac{1}{\varepsilon}\Bigg(\frac{41}{3} - \frac{2}{3}\zeta_2\Bigg) + \Bigg(-\frac{605}{108} + 10\zeta_3 + \frac{5}{9}\zeta_2\Bigg) + \varepsilon\Bigg(-\frac{21557}{1296} \\
&-\frac{182}{9}\zeta_3 + \frac{145}{108}\zeta_2 - \frac{57}{20}\zeta_2^2\Bigg) + \varepsilon^2\Bigg(\frac{320813}{5184} + \frac{71}{10}\zeta_5 + \frac{6407}{216}\zeta_3 - \frac{3617}{648}\zeta_2 \\
&-\frac{43}{18}\zeta_2\zeta_3 + \frac{1099}{180}\zeta_2^2\Bigg) + C_A^2\Bigg[\frac{1}{\varepsilon^4}\Bigg(32\Bigg) + \frac{1}{\varepsilon^3}\Bigg(-44\Bigg) + \frac{1}{\varepsilon^2}\Bigg(\frac{226}{3} - 4\zeta_2\Bigg) \\
&+\frac{1}{\varepsilon}\Bigg(-81 + \frac{50}{3}\zeta_3 + \frac{11}{3}\zeta_2\Bigg) + \Bigg(\frac{5249}{108} - 11\zeta_3 - \frac{67}{18}\zeta_2 - \frac{21}{5}\zeta_2^2\Bigg) + \varepsilon\Bigg(\frac{59009}{1296} \\
&-\frac{71}{10}\zeta_5 + \frac{433}{18}\zeta_3 - \frac{337}{108}\zeta_2 - \frac{23}{6}\zeta_2\zeta_3 + \frac{99}{40}\zeta_2^2\Bigg) + \varepsilon^2\Bigg(-\frac{1233397}{5184} + \frac{759}{20}\zeta_5 \\
&-\frac{8855}{216}\zeta_3 + \frac{901}{36}\zeta_3^2 + \frac{12551}{648}\zeta_2 + \frac{77}{36}\zeta_2\zeta_3 - \frac{4843}{720}\zeta_2^2 + \frac{2313}{280}\zeta_2^3\Bigg)\Bigg] \qquad (A.3)
\end{aligned}$$

$$\begin{aligned}
\hat{F}_q^{T,(1)} &= C_F\Bigg[\frac{1}{\varepsilon^2}\Bigg(-8\Bigg) + \frac{1}{\varepsilon}\Bigg(6\Bigg) + \Bigg(-10 + \zeta_2\Bigg) + \varepsilon\Bigg(12 - \frac{7}{3}\zeta_3 - \frac{3}{4}\zeta_2\Bigg) \\
&+\varepsilon^2\Bigg(-13 + \frac{7}{4}\zeta_3 + \frac{5}{4}\zeta_2 + \frac{47}{80}\zeta_2^2\Bigg) + \varepsilon^3\Bigg(\frac{27}{2} - \frac{31}{20}\zeta_5 - \frac{35}{12}\zeta_3 - \frac{3}{2}\zeta_2 \\
&+\frac{7}{24}\zeta_2\zeta_3 - \frac{141}{320}\zeta_2^2\Bigg) + \varepsilon^4\Bigg(-\frac{55}{4} + \frac{93}{80}\zeta_5 + \frac{7}{2}\zeta_3 - \frac{49}{144}\zeta_3^2 + \frac{13}{8}\zeta_2 \\
&-\frac{7}{32}\zeta_2\zeta_3 + \frac{47}{64}\zeta_2^2 + \frac{949}{4480}\zeta_2^3\Bigg)\Bigg] \qquad (A.4)
\end{aligned}$$

$$
\begin{aligned}
\hat{F}_q^{T,(2)} \;=\;& C_F n_f\Bigg[\frac{1}{\varepsilon^3}\Bigg(-\frac{8}{3}\Bigg)+\frac{1}{\varepsilon^2}\Bigg(\frac{56}{9}\Bigg)+\frac{1}{\varepsilon}\Bigg(-\frac{425}{27}-\frac{2}{3}\zeta_2\Bigg)+\Bigg(\frac{9989}{324}-\frac{26}{9}\zeta_3+\frac{38}{9}\zeta_2\Bigg)\\
&+\varepsilon\Bigg(-\frac{202253}{3888}+\frac{2}{27}\zeta_3-\frac{989}{108}\zeta_2+\frac{41}{60}\zeta_2^2\Bigg)+\varepsilon^2\Bigg(\frac{3788165}{46656}-\frac{121}{30}\zeta_5-\frac{935}{324}\zeta_3\\
&+\frac{22937}{1296}\zeta_2-\frac{13}{18}\zeta_2\zeta_3+\frac{97}{180}\zeta_2^2\Bigg)\Bigg]+C_F^2\Bigg[\frac{1}{\varepsilon^4}\Bigg(32\Bigg)+\frac{1}{\varepsilon^3}\Bigg(-48\Bigg)\\
&+\frac{1}{\varepsilon^2}\Bigg(98-8\zeta_2\Bigg)+\frac{1}{\varepsilon}\Bigg(-\frac{309}{2}+\frac{128}{3}\zeta_3\Bigg)+\Bigg(\frac{5317}{24}-90\zeta_3+\frac{41}{2}\zeta_2-13\zeta_2^2\Bigg)\\
&+\varepsilon\Bigg(-\frac{28127}{96}+\frac{92}{5}\zeta_5+\frac{1327}{6}\zeta_3-\frac{1495}{24}\zeta_2-\frac{56}{3}\zeta_2\zeta_3+\frac{173}{6}\zeta_2^2\Bigg)\\
&+\varepsilon^2\Bigg(\frac{1244293}{3456}-\frac{311}{10}\zeta_5-\frac{34735}{72}\zeta_3+\frac{652}{9}\zeta_3^2+\frac{38543}{288}\zeta_2+\frac{193}{6}\zeta_2\zeta_3-\frac{10085}{144}\zeta_2^2\\
&+\frac{223}{20}\zeta_2^3\Bigg)\Bigg]+C_AC_F\Bigg[\frac{1}{\varepsilon^3}\Bigg(\frac{44}{3}\Bigg)+\frac{1}{\varepsilon^2}\Bigg(-\frac{332}{9}+4\zeta_2\Bigg)+\frac{1}{\varepsilon}\Bigg(\frac{4921}{54}-26\zeta_3\\
&+\frac{11}{3}\zeta_2\Bigg)+\Bigg(-\frac{120205}{648}+\frac{755}{9}\zeta_3-\frac{251}{9}\zeta_2+\frac{44}{5}\zeta_2^2\Bigg)+\varepsilon\Bigg(\frac{2562925}{7776}-\frac{51}{2}\zeta_5\\
&-\frac{5273}{27}\zeta_3+\frac{14761}{216}\zeta_2+\frac{89}{6}\zeta_2\zeta_3-\frac{3299}{120}\zeta_2^2\Bigg)+\varepsilon^2\Bigg(-\frac{50471413}{93312}+\frac{3971}{60}\zeta_5\\
&+\frac{282817}{648}\zeta_3-\frac{569}{12}\zeta_3^2-\frac{351733}{2592}\zeta_2-\frac{1069}{36}\zeta_2\zeta_3+\frac{7481}{120}\zeta_2^2-\frac{809}{280}\zeta_2^3\Bigg)\Bigg]
\end{aligned}
\tag{A.5}
$$

Appendix B

B.1 Harmonic polylogarithms

In this section, we briefly describe the definition and properties of HPL and 2dHPL. HPL is represented by $H(\vec{m}_w; y)$ with a w-dimensional vector $\vec{m}_w$ of parameters and its argument y. w is called the weight of the HPL. The elements of $\vec{m}_w$ belong to $\{1, 0, -1\}$ through which the following rational functions are represented

$$f(1; y) \equiv \frac{1}{1-y}, \qquad f(0; y) \equiv \frac{1}{y}, \qquad f(-1; y) \equiv \frac{1}{1+y}\,. \tag{B.1}$$

The weight 1 ($w = 1$) HPLs are defined by

$$H(1, y) \equiv -\ln(1-y), \qquad H(0, y) \equiv \ln y, \qquad H(-1, y) \equiv \ln(1+y)\,. \tag{B.2}$$

For $w > 1$, $H(m, \vec{m}_w; y)$ is defined by

$$H(m, \vec{m}_w; y) \equiv \int_0^y dx\ f(m, x)\ H(\vec{m}_w; x), \qquad\qquad m \in 0, \pm 1\,. \tag{B.3}$$

The 2dHPLs are defined in the same way as eqn.(B.3) with the new elements $\{2, 3\}$ in $\vec{m}_w$ representing a new class of rational functions

$$f(2; y) \equiv f(1-z; y) \equiv \frac{1}{1-y-z}, \qquad f(3; y) \equiv f(z; y) \equiv \frac{1}{y+z} \tag{B.4}$$

and correspondingly with the weight 1 ($w = 1$) 2dHPLs

$$H(2, y) \equiv -\ln\left(1 - \frac{y}{1-z}\right), \qquad H(3, y) \equiv \ln\left(\frac{y+z}{z}\right). \tag{B.5}$$

B.1.1 Properties

Shuffle algebra : A product of two HPL with weights w_1 and w_2 of the same argument y is a combination of HPLs with weight $(w_1 + w_2)$ and argument y, such that all possible permutations of the elements of $\vec{m}_{w_1}$ and $\vec{m}_{w_2}$ are considered preserving the relative orders of the elements of $\vec{m}_{w_1}$ and $\vec{m}_{w_2}$,

$$H(\vec{m}_{w_1}; y)H(\vec{m}_{w_2}; y) = \sum_{\vec{m}_w \,=\, \vec{m}_{w_1} \uplus \vec{m}_{w_2}} H(\vec{m}_w; y). \tag{B.6}$$

Integration-by-parts identities : The ordering of the elements of $\vec{m}_w$ in an HPL with weight w and argument y can be reversed using integration-by-parts and in the process, some products of two HPLs are generated in the following way

$$\begin{aligned} H(\vec{m}_w; y) \equiv H(m_1, m_2, ..., m_w; y) &= H(m_1, y)H(m_2, ..., m_w; y) \\ &- H(m_2, m_1, y)H(m_3, ..., m_w; y) \\ &+ ... + (-1)^{w+1} H(m_w, ..., m_2, m_1; y)\,. \end{aligned} \tag{B.7}$$

Appendix C

C.1 One-loop coefficients

$$
\begin{aligned}
\mathcal{A}^{(1)}_{1;C_A} &= -\frac{4}{stu}\left(s^4+2s^3(t+u)+3s^2\left(t^2+u^2\right)+2s\left(t^3+u^3\right)+t^4+u^4\right)\\
\mathcal{A}^{(1)}_{1;n_f} &= -\frac{2}{s}\left(t^2+u^2\right)\\
\mathcal{A}^{(1)}_{2;C_A} &= \Big\{-6(t+u)^4\Big(11s^8+22(3t+u)s^7+\big(187t^2+64ut+33u^2\big)s^6+\big(330t^3+60ut^2+78u^2t+22u^3\big)s^5+\big(396t^4+14ut^3\\
&+51u^2t^2+36u^3t+11u^4\big)s^4+2t\big(165t^4+7ut^3+6u^2t^2+7u^3t+10u^4\big)s^3+t^2\big(187t^4+60ut^3+51u^2t^2+14u^3t+24u^4\big)s^2\\
&+2t^3\big(33t^4+32ut^3+39u^2t^2+18u^3t+10u^4\big)s+11t^4\big(t^2+ut+u^2\big)^2\Big)H(0,y)(s+u)^4-36(s+t)^4(t+u)^4\Big(s^4+2(t+u)s^3\\
&+3\big(t^2+u^2\big)s^2+2\big(t^3+u^3\big)s+t^4+u^4\Big)H(0,y)H(0,z)(s+u)^4+6(s+t)^4\Big(\big(11t^4+20ut^3+24u^2t^2+20u^3t+11u^4\big)s^4\\
&+2\big(11t^5+18ut^4+7u^2t^3+7u^3t^2+18u^4t+11u^5\big)s^3+3(t+u)^2\big(11t^4+4ut^3-2u^2t^2+4u^3t+11u^4\big)s^2+2(t+u)^3\big(11t^4\\
&-ut^3-u^3t+11u^4\big)s+11(t+u)^4\big(t^2+ut+u^2\big)^2\Big)H(1,z)(s+u)^4+36(s+t)^4(t+u)^4\Big(s^4+2ts^3+3t^2s^2+2t^3s+\big(t^2+ut\\
&+u^2\big)^2\Big)H(0,y)H(1,z)(s+u)^4+6(s+t)^4\Big(\big(11t^4+20ut^3+24u^2t^2+20u^3t+11u^4\big)s^4+2\big(11t^5+18ut^4+7u^2t^3+7u^3t^2\\
&+18u^4t+11u^5\big)s^3+3(t+u)^2\big(11t^4+4ut^3-2u^2t^2+4u^3t+11u^4\big)s^2+2(t+u)^3\big(11t^4-ut^3-u^3t+11u^4\big)s+11(t+u)^4\big(t^2\\
&+ut+u^2\big)^2\Big)H(2,y)(s+u)^4+36(s+t)^4(t+u)^4\Big(s^4+2us^3+3u^2s^2+2u^3s+\big(t^2+ut+u^2\big)^2\Big)H(0,z)H(2,y)(s+u)^4\\
&-36(s+t)^4(t+u)^4\Big(2s^4+2(t+u)s^3+3\big(t^2+u^2\big)s^2+2\big(t^3+u^3\big)s+2\big(t^2+ut+u^2\big)^2\Big)H(1,z)H(3,y)(s+u)^4\\
&-36(s+t)^4(t+u)^4\Big(s^4+2ts^3+3t^2s^2+2t^3s+\big(t^2+ut+u^2\big)^2\Big)H(0,1,z)(s+u)^4+36(s+t)^4(t+u)^4\Big(s^4+2ts^3\\
&+3t^2s^2+2t^3s+\big(t^2+ut+u^2\big)^2\Big)H(0,2,y)(s+u)^4-36(s+t)^4(t+u)^4\Big(2s^4+2(2t+u)s^3+3\big(2t^2+u^2\big)s^2+2\big(2t^3\\
&+u^3\big)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u\Big)H(1,0,y)(s+u)^4-36(s+t)^4(t+u)^4\Big(s^4+2(t+u)s^3+3\big(t^2+u^2\big)s^2+2\big(t^3\\
&+u^3\big)s+t^4+u^4\Big)H(1,0,z)(s+u)^4+36(s+t)^4(t+u)^4\Big(s^4+2ts^3+3t^2s^2+2t^3s+\big(t^2+ut+u^2\big)^2\Big)H(2,0,y)(s+u)^4\\
&-36(s+t)^4(t+u)^4\Big(2s^4+2(t+u)s^3+3\big(t^2+u^2\big)s^2+2\big(t^3+u^3\big)s+2\big(t^2+ut+u^2\big)^2\Big)H(3,2,y)(s+u)^4\\
&=-2(s+t)(t+u)\Big(203(t+u)^3s^{10}+2\big(506t^4+1988ut^3+2973u^2t^2+1988u^3t+506u^4\big)s^9+3\big(807t^5+3709ut^4+7123u^2t^3\\
&+7123u^3t^2+3709u^4t+807u^5\big)s^8+3\big(1208t^6+6186ut^5+13887u^2t^4+17836u^3t^3+13887u^4t^2+6186u^5t+1208u^6\big)s^7\\
&+\big(3624t^7+21596ut^6+54318u^2t^5+80567u^3t^4+80567u^4t^3+54318u^5t^2+21596u^6t+3624u^7\big)s^6+3\big(807t^8+6186ut^7\\
&+18106u^2t^6+29776u^3t^5+34116u^4t^4+29776u^5t^3+18106u^6t^2+6186u^7t+807u^8\big)s^5+\big(1012t^9+11127ut^8+41661u^2t^7\\
&+80567u^3t^6+102348u^4t^5+102348u^5t^4+80567u^6t^3+41661u^7t^2+11127u^8t+1012u^9\big)s^4+\big(203t^{10}+3976ut^9+21369u^2t^8\\
&+53508u^3t^7+80567u^4t^6+89328u^5t^5+80567u^6t^4+53508u^7t^3+21369u^8t^2+3976u^9t+203u^{10}\big)s^3+3tu\big(203t^9+1982ut^8\\
&+7123u^2t^7+13887u^3t^6+18106u^4t^5+18106u^5t^4+13887u^6t^3+7123u^7t^2+1982u^8t+203u^9\big)s^2+t^2u^2(t+u)^2\big(609t^6\\
&+2758ut^5+5002u^2t^4+5796u^3t^3+5002u^4t^2+2758u^5t+609u^6\big)s+t^3u^3(t+u)^3\big(203t^4+403ut^3+603u^2t^2+403u^3t\\
&+203u^4\big)\Big)(s+u)-6(s+t)^4(t+u)^4\Big(11s^8+22(t+3u)s^7+\big(33t^2+64ut+187u^2\big)s^6+\big(22t^3+78ut^2+60u^2t+330u^3\big)s^5\\
&+\big(11t^4+36ut^3+51u^2t^2+14u^3t+396u^4\big)s^4+2u\big(10t^4+7ut^3+6u^2t^2+7u^3t+165u^4\big)s^3+u^2\big(24t^4+14ut^3+51u^2t^2+60u^3t\\
&+187u^4\big)s^2+2u^3\big(10t^4+18ut^3+39u^2t^2+32u^3t+33u^4\big)s+11u^4\big(t^2+ut+u^2\big)^2\Big)H(0,z)\Big\}\Big/\Big(9st(s+t)^4u(s+u)^4(t+u)^4\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(1)}_{2;n_f} = \{ & - 18u(s+u)^4(t+u)^4 (s^2 - ts + t^2 + u^2) H(1,0,y)(s+t)^5 + 6\ (t+u)^4 (2s^8 + 4(t+3u)s^7 + 2(3t^2 + 5ut + 17u^2)\ s^6 \\
& + (4t^3 + 15ut^2 + 6u^2t + 60u^3)s^5 + 2(t^4 + 12u^2t^2 - 2\ u^3t + 36u^4)s^4 + 2u(t^4 - 2ut^3 + 15u^2t^2 - 2u^3t + 30\ u^4)s^3 \\
& + 2u^2(3t^4 - 2ut^3 + 12u^2t^2 + 3u^3t + 17u^4)\ s^2 + (12u^7 + 10tu^6 + 15t^2u^5 + 2t^4u^3)s + 2u^4(t^2 + u\ t + u^2)^2) H(0,z)(s+t)^4 \\
& - 18tu(s+u)^4(t+u)^4\ (t^2 + u^2) H(0,y)H(0,z)(s+t)^4 - 6(s+u)^4(2(t^4 + u\ t^3 + 3u^2t^2 + u^3t + u^4)s^4 + 4(t^5 - u^2t^3 \\
& - u^3t^2 + u^5)\ s^3 + 3(t+u)^2(2t^4 + ut^3 + 4u^2t^2 + u^3t + 2u^4)s^2 + 2(t+u)^3\ (2t^4 - ut^3 - u^3t + 2u^4)s + 2(t+u)^4(t^2 + u\ t \\
& + u^2)^2) H(1,z)(s+t)^4 + 18su(s+u)^4(t+u)^4\ (s^2 + u^2) H(0,y)H(1,z)(s+t)^4 - 6(s+u)^4(2(t^4 + u\ t^3 + 3u^2t^2 + u^3t \\
& + u^4)s^4 + 4(t^5 - u^2t^3 - u^3t^2 + u^5)\ s^3 + 3(t+u)^2(2t^4 + ut^3 + 4u^2t^2 + u^3t + 2u^4)s^2 + 2(t+u)^3\ (2t^4 - ut^3 - u^3t + 2u^4)s \\
& + 2(t+u)^4(t^2 + u\ t + u^2)^2) H(2,y)(s+t)^4 + 18st(s^2 + t^2)(s+u)^4\ (t+u)^4 H(0,z)H(2,y)(s+t)^4 - 18s(s+u)^4(t+u)^5(s^2 \\
& + t^2 + u^2 - t\ u) H(1,z)H(3,y)(s+t)^4 - 18su(s+u)^4(t+u)^4(s^2 + u^2)\ H(0,1,z)(s+t)^4 + 18su(s+u)^4(t+u)^4(s^2 \\
& + u^2) H(0,2,y)\ (s+t)^4 - 18tu(s+u)^4(t+u)^4(t^2 + u^2) H(1,0,z)(s+t)^4 + 18su\ (s+u)^4(t+u)^4(s^2 + u^2) H(2,0,y)(s+t)^4 \\
= & - 18s(s+u)^4(t+u)^5\ (s^2 + t^2 + u^2 - tu) H(3,2,y)(s+t)^4 + 2(s+u)(t+u)(35(t+u)^3\ s^{10} + 2(86t^4 + 335ut^3 + 507u^2t^2 \\
& + 335u^3t + 86u^4)s^9 + 3\ (135t^5 + 607ut^4 + 1177u^2t^3 + 1177u^3t^2 + 607u^4t + 135u^5)\ s^8 + 3(200t^6 + 990ut^5 + 2211u^2t^4 \\
& + 2860u^3t^3 + 2211u^4t^2 + 990u^5\ t + 200u^6)s^7 + (600t^7 + 3428ut^6 + 8436u^2t^5 + 12593u^3\ t^4 + 12593u^4t^3 + 8436u^5t^2 \\
& + 3428u^6t + 600u^7)s^6 + 3(135\ t^8 + 990ut^7 + 2812u^2t^6 + 4612u^3t^5 + 5328u^4t^4 + 4612u^5t^3 + 2812u^6\ t^2 + 990u^7t + 135u^8)s^5
\end{aligned}
$$

$$
\begin{aligned}
= & + (172t^9 + 1821ut^8 + 6633u^2t^7 + 12593\ u^3t^6 + 15984u^4t^5 + 15984u^5t^4 + 12593u^6t^3 + 6633u^7t^2 + 1821u^8\ t + 172u^9)s^4 \\
& + (35t^{10} + 670ut^9 + 3531u^2t^8 + 8580u^3\ t^7 + 12593u^4t^6 + 13836u^5t^5 + 12593u^6t^4 + 8580u^7t^3 + 3531u^8t^2 + 670\ u^9t \\
& + 35u^{10})s^3 + 3tu(35t^9 + 338ut^8 + 1177u^2t^7 + 2211\ u^3t^6 + 2812u^4t^5 + 2812u^5t^4 + 2211u^6t^3 + 1177u^7t^2 + 338u^8t + 35\ u^9)s^2 \\
& + t^2u^2(t+u)^2(105t^6 + 460ut^5 + 796u^2t^4 + 918u^3\ t^3 + 796u^4t^2 + 460u^5t + 105u^6)s + t^3u^3(t+u)^3(35t^4 + 67u\ t^3 \\
& + 99u^2t^2 + 67u^3t + 35u^4)))(s+t) + 6(s+u)^4(t+u)^4(2\ s^8 + 4(3t+u)s^7 + 2(17t^2 + 5ut + 3u^2)s^6 + (60t^3 + 6u\ t^2 + 15u^2t \\
& + 4u^3)s^5 + 2(36t^4 - 2ut^3 + 12u^2t^2 + u^4)\ s^4 + 2t(30t^4 - 2ut^3 + 15u^2t^2 - 2u^3t + u^4)s^3 + 2t^2\ (17t^4 + 3ut^3 + 12u^2t^2 \\
& - 2u^3t + 3u^4)s^2 + (12t^7 + 10u\ t^6 + 15u^2t^5 + 2u^4t^3)s + 2t^4(t^2 + ut + u^2)^2)\ H(0,y) \} \Big/ (9st(s+t)^4u(s+u)^4(t+u)^4)
\end{aligned}
$$

C.2 Two-loop coefficients

$$
\begin{aligned}
\mathcal{A}^{(2)}_{1;C_A^2} &= \frac{11}{stu}\left(s^4 + 2s^3(t+u) + 3s^2\left(t^2 + u^2\right) + 2s\ \left(t^3 + u^3\right) + t^4 - 2t^3u - 3t^2u^2 - 2tu^3 + u^4\right) \\
\mathcal{A}^{(2)}_{1;C_A n_f} &= -\frac{1}{stu}\left(2s^4 + 4s^3(t+u) + 6s^2\left(t^2 + u^2\right) + 4s\ \left(t^3 + u^3\right) + 2t^4 - 15t^3u - 6t^2u^2 - 15tu^3 + 2u^4\right) \\
\mathcal{A}^{(2)}_{1;C_F n_f} &= 0 \\
\mathcal{A}^{(2)}_{1;n_f^2} &= -\frac{2}{s}\left(t^2 + u^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{2;C_A^2} = \{ & 33(t+u)^4(11s^8 + 22(3t+u)s^7 + (187t^2 + 64ut + 33\ u^2)s^6 + (330t^3 + 60ut^2 + 78u^2t + 22u^3)s^5 + (396\ t^4 + 14ut^3 + 51u^2t^2 \\
= & + 36u^3t + 11u^4)s^4 + 2t(165t^4 + 7u\ t^3 + 6u^2t^2 + 7u^3t + 10u^4)s^3 + t^2(187t^4 + 60ut^3 + 51u^2\ t^2 + 14u^3t + 24u^4)s^2 + 2t^3(33t^4 + 32ut^3 \\
& + 39u^2t^2 + 18u^3\ t + 10u^4)s + 11t^4(t^2 + ut + u^2)^2) H(0,y)\ (s+u)^4 + 198(s+t)^4(t+u)^4(s^4 + 2(t+u)s^3 + 3(t^2 + u^2)\ s^2 + 2(t^3 \\
& + u^3)s + t^4 + u^4) H(0,y)H(0,z)(s+u)^4 - 33(s+t)^4\ ((11t^4 + 20ut^3 + 24u^2t^2 + 20u^3t + 11u^4)s^4 + 2(11\ t^5 + 18ut^4 + 7u^2t^3 + 7u^3t^2 \\
& + 18u^4t + 11u^5)s^3 + 3(t+u)^2\ (11t^4 + 4ut^3 - 2u^2t^2 + 4u^3t + 11u^4)s^2 + 2(t+u)^3(11\ t^4 - ut^3 - u^3t + 11u^4)s + 11(t+u)^4(t^2 + ut \\
& + u^2)^2)\ H(1,z)(s+u)^4 - 198(s+t)^4(t+u)^4(s^4 + 2ts^3 + 3t^2s^2 + 2t^3\ s + (t^2 + ut + u^2)^2) H(0,y)H(1,z)(s+u)^4 - 33(s+t)^4\ ((11t^4 \\
& + 20ut^3 + 24u^2t^2 + 20u^3t + 11u^4)s^4 + 2(11\ t^5 + 18ut^4 + 7u^2t^3 + 7u^3t^2 + 18u^4t + 11u^5)s^3 + 3(t+u)^2\ (11t^4 + 4ut^3 - 2u^2t^2 + 4u^3t \\
& + 11u^4)s^2 + 2(t+u)^3(11\ t^4 - ut^3 - u^3t + 11u^4)s + 11(t+u)^4(t^2 + ut + u^2)^2)\ H(2,y)(s+u)^4 - 198(s+t)^4(t+u)^4(s^4 + 2us^3 \\
& + 3u^2s^2 + 2u^3\ s + (t^2 + ut + u^2)^2) H(0,z)H(2,y)(s+u)^4 + 198(s+t)^4\ (t+u)^4(2s^4 + 2(t+u)s^3 + 3(t^2 + u^2)s^2 + 2\ (t^3 + u^3)s \\
& + 2(t^2 + ut + u^2)^2) H(1,z)H(3,y)\ (s+u)^4 + 198(s+t)^4(t+u)^4(s^4 + 2ts^3 + 3t^2s^2 + 2t^3s + (t^2 + u\ t + u^2)^2) H(0,1,z)(s+u)^4
\end{aligned}
$$

$$= -198(s+t)^4(t+u)^4\Big(s^4+2t\ s^3+3t^2s^2+2t^3s+\Big(t^2+ut+u^2\Big)^2\Big)H(0,2,y)(s+u)^4+198\ (s+t)^4(t+u)^4\Big(2s^4+2(2t+u)s^3$$

$$+3\Big(2t^2+u^2\Big)s^2+2\ \Big(2t^3+u^3\Big)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u\Big)\ H(1,0,y)(s+u)^4+198(s+t)^4(t+u)^4\Big(s^4+2(t+u)s^3$$

$$+3\ \Big(t^2+u^2\Big)s^2+2\Big(t^3+u^3\Big)s+t^4+u^4\Big)H(1,0,z)\ (s+u)^4-198(s+t)^4(t+u)^4\Big(s^4+2ts^3+3t^2s^2+2t^3s+\Big(t^2+u\ t$$

$$+u^2\Big)^2\Big)H(2,0,y)(s+u)^4+198(s+t)^4(t+u)^4\Big(2s^4+2\ (t+u)s^3+3\Big(t^2+u^2\Big)s^2+2\Big(t^3+u^3\Big)s+2\Big(t^2+u\ t+u^2\Big)^2\Big)H(3,2,y)(s+u)^4$$

$$= +(s+t)(t+u)\Big(1939(t+u)^3\ s^{10}+\Big(9662t^4+37856ut^3+56586u^2t^2+37856u^3t+9662u^4\Big)\ s^9+3\Big(7701t^5+35213ut^4+67475u^2t^3$$

$$+67475u^3t^2+35213u^4\ t+7701u^5\Big)s^8+3\Big(11524t^6+58638ut^5+131001u^2t^4+167972\ u^3t^3+131001u^4t^2+58638u^5t+11524u^6\Big)s^7$$

$$+\Big(34572\ t^7+204628ut^6+511062u^2t^5+754525u^3t^4+754525u^4t^3+511062u^5\ t^2+204628u^6t+34572u^7\Big)s^6+3\Big(7701t^8+58638ut^7$$

$$+170354\ u^2t^6+278144u^3t^5+317652u^4t^4+278144u^5t^3+170354u^6t^2+58638\ u^7t+7701u^8\Big)s^5+\Big(9662t^9+105639ut^8+393003u^2t^7$$

$$+754525\ u^3t^6+952956u^4t^5+952956u^5t^4+754525u^6t^3+393003u^7t^2+105639\ u^8t+9662u^9\Big)s^4+\Big(1939t^{10}+37856ut^9+202425u^2\ t^8$$

$$+503916u^3t^7+754525u^4t^6+834432u^5t^5+754525u^6t^4+503916u^7\ t^3+202425u^8t^2+37856u^9t+1939u^{10}\Big)s^3+3tu\Big(1939\ t^9+18862ut^8$$

$$+67475u^2t^7+131001u^3t^6+170354u^4t^5+170354u^5\ t^4+131001u^6t^3+67475u^7t^2+18862u^8t+1939u^9\Big)s^2+t^2u^2\ (t+u)^2\Big(5817t^6$$

$$+26222ut^5+47378u^2t^4+54936u^3t^3+47378u^4\ t^2+26222u^5t+5817u^6\Big)s+t^3u^3(t+u)^3\Big(1939t^4+3845u\ t^3+5751u^2t^2+3845u^3t$$

$$+1939u^4\Big)\Big)(s+u)+33(s+t)^4(t+u)^4\ \Big(11s^8+22(t+3u)s^7+\Big(33t^2+64ut+187u^2\Big)s^6+\Big(22\ t^3+78ut^2+60u^2t+330u^3\Big)s^5$$

$$+\Big(11t^4+36ut^3+51u^2t^2+14\ u^3t+396u^4\Big)s^4+2u\Big(10t^4+7ut^3+6u^2t^2+7u^3t+165\ u^4\Big)s^3+u^2\Big(24t^4+14ut^3+51u^2t^2+60u^3t$$

$$+187u^4\Big)\ s^2+2u^3\Big(10t^4+18ut^3+39u^2t^2+32u^3t+33u^4\Big)s+11u^4\ \Big(t^2+ut+u^2\Big)^2\Big)H(0,z)\Big\}\Big/\Big(9st(s+t)^4u(s+u)^4(t+u)^4\Big)$$

$$\mathcal{A}^{(2)}_{2;C_A n_f} = -\Big\{3(t+u)^4\Big(44s^8+88(3t+u)s^7+2\Big(374t^2+119ut+66\ u^2\Big)s^6+\Big(1320t^3+186ut^2+321u^2t+88u^3\Big)s^5+2\ \Big(792t^4-8ut^3$$

$$= +183u^2t^2+36u^3t+22u^4\Big)s^4+2t\Big(660\ t^4-8ut^3+177u^2t^2-8u^3t+31u^4\Big)s^3+2t^2\Big(374t^4+93u\ t^3+183u^2t^2-8u^3t+57u^4\Big)s^2$$

$$+t^3\Big(264t^4+238ut^3+321\ u^2t^2+72u^3t+62u^4\Big)s+44t^4\Big(t^2+ut+u^2\Big)^2\Big)\ H(0,y)(s+u)^4+9(s+t)^4(t+u)^4\Big(4s^4+8(t+u)s^3$$

$$+12\ \Big(t^2+u^2\Big)s^2+8\Big(t^3+u^3\Big)s+4t^4+4u^4-11tu^3-11\ t^3u\Big)H(0,y)H(0,z)(s+u)^4-3(s+t)^4\Big(2\Big(22t^4+31u\ t^3+57u^2t^2$$

$$+31u^3t+22u^4\Big)s^4+8\Big(11t^5+9ut^4-2u^2t^3-2\ u^3t^2+9u^4t+11u^5\Big)s^3+3(t+u)^2\Big(44t^4+19ut^3+40u^2\ t^2+19u^3t+44u^4\Big)s^2$$

$$+2(t+u)^3\Big(44t^4-13ut^3-13u^3t+44\ u^4\Big)s+44(t+u)^4\Big(t^2+ut+u^2\Big)^2\Big)H(1,z)(s+u)^4-9\ (s+t)^4(t+u)^4\Big(4s^4+(8t-11u)s^3$$

$$+12t^2s^2+\Big(8t^3-11\ u^3\Big)s+4\Big(t^2+ut+u^2\Big)^2\Big)H(0,y)H(1,z)(s+u)^4-3\ (s+t)^4\Big(2\Big(22t^4+31ut^3+57u^2t^2+31u^3t+22u^4\Big)\ s^4$$

$$+8\Big(11t^5+9ut^4-2u^2t^3-2u^3t^2+9u^4t+11u^5\Big)s^3+3\ (t+u)^2\Big(44t^4+19ut^3+40u^2t^2+19u^3t+44u^4\Big)s^2+2\ (t+u)^3\Big(44t^4$$

$$-13ut^3-13u^3t+44u^4\Big)s+44(t+u)^4\ \Big(t^2+ut+u^2\Big)^2\Big)H(2,y)(s+u)^4-9(s+t)^4(t+u)^4\Big(4\ s^4+(8u-11t)s^3+12u^2s^2+\Big(8u^3$$

$$-11t^3\Big)s+4\Big(t^2+u\ t+u^2\Big)^2\Big)H(0,z)H(2,y)(s+u)^4+9(s+t)^4(t+u)^4\Big(8s^4-3\ (t+u)s^3+12\Big(t^2+u^2\Big)s^2-3\Big(t^3+u^3\Big)s$$

$$+8\Big(t^2+u\ t+u^2\Big)^2\Big)H(1,z)H(3,y)(s+u)^4+9(s+t)^4(t+u)^4\Big(4s^4+(8\ t-11u)s^3+12t^2s^2+\Big(8t^3-11u^3\Big)s+4\Big(t^2+u\ t$$

$$+u^2\Big)^2\Big)H(0,1,z)(s+u)^4-9(s+t)^4(t+u)^4\Big(4s^4+(8t-11\ u)s^3+12t^2s^2+\Big(8t^3-11u^3\Big)s+4\Big(t^2+u\ t+u^2\Big)^2\Big)H(0,2,y)(s+u)^4$$

$$+9(s+t)^4(t+u)^4\Big(8s^4+(16t-3\ u)s^3+12\Big(2t^2+u^2\Big)s^2+\Big(16t^3-3u^3\Big)s+8t^4+8\ u^4-3tu^3+12t^2u^2-3t^3u\Big)H(1,0,y)(s+u)^4$$

$$+9(s+t)^4(t+u)^4\ \Big(4s^4+8(t+u)s^3+12\Big(t^2+u^2\Big)s^2+8\Big(t^3+u^3\Big)\ s+4t^4+4u^4-11tu^3-11t^3u\Big)H(1,0,z)(s+u)^4$$

$$-9(s+t)^4(t+u)^4\ \Big(4s^4+(8t-11u)s^3+12t^2s^2+\Big(8t^3-11u^3\Big)s+4\ \Big(t^2+ut+u^2\Big)^2\Big)H(2,0,y)(s+u)^4+9(s+t)^4(t+u)^4\Big(8\ s^4$$

$$-3(t+u)s^3+12\Big(t^2+u^2\Big)s^2-3\Big(t^3+u^3\Big)s+8\ \Big(t^2+ut+u^2\Big)^2\Big)H(3,2,y)(s+u)^4+(s+t)(t+u)\Big(499\ (t+u)^3s^{10}+2\Big(1228t^4+4741ut^3$$

$$= +7143u^2t^2+4741u^3t+1228\ u^4\Big)s^9+3\Big(1931t^5+8547ut^4+16389u^2t^3+16389u^3t^2+8547\ u^4t+1931u^5\Big)s^8+3\Big(2864t^6$$

$$+13918ut^5+30487u^2t^4+39100\ u^3t^3+30487u^4t^2+13918u^5t+2864u^6\Big)s^7+\Big(8592t^7+48196u\ t^6+115584u^2t^5+168841u^3t^4$$

$$+168841u^4t^3+115584u^5t^2+48196u^6\ t+8592u^7\Big)s^6+3\Big(1931t^8+13918ut^7+38528u^2t^6+61228u^3\ t^5+69608u^4t^4$$

$$+61228u^5t^3+38528u^6t^2+13918u^7t+1931u^8\Big)\ s^5+\Big(2456t^9+25641ut^8+91461u^2t^7+168841u^3t^6+208824u^4\ t^5$$

$$+208824u^5t^4+168841u^6t^3+91461u^7t^2+25641u^8t+2456u^9\Big)\ s^4+\Big(499t^{10}+9482ut^9+49167u^2t^8+117300u^3t^7+168841u^4\ t^6$$

$$+183684u^5t^5+168841u^6t^4+117300u^7t^3+49167u^8t^2+9482u^9\ t+499u^{10}\Big)s^3+3tu\Big(499t^9+4762ut^8+16389u^2t^7+30487\ u^3t^6$$

$$+38528u^4t^5+38528u^5t^4+30487u^6t^3+16389u^7t^2+4762u^8\ t+499u^9\Big)s^2+t^2u^2(t+u)^2\Big(1497t^6+6488ut^5+11168u^2\ t^4$$

$$+12930u^3t^3+11168u^4t^2+6488u^5t+1497u^6\Big)s+t^3u^3(t+u)^3\ \Big(499t^4+959ut^3+1419u^2t^2+959u^3t+499u^4\Big)\Big)(s+u)$$

$$+3\ (s+t)^4(t+u)^4\Big(44s^8+88(t+3u)s^7+2\Big(66t^2+119ut+374\ u^2\Big)s^6+\Big(88t^3+321ut^2+186u^2t+1320u^3\Big)s^5+2\ \Big(22t^4$$

$$+36ut^3+183u^2t^2-8u^3t+792u^4\Big)s^4+2u\Big(31\ t^4-8ut^3+177u^2t^2-8u^3t+660u^4\Big)s^3+2u^2\Big(57t^4-8u\ t^3+183u^2t^2+93u^3t$$

$$+374u^4\Big)s^2+u^3\Big(62t^4+72ut^3+321\ u^2t^2+238u^3t+264u^4\Big)s+44u^4\Big(t^2+ut+u^2\Big)^2\Big)\ H(0,z)\Big\}\Big/\Big(9st(s+t)^4u(s+u)^4(t+u)^4\Big)$$

$$\mathcal{A}^{(2)}_{2;C_F n_f} = -\Big\{8\Big(s^4+2s^3(t+u)+3s^2\Big(t^2+u^2\Big)+2s\ \Big(t^3+u^3\Big)+\Big(t^2+tu+u^2\Big)^2\Big)\Big\}\Big/\Big(stu\Big)$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{2;n_f^2} = \Big\{ & -18u(s+u)^4(t+u)^4\Big(s^2-ts+t^2+u^2\Big)H(1,0,y)(s+t)^5+6\ (t+u)^4\Big(2s^8+4(t+3u)s^7+2\Big(3t^2+5ut+17u^2\Big)\ s^6 \\
=+ & \Big(4t^3+15ut^2+6u^2t+60u^3\Big)s^5+2\Big(t^4+12u^2t^2-2\ u^3t+36u^4\Big)s^4+2u\Big(t^4-2ut^3+15u^2t^2-2u^3t+30\ u^4\Big)s^3+2u^2\Big(3t^4-2ut^3 \\
& +12u^2t^2+3u^3t+17u^4\Big)\ s^2+\Big(12u^7+10tu^6+15t^2u^5+2t^4u^3\Big)s+2u^4\Big(t^2+u\ t+u^2\Big)^2\Big)H(0,z)(s+t)^4-18tu(s+u)^4(t+u)^4\ \Big(t^2 \\
& +u^2\Big)H(0,y)H(0,z)(s+t)^4-6(s+u)^4\Big(2\Big(t^4+u\ t^3+3u^2t^2+u^3t+u^4\Big)s^4+4\Big(t^5-u^2t^3-u^3t^2+u^5\Big)\ s^3+3(t+u)^2\Big(2t^4+ut^3 \\
& +4u^2t^2+u^3t+2u^4\Big)s^2+2(t+u)^3\ \Big(2t^4-ut^3-u^3t+2u^4\Big)s+2(t+u)^4\Big(t^2+u\ t+u^2\Big)^2\Big)H(1,z)(s+t)^4+18su(s+u)^4(t+u)^4\ \Big(s^2 \\
& +u^2\Big)H(0,y)H(1,z)(s+t)^4-6(s+u)^4\Big(2\Big(t^4+u\ t^3+3u^2t^2+u^3t+u^4\Big)s^4+4\Big(t^5-u^2t^3-u^3t^2+u^5\Big)\ s^3+3(t+u)^2\Big(2t^4 \\
& +ut^3+4u^2t^2+u^3t+2u^4\Big)s^2+2(t+u)^3\ \Big(2t^4-ut^3-u^3t+2u^4\Big)s+2(t+u)^4\Big(t^2+u\ \ t+u^2\Big)^2\Big)H(2,y)(s+t)^4 \\
& +18st\Big(s^2+t^2\Big)(s+u)^4\ \ (t+u)^4H(0,z)H(2,y)(s+t)^4-18s(s+u)^4(t+u)^5\Big(s^2+t^2+u^2-t\ \ u\Big)H(1,z)H(3,y)(s+t)^4 \\
& -18su(s+u)^4(t+u)^4\Big(s^2+u^2\Big)\ H(0,1,z)(s+t)^4+18su(s+u)^4(t+u)^4\Big(s^2+u^2\Big)H(0,2,y)\ (s+t)^4-18tu(s+u)^4(t+u)^4\Big(t^2 \\
& +u^2\Big)H(1,0,z)(s+t)^4+18su\ (s+u)^4(t+u)^4\Big(s^2+u^2\Big)H(2,0,y)(s+t)^4-18s(s+u)^4(t+u)^5\ \Big(s^2+t^2+u^2-tu\Big)H(3,2,y)(s+t)^4 \\
& +6(s+u)(t+u)\Big(5(t+u)^3\ s^{10}+6\Big(4t^4+15ut^3+23u^2t^2+15u^3t+4u^4\Big)s^9+\Big(55\ t^5+227ut^4+437u^2t^3+437u^3t^2+227u^4t+55u^5\Big)s^8 \\
=+ & \Big(80\ t^6+350ut^5+731u^2t^4+940u^3t^3+731u^4t^2+350u^5t+80u^6\Big)\ s^7+\Big(80t^7+396ut^6+852u^2t^5+1211u^3t^4+1211u^4t^3+852u^5\ t^2 \\
& +396u^6t+80u^7\Big)s^6+\Big(55t^8+350ut^7+852u^2t^6+1252u^3\ t^5+1408u^4t^4+1252u^5t^3+852u^6t^2+350u^7t+55u^8\Big)\ s^5+\Big(24t^9+227ut^8 \\
& +731u^2t^7+1211u^3t^6+1408u^4t^5+1408u^5\ \ t^4+1211u^6t^3+731u^7t^2+227u^8t+24u^9\Big)s^4+\Big(5t^{10}+90u\ \ t^9+437u^2t^8+940u^3t^7 \\
& +1211u^4t^6+1252u^5t^5+1211u^6t^4+940u^7\ \ t^3+437u^8t^2+90u^9t+5u^{10}\Big)s^3+tu\Big(15t^9+138ut^8+437\ u^2t^7+731u^3t^6+852u^4t^5 \\
& +852u^5t^4+731u^6t^3+437u^7t^2+138u^8\ \ t+15u^9\Big)s^2+t^2u^2(t+u)^2\Big(15t^6+60ut^5+92u^2t^4+106u^3\ \ t^3+92u^4t^2+60u^5t+15u^6\Big)s \\
& +t^3u^3(t+u)^3\Big(5t^4+9u\ t^3+13u^2t^2+9u^3t+5u^4\Big)\Big)(s+t)+6(s+u)^4(t+u)^4\Big(2\ s^8+4(3t+u)s^7+2\Big(17t^2+5ut+3u^2\Big)s^6+\Big(60t^3 \\
& +6u\ \ t^2+15u^2t+4u^3\Big)s^5+2\Big(36t^4-2ut^3+12u^2t^2+u^4\Big)\ \ s^4+2t\Big(30t^4-2ut^3+15u^2t^2-2u^3t+u^4\Big)s^3+2t^2\ \ \Big(17t^4+3ut^3 \\
& +12u^2t^2-2u^3t+3u^4\Big)s^2+\Big(12t^7+10u\ \ t^6+15u^2t^5+2u^4t^3\Big)s+2t^4\Big(t^2+ut+u^2\Big)^2\Big)\ \ H(0,y)\Big\}\Big/\Big(9st(s+t)^4u(s+u)^4(t+u)^4\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{3;C_A^2} &= \Big\{49s^4+98s^3\ \ (t+u)+147s^2\Big(t^2+u^2\Big)+98s\Big(t^3+u^3\Big)+49t^4-450t^3\ \ u+165t^2u^2-450tu^3+49u^4\Big\}\Big/\Big(5stu\Big) \\
\mathcal{A}^{(2)}_{3;C_An_f} &= \Big\{353t^2-588tu+353u^2\Big\}\Big/\Big(5s\Big) \\
\mathcal{A}^{(2)}_{3;C_Fn_f} &= -18\Big\{t^2+u^2\Big\}\Big/\Big(s\Big) \\
\mathcal{A}^{(2)}_{3;n_f^2} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{4;C_A^2} = \Big\{ & 9s^2t^2u^2\Big(231s^{16}+1896(t+u)s^{15}+6\Big(1205t^2+2582u\ \ t+1205u^2\Big)s^{14}+2\Big(8485t^3+28497ut^2+28497u^2t+8485\ \ u^3\Big)s^{13} \\
=+ & \Big(27247t^4+132002ut^3+187458u^2t^2+132002u^3\ \ t+27247u^4\Big)s^{12}+12\Big(2609t^5+18137ut^4+32471u^2t^3+32471\ u^3t^2+18137u^4t \\
& +2609u^5\Big)s^{11}+\Big(26071t^6+265278u\ t^5+584223u^2t^4+758272u^3t^3+584223u^4t^2+265278u^5t+26071\ u^6\Big)s^{10}+2\Big(7765t^7+118994ut^6 \\
& +329001u^2t^5+544812u^3\ t^4+544812u^4t^3+329001u^5t^2+118994u^6t+7765u^7\Big)s^9+3\ \Big(2110t^8+50618ut^7+186907u^2t^6+405032u^3t^5 \\
& +468376u^4\ \ t^4+405032u^5t^3+186907u^6t^2+50618u^7t+2110u^8\Big)s^8+4\ \Big(398t^9+16089ut^8+86178u^2t^7+272710u^3t^6+333261u^4t^5 \\
& +333261\ u^5t^4+272710u^6t^3+86178u^7t^2+16089u^8t+398u^9\Big)\ s^7+\Big(187t^{10}+16046ut^9+137715u^2t^8+724384u^3t^7+1065452u^4\ t^6 \\
& +947616u^5t^5+1065452u^6t^4+724384u^7t^3+137715u^8t^2+16046u^9\ \ t+187u^{10}\Big)s^6+6tu\Big(299t^9+5160ut^8+51597u^2t^7+106788\ u^3t^6 \\
& +101934u^4t^5+101934u^5t^4+106788u^6t^3+51597u^7t^2+5160u^8\ \ t+299u^9\Big)s^5+t^2u^2\Big(2973t^8+74726ut^7+236013u^2\ t^6+329784u^3t^5 \\
& +314484u^4t^4+329784u^5t^3+236013u^6t^2+74726u^7\ \ t+2973u^8\Big)s^4+12t^3u^3\Big(651t^7+3762ut^6+9502u^2t^5+11138\ u^3t^4+11138u^4t^3 \\
& +9502u^5t^2+3762u^6t+651u^7\Big)s^3+3t^4u^4\ \ \Big(1027t^6+7446ut^5+12323u^2t^4+11392u^3t^3+12323u^4t^2+7446u^5\ \ t+1027u^6\Big)s^2 \\
& +6t^5u^5\Big(315t^5+904ut^4+837u^2t^3+837u^3\ t^2+904u^4t+315u^5\Big)s+t^6u^6\Big(215t^4+250ut^3+357u^2\ t^2+250u^3t+215u^4\Big)\Big)(t+u)^6 \\
=- & 108s^2t^2(s+t)^6u^2(s+u)^6\ \ \Big(s^4+2(t+u)s^3+3\Big(t^2+u^2\Big)s^2+2\Big(t^3+u^3\Big)\ s+t^4+u^4-12tu^3+18t^2u^2-24t^3u\Big)H(0,y)(t+u)^6 \\
& -108s^2t^2\ \ (s+t)^6u^2(s+u)^6\Big(s^4+2(t+u)s^3+3\Big(t^2+u^2\Big)s^2+2\ \ \Big(t^3+u^3\Big)s+t^4+u^4-24tu^3+18t^2u^2-12t^3u\Big)H(0,z)\ \ (t+u)^6 \\
& +216s^2t^2(s+t)^6u^3(s+u)^6\Big(2s^3-6us^2+10u^2s+3t\ \ \Big(2t^2-3ut+4u^2\Big)\Big)H(1,y)(t+u)^6+108s^2t^2(s+t)^6u^2\ \ (s+u)^6\Big(s^4+2(t-u)s^3 \\
& +3\Big(t^2+5u^2\Big)s^2+2\Big(t^3-9\ u^3\Big)s+t^4+u^4+18tu^3-9t^2u^2+30t^3u\Big)H(1,z)(t+u)^6+108\ \ s^2t^2(s+t)^6u^2(s+u)^6\Big(s^4-2(t+u)s^3 \\
& +15\Big(t^2+u^2\Big)\ \ s^2-18\Big(t^3+u^3\Big)s+t^4+u^4+6tu^3+9t^2u^2+6t^3u\Big)\ \ H(2,y)(t+u)^6\Big\}\Big/\Big(27s^3t^3u^3(s+t)^6(s+u)^6(t+u)^6\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{4;C_A n_f} =& \Big\{9s^2t^2u^2\Big(54s^{16}+420(t+u)s^{15}+24\Big(64t^2+93ut+64u^2\Big)s^{14}+\Big(3520t^3+6099ut^2+6099u^2t+3520u^3\Big)s^{13} \\
=&+2\Big(2821t^4+4943ut^3+9330u^2t^2+4943u^3t+2821u^4\Big)s^{12}+6\Big(1106t^5+1697ut^4+6163u^2t^3+6163u^3t^2+1697u^4t+1106u^5\Big)s^{11} \\
&+\Big(5810t^6+7365ut^5+65076u^2t^4+38726u^3t^3+65076u^4t^2+7365u^5t+5810u^6\Big)s^{10}+\Big(3736t^7+3503ut^6+105864u^2t^5+17977u^3t^4 \\
&+17977u^4t^3+105864u^5t^2+3503u^6t+3736u^7\Big)s^9+3\Big(560t^8+208ut^7+38974u^2t^6+8059u^3t^5-15308u^4t^4+8059u^5t^3+38974u^6t^2 \\
&+208u^7t+560u^8\Big)s^8+\Big(472t^9+60ut^8+81318u^2t^7-8215u^3t^6+18453u^4t^5+18453u^5t^4-8215u^6t^3+81318u^7t^2+60u^8t \\
&+472u^9\Big)s^7+\Big(62t^{10}+229ut^9+37632u^2t^8-50045u^3t^7+25292u^4t^6+177060u^5t^5+25292u^6t^4-50045u^7t^3+37632u^8t^2 \\
&+229u^9t+62u^{10}\Big)s^6+3tu\Big(24t^9+3679ut^8-12820u^2t^7+195u^3t^6+50191u^4t^5+50191u^5t^4+195u^6t^3-12820u^7t^2+3679u^8t \\
&+24u^9\Big)s^5+2t^2u^2\Big(735t^8-6322ut^7+3033u^2t^6+30537u^3t^5+41982u^4t^4+30537u^5t^3+3033u^6t^2-6322u^7t+735u^8\Big)s^4 \\
&+t^3u^3\Big(-1700t^7+5703ut^6+17862u^2t^5+21167u^3t^4+21167u^4t^3+17862u^5t^2+5703u^6t-1700u^7\Big)s^3+12t^4u^4\Big(109t^6 \\
&+190ut^5+297u^2t^4+423u^3t^3+297u^4t^2+190u^5t+109u^6\Big)s^2-6t^5u^5\Big(12t^5-24ut^4-209u^2t^3-209u^3t^2-24u^4t+12u^5\Big)s \\
=&+t^6u^6\Big(20t^4+131ut^3+174u^2t^2+131u^3t+20u^4\Big)\Big)(t+u)^6-162s^2t^3(s+t)^6u^3(s+u)^6\Big(8t^2-21ut \\
&+17u^2\Big)H(0,y)(t+u)^6-162s^2t^3(s+t)^6u^3(s+u)^6\Big(17t^2-21ut+8u^2\Big)H(0,z)(t+u)^6-54s^2t^2(s+t)^6u^3(s+u)^6\Big(41s^3 \\
&-42us^2+5u^2s+t\Big(50t^2-63ut+23u^2\Big)\Big)H(1,y)(t+u)^6+54s^2t^2(s+t)^6u^3(s+u)^6\Big(41s^3-42us^2 \\
&+5u^2s+t\Big(-25t^2+63ut-52u^2\Big)\Big)H(1,z)(t+u)^6+54s^2t^2(s+t)^6u^2(s+u)^6\Big(41(t+u)s^3-42\Big(t^2 \\
&+u^2\Big)s^2+5\Big(t^3+u^3\Big)s-2tu\Big(t^2+u^2\Big)\Big)H(2,y)(t+u)^6\Big\}\Big/\Big(27s^3t^3u^3(s+t)^6(s+u)^6(t+u)^6\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{4;C_F n_f} =& \Big\{432s^3t^2(s+t)^6u^2(s+u)^6\Big(-3s^2+t^2+u^2-tu\Big)H(2,y)(t+u)^7-18s^2t^2u^2\Big(48s^{16}+384(t+u)s^{15}+12\Big(120t^2+227ut \\
=&+120u^2\Big)s^{14}+\Big(3360t^3+9087ut^2+9087u^2t+3360u^3\Big)s^{13}+3\Big(1808t^4+6521ut^3+8604u^2t^2+6521u^3t+1808u^4\Big)s^{12}+\Big(6336t^5 \\
&+30735ut^4+47598u^2t^3+47598u^3t^2+30735u^4t+6336u^5\Big)s^{11}+\Big(5424t^6+36543ut^5+67614u^2t^4+73464u^3t^3+67614u^4t^2 \\
&+36543u^5t+5424u^6\Big)s^{10}+\Big(3360t^7+32505ut^6+77922u^2t^5+91295u^3t^4+91295u^4t^3+77922u^5t^2+32505u^6t+3360u^7\Big)s^9 \\
&+3\Big(480t^8+6939ut^7+23404u^2t^6+34497u^3t^5+31012u^4t^4+34497u^5t^3+23404u^6t^2+6939u^7t+480u^8\Big)s^8+3\Big(128t^9+2999ut^8 \\
&+15226u^2t^7+33508u^3t^6+30137u^4t^5+30137u^5t^4+33508u^6t^3+15226u^7t^2+2999u^8t+128u^9\Big)s^7+\Big(48t^{10}+2337ut^9+19494u^2t^8 \\
&+70630u^3t^7+86678u^4t^6+65250u^5t^5+86678u^6t^4+70630u^7t^3+19494u^8t^2+2337u^9t+48u^{10}\Big)s^6+3tu\Big(92t^9+1601ut^8+10417u^2t^7 \\
&+20343u^3t^6+17708u^4t^5+17708u^5t^4+20343u^6t^3+10417u^7t^2+1601u^8t+92u^9\Big)s^5+3t^2u^2\Big(172t^8+2556ut^7+8524u^2t^6+11797u^3t^5 \\
&+10692u^4t^4+11797u^5t^3+8524u^6t^2+2556u^7t+172u^8\Big)s^4+t^3u^3\Big(788t^7+5538ut^6+14103u^2t^5+16252u^3t^4+16252u^4t^3+14103u^5t^2 \\
&+5538u^6t+788u^7\Big)s^3+3t^4u^4\Big(148t^6+959ut^5+1726u^2t^4+1748u^3t^3+1726u^4t^2+959u^5t+148u^6\Big)s^2+3t^5u^5\Big(72t^5+265ut^4 \\
&+347u^2t^3+347u^3t^2+265u^4t+72u^5\Big)s+2t^6u^6\Big(16t^4+39ut^3+54u^2t^2+39u^3t+16u^4\Big)\Big)(t+u)^6+108s^2t^3(s+t)^6u^3(s+u)^6\Big(t^2 \\
&+13u^2\Big)H(0,y)(t+u)^6+108s^2t^3(s+t)^6u^3(s+u)^6\Big(13t^2+u^2\Big)H(0,z)(t+u)^6+108s^2t^2(s+t)^6u^3(s+u)^6\Big(12s^3-4u^2s+13t^3 \\
&+tu^2\Big)H(1,y)(t+u)^6-108s^2t^2(s+t)^6u^3(s+u)^6\Big(12s^3-4u^2s-t\Big(t^2+13u^2\Big)\Big)H(1,z)(t+u)^6\Big\}\Big/\Big(27s^3t^3u^3(s+t)^6(s+u)^6(t+u)^6\Big)
\end{aligned}
$$

$$
\mathcal{A}^{(2)}_{4;n_f^2} = 0
$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{5;C_A^2} =& \Big\{9s^2t^2(s+t)^2u^2\Big(55s^{14}+(330t+416u)s^{13}+\Big(935t^2+1844ut+1500u^2\Big)s^{12}+30\Big(55t^3+135ut^2+167u^2t+115u^3\Big)s^{11} \\
=&+\Big(1980t^4+5670ut^3+8826u^2t^2+8462u^3t+5643u^4\Big)s^{10}+2\Big(825t^5+2487ut^4+6495u^2t^3+3799u^3t^2+5061u^4t+3426u^5\Big)s^9 \\
&+\Big(935t^6+2568ut^5+15141u^2t^4+4070u^3t^3-1902u^4t^2+9678u^5t+6231u^6\Big)s^8+2\Big(165t^7+443ut^6+6261u^2t^5-139u^3t^4-8575u^4t^3 \\
&-4065u^5t^2+3881u^6t+2085u^7\Big)s^7+\Big(55t^8+334ut^7+7728u^2t^6-6262u^3t^5-17396u^4t^4-22194u^5t^3-7172u^6t^2+5130u^7t \\
&+1950u^8\Big)s^6+2u\Big(45t^8+1695ut^7-3483u^2t^6-4305u^3t^5-4851u^4t^4-8249u^5t^3-1455u^6t^2+1362u^7t+284u^8\Big)s^5+u^2\Big(717t^8 \\
&-2782ut^7-1182u^2t^6+5322u^3t^5-4938u^4t^4-7502u^5t^3+711u^6t^2+998u^7t+77u^8\Big)s^4-2tu^3\Big(158t^7-735ut^6-2541u^2t^5 \\
&-2255u^3t^4+577u^4t^3+150u^5t^2-512u^6t-82u^7\Big)s^3+t^2u^4\Big(717t^6+1098ut^5+2632u^2t^4+2622u^3t^3+1308u^4t^2+852u^5t+234u^6\Big)s^2 \\
&+2t^3u^5\Big(45t^5+157ut^4+619u^2t^3+516u^3t^2+340u^4t+102u^5\Big)s+t^4u^6\Big(55t^4+134ut^3+207u^2t^2+134u^3t+105u^4\Big)\Big)H(0,y)(t+u)^6
\end{aligned}
$$

$$= + 9s^2t^2u^2(s+u)^2\Big(55s^{14} + (416\ t + 330u)s^{13} + \Big(1500t^2 + 1844ut + 935u^2\Big)s^{12} + 30\Big(115\ t^3 + 167ut^2 + 135u^2t + 55u^3\Big)s^{11} + \Big(5643t^4$$

$$+ 8462ut^3 + 8826\ u^2t^2 + 5670u^3t + 1980u^4\Big)s^{10} + 2\Big(3426t^5 + 5061ut^4 + 3799\ u^2t^3 + 6495u^3t^2 + 2487u^4t + 825u^5\Big)s^9 + \Big(6231t^6$$

$$+ 9678u\ t^5 - 1902u^2t^4 + 4070u^3t^3 + 15141u^4t^2 + 2568u^5t + 935u^6\Big)s^8 + 2\ \Big(2085t^7 + 3881ut^6 - 4065u^2t^5 - 8575u^3t^4 - 139u^4t^3$$

$$+ 6261u^5\ t^2 + 443u^6t + 165u^7\Big)s^7 + \Big(1950t^8 + 5130ut^7 - 7172u^2\ t^6 - 22194u^3t^5 - 17396u^4t^4 - 6262u^5t^3 + 7728u^6t^2 + 334u^7t$$

$$+ 55\ u^8\Big)s^6 + 2t\Big(284t^8 + 1362ut^7 - 1455u^2t^6 - 8249u^3t^5 - 4851\ u^4t^4 - 4305u^5t^3 - 3483u^6t^2 + 1695u^7t + 45u^8\Big)s^5 + t^2\Big(77\ t^8$$

$$+ 998ut^7 + 711u^2t^6 - 7502u^3t^5 - 4938u^4t^4 + 5322u^5t^3 - 1182u^6\ t^2 - 2782u^7t + 717u^8\Big)s^4 + 2t^3u\Big(82t^7 + 512ut^6 - 150u^2\ t^5 - 577u^3t^4$$

$$+ 2255u^4t^3 + 2541u^5t^2 + 735u^6t - 158u^7\Big)s^3 + t^4\ u^2\Big(234t^6 + 852ut^5 + 1308u^2t^4 + 2622u^3t^3 + 2632u^4t^2 + 1098u^5\ t + 717u^6\Big)s^2$$

$$+ 2t^5u^3\Big(102t^5 + 340ut^4 + 516u^2t^3 + 619u^3\ t^2 + 157u^4t + 45u^5\Big)s + t^6u^4\Big(105t^4 + 134ut^3 + 207u^2\ t^2 + 134u^3t + 55u^4\Big)\Big)H(0,z)(t+u)^6$$

$$+ 108s^2t^2(s+t)^6u^2\ (s+u)^6\Big(3s^4 + 6(t+u)s^3 + 9\Big(t^2+u^2\Big)s^2 + 6\ \Big(t^3+u^3\Big)s + 3t^4 + 3u^4 - 4tu^3 - 6t^2u^2 - 4t^3u\Big)H(0,y)\ H(0,z)(t+u)^6$$

$$- 18s^2t^2(s+t)^2u^2(s+u)^6\Big(3s^8 + 2(51t+u)\ s^7 + \Big(486t^2 + 200ut + 30u^2\Big)s^6 + 6\Big(179t^3 + 130ut^2 + 96u^2\ t + 7u^3\Big)s^5 + 2\Big(687t^4 + 677ut^3$$

$$+ 945u^2t^2 + 265u^3t + 29\ u^4\Big)s^4 + 2t\Big(537t^4 + 677ut^3 + 1344u^2t^2 + 658u^3t + 170\ u^4\Big)s^3 + 2t^2\Big(243t^4 + 390ut^3 + 945u^2t^2 + 658u^3t$$

$$+ 270\ u^4\Big)s^2 + 2t^3\Big(51t^4 + 100ut^3 + 288u^2t^2 + 265u^3t + 170\ u^4\Big)s + t^4\Big(3t^4 + 2ut^3 + 30u^2t^2 + 42u^3t + 58\ u^4\Big)\Big)H(1,y)(t+u)^6$$

$$= - 216s^2t^2(s+t)^6u^3(s+u)^6\Big(2\ s^3 - 3us^2 + 4u^2s + t\Big(2t^2 - 3ut + 4u^2\Big)\Big)H(0,z)H(1,y)\ (t+u)^6 - 108s^2t^2(s+t)^6u^2(s+u)^6\Big(3s^4$$

$$+ (6t-8u)s^3 + \Big(9\ t^2 + 6u^2\Big)s^2 + \Big(6t^3 - 4u^3\Big)s + 3t^4 + 3u^4 + 12tu^3 + 6t^2\ u^2 + 16t^3u\Big)H(0,y)H(1,z)(t+u)^6$$

$$- 1296s^2t^2(s+t)^6u^3(s+u)^6\ \Big(s^3 - us^2 + u^2s + t\Big(t^2 - ut + u^2\Big)\Big)H(1,y)H(1,z)\ (t+u)^6 - 108s^2t^2(s+t)^6u^2(s+u)^6\Big(3s^4 + (2u-8t)s^3$$

$$+ 3\Big(2\ t^2 + 5u^2\Big)s^2 - 2\Big(2t^3 + u^3\Big)s + 3t^4 + 3u^4 + 8tu^3 + 12t^2\ u^2 + 8t^3u\Big)H(0,z)H(2,y)(t+u)^6 - 108s^2t^2(s+t)^6u^2(s+u)^6\ \Big(8s^4$$

$$+ (4t-2u)s^3 + 9\Big(2t^2 + 3u^2\Big)s^2 - 2u^3s + 8t^4 + 8\ u^4 + 18tu^3 + 27t^2u^2 + 18t^3u\Big)H(1,z)H(2,y)(t+u)^6 - 108s^2t^2\ (s+t)^6u^2(s+u)^6\Big(2s^4$$

$$+ 10(t+u)s^3 - 3\Big(t^2+u^2\Big)s^2 + 6\ \Big(t^3+u^3\Big)s + 2\Big(t^2+ut+u^2\Big)^2\Big)H(1,z)H(3,y)\ (t+u)^6 - 108s^2t^3(s+t)^6u^3(s+u)^6\Big(6t^2 - 9ut$$

$$+ 2u^2\Big)\ H(0,0,y)(t+u)^6 - 108s^2t^3(s+t)^6u^3(s+u)^6\Big(2t^2 - 9ut + 6\ u^2\Big)H(0,0,z)(t+u)^6 + 432s^2t^2(s+t)^6u^2(s+u)^6\Big(2s^4$$

$$+ 2(2\ t+u)s^3 + 3\Big(2t^2+u^2\Big)s^2 + 2\Big(2t^3+u^3\Big)s + 2t^4 + 2\ u^4 + 2tu^3 + 3t^2u^2 + 2t^3u\Big)H(0,1,y)(t+u)^6 + 108s^2t^2(s+t)^6\ u^2(s+u)^6\Big(3s^4$$

$$+ (6t-2u)s^3 + 3\Big(3t^2+5u^2\Big)s^2 + 2\ \Big(3t^3+u^3\Big)s + 3t^4 + 3u^4 - 14tu^3 + 9t^2u^2 - 14t^3u\Big)\ H(0,1,z)(t+u)^6 - 108s^2t^2(s+t)^6u^2(s+u)^6\Big(3s^4$$

$$+ 8(t-u)s^3 + 6\ \Big(2t^2+u^2\Big)s^2 + \Big(8t^3 - 4u^3\Big)s + 3\Big(t^4 + 6u\ t^3 - u^2t^2 + 6u^3t + u^4\Big)\Big)H(0,2,y)(t+u)^6 + 108s^2t^2(s+t)^6\ u^2(s+u)^6\Big(6s^4$$

$$= + 2(6t-u)s^3 + 3\Big(6t^2+5u^2\Big)s^2 + 2\ \Big(6t^3+u^3\Big)s + 6t^4 + 6u^4 + 6tu^3 + 9t^2u^2 + 6t^3u\Big)\ H(1,0,y)(t+u)^6 + 108s^2t^2(s+t)^6u^2(s+u)^6\Big(3s^4$$

$$+ 2(3t+5u)\ s^3 + 3\Big(3t^2+u^2\Big)s^2 + 2\Big(3t^3+7u^3\Big)s + 3t^4 + 3u^4 - 2\ tu^3 - 3t^2u^2 - 2t^3u\Big)H(1,0,z)(t+u)^6 + 216s^2t^2(s+t)^6u^2\ (s+u)^6\Big(4s^4$$

$$+ 8(t+u)s^3 + 3\Big(4t^2+u^2\Big)s^2 + \Big(8t^3 + 6\ u^3\Big)s + 4t^4 + 4u^4 + 6tu^3 + 3t^2u^2 + 8t^3u\Big)H(1,1,y)\ (t+u)^6 + 108s^2t^2(s+t)^6u^3(s+u)^6\Big(18s^3$$

$$- 3us^2 + 18u^2s - t\ \Big(18t^2 + 9ut + 14u^2\Big)\Big)H(1,1,z)(t+u)^6 - 1296s^2t^2\ (s+t)^6u^3(s+u)^6\Big(s^3 - us^2 + u^2s + t\Big(t^2 - ut$$

$$+ u^2\Big)\Big)\ H(1,2,y)(t+u)^6 - 108s^2t^2(s+t)^6u^2(s+u)^6\Big(3s^4 + 2(t-4u)s^3 + 3\ \Big(5t^2 + 2u^2\Big)s^2 - 2\Big(t^3 + 2u^3\Big)s + 3t^4 + 3u^4 + 8t\ u^3$$

$$+ 12t^2u^2 + 8t^3u\Big)H(2,0,y)(t+u)^6 - 108s^2t^2(s+t)^6u^3\ (s+u)^6\Big(6s^3 - 9us^2 + 2u^2s + t\Big(2t^2 + 3ut + 2u^2\Big)\Big)\ H(2,1,y)(t+u)^6$$

$$- 108s^2t^2(s+t)^6u^2(s+u)^6\Big(8s^4 - 2(t+u)s^3 + 27\ \Big(t^2+u^2\Big)s^2 - 2\Big(t^3+u^3\Big)s + 8\Big(t^2 + u\ t + u^2\Big)^2\Big)H(2,2,y)(t+u)^6$$

$$- 108s^2t^2(s+t)^6u^2(s+u)^6\ \Big(2s^4 + 10(t+u)s^3 - 3\Big(t^2+u^2\Big)s^2 + 6\Big(t^3+u^3\Big)\ s + 2\Big(t^2+ut+u^2\Big)^2\Big)H(3,2,y)(t+u)^6$$

$$= + 9t^2(s+t)u^2(s+u)\ \Big(366(t+u)^6s^{16} + 12\Big(213t^7 + 1475ut^6 + 4463u^2t^5 + 7365u^3\ t^4 + 7365u^4t^3 + 4463u^5t^2 + 1475u^6t + 213u^7\Big)s^{15}$$

$$+ 2(t+u)^2\ \Big(4186t^6 + 23989ut^5 + 61240u^2t^4 + 77078u^3t^3 + 61240u^4t^2 + 23989\ u^5t + 4186u^6\Big)s^{14} + 2\Big(8526t^9 + 72111ut^8 + 283103u^2\ t^7$$

$$+ 646677u^3t^6 + 948359u^4t^5 + 948359u^5t^4 + 646677u^6t^3 + 283103u^7\ t^2 + 72111u^8t + 8526u^9\Big)s^{13} + (t+u)^2\Big(23886t^8 + 172264u\ t^7$$

$$+ 582649u^2t^6 + 1131132u^3t^5 + 1344162u^4t^4 + 1131132u^5t^3 + 582649\ u^6t^2 + 172264u^7t + 23886u^8\Big)s^{12} + \Big(23866t^{11} + 243086u\ t^{10}$$

$$+ 1157425u^2t^9 + 3382695u^3t^8 + 6575795u^4t^7 + 8973837u^5\ t^6 + 8973837u^6t^5 + 6575795u^7t^4 + 3382695u^8t^3 + 1157425u^9t^2$$

$$+ 243086\ u^{10}t + 23866u^{11}\Big)s^{11} + (t+u)^2\Big(17016t^{10} + 164440u\ t^9 + 724400u^2t^8 + 1917401u^3t^7 + 3236864u^4t^6 + 3710558u^5t^5$$

$$+ 3236864\ u^6t^4 + 1917401u^7t^3 + 724400u^8t^2 + 164440u^9t + 17016u^{10}\Big)\ s^{10} + 2\Big(4176t^{13} + 58901ut^{12} + 382615u^2t^{11} + 1458232u^3\ t^{10}$$

$$+ 3634444u^4t^9 + 6299163u^5t^8 + 8074581u^6t^7 + 8074581u^7\ t^6 + 6299163u^8t^5 + 3634444u^9t^4 + 1458232u^{10}t^3 + 382615u^{11}\ t^2$$

$$+ 58901u^{12}t + 4176u^{13}\Big)s^9 + (t+u)^2\Big(2552t^{12} + 42616u\ t^{11} + 321337u^2t^{10} + 1192960u^3t^9 + 2636700u^4t^8 + 3851544u^5\ t^7$$

$$+ 4187542u^6t^6 + 3851544u^7t^5 + 2636700u^8t^4 + 1192960u^9t^3 + 321337\ u^{10}t^2 + 42616u^{11}t + 2552u^{12}\Big)s^8 + \Big(366t^{15} + 11590u\ t^{14}$$

$$+ 150117u^2t^{13} + 868535u^3t^{12} + 2906129u^4t^{11} + 6451075u^5\ t^{10} + 10280514u^6t^9 + 12609962u^7t^8 + 12609962u^8t^7 + 10280514u^9\ t^6$$

$$= + 6451075u^{10}t^5 + 2906129u^{11}t^4 + 868535u^{12}t^3 + 150117u^{13}t^2 + 11590u^{14}t + 366u^{15}) s^7 + tu(t+u)^2 (1254t^{12} + 30458ut^{11}$$

$$+ 192025u^2t^{10} + 634366u^3t^9 + 1369170u^4t^8 + 2072140u^5t^7 + 2324670u^6t^6 + 2072140u^7t^5 + 1369170u^8t^4 + 634366u^9t^3 + 192025u^{10}t^2$$

$$+ 30458u^{11}t + 1254u^{12}) s^6 + 2t^2u^2 (1605t^{13} + 19222ut^{12} + 110551u^2t^{11} + 398318u^3t^{10} + 989437u^4t^9 + 1764628u^5t^8 + 2329703u^6t^7$$

$$+ 2329703u^7t^6 + 1764628u^8t^5 + 989437u^9t^4 + 398318u^{10}t^3 + 110551u^{11}t^2 + 19222u^{12}t + 1605u^{13}) s^5 + t^3u^3(t+u)^2 (1678t^{10}$$

$$+ 19862ut^9 + 93582u^2t^8 + 253073u^3t^7 + 439352u^4t^6 + 520162u^5t^5 + 439352u^6t^4 + 253073u^7t^3 + 93582u^8t^2 + 19862u^9t + 1678u^{10}) s^4$$

$$+ t^4u^4(t+u)^3 (1260t^8 + 9096ut^7 + 30583u^2t^6 + 61244u^3t^5 + 75426u^4t^4 + 61244u^5t^3 + 30583u^6t^2 + 9096u^7t + 1260u^8) s^3$$

$$+ t^5u^5(t+u)^4 (324t^6 + 2168ut^5 + 6197u^2t^4 + 8622u^3t^3 + 6197u^4t^2 + 2168u^5t + 324u^6) s^2 + 10t^6u^6(t+u)^7 (14t^2 + 27ut + 14u^2) s$$

$$+ 40t^7u^7(t+u)^6 (t^2 + ut + u^2))) + 9s^2t^2u^2 (3 (11t^6 + 130ut^5 + 141u^2t^4 + 604u^3t^3 + 141u^4t^2 + 130u^5t + 11u^6) s^{16} + 8 (45t^7 + 521ut^6$$

$$+ 1056u^2t^5 + 2598u^3t^4 + 2488u^4t^3 + 858u^5t^2 + 411u^6t + 23u^7) s^{15} + 6 (285t^8 + 3120ut^7 + 8470u^2t^6 + 17852u^3t^5 + 25104u^4t^4$$

$$= + 15024u^5t^3 + 5642u^6t^2 + 1908u^7t + 83u^8) s^{14} + 2 (2375t^9 + 24561ut^8 + 80646u^2t^7 + 164370u^3t^6 + 268500u^4t^5 + 242676u^5t^4$$

$$+ 112746u^6t^3 + 43806u^7t^2 + 11685u^8t + 539u^9) s^{13} + (8669t^{10} + 85044ut^9 + 320811u^2t^8 + 677420u^3t^7 + 1161930u^4t^6 + 1394268u^5t^5$$

$$+ 895510u^6t^4 + 373044u^7t^3 + 149697u^8t^2 + 34384u^9t + 2343u^{10}) s^{12} + 12 (915t^{11} + 8573ut^{10} + 35593u^2t^9 + 81037u^3t^8 + 140831u^4t^7$$

$$+ 205671u^5t^6 + 184091u^6t^5 + 94279u^7t^4 + 41772u^8t^3 + 16990u^9t^2 + 3718u^{10}t + 370u^{11}) s^{11} + (9845t^{12} + 88872ut^{11} + 394182u^2t^{10}$$

$$+ 982076u^3t^9 + 1680477u^4t^8 + 2723040u^5t^7 + 3377880u^6t^6 + 2469456u^7t^5 + 1323771u^8t^4 + 704312u^9t^3 + 264210u^{10}t^2 + 55188u^{11}t$$

$$+ 6179u^{12}) s^{10} + 2 (3095t^{13} + 27024ut^{12} + 127632u^2t^{11} + 352506u^3t^{10} + 563960u^4t^9 + 824355u^5t^8 + 1409056u^6t^7 + 1665520u^7t^6$$

$$+ 1351875u^8t^5 + 966790u^9t^4 + 514400u^{10}t^3 + 159702u^{11}t^2 + 28798u^{12}t + 2919u^{13}) s^9 + 3 (870t^{14} + 7314ut^{13} + 37677u^2t^{12}$$

$$= + 112780u^3t^{11} + 143003u^4t^{10} + 53282u^5t^9 + 249844u^6t^8 + 845260u^7t^7 + 1239890u^8t^6 + 1252614u^9t^5 + 905087u^{10}t^4 + 401760u^{11}t^3$$

$$+ 101733u^{12}t^2 + 14670u^{13}t + 1176u^{14}) s^8 + 4 (166t^{15} + 1359ut^{14} + 8343u^2t^{13} + 18170u^3t^{12} - 29355u^4t^{11} - 193428u^5t^{10}$$

$$- 186082u^6t^9 + 373479u^7t^8 + 1074360u^8t^7 + 1349559u^9t^6 + 1156713u^{10}t^5 + 673434u^{11}t^4 + 244731u^{12}t^3 + 51600u^{13}t^2 + 5604u^{14}t$$

$$+ 307u^{15}) s^7 + (77t^{16} + 712ut^{15} + 7926u^2t^{14} - 22420u^3t^{13} - 310304u^4t^{12} - 964008u^5t^{11} - 1004190u^6t^{10} + 1290836u^7t^9$$

$$+ 5018184u^8t^8 + 6920472u^9t^7 + 6102774u^{10}t^6 + 3900588u^{11}t^5 + 1764592u^{12}t^4 + 528392u^{13}t^3 + 92178u^{14}t^2 + 6772u^{15}t$$

$$+ 187u^{16}) s^6 + 6\, tu (5t^{15} + 338ut^{14} - 3532u^2t^{13} - 33260u^3t^{12} - 97374u^4t^{11} - 95800u^5t^{10} + 181294u^6t^9 + 739773u^7t^8$$

$$+ 1163599u^8t^7 + 1123652u^9t^6 + 758126u^{10}t^5 + 368632u^{11}t^4 + 127306u^{12}t^3 + 29970u^{13}t^2 + 4032u^{14}t + 151u^{15}) s^5 + t^2u^2 (327t^{14}$$

$$- 6804ut^{13} - 59454u^2t^{12} - 167472u^3t^{11} - 99128u^4t^{10} + 699492u^5t^9 + 2552097u^6t^8 + 4572060u^7t^7 + 5195055u^8t^6 + 4069272u^9t^5$$

$$+ 2242188u^{10}t^4 + 847572u^{11}t^3 + 212250u^{12}t^2 + 33688u^{13}t + 2793u^{14}) s^4 + 4t^3u^3 (- 233t^{13} - 1616ut^{12} - 3051u^2t^{11} + 10663u^3t^{10}$$

$$= + 80192u^4t^9 + 249270u^5t^8 + 483611u^6t^7 + 633999u^7t^6 + 583800u^8t^5 + 382847u^9t^4 + 174178u^{10}t^3 + 51444u^{11}t^2 + 8727u^{12}t$$

$$+ 617u^{13}) s^3 + 3t^4u^4(t+u)^2 (109t^{10} + 662ut^9 + 4205u^2t^8 + 17972u^3t^7 + 46152u^4t^6 + 74668u^5t^5 + 76440u^6t^4 + 53188u^7t^3$$

$$+ 24579u^8t^2 + 6934u^9t + 931u^{10}) s^2 + 2t^5u^5(t+u)^3 (15t^8 + 481ut^7 + 3156u^2t^6 + 8739u^3t^5 + 13780u^4t^4 + 12711u^5t^3$$

$$+ 7500u^6t^2 + 2637u^7t + 453u^8) s + t^6u^6 (t+u)^6 (77t^4 + 270ut^3 + 477u^2t^2 + 350u^3t + 187u^4))) H(1,z) + 9s^2t^2u^2 (3 (13t^6$$

$$+ 142ut^5 + 171u^2t^4 + 644u^3t^3 + 171u^4t^2 + 142u^5t + 13u^6) s^{16} + 8 (50t^7 + 578ut^6 + 1293u^2t^5 + 3103u^3t^4 + 3103u^4t^3$$

$$+ 1293u^5t^2 + 578u^6t + 50u^7) s^{15} + 12 (157t^8 + 1789ut^7 + 5420u^2t^6 + 12039u^3t^5 + 17342u^4t^4 + 12039u^5t^3 + 5420u^6t^2 + 1789u^7t$$

$$+ 157u^8) s^{14} + (5374t^9 + 59826ut^8 + 223092u^2t^7 + 513996u^3t^6 + 871536u^4t^5 + 871536u^5t^4 + 513996u^6t^3 + 223092u^7t^2$$

$$+ 59826u^8t + 5374u^9) s^{13} + (10359t^{10} + 114484ut^9 + 503079u^2t^8 + 1279432u^3t^7 + 2390090u^4t^6 + 3044424u^5t^5 + 2390090u^6t^4$$

$$+ 1279432u^7t^3 + 503079u^8t^2 + 114484u^9t + 10359u^{10}) s^{12} + 12 (1186t^{11} + 13429ut^{10} + 67778u^2t^9 + 197005u^3t^8 + 403359u^4t^7$$

$$+ 605123u^5t^6 + 605123u^6t^5 + 403359u^7t^4 + 197005u^8t^3 + 67778u^9t^2 + 13429u^{10}t + 1186u^{11}) s^{11} + (14195t^{12} + 171720ut^{11}$$

$$+ 991722u^2t^{10} + 3343932u^3t^9 + 7591317u^4t^8 + 12777132u^5t^7 + 15424700u^6t^6 + 12777132u^7t^5 + 7591317u^8t^4 + 3343932u^9t^3$$

$$+ 991722u^{10}t^2 + 171720u^{11}t + 14195u^{12}) s^{10} + 2 (5067t^{13} + 68848ut^{12} + 464430u^2t^{11} + 1834210u^3t^{10} + 4687302u^4t^9$$

$$+ 8645967u^5t^8 + 11943168u^6t^7 + 11943168u^7t^6 + 8645967u^8t^5 + 4687302u^9t^4 + 1834210u^{10}t^3 + 464430u^{11}t^2 + 68848u^{12}t$$

$$+ 5067u^{13}) s^9 + 3 (1638t^{14} + 26822ut^{13} + 219527u^2t^{12} + 1022692u^3t^{11} + 2994269u^4t^{10} + 6115342u^5t^9 + 9403374u^6t^8$$

$$= + 10916304u^{7}\ t^{7} + 9403374u^{8}t^{6} + 6115342u^{9}t^{5} + 2994269u^{10}t^{4} + 1022692u^{11}\ t^{3} + 219527u^{12}t^{2} + 26822u^{13}t + 1638u^{14}\Big)s^{8}$$

$$+ 4\Big(361\ t^{15} + 8109ut^{14} + 85470u^{2}t^{13} + 471328u^{3}t^{12} + 1600674u^{4}\ t^{11} + 3733632u^{5}t^{10} + 6488383u^{6}t^{9} + 8627643u^{7}t^{8} + 8627643u^{8}\ t^{7}$$

$$+ 6488383u^{9}t^{6} + 3733632u^{10}t^{5} + 1600674u^{11}t^{4} + 471328u^{12}\ t^{3} + 85470u^{13}t^{2} + 8109u^{14}t + 361u^{15}\Big)s^{7} + \Big(193\ t^{16} + 8108ut^{15}$$

$$+ 123366u^{2}t^{14} + 816896u^{3}t^{13} + 3259172u^{4}\ t^{12} + 8952972u^{5}t^{11} + 18149594u^{6}t^{10} + 27988696u^{7}t^{9} + 32478774u^{8}\ t^{8} + 27988696u^{9}t^{7}$$

$$+ 18149594u^{10}t^{6} + 8952972u^{11}t^{5} + 3259172u^{12}\ t^{4} + 816896u^{13}t^{3} + 123366u^{14}t^{2} + 8108u^{15}t + 193u^{16}\Big)\ s^{6} + 6tu\Big(157t^{15} + 4612ut^{14}$$

$$+ 39024u^{2}t^{13} + 191670u^{3}\ t^{12} + 643658u^{4}t^{11} + 1557030u^{5}t^{10} + 2799334u^{6}t^{9} + 3770803u^{7}\ t^{8} + 3770803u^{8}t^{7} + 2799334u^{9}t^{6}$$

$$+ 1557030u^{10}t^{5} + 643658u^{11}\ t^{4} + 191670u^{12}t^{3} + 39024u^{13}t^{2} + 4612u^{14}t + 157u^{15}\Big)\ s^{5} + t^{2}u^{2}\Big(2883t^{14} + 38608ut^{13} + 269730u^{2}t^{12}$$

$$+ 1182108u^{3}\ t^{11} + 3470348u^{4}t^{10} + 7219608u^{5}t^{9} + 11105895u^{6}t^{8} + 12818744u^{7}\ t^{7} + 11105895u^{8}t^{6} + 7219608u^{9}t^{5} + 3470348u^{10}t^{4}$$

$$+ 1182108u^{11}\ t^{3} + 269730u^{12}t^{2} + 38608u^{13}t + 2883u^{14}\Big)s^{4} + 4t^{3}u^{3}\ \Big(647t^{13} + 9737ut^{12} + 60783u^{2}t^{11} + 220492u^{3}t^{10} + 533350u^{4}\ t^{9}$$

$$+ 931704u^{5}t^{8} + 1224463u^{6}t^{7} + 1224463u^{7}t^{6} + 931704u^{8}t^{5} + 533350u^{9}\ t^{4} + 220492u^{10}t^{3} + 60783u^{11}t^{2} + 9737u^{12}t + 647u^{13}\Big)s^{3}$$

$$+ 3\ t^{4}u^{4}(t+u)^{2}\Big(961t^{10} + 7506ut^{9} + 28145u^{2}t^{8} + 66084u^{3}\ t^{7} + 107838u^{4}t^{6} + 127716u^{5}t^{5} + 107838u^{6}t^{4} + 66084u^{7}t^{3} + 28145u^{8}\ t^{2}$$

$$+ 7506u^{9}t + 961u^{10}\Big)s^{2} + 6t^{5}u^{5}(t+u)^{3}\Big(157t^{8} + 937u\ t^{7} + 2766u^{2}t^{6} + 5043u^{3}t^{5} + 6226u^{4}t^{4} + 5043u^{5}t^{3} + 2766u^{6}t^{2} + 937u^{7}\ t$$

$$+ 157u^{8}\Big)s + t^{6}u^{6}(t+u)^{6}\Big(193t^{4} + 354ut^{3} + 537u^{2}t^{2} + 354\ u^{3}t + 193u^{4}\Big)\Big)H(2,y)\Big\}\Big/\Big(27s^{3}t^{3}u^{3}(s+t)^{6}(s+u)^{6}(t+u)^{6}\Big)$$

$$\mathcal{A}^{(2)}_{5;C_A n_f} = \Big\{216s^{2}t^{2}u^{3}(s+u)^{6}(t+u)^{6}\Big(s^{2} - ts + t^{2} + u^{2}\Big)H(0,1,y)\ (s+t)^{7} + 162s^{2}t^{3}u^{3}(s+u)^{6}(t+u)^{6}\Big(t^{2} + u^{2}\Big)H(0,y)H(0,z)\ (s+t)^{6}$$

$$= + 54s^{2}t^{2}u^{3}(s+u)^{6}(t+u)^{6}\Big(16s^{3} - 21us^{2} + 7u^{2}s + t\ \Big(16t^{2} - 21ut + 7u^{2}\Big)\Big)H(0,z)H(1,y)(s+t)^{6} - 54s^{2}t^{2}u^{3}\ (s+u)^{6}(t+u)^{6}\Big(10s^{3}$$

$$- 21us^{2} + 19u^{2}s + t\Big(- 8t^{2} + 21ut - 17\ u^{2}\Big)\Big)H(0,y)H(1,z)(s+t)^{6} + 1134s^{2}t^{2}u^{3}(s+u)^{6}(t+u)^{6}\ \Big(s^{3} - 2us^{2}$$

$$+ u^{2}s + t(t-u)^{2}\Big)H(1,y)H(1,z)(s+t)^{6} + 54s^{2}t^{2}\ u^{2}(s+u)^{6}(t+u)^{6}\Big(- 2(5t+8u)s^{3} + 21\Big(t^{2} + u^{2}\Big)\ s^{2} - \Big(19t^{3}$$

$$+ 7u^{3}\Big)s + tu\Big(t^{2} + u^{2}\Big)\Big)H(0,z)\ H(2,y)(s+t)^{6} + 54s^{2}t^{2}u^{2}(s+u)^{6}(t+u)^{6}\Big(- 4(5t+6u)s^{3} + 21\ \Big(t^{2} + 2u^{2}\Big)s^{2}$$

$$- \Big(11t^{3} + 24u^{3}\Big)s + tu\ \Big(t^{2} + u^{2}\Big)\Big)H(1,z)H(2,y)(s+t)^{6} + 162s^{3}t^{2}u^{2}(s+u)^{6}\ (t+u)^{6}\Big(2(t+u)s^{2} - 7\Big(t^{2}$$

$$+ u^{2}\Big)s + 5\ \Big(t^{3} + u^{3}\Big)\Big)H(1,z)H(3,y)(s+t)^{6} + 162s^{2}t^{3}u^{3}(s+u)^{6}\ (t+u)^{6}\Big(2t^{2} - 7ut + 5u^{2}\Big)H(0,0,y)(s+t)^{6}$$

$$= + 162s^{2}t^{3}u^{3}\ (s+u)^{6}(t+u)^{6}\Big(5t^{2} - 7ut + 2u^{2}\Big)H(0,0,z)(s+t)^{6} + 54s^{2}t^{2}\ u^{3}(s+u)^{6}(t+u)^{6}\Big(7s^{3} - 21us^{2} + 16u^{2}s + t\Big(25t^{2} - 42ut$$

$$+ 25\ u^{2}\Big)\Big)H(0,1,z)(s+t)^{6} + 54s^{2}t^{2}u^{2}(s+u)^{6}(t+u)^{6}\ \Big((t-10u)s^{3} + 21u^{2}s^{2} + \Big(t^{3} - 19u^{3}\Big)s + 21t(t-u)^{2}\ u\Big)H(0,2,y)(s+t)^{6}$$

$$+ 54s^{2}t^{2}u^{3}(s+u)^{6}(t+u)^{6}\Big(10s^{3} - 21u\ s^{2} + 19u^{2}s + t^{3} + tu^{2}\Big)H(1,0,y)(s+t)^{6} - 54s^{2}t^{2}u^{3}(s+u)^{6}\ (t+u)^{6}\Big(16s^{3} - 21us^{2} + 7u^{2}s$$

$$- 2t\Big(t^{2} + u^{2}\Big)\Big)\ H(1,0,z)(s+t)^{6} - 54s^{2}t^{2}u^{3}(s+u)^{6}(t+u)^{6}\Big(s^{3} - 21us^{2} + 10u^{2}\ s + t\Big(t^{2} - 21ut + 10u^{2}\Big)\Big)H(1,1,y)(s+t)^{6}$$

$$- 162s^{2}t^{2}u^{3}\ (s+u)^{6}(t+u)^{6}\Big(8s^{3} - 14us^{2} + 8u^{2}s + t\Big(- 4t^{2} + 7ut - 7\ u^{2}\Big)\Big)H(1,1,z)(s+t)^{6} + 1134s^{2}t^{2}u^{3}(s+u)^{6}(t+u)^{6}\ \Big(s^{3}$$

$$- 2us^{2} + u^{2}s + t(t-u)^{2}\Big)H(1,2,y)(s+t)^{6} + 54s^{2}t^{2}u^{2}\ (s+u)^{6}(t+u)^{6}\Big(- 2(8t+5u)s^{3} + 21\Big(t^{2} + u^{2}\Big)s^{2} - \Big(7\ t^{3}$$

$$+ 19u^{3}\Big)s + tu\Big(t^{2} + u^{2}\Big)\Big)H(2,0,y)(s+t)^{6} + 54s^{2}\ t^{2}u^{3}(s+u)^{6}(t+u)^{6}\Big(4s^{3} - 21us^{2} + 13u^{2}s + t^{3} + tu^{2}\Big)\ H(2,1,y)(s+t)^{6}$$

$$- 324s^{3}t^{2}u^{2}(s+u)^{6}(t+u)^{6}\Big(4(t+u)s^{2} - 7\ \Big(t^{2} + u^{2}\Big)s + 4\Big(t^{3} + u^{3}\Big)\Big)H(2,2,y)(s+t)^{6} + 162\ s^{3}t^{2}u^{2}(s+u)^{6}(t+u)^{6}\Big(2(t+u)s^{2}$$

$$- 7\Big(t^{2} + u^{2}\Big)s + 5\ \Big(t^{3} + u^{3}\Big)\Big)H(3,2,y)(s+t)^{6} - 9s^{2}t^{2}u^{2}(t+u)^{6}\Big(10\ s^{14} + (60t + 74u)s^{13} + 2\Big(85t^{2} + 94ut + 129u^{2}\Big)s^{12} + 3\ \Big(100t^{3}$$

$$+ 74ut^{2} + 123u^{2}t + 190u^{3}\Big)s^{11} + \Big(360t^{4} + 199u\ t^{3} + 954u^{2}t^{2} + 467u^{3}t + 900u^{4}\Big)s^{10} + \Big(300t^{5} - 194ut^{4} + 4854\ u^{2}t^{3} + 635u^{3}t^{2} + 348u^{4}t$$

$$+ 1074u^{5}\Big)s^{9} + 2\Big(85t^{6} - 429u\ t^{5} + 4734u^{2}t^{4} + 2414u^{3}t^{3} - 498u^{4}t^{2} + 135u^{5}t + 492u^{6}\Big)\ s^{8} + \Big(60t^{7} - 874ut^{6} + 8919u^{2}t^{5} + 5677u^{3}t^{4}$$

$$+ 1791u^{4}t^{3} - 159u^{5}\ t^{2} + 165u^{6}t + 678u^{7}\Big)s^{7} + \Big(10t^{8} - 337ut^{7} + 4848u^{2}t^{6} - 2634\ u^{3}t^{5} + 4506u^{4}t^{4} + 5835u^{5}t^{3} + 288u^{6}t^{2} + 39u^{7}t$$

$$+ 330u^{8}\Big)s^{6} + u\ \Big(- 36t^{8} + 1746ut^{7} - 7930u^{2}t^{6} + 1614u^{3}t^{5} + 14679u^{4}t^{4} + 3583u^{5}\ t^{3} - 339u^{6}t^{2} + 150u^{7}t + 100u^{8}\Big)s^{5} + u^{2}\Big(360t^{8}$$

$$= - 4417u\ t^{7} - 1014u^{2}t^{6} + 13797u^{3}t^{5} + 8278u^{4}t^{4} - 689u^{5}t^{3} + 306u^{6}t^{2} + 172u^{7}\ t + 14u^{8}\Big)s^{4} + tu^{3}\Big(- 708t^{7} - 45ut^{6} + 4974u^{2}t^{5}$$

$$+ 6885u^{3}\ t^{4} + 1171u^{4}t^{3} - 108u^{5}t^{2} + 489u^{6}t + 44u^{7}\Big)s^{3} + 3t^{2}u^{4}\Big(120\ t^{6} + 114ut^{5} + 580u^{2}t^{4} + 562u^{3}t^{3} + 152u^{4}t^{2} + 117u^{5}t + 42u^{6}\Big)\ s^{2}$$

$$+ t^{3}u^{5}\Big(- 36t^{5} - 53ut^{4} + 526u^{2}t^{3} + 402u^{3}t^{2} + 271u^{4}t + 104\ u^{5}\Big)s + t^{4}u^{6}\Big(10t^{4} + 40ut^{3} + 72u^{2}t^{2} + 49u^{3}t + 56\ u^{4}\Big)\Big)H(0,y)(s+t)^{2}$$

$$+ 9s^{2}t^{2}u^{2}(s+u)^{6}(t+u)^{6}\Big(6\ s^{8} + (78t + 22u)s^{7} + \Big(324t^{2} + 202ut + 42u^{2}\Big)s^{6} + 3\Big(230\ t^{3} + 202ut^{2} + 186u^{2}t + 7u^{3}\Big)s^{5} + \Big(876t^{4}$$

$$+ 962ut^{3} + 1710u^{2}\ t^{2} + 103u^{3}t + 58u^{4}\Big)s^{4} + 2t\Big(345t^{4} + 481ut^{3} + 1194u^{2}t^{2} + 38\ u^{3}t + 122u^{4}\Big)s^{3} + 2t^{2}\Big(162t^{4} + 303ut^{3} + 855u^{2}t^{2}$$

$$+ 38u^{3}\ t + 162u^{4}\Big)s^{2} + t^{3}\Big(78t^{4} + 202ut^{3} + 558u^{2}t^{2} + 103u^{3}t + 244\ u^{4}\Big)s + t^{4}\Big(6t^{4} + 22ut^{3} + 42u^{2}t^{2} + 21u^{3}t + 58\ u^{4}\Big)\Big)H(1,y)(s+t)^{2}$$

$$
= -\frac{3}{2}t^2u^2(s+u)\Big(462(t+u)^6\ s^{16}+6\Big(533t^7+3647ut^6+11355u^2t^5+18041u^3t^4+18041u^4\ t^3+11355u^5t^2+3647u^6t+533u^7\Big)s^{15}
$$

$$
+6(t+u)^2\Big(1725\ t^6+9572ut^5+26881u^2t^4+27580u^3t^3+26881u^4t^2+9572u^5t+1725\ u^6\Big)s^{14}+\Big(20814t^9+169590ut^8+698271u^2t^7
$$

$$
+1572623u^3\ t^6+2180718u^4t^5+2180718u^5t^4+1572623u^6t^3+698271u^7t^2+169590\ u^8t+20814u^9\Big)s^{13}+(t+u)^2\Big(28872t^8+187896ut^7
$$

$$
+702435\ u^2t^6+1379212u^3t^5+1402562u^4t^4+1379212u^5t^3+702435u^6\ t^2+187896u^7t+28872u^8\Big)s^{12}+\Big(28752t^{11}+247032u\ t^{10}
$$

$$
+1200159u^2t^9+3905948u^3t^8+7929314u^4t^7+10661099u^5\ t^6+10661099u^6t^5+7929314u^7t^4+3905948u^8t^3+1200159u^9t^2
$$

$$
+247032\ u^{10}t+28752u^{11}\Big)s^{11}+(t+u)^2\Big(20598t^{10}+137298u\ t^9+671079u^2t^8+2337710u^3t^7+4295427u^4t^6+4577312u^5t^5
$$

$$
+4295427\ u^6t^4+2337710u^7t^3+671079u^8t^2+137298u^9t+20598u^{10}\Big)\ s^{10}+\Big(10230t^{13}+94398ut^{12}+670143u^2t^{11}+3176397u^3\ t^{10}
$$

$$
+9253014u^4t^9+16795202u^5t^8+21138216u^6t^7+21138216u^7\ t^6+16795202u^8t^5+9253014u^9t^4+3176397u^{10}t^3+670143u^{11}\ t^2
$$

$$
+94398u^{12}t+10230u^{13}\Big)s^9+(t+u)^2\Big(3174t^{12}+28308u\ t^{11}+353829u^2t^{10}+1598364u^3t^9+3798910u^4t^8+5149552u^5\ t^7
$$

$$
+4884942u^6t^6+5149552u^7t^5+3798910u^8t^4+1598364u^9t^3+353829\ u^{10}t^2+28308u^{11}t+3174u^{12}\Big)s^8+\Big(462t^{15}+7482u\ t^{14}
$$

$$
+190719u^2t^{13}+1263290u^3t^{12}+4354796u^4t^{11}+9652847u^5\ t^{10}+14817008u^6t^9+17262724u^7t^8+17262724u^8t^7+14817008u^9\ t^6
$$

$$
+9652847u^{10}t^5+4354796u^{11}t^4+1263290u^{12}t^3+190719u^{13}\ t^2+7482u^{14}t+462u^{15}\Big)s^7+tu(t+u)^2\Big(690t^{12}+52407u\ t^{11}
$$

$$
=+300320u^2t^{10}+958425u^3t^9+2341562u^4t^8+3965828u^5\ t^7+4535936u^6t^6+3965828u^7t^5+2341562u^8t^4+958425u^9t^3+300320\ u^{10}t^2
$$

$$
+52407u^{11}t+690u^{12}\Big)s^6+t^2u^2\Big(6786\ t^{13}+57592ut^{12}+322024u^2t^{11}+1440278u^3t^{10}+4550721u^4\ t^9+9574691u^5t^8+13690372u^6t^7
$$

$$
+13690372u^7t^6+9574691u^8\ t^5+4550721u^9t^4+1440278u^{10}t^3+322024u^{11}t^2+57592u^{12}t+6786\ u^{13}\Big)s^5+t^3u^3(t+u)^2\Big(74t^{10}
$$

$$
+34768ut^9+270574u^2\ t^8+947815u^3t^7+1872940u^4t^6+2304682u^5t^5+1872940u^6t^4+947815\ u^7t^3+270574u^8t^2+34768u^9t
$$

$$
+74u^{10}\Big)s^4+t^4u^4(t+u)^3\ \Big(4432t^8+44590ut^7+177657u^2t^6+382429u^3t^5+481348u^4\ t^4+382429u^5t^3+177657u^6t^2+44590u^7t
$$

$$
+4432u^8\Big)s^3+t^5u^5\ (t+u)^4\Big(2948t^6+20395ut^5+53828u^2t^4+72306u^3t^3+53828u^4\ t^2+20395u^5t+2948u^6\Big)s^2+720t^6u^6(t+u)^7\Big(2t^2
$$

$$
+3ut+2\ u^2\Big)s+288t^7u^7(t+u)^6\Big(t^2+ut+u^2\Big)\Big)(s+t)-9s^2\ t^2u^2(s+u)^2(t+u)^6\Big(10s^{14}+(74t+60u)s^{13}+2\Big(129\ t^2+94ut+85u^2\Big)s^{12}
$$

$$
+3\Big(190t^3+123ut^2+74u^2t+100\ u^3\Big)s^{11}+\Big(900t^4+467ut^3+954u^2t^2+199u^3t+360\ u^4\Big)s^{10}+\Big(1074t^5+348ut^4+635u^2t^3
$$

$$
+4854u^3t^2-194u^4\ t+300u^5\Big)s^9+2\Big(492t^6+135ut^5-498u^2t^4+2414u^3t^3+4734\ u^4t^2-429u^5t+85u^6\Big)s^8+\Big(678t^7+165ut^6
$$

$$
-159u^2t^5+1791\ u^3t^4+5677u^4t^3+8919u^5t^2-874u^6t+60u^7\Big)s^7+\Big(330\ t^8+39ut^7+288u^2t^6+5835u^3t^5+4506u^4t^4-2634u^5t^3
$$

$$
+4848u^6\ t^2-337u^7t+10u^8\Big)s^6+t\Big(100t^8+150ut^7-339u^2t^6+3583\ u^3t^5+14679u^4t^4+1614u^5t^3-7930u^6t^2+1746u^7t-36u^8\Big)\ s^5
$$

$$
+t^2\Big(14t^8+172ut^7+306u^2t^6-689u^3t^5+8278u^4t^4+13797\ u^5t^3-1014u^6t^2-4417u^7t+360u^8\Big)s^4+t^3u\Big(44t^7+489u\ t^6-108u^2t^5
$$

$$
+1171u^3t^4+6885u^4t^3+4974u^5t^2-45u^6t-708\ u^7\Big)s^3+3t^4u^2\Big(42t^6+117ut^5+152u^2t^4+562u^3t^3+580\ u^4t^2+114u^5t+120u^6\Big)s^2
$$

$$
=+t^5u^3\Big(104t^5+271ut^4+402u^2\ t^3+526u^3t^2-53u^4t-36u^5\Big)s+t^6u^4\Big(56t^4+49ut^3+72\ u^2t^2+40u^3t+10u^4\Big)\Big)H(0,z)
$$

$$
-9s^2t^2u^2\Big(2\Big(3\ t^6+60ut^5-78u^2t^4+490u^3t^3-78u^4t^2+60u^5t+3u^6\Big)\ s^{16}+\Big(72t^7+1289ut^6+414u^2t^5+8005u^3t^4+7845u^4t^3
$$

$$
+126u^5\ t^2+1129u^6t+40u^7\Big)s^{15}+6\Big(60t^8+900ut^7+1076u^2\ t^6+4412u^3t^5+9687u^4t^4+3894u^5t^3+558u^6t^2+678u^7t+23\ u^8\Big)s^{14}
$$

$$
+\ \Big(1024t^9+12954ut^8+24576u^2t^7+46761u^3\ t^6+163314u^4t^5+154677u^5t^4+29274u^6t^3+11781u^7t^2+8316u^8t+331\ u^9\Big)s^{13}
$$

$$
+\ \Big(1874t^{10}+20222ut^9+53220u^2t^8+61201u^3\ t^7+246846u^4t^6+451965u^5t^5+210152u^6t^4+17091u^7t^3+26370u^8\ t^2+11421u^9t
$$

$$
+638u^{10}\Big)s^{12}+3\Big(780t^{11}+7249u\ t^{10}+25014u^2t^9+30013u^3t^8+86909u^4t^7+276042u^5t^6+269826u^6\ t^5+71475u^7t^4+13798u^8t^3
$$

$$
+15298u^9t^2+4109u^{10}t+359\ u^{11}\Big)s^{11}+\Big(2042t^{12}+16125ut^{11}+76122u^2\ t^{10}+155760u^3t^9+296460u^4t^8+1068120u^5t^7
$$

$$
+1786074u^6t^6+1133742\ u^7t^5+352110u^8t^4+166875u^9t^3+68808u^{10}t^2+12114u^{11}t+1520\ u^{12}\Big)s^{10}+\Big(1240t^{13}+7152ut^{12}
$$

$$
+58566u^2t^{11}+232980\ u^3t^{10}+464512u^4t^9+1090308u^5t^8+2341921u^6t^7+2520886u^7\ t^6+1455108u^8t^5+740077u^9t^4+345463u^{10}t^3
$$

$$
+84624u^{11}t^2+10790\ u^{12}t+1573u^{13}\Big)s^9+3\Big(168t^{14}+208ut^{13}+10592u^2\ t^{12}+73981u^3t^{11}+192464u^4t^{10}+350678u^5t^9+731518u^6\ t^8
$$

$$= + 1095507u^7t^7 + 938868u^8t^6 + 596751u^9t^5 + 345880u^{10}t^4 + 133260\ u^{11}t^3 + 25570u^{12}t^2 + 2627u^{13}t + 352u^{14}\Big)s^8 + \Big(124\ t^{15}$$

$$- 1050ut^{14} + 11760u^2t^{13} + 109681u^3t^{12} + 348969u^4\ t^{11} + 744780u^5t^{10} + 1812415u^6t^9 + 3571941u^7t^8 + 4045692u^8\ t^7 + 2838503u^9t^6$$

$$+ 1635744u^{10}t^5 + 818670u^{11}t^4 + 274406u^{12}\ t^3 + 50556u^{13}t^2 + 4374u^{14}t + 403u^{15}\Big)s^7 + \Big(14t^{16} - 459\ ut^{15} + 4194u^2t^{14} + 16667u^3t^{13}$$

$$+ 21936u^4t^{12} + 169446u^5\ t^{11} + 1113116u^6t^{10} + 3298442u^7t^9 + 4969926u^8t^8 + 4145565u^9\ t^7 + 2172862u^{10}t^6 + 899067u^{11}t^5$$

$$+ 350974u^{12}t^4 + 118004u^{13}\ t^3 + 24360u^{14}t^2 + 1596u^{15}t + 66u^{16}\Big)s^6 + 3tu\Big(- 20\ t^{15} + 510ut^{14} - 2846u^2t^{13} - 25726u^3t^{12}$$

$$- 36057u^4t^{11} + 137862\ u^5t^{10} + 687022u^6t^9 + 1363652u^7t^8 + 1488125u^8t^7 + 962136u^9\ t^6 + 387285u^{10}t^5 + 104624u^{11}t^4 + 27898u^{12}t^3$$

$$+ 10766u^{13}\ t^2 + 2505u^{14}t + 88u^{15}\Big)s^5 + t^2u^2\Big(276t^{14} - 4915u\ t^{13} - 35832u^2t^{12} - 60573u^3t^{11} + 131822u^4t^{10} + 839823u^5\ t^9$$

$$+ 1989780u^6t^8 + 2753037u^7t^7 + 2394378u^8t^6 + 1355826u^9t^5 + 497554\ u^{10}t^4 + 108936u^{11}t^3 + 16362u^{12}t^2 + 4782u^{13}t + 1056\ u^{14}\Big)s^4$$

$$+ t^3u^3\Big(- 820t^{13} - 4665ut^{12} - 3330u^2\ t^{11} + 50014u^3t^{10} + 234315u^4t^9 + 583398u^5t^8 + 954377u^6t^7 + 1055745\ u^7t^6 + 809880u^8t^5$$

$$+ 443150u^9t^4 + 168838u^{10}t^3 + 40614u^{11}\ t^2 + 5192u^{12}t + 196u^{13}\Big)s^3 + 3t^4u^4(t + u)^2\Big(92\ t^{10} + 476ut^9 + 2090u^2t^8 + 7269u^3t^7$$

$$+ 17036u^4t^6 + 25733u^5t^5 + 22780\ u^6t^4 + 14183u^7t^3 + 6446u^8t^2 + 2047u^9t + 352u^{10}\Big)s^2 - t^5u^5\ (t + u)^3\Big(60t^8 + 29ut^7 - 1449u^2t^6$$

$$- 4932u^3t^5 - 8177u^4t^4 - 6609\ u^5t^3 - 3504u^6t^2 - 1214u^7t - 264u^8\Big)s + t^6u^6(t + u)^6\Big(14\ t^4 + 90ut^3 + 126u^2t^2 + 89u^3t + 66u^4\Big)\Big)H(1, z)$$

$$= - 9s^2t^2u^2\ \Big(2\Big(6t^6 + 78ut^5 - 33u^2t^4 + 550u^3t^3 - 33u^4t^2 + 78u^5t + 6\ u^6\Big)s^{16} + \Big(130t^7 + 1727ut^6 + 1824u^2t^5 + 10515u^3t^4$$

$$+ 10515\ u^4t^3 + 1824u^5t^2 + 1727u^6t + 130u^7\Big)s^{15} + 6\Big(104t^8 + 1295u\ t^7 + 2603u^2t^6 + 7743u^3t^5 + 14182u^4t^4 + 7743u^5t^3$$

$$+ 2603u^6t^2 + 1295\ u^7t + 104u^8\Big)s^{14} + \Big(1747t^9 + 20760ut^8 + 60177u^2t^7 + 138345\ u^3t^6 + 311235u^4t^5 + 311235u^5t^4 + 138345u^6t^3$$

$$+ 60177u^7t^2 + 20760u^8\ t + 1747u^9\Big)s^{13} + \Big(3218t^{10} + 37773ut^9 + 146652u^2t^8 + 339358\ u^3t^7 + 769808u^4t^6 + 1107894u^5t^5$$

$$+ 769808u^6t^4 + 339358u^7t^3 + 146652\ u^8t^2 + 37773u^9t + 3218u^{10}\Big)s^{12} + 3\Big(1403t^{11} + 16697u\ t^{10} + 82856u^2t^9 + 227499u^3t^8$$

$$+ 514057u^4t^7 + 898468u^5t^6 + 898468u^6\ t^5 + 514057u^7t^4 + 227499u^8t^3 + 82856u^9t^2 + 16697u^{10}t + 1403\ u^{11}\Big)s^{11} + 2\Big(2050t^{12}$$

$$+ 24939ut^{11} + 154530u^2\ t^{10} + 531155u^3t^9 + 1271385u^4t^8 + 2420490u^5t^7 + 3112438u^6\ t^6 + 2420490u^7t^5 + 1271385u^8t^4$$

$$+ 531155u^9t^3 + 154530u^{10}t^2 + 24939\ u^{11}t + 2050u^{12}\Big)s^{10} + \Big(2989t^{13} + 37142ut^{12} + 287298\ u^2t^{11} + 1240898u^3t^{10} + 3321201u^4t^9$$

$$+ 6609462u^5t^8 + 9856850u^6\ t^7 + 9856850u^7t^6 + 6609462u^8t^5 + 3321201u^9t^4 + 1240898u^{10}\ t^3 + 287298u^{11}t^2 + 37142u^{12}t$$

$$+ 2989u^{13}\Big)s^9 + 3\Big(514\ t^{14} + 6775ut^{13} + 65664u^2t^{12} + 346961u^3t^{11} + 1076100u^4\ t^{10} + 2314869u^5t^9 + 3821756u^6t^8 + 4598458u^7t^7$$

$$+ 3821756u^8\ t^6 + 2314869u^9t^5 + 1076100u^{10}t^4 + 346961u^{11}t^3 + 65664u^{12}\ t^2 + 6775u^{13}t + 514u^{14}\Big)s^8 + \Big(493t^{15} + 8076ut^{14}$$

$$+ 98952\ u^2t^{13} + 596673u^3t^{12} + 2146416u^4t^{11} + 5342982u^5t^{10} + 10174467\ u^6t^9 + 14554545u^7t^8 + 14554545u^8t^7 + 10174467u^9t^6$$

$$+ 5342982u^{10}\ t^5 + 2146416u^{11}t^4 + 596673u^{12}t^3 + 98952u^{13}t^2 + 8076u^{14}t + 493\ u^{15}\Big)s^7 + \Big(72t^{16} + 2194ut^{15} + 36630u^2t^{14}$$

$$= + 227075u^3\ t^{13} + 910630u^4t^{12} + 2784993u^5t^{11} + 6611664u^6t^{10} + 11660494u^7\ t^9 + 14240640u^8t^8 + 11660494u^9t^7 + 6611664u^{10}t^6$$

$$+ 2784993u^{11}\ t^5 + 910630u^{12}t^4 + 227075u^{13}t^3 + 36630u^{14}t^2 + 2194u^{15}t + 72\ u^{16}\Big)s^6 + 3tu\Big(100t^{15} + 3071ut^{14} + 18464u^2\ t^{13}$$

$$+ 80084u^3t^{12} + 323267u^4t^{11} + 1009711u^5t^{10} + 2219756u^6\ t^9 + 3327843u^7t^8 + 3327843u^8t^7 + 2219756u^9t^6 + 1009711u^{10}\ t^5$$

$$+ 323267u^{11}t^4 + 80084u^{12}t^3 + 18464u^{13}t^2 + 3071u^{14}t + 100\ u^{15}\Big)s^5 + t^2u^2\Big(1146t^{14} + 7452ut^{13} + 43332u^2\ t^{12} + 256857u^3t^{11}$$

$$+ 1020516u^4t^{10} + 2637270u^5t^9 + 4640688u^6\ t^8 + 5609726u^7t^7 + 4640688u^8t^6 + 2637270u^9t^5 + 1020516u^{10}\ t^4 + 256857u^{11}t^3$$

$$+ 43332u^{12}t^2 + 7452u^{13}t + 1146u^{14}\Big)\ s^4 + t^3u^3\Big(316t^{13} + 7702ut^{12} + 60600u^2t^{11} + 260422u^3\ t^{10} + 721307u^4t^9 + 1402338u^5t^8$$

$$+ 1962295u^6t^7 + 1962295u^7\ t^6 + 1402338u^8t^5 + 721307u^9t^4 + 260422u^{10}t^3 + 60600u^{11}t^2 + 7702\ u^{12}t + 316u^{13}\Big)s^3$$

$$+ 3t^4u^4(t + u)^2\Big(382t^{10} + 2457u\ t^9 + 8650u^2t^8 + 21232u^3t^7 + 37622u^4t^6 + 46842u^5t^5 + 37622u^6\ t^4 + 21232u^7t^3 + 8650u^8t^2$$

$$+ 2457u^9t + 382u^{10}\Big)s^2 + t^5u^5\ (t + u)^3\Big(300t^8 + 1544ut^7 + 4776u^2t^6 + 9573u^3t^5 + 12690u^4\ t^4 + 9573u^5t^3 + 4776u^6t^2$$

$$+ 1544u^7t + 300u^8\Big)s + 3t^6u^6(t + u)^6\ \Big(24t^4 + 37ut^3 + 56u^2t^2 + 37u^3t + 24u^4\Big)\Big)H(2, y)\Big\}\Big/\Big(27s^3t^3u^3(s + t)^6(s + u)^6(t + u)^6\Big)$$

$$\mathcal{A}^{(2)}_{5;C_F n_f} = \Big\{ - 432s^2t^2\Big(s^2 - ts + t^2\Big)u^3(s + u)^6(t + u)^6H(0, z)H(1, y)\ (s + t)^7 - 324s^2t^2u^3(s + u)^6(t + u)^6\Big(s^2 - ts$$

$$= + t^2 + u^2\Big)H(1, y)\ H(1, z)(s + t)^7 - 108s^2t^2u^3(s + u)^6(t + u)^6\Big(s^2 - ts + t^2 - 3\ u^2\Big)H(1, 1, y)(s + t)^7$$

$$- 324s^2t^2u^3(s + u)^6(t + u)^6\Big(s^2 - t\ s + t^2 + u^2\Big)H(1, 2, y)(s + t)^7 + 108s^2t^3u^3(s + u)^6(t + u)^6\ \Big(t^2 + u^2\Big)H(0, y)H(0, z)(s + t)^6$$

$$+ 432s^2(s - t)t^2u^5(s + u)^6\ (t + u)^6H(0, y)H(1, z)(s + t)^6 + 432s^3t^2u^2(s + u)^6(t + u)^6\Big(t^3 + s^2\ u\Big)H(0, z)H(2, y)(s + t)^6$$

$$=+108s^3t^2u^2(s+u)^6(t+u)^6(3u^3+s^2(4t+3u))H(1,z)H(2,y)(s+t)^6-432s^3t^2u^2(s+u)^6(t+u)^7(t^2-ut+u^2)H(1,z)H(3,y)(s+t)^6$$
$$-432s^2t^3u^5(s+u)^6(t+u)^6H(0,0,y)(s+t)^6-432s^2t^5u^3(s+u)^6(t+u)^6H(0,0,z)(s+t)^6-108s^2t^2u^3(s+u)^6(t+u)^6(3t^3+3u^2t$$
$$+4su^2)H(0,1,z)(s+t)^6+108s^2t^2u^3(s+u)^6(t+u)^6(-3t^3-3u^2t+4su^2)H(0,2,y)(s+t)^6+108s^2t^2u^3(s+u)^6(t+u)^6(t^3$$
$$+u^2t-4su^2)H(1,0,y)(s+t)^6+108s^2t^2u^3(s+u)^6(t+u)^6(4s^3+t^3+tu^2)H(1,0,z)(s+t)^6+108s^2t^2u^3(s+u)^6(t+u)^6(3s^3$$
$$+3u^2s-4tu^2)H(1,1,z)(s+t)^6+432s^3t^2u^2(s+u)^6(t+u)^6(u^3+s^2t)H(2,0,y)(s+t)^6+108s^3t^2u^3(s+u)^6(t+u)^6(s^2$$
$$-3u^2)H(2,1,y)(s+t)^6+324s^3t^2u^2(s+u)^6(t+u)^7(s^2+t^2+u^2-tu)H(2,2,y)(s+t)^6-432s^3t^2u^2(s+u)^6(t+u)^7(t^2-ut$$
$$+u^2)H(3,2,y)(s+t)^6-18s^2t^2u^3(s+u)^6(t+u)^6(9(t+2u)s^4+9(3t^2+6ut+2u^2)s^3+(27t^3+72ut^2+12u^2t+16u^3)s^2+t(9t^3$$
$$+54ut^2+12u^2t+20u^3)s+2t^2u(9t^2+9ut+8u^2))H(1,y)(s+t)^4-18s^2t^3u^3(t+u)^6(18s^{11}+51(t+2u)s^{10}+(33t^2+360ut$$
$$+203u^2)s^9+(-27t^3+498ut^2+705u^2t+156u^3)s^8+(-39t^4+420ut^3+776u^2t^2+546u^3t-63u^4)s^7-3(4t^5-108ut^4-91u^2t^3$$
$$=-106u^3t^2+21u^4t+98u^5)s^6-3u(-64t^5-9ut^4+98u^2t^3+142u^3t^2+236u^4t+121u^5)s^5-u(-48t^6-44ut^5+174u^2t^4+711u^3t^3$$
$$+1102u^4t^2+897u^5t+240u^6)s^4-u^2(-16t^6-100ut^5+333u^2t^4+948u^3t^3+1084u^4t^2+558u^5t+83u^6)s^3-3u^3(-16t^6+8ut^5$$
$$+124u^2t^4+199u^3t^3+182u^4t^2+60u^5t+4u^6)s^2-tu^5(52t^4+135ut^3+186u^2t^2+147u^3t+24u^4)s-2t^2u^6(4t^3+9ut^2+13u^2t$$
$$+8u^3))H(0,y)(s+t)^3-12t^3u^3(s+u)(24tu(3t^3+4ut^2+4u^2t+3u^3)s^{15}-6(t+u)^2(19t^4-7ut^3+86u^2t^2-7u^3t+19u^4)s^{14}$$
$$-(528t^7+2241ut^6+5905u^2t^5+10806u^3t^4+10806u^4t^3+5905u^5t^2+2241u^6t+528u^7)s^{13}-(t+u)^2(999t^6+3708ut^5+7757u^2t^4$$
$$+13636u^3t^3+7757u^4t^2+3708u^5t+999u^6)s^{12}-(945t^9+6444ut^8+18478u^2t^7+38524u^3t^6+60985u^4t^5+60985u^5t^4+38524u^6t^3$$
$$+18478u^7t^2+6444u^8t+945u^9)s^{11}-(t+u)^2(408t^8+1419ut^7-737u^2t^6-1221u^3t^5+6934u^4t^4-1221u^5t^3-737u^6t^2+1419u^7t$$
$$+408u^8)s^{10}+2(-9t^{11}+1365ut^{10}+9543u^2t^9+28308u^3t^8+43963u^4t^7+44166u^5t^6+44166u^6t^5+43963u^7t^4+28308u^8t^3$$
$$+9543u^9t^2+1365u^{10}t-9u^{11})s^9+(t+u)^2(33t^{10}+3726ut^9+16908u^2t^8+31720u^3t^7+24859u^4t^6+8028u^5t^5+24859u^6t^4$$
$$+31720u^7t^3+16908u^8t^2+3726u^9t+33u^{10})s^8+(3t^{13}+2076ut^{12}+13514u^2t^{11}+34430u^3t^{10}+33548u^4t^9-29287u^5t^8$$
$$-114020u^6t^7-114020u^7t^6-29287u^8t^5+33548u^9t^4+34430u^{10}t^3+13514u^{11}t^2+2076u^{12}t+3u^{13})s^7+tu(t+u)^2(615t^{10}$$
$$=+2129ut^9-2862u^2t^8-27958u^3t^7-70915u^4t^6-96406u^5t^5-70915u^6t^4-27958u^7t^3-2862u^8t^2+2129u^9t+615u^{10})s^6$$
$$+tu(87t^{13}+151ut^{12}-4613u^2t^{11}-33895u^3t^{10}-117246u^4t^9-252106u^5t^8-365426u^6t^7-365426u^7t^6-252106u^8t^5-117246u^9t^4$$
$$-33895u^{10}t^3-4613u^{11}t^2+151u^{12}t+87u^{13})s^5-t^2u^2(t+u)^2(61t^{10}+1421ut^9+9539u^2t^8+31118u^3t^7+60284u^4t^6+74918u^5t^5$$
$$+60284u^6t^4+31118u^7t^3+9539u^8t^2+1421u^9t+61u^{10})s^4-t^3u^3(t+u)^3(119t^8+1631ut^7+6699u^2t^6+14006u^3t^5+17750u^4t^4$$
$$+14006u^5t^3+6699u^6t^2+1631u^7t+119u^8)s^3-t^4u^4(t+u)^4(133t^6+842ut^5+1972u^2t^4+2532u^3t^3+1972u^4t^2+842u^5t+133u^6)s^2$$
$$-6t^5u^5(t+u)^7(8t^2+9ut+8u^2)s-6t^6u^6(t+u)^6(t^2+ut+u^2))(s+t)-18s^2t^3u^3(s+u)^3(t+u)^6(18s^{11}+51(2t+u)s^{10}+(203t^2$$
$$+360ut+33u^2)s^9+3(52t^3+235ut^2+166u^2t-9u^3)s^8+(-63t^4+546ut^3+776u^2t^2+420u^3t-39u^4)s^7-3(98t^5+21ut^4-106u^2t^3$$
$$-91u^3t^2-108u^4t+4u^5)s^6-3t(121t^5+236ut^4+142u^2t^3+98u^3t^2-9u^4t-64u^5)s^5-t(240t^6+897ut^5+1102u^2t^4+711u^3t^3$$
$$+174u^4t^2-44u^5t-48u^6)s^4-t^2(83t^6+558ut^5+1084u^2t^4+948u^3t^3+333u^4t^2-100u^5t-16u^6)s^3-3t^3(4t^6+60ut^5+182u^2t^4$$
$$+199u^3t^3+124u^4t^2+8u^5t-16u^6)s^2-t^5u(24t^4+147ut^3+186u^2t^2+135u^3t+52u^4)s-2t^6u^2(8t^3+13ut^2+9u^2t+4u^3))H(0,z)$$
$$=+18s^2t^2u^3(16t^2u(3t^2+ut+3u^2)s^{16}-4t(t^5-96ut^4-125u^2t^3-125u^3t^2-96u^4t+u^5)s^{15}+3(-21t^7+376ut^6+969u^2t^5$$
$$+1014u^3t^4+1053u^4t^3+460u^5t^2+15u^6t+6u^7)s^{14}+3(-91t^8+434ut^7+2445u^2t^6+3548u^3t^5+4121u^4t^4+3594u^5t^3$$
$$+1259u^6t^2+200u^7t+42u^8)s^{13}+(-507t^9-228ut^8+8958u^2t^7+21034u^3t^6+29016u^4t^5+37554u^5t^4+27842u^6t^3$$
$$+10398u^7t^2+2643u^8t+394u^9)s^{12}+3(-139t^{10}-658ut^9+1402u^2t^8+7968u^3t^7+14642u^4t^6+25576u^5t^5+31288u^6t^4$$
$$+20652u^7t^3+8197u^8t^2+2094u^9t+242u^{10})s^{11}+(-33t^{11}-1944ut^{10}-2626u^2t^9+15924u^3t^8+52455u^4t^7+115368u^5t^6$$
$$+193671u^6t^5+189594u^7t^4+109994u^8t^3+41340u^9t^2+9291u^{10}t+870u^{11})s^{10}+(229t^{12}-1206ut^{11}-7826u^2t^{10}+2016u^3t^9$$
$$+60801u^4t^8+168742u^5t^7+312211u^6t^6+377892u^7t^5+285136u^8t^4+142036u^9t^3+47343u^{10}t^2+9112u^{11}t+698u^{12})s^9$$
$$+3(69t^{13}-396ut^{12}-3542u^2t^{11}-3876u^3t^{10}+19403u^4t^9+74332u^5t^8+145298u^6t^7+193458u^7t^6+172121u^8t^5+103198u^9t^4$$
$$+43468u^{10}t^3+12394u^{11}t^2+2015u^{12}t+122u^{13})s^8+(81t^{14}-1338ut^{13}-9602u^2t^{12}-15144u^3t^{11}+41583u^4t^{10}+217006u^5t^9$$
$$+472374u^6t^8+692016u^7t^7+711667u^8t^6+501642u^9t^5+246150u^{10}t^4+85492u^{11}t^3+19953u^{12}t^2+2646u^{13}t+114u^{14})s^7$$

$$
\begin{aligned}
&=+\Big(12t^{15}-936ut^{14}-5785u^2t^{13}-9734\,u^3t^{12}+23766u^4t^{11}+151768u^5t^{10}+374411u^6t^9+611526u^7\,t^8+725603u^8t^7+609872u^9t^6\\
&+355422u^{10}t^5+143724u^{11}t^4+39635\,u^{12}t^3+7044u^{13}t^2+696u^{14}t+16u^{15}\Big)s^6+3tu\,\Big(-112t^{14}-711ut^{13}-1428u^2t^{12}+2680u^3t^{11}\\
&+24408u^4\,t^{10}+73333u^5t^9+137580u^6t^8+184601u^7t^7+179614u^8t^6+124026u^9\,t^5+59992u^{10}t^4+19831u^{11}t^3+4174u^{12}t^2\\
&+496u^{13}t+28\,u^{14}\Big)s^5+t^2u\Big(-48t^{14}-380ut^{13}-1422u^2t^{12}-453\,u^3t^{11}+19778u^4t^{10}+90336u^5t^9+213354u^6t^8+330336u^7\,t^7\\
&+360906u^8t^6+281670u^9t^5+156716u^{10}t^4+61605u^{11}t^3+16236\,u^{12}t^2+2470u^{13}t+144u^{14}\Big)s^4+t^3u^2\Big(-16t^{13}-340\,ut^{12}\\
&-1209u^2t^{11}+1822u^3t^{10}+23522u^4t^9+76164u^5t^8+142722\,u^6t^7+179964u^7t^6+159444u^8t^5+99732u^9t^4+43795u^{10}t^3+13194\,u^{11}t^2\\
&+2510u^{12}t+232u^{13}\Big)s^3-3t^4u^3(t+u)^2\Big(16\,t^{10}+56ut^9-124u^2t^8-1147u^3t^7-3208u^4t^6-5170u^5t^5-5528u^6\,t^4-3835u^7t^3\\
&-1676u^8t^2-424u^9t-48u^{10}\Big)s^2+t^5u^5(t+u)^3\,\Big(52t^7+321ut^6+921u^2t^5+1569u^3t^4+1779u^4t^3+1230u^5\,t^2+484u^6t+84u^7\Big)s\\
&+2t^6u^6(t+u)^7\Big(4t^2+5ut+8\,u^2\Big)\Big)H(1,z)+18s^2t^2u^2\Big(16t^2u^2\Big(3t^2+ut+3\,u^2\Big)s^{16}-4tu\Big(t^5-96ut^4-125u^2t^3-125u^3t^2-96u^4\,t\\
&+u^5\Big)s^{15}+6\Big(3t^8+9ut^7+242u^2t^6+567u^3t^5+582u^4\,t^4+567u^5t^3+242u^6t^2+9u^7t+3u^8\Big)s^{14}+3\Big(42t^9+221u\,t^8+1445u^2t^7\\
&+4311u^3t^6+5693u^4t^5+5693u^5t^4+4311u^6t^3+1445\,u^7t^2+221u^8t+42u^9\Big)s^{13}+2\Big(197t^{10}+1416ut^9+6153\,u^2t^8+18148u^3t^7\\
&+29594u^4t^6+32160u^5t^5+29594u^6t^4+18148u^7\,t^3+6153u^8t^2+1416u^9t+197u^{10}\Big)s^{12}+3\Big(242t^{11}+2199\,ut^{10}+9451u^2t^9+27134u^3t^8\\
&=+50612u^4t^7+62634u^5t^6+62634u^6\,t^5+50612u^7t^4+27134u^8t^3+9451u^9t^2+2199u^{10}t+242u^{11}\Big)\,s^{11}+2\Big(435t^{12}+4803ut^{11}\\
&+23010u^2t^{10}+69466u^3\,t^9+145491u^4t^8+210555u^5t^7+230976u^6t^6+210555u^7t^5+145491u^8\,t^4+69466u^9t^3+23010u^{10}t^2\\
&+4803u^{11}t+435u^{12}\Big)\,s^{10}+\Big(698t^{13}+9301ut^{12}+51105u^2t^{11}+170974u^3\,t^{10}+406704u^4t^9+698997u^5t^8+884941u^6t^7+884941u^7t^6\\
&+698997\,u^8t^5+406704u^9t^4+170974u^{10}t^3+51105u^{11}t^2+9301u^{12}t+698\,u^{13}\Big)s^9+6\Big(61t^{14}+1018ut^{13}+6515u^2t^{12}+24975u^3\,t^{11}\\
&+68497u^4t^{10}+139578u^5t^9+209143u^6t^8+237090u^7t^7+209143\,u^8t^6+139578u^9t^5+68497u^{10}t^4+24975u^{11}t^3+6515u^{12}\,t^2\\
&+1018u^{13}t+61u^{14}\Big)s^8+\Big(114t^{15}+2655ut^{14}+20511\,u^2t^{13}+93946u^3t^{12}+304122u^4t^{11}+729081u^5t^{10}+1284397u^6\,t^9+1678662u^7t^8\\
&+1678662u^8t^7+1284397u^9t^6+729081u^{10}t^5+304122\,u^{11}t^4+93946u^{12}t^3+20511u^{13}t^2+2655u^{14}t+114u^{15}\Big)\,s^7+2\Big(8t^{16}+348ut^{15}\\
&+3558u^2t^{14}+20893u^3t^{13}+82679u^4\,t^{12}+233298u^5t^{11}+478228u^6t^{10}+720901u^7t^9+821694u^8\,t^8+720901u^9t^7+478228u^{10}t^6\\
&+233298u^{11}t^5+82679u^{12}\,t^4+20893u^{13}t^3+3558u^{14}t^2+348u^{15}t+8u^{16}\Big)s^6+3tu\,\Big(28t^{15}+496ut^{14}+4255u^2t^{13}+21403u^3t^{12}\\
&=+71760u^4\,t^{11}+172018u^5t^{10}+302499u^6t^9+397333u^7t^8+397333u^8t^7+302499\,u^9t^6+172018u^{10}t^5+71760u^{11}t^4+21403u^{12}t^3\\
&+4255u^{13}\,t^2+496u^{14}t+28u^{15}\Big)s^5+2t^2u^2\Big(72t^{14}+1235u\,t^{13}+8343u^2t^{12}+34020u^3t^{11}+97435u^4t^{10}+204801u^5\,t^9+317982u^6t^8\\
&+367512u^7t^7+317982u^8t^6+204801u^9t^5+97435u^{10}\,t^4+34020u^{11}t^3+8343u^{12}t^2+1235u^{13}t+72u^{14}\Big)s^4+t^3\,u^3\Big(232t^{13}\\
&+2510ut^{12}+13689u^2t^{11}+49393u^3t^{10}+127070\,u^4t^9+236640u^5t^8+321522u^6t^7+321522u^7t^6+236640u^8t^5+127070\,u^9t^4\\
&+49393u^{10}t^3+13689u^{11}t^2+2510u^{12}t+232u^{13}\Big)\,s^3+6t^4u^4(t+u)^2\Big(24t^{10}+212ut^9+892u^2t^8+2315u^3\,t^7+4004u^4t^6+4762u^5t^5\\
&+4004u^6t^4+2315u^7t^3+892u^8t^2+212u^9\,t+24u^{10}\Big)s^2+t^5u^5(t+u)^3\Big(84t^8+484ut^7+1347u^2\,t^6+2364u^3t^5+2802u^4t^4+2364u^5t^3\\
&+1347u^6t^2+484u^7t+84\,u^8\Big)s+2t^6u^6(t+u)^6\Big(8t^4+13ut^3+18u^2t^2+13u^3t+8\,u^4\Big)\Big)H(2,y)\Big\}\Big/\Big(27s^3t^3u^3(s+t)^6(s+u)^6(t+u)^6\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{5;n_f^2}&=2\Big\{t^3uH(2,y)+t^3uH(1,z)-tu\,\Big(t^2+u^2\Big)H(0,y)-tu\Big(t^2+u^2\Big)H(0,z)+tu^3H(2,y)+t\,u^3H(1,z)\\
&+s^4+2s^3t+2s^3u+3s^2t^2+3s^2u^2+2st^3+2su^3+t^4+2t^3\,u+3t^2u^2+2tu^3+u^4\Big\}\Big/\Big(3stu\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_{6;C_A^2}&=\Big\{3s^2t^2(s+t)^6u^2(t+u)^2\Big(\Big(363t^4+492ut^3+504u^2t^2+492\,u^3t+363u^4\Big)s^4+2\Big(363t^5+479ut^4-58u^2t^3-58u^3t^2+479\,u^4t\\
&=+363u^5\Big)s^3+9(t+u)^2\Big(121t^4-10ut^3-134u^2t^2-10u^3\,t+121u^4\Big)s^2+6(t+u)^3\Big(121t^4+35ut^3+12u^2t^2+35u^3\,t+121u^4\Big)s\\
&+363(t+u)^4\Big(t^2+ut+u^2\Big)^2\Big)H(1,z)\,H(2,y)(s+u)^6+3s^2t^2(s+t)^2u^2(t+u)^6\Big(363s^8+726(3t+u)s^7+3\,\Big(2057t^2+796ut+363u^2\Big)s^6\\
&+6\Big(1815t^3+480ut^2+348u^2\,t+121u^3\Big)s^5+\Big(13068t^4+1782ut^3-297u^2t^2+958u^3t+363\,u^4\Big)s^4+2t\Big(5445t^4+891ut^3-1296u^2t^2\\
&-58u^3t+246\,u^4\Big)s^3+t^2\Big(6171t^4+2880ut^3-297u^2t^2-116u^3t+504\,u^4\Big)s^2+2t^3\Big(1089t^4+1194ut^3+1044u^2t^2+479u^3t+246\,u^4\Big)s\\
&+363t^4\Big(t^2+ut+u^2\Big)^2\Big)H(0,0,y)(s+u)^6+36\,s^2t^2(s+t)^2u^2(t+u)^6\Big(22s^8+44(3t+u)s^7+2\Big(187t^2+70u\,t+33u^2\Big)s^6+\Big(660t^3\\
&+147ut^2+246u^2t+44u^3\Big)\,s^5+\Big(792t^4+41ut^3+360u^2t^2+115u^3t+22u^4\Big)s^4+t\,\Big(660t^4+2ut^3+246u^2t^2+95u^3t+64u^4\Big)s^3\\
&+t^2\Big(374\,t^4+54ut^3+84u^2t^2+9u^3t+84u^4\Big)s^2+t^3\Big(132t^4+63u\,t^3+36u^2t^2-11u^3t+54u^4\Big)s+t^4\Big(22t^4+21ut^3+18u^2\,t^2+4u^3t\\
&+18u^4\Big)\Big)H(0,z)H(0,0,y)(s+u)^6-36s^2t^2(s+t)^2\,u^2(t+u)^6\Big(22s^8+3(44t+7u)s^7+\Big(374t^2+63ut+18u^2\Big)\,s^6+\Big(660t^3+54ut^2\\
&+36u^2t+4u^3\Big)s^5+\Big(792t^4+2u\,t^3+84u^2t^2-11u^3t+18u^4\Big)s^4+t\Big(660t^4+41ut^3+246u^2\,t^2+9u^3t+54u^4\Big)s^3+t^2\Big(374t^4+147ut^3\\
&+360u^2t^2+95u^3\,t+84u^4\Big)s^2+t^3\Big(132t^4+140ut^3+246u^2t^2+115u^3t+64\,u^4\Big)s+22t^4\Big(t^2+ut+u^2\Big)^2\Big)H(1,z)H(0,0,y)\,(s+u)^6
\end{aligned}
$$

$$= + 108 s^2 t^2 (s+t)^6 u^2 (t+u)^6 \Big(2s^4 + 4(t+u)s^3 + 6(t^2+u^2)s^2 + 4(t^3+u^3)s + 2t^4 + 2u^4 - 2tu^3 - 3t^2 u^2$$
$$- 2t^3 u\Big) H(0,0,y)H(0,0,z)(s+u)^6 + 18 s^2 t^2 (s+t)^6 u^2 (t+u)^2 \Big(\Big(44t^4 + 296ut^3 + 486u^2t^2 + 316u^3 t + 58u^4\Big)s^4 + 2\Big(15t^5 + 247ut^4$$
$$+ 634u^2t^3 + 646u^3t^2 + 271u^4 t + 27u^5\Big)s^3 + 3(t+u)^2\Big(5t^4 + 150ut^3 + 228u^2t^2 + 146u^3 t + 15u^4\Big)s^2 - 2(t+u)^3\Big(2t^4 - 91ut^3 - 90u^2t^2$$
$$- 97u^3 t - 4u^4\Big)s + (t+u)^4\Big(11t^4 + 94ut^3 + 105u^2t^2 + 72u^3 t + 6u^4\Big)\Big)H(3,y)H(0,1,z)(s+u)^6 - 108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4$$
$$+ 2(2t+u)s^3 + 3\Big(2t^2+u^2\Big)s^2 + 2\Big(2t^3+u^3\Big)s + 2\Big(t^4 - 2ut^3 + 6u^2t^2 + u^4\Big)\Big)H(0,0,y)H(0,1,z)(s+u)^6$$
$$= + 36 s^2t^2(s+t)^6u^2(t+u)^2\Big(2\Big(11t^4+32ut^3+42u^2t^2+27u^3t+9u^4\Big)s^4 + \Big(44t^5+115ut^4+95u^2t^3+9u^3t^2-11u^4t$$
$$+ 4u^5\Big)s^3 + 6(t+u)^2\Big(11t^4+19ut^3+11u^2t^2+3u^4\Big)s^2 + (t+u)^3\Big(44t^4+8ut^3-9u^2t^2+21u^4\Big)s + 22(t+u)^4\Big(t^2+ut$$
$$+ u^2\Big)^2\Big)H(1,z)H(0,2,y)(s+u)^6 - 108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+2(t+2u)s^3+3\Big(t^2+2u^2\Big)s^2+2\Big(t^3+2u^3\Big)s+2\Big(t^4+6u^2t^2$$
$$- 2u^3t + u^4\Big)\Big)H(0,0,z)H(0,2,y)(s+u)^6 + 108s^2t^2(s+t)^6u^2(t+u)^6\Big(-2(t-2u)s^3 - 3\Big(t^2-2u^2\Big)s^2 - 2\Big(t^3-2u^3\Big)s + 12tu\Big(t^2-ut$$
$$+ u^2\Big)\Big)H(0,1,z)H(0,2,y)(s+u)^6 + 216s^2t^2(s+t)^6u^2(t+u)^6\Big(s^4+2us^3+3u^2s^2+2u^3s+\Big(t^2+ut+u^2\Big)^2\Big)H(0,1,z)H(0,3,y)(s+u)^6$$
$$+ 18s^2t^2(s+t)^3u^2(t+u)^6\Big(3s^7+(6u-9t)s^6+\Big(-48t^2+12ut+15u^2\Big)s^5 - \Big(56t^3+6ut^2+63u^2t-12u^3\Big)s^4 - t^2\Big(7t^2+36ut+222u^2\Big)s^3$$
$$+ 3t\Big(11t^4-14ut^3-70u^2t^2-8u^3t-8u^4\Big)s^2 + t^2\Big(28t^4-24ut^3-81u^2t^2-24u^3t-30u^4\Big)s + t^3\Big(8t^4-6ut^3-15u^2t^2-12u^3t$$
$$= - 14u^4\Big)\Big)H(0,z)H(1,0,y)(s+u)^6 - 18s^2t^2(s+t)^3u^2(t+u)^6\Big(8s^7+(28t-6u)s^6+3\Big(11t^2-8ut-5u^2\Big)s^5 - \Big(7t^3+42ut^2+81u^2t$$
$$+ 12u^3\Big)s^4 - 2\Big(28t^4+18ut^3+105u^2t^2+12u^3t+7u^4\Big)s^3 - 6t\Big(8t^4+ut^3+37u^2t^2+4u^3t+5u^4\Big)s^2 - 3\Big(3t^6-4ut^5+21u^2t^4+8u^4t^2\Big)s$$
$$+ 3t^4\Big(t^3+2ut^2+5u^2t+4u^3\Big)\Big)H(1,z)H(1,0,y)(s+u)^6 + 108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+2(2t+u)s^3+3\Big(2t^2+u^2\Big)s^2+2\Big(2t^3+u^3\Big)s$$
$$+ 2t^4+2u^4-2tu^3+9t^2u^2-6t^3u\Big)H(0,1,z)H(1,0,y)(s+u)^6 - 18s^2t^2(s+t)^6u^2(t+u)^3\Big(2t\Big(7t^2+15ut+12u^2\Big)s^4+12\Big(t^4+2ut^3$$
$$+ 2u^2t^2-u^4\Big)s^3 + 3(t+u)^2\Big(5t^3+17ut^2+31u^2t-5u^3\Big)s^2 + 6(t+u)^3\Big(t^3+ut^2+u^2t-u^3\Big)s - (t+u)^3\Big(8t^4+4ut^3-3u^2t^2-18u^3t$$
$$+ 3u^4\Big)\Big)H(3,y)H(1,0,z)(s+u)^6 - 108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+4(t-u)s^3+6\Big(t^2+2u^2\Big)s^2+4t^3s+2t^4+2u^4+2tu^3+3t^2u^2$$
$$+ 2t^3u\Big)H(0,0,y)H(1,0,z)(s+u)^6 - 216s^3t^2(s+t)^6u^2(t+u)^6\Big(2(t-2u)s^2+3\Big(t^2+u^2\Big)s+2\Big(t^3-u^3\Big)\Big)H(0,2,y)H(1,0,z)(s+u)^6$$
$$+ 108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+2(t+3u)s^3+3\Big(t^2-u^2\Big)s^2+2\Big(t^3+5u^3\Big)s+2\Big(t^2+ut+u^2\Big)^2\Big)H(0,3,y)H(1,0,z)(s+u)^6$$
$$= + 108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+(4t-6u)s^3+\Big(6t^2+9u^2\Big)s^2+\Big(4t^3-2u^3\Big)s+2t^4+2u^4+2tu^3+3t^2u^2$$
$$+ 2t^3u\Big)H(1,0,y)H(1,0,z)(s+u)^6 + 3s^2t^2(s+t)^6u^2(t+u)^2\Big(\Big(363t^4+492ut^3+504u^2t^2+492u^3t+363u^4\Big)s^4+2\Big(363t^5+479ut^4$$
$$- 58u^2t^3-58u^3t^2+479u^4t+363u^5\Big)s^3+9(t+u)^2\Big(121t^4-10ut^3-134u^2t^2-10u^3t+121u^4\Big)s^2+6(t+u)^3\Big(121t^4+35ut^3+12u^2t^2$$
$$+ 35u^3t+121u^4\Big)s + 363(t+u)^4\Big(t^2+ut+u^2\Big)^2\Big)H(1,1,z)(s+u)^6 + 36s^2t^2(s+t)^6u^2(t+u)^2\Big(2\Big(11t^4+32ut^3+42u^2t^2+27u^3t$$
$$+ 9u^4\Big)s^4+\Big(44t^5+115ut^4+95u^2t^3+9u^3t^2-11u^4t+4u^5\Big)s^3+6(t+u)^2\Big(11t^4+19ut^3+11u^2t^2+3u^4\Big)s^2+(t+u)^3\Big(44t^4+8ut^3$$
$$- 9u^2t^2+21u^4\Big)s+22(t+u)^4\Big(t^2+ut+u^2\Big)^2\Big)H(0,y)H(1,1,z)(s+u)^6 - 36s^2t^2(s+t)^6u^2(t+u)^2\Big(2\Big(20t^4+59ut^3+84u^2t^2+59u^3t$$
$$+ 20u^4\Big)s^4+8\Big(6t^5+13ut^4+13u^2t^3+13u^3t^2+13u^4t+6u^5\Big)s^3+6(t+u)^2\Big(14t^4+19ut^3+22u^2t^2+19u^3t+14u^4\Big)s^2+(t+u)^3\Big(65t^4$$
$$+ 8ut^3-18u^2t^2+8u^3t+65u^4\Big)s+44(t+u)^4\Big(t^2+ut+u^2\Big)^2\Big)H(3,y)H(1,1,z)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4$$
$$+ (4t-2u)s^3+\Big(6t^2-3u^2\Big)s^2+\Big(4t^3-2u^3\Big)s+2\Big(t^2+ut+u^2\Big)^2\Big)H(0,0,y)H(1,1,z)(s+u)^6 - 108s^2t^2(s+t)^6u^2(t+u)^6\Big(4s^4$$
$$+ (6t-2u)s^3+9\Big(t^2+u^2\Big)s^2+6\Big(t^3-u^3\Big)s+4\Big(t^2+ut+u^2\Big)^2\Big)H(0,3,y)H(1,1,z)(s+u)^6+216s^2t^2(s+t)^6u^3(t+u)^6\Big(2s^3$$
$$- 3us^2+4u^2s+6t\Big(t^2-ut+u^2\Big)\Big)H(0,1,z)H(1,2,y)(s+u)^6 - 216s^2t^2(s+t)^6u^3(t+u)^6\Big(6s^3-6us^2+6u^2s+t\Big(2t^2-3ut$$
$$= + 4u^2\Big)\Big)H(1,0,z)H(1,2,y)(s+u)^6+36s^2t^2(s+t)^6u^2(t+u)^2\Big(2\Big(11t^4+32ut^3+42u^2t^2+27u^3t+9u^4\Big)s^4+\Big(44t^5+115ut^4+95u^2t^3$$
$$+ 9u^3t^2-11u^4t+4u^5\Big)s^3+6(t+u)^2\Big(11t^4+19ut^3+11u^2t^2+3u^4\Big)s^2+(t+u)^3\Big(44t^4+8ut^3-9u^2t^2+21u^4\Big)s+22(t+u)^4\Big(t^2$$
$$+ ut+u^2\Big)^2\Big)H(1,z)H(2,0,y)(s+u)^6 - 108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4-4(t-u)s^3+6\Big(2t^2+u^2\Big)s^2+4u^3s+2t^4+2u^4+2tu^3$$
$$+ 3t^2u^2+2t^3u\Big)H(0,0,z)H(2,0,y)(s+u)^6 - 108s^2t^2(s+t)^6u^2(t+u)^6\Big(4s^4+2(5t+2u)s^3+3\Big(t^2+2u^2\Big)s^2+2\Big(7t^3+2u^3\Big)s+4t^4$$
$$+ 4u^4+4tu^3+6t^2u^2+4t^3u\Big)H(0,1,z)H(2,0,y)(s+u)^6 - 108s^2t^2(s+t)^6u^2(t+u)^6\Big(4s^4+6ts^3+9\Big(t^2+2u^2\Big)s^2+\Big(6t^3+4u^3\Big)s$$
$$+ 4t^4+4u^4+6tu^3+9t^2u^2+6t^3u\Big)H(1,0,z)H(2,0,y)(s+u)^6 + 3s^2t^2(s+t)^6u^2(t+u)^2\Big(\Big(363t^4+492ut^3+504u^2t^2+492u^3t$$
$$+ 363u^4\Big)s^4+2\Big(363t^5+479ut^4-58u^2t^3-58u^3t^2+479u^4t+363u^5\Big)s^3+9(t+u)^2\Big(121t^4-10ut^3-134u^2t^2-10u^3t+121u^4\Big)s^2$$
$$+ 6(t+u)^3\Big(121t^4+35ut^3+12u^2t^2+35u^3t+121u^4\Big)s+363(t+u)^4\Big(t^2+ut+u^2\Big)^2\Big)H(2,2,y)(s+u)^6+36s^2t^2(s+t)^6u^2(t+u)^2\Big(2\Big(9t^4$$
$$+ 27ut^3+42u^2t^2+32u^3t+11u^4\Big)s^4+\Big(4t^5-11ut^4+9u^2t^3+95u^3t^2+115u^4t+44u^5\Big)s^3+6(t+u)^2\Big(3t^4+11u^2t^2$$
$$+ 19u^3t+11u^4\Big)s^2+(t+u)^3\Big(21t^4-9u^2t^2+8u^3t+44u^4\Big)s+22(t+u)^4\Big(t^2+ut+u^2\Big)^2\Big)H(0,z)H(2,2,y)(s+u)^6$$
$$+ 108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4-2(t-2u)s^3-3\Big(t^2-2u^2\Big)s^2-2\Big(t^3-2u^3\Big)s+2\Big(t^2+ut+u^2\Big)^2\Big)H(0,0,z)H(2,2,y)(s+u)^6$$
$$= + 432s^2t^2(s+t)^6u^2(t+u)^6\Big(s^4+3\Big(t^2+u^2\Big)s^2-\Big(t^3+u^3\Big)s+\Big(t^2+ut+u^2\Big)^2\Big)H(0,1,z)H(2,2,y)(s+u)^6$$
$$- 108s^2t^2(s+t)^6u^2(t+u)^6\Big(4s^4-2(t+2u)s^3+3\Big(5t^2+8u^2\Big)s^2+2\Big(t^3-2u^3\Big)s+4\Big(t^2+ut+u^2\Big)^2\Big)H(1,0,z)H(2,2,y)(s+u)^6$$
$$+ 324s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+2(t+u)s^3+3\Big(t^2+u^2\Big)s^2+2\Big(t^3+u^3\Big)s+2\Big(t^2+ut+u^2\Big)^2\Big)H(0,1,z)H(2,3,y)(s+u)^6$$

$$= - 108s^2t^2(s+t)^6u^2 (t+u)^6\Big(2s^4+2(t+u)s^3+3\Big(t^2+u^2\Big)s^2+2 \Big(t^3+u^3\Big)s+2\Big(t^2+ut+u^2\Big)^2\Big)H(1,0,z)H(2,3,y) (s+u)^6$$

$$- 18s^2t^2(s+t)^6u^2(t+u)^3\Big(2u\Big(12t^2+15ut+7 u^2\Big)s^4-12\Big(t^4-2u^2t^2-2u^3t-u^4\Big)s^3-3(t+u)^2 \Big(5t^3$$

$$- 31ut^2-17u^2t-5u^3\Big)s^2-6(t+u)^3\Big(t^3-ut^2-u^2 t-u^3\Big)s-(t+u)^3\Big(3t^4-18ut^3-3u^2t^2+4u^3t$$

$$+ 8 u^4\Big)\Big)H(1,z)H(3,0,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6 \Big(2s^4+2(t+u)s^3+3\Big(t^2+u^2\Big)s^2+2\Big(t^3+u^3\Big) s+2\Big(t^2+ut$$

$$+ u^2\Big)^2\Big)H(0,1,z)H(3,0,y)(s+u)^6-108s^2t^2 (s+t)^6u^2(t+u)^6\Big(2s^4+10(t+u)s^3-3\Big(t^2+u^2\Big)s^2+6 \Big(t^3+u^3\Big)s+2\Big(t^2+ut$$

$$+ u^2\Big)^2\Big)H(1,0,z)H(3,0,y) (s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+2ts^3+9t^2s^2+2 \Big(t^2+ut+u^2\Big)^2\Big)H(1,1,z)H(3,0,y)(s+u)^6$$

$$= - 18s^2t^2 (s+t)^6u^2(t+u)^3\Big(2t\Big(7t^2+15ut+12u^2\Big)s^4+12 \Big(t^4+2ut^3+2u^2t^2-u^4\Big)s^3+3(t+u)^2\Big(5t^3+17ut^2+31 u^2t$$

$$- 5u^3\Big)s^2+6(t+u)^3\Big(t^3+ut^2+u^2t-u^3\Big)s-(t+u)^3 \Big(8t^4+4ut^3-3u^2t^2-18u^3t+3u^4\Big)\Big)H(0,z)H(3,2,y) (s+u)^6$$

$$- 36s^2t^2(s+t)^6u^2(t+u)^2\Big(2\Big(20t^4+59ut^3+84u^2 t^2+59u^3t+20u^4\Big)s^4+8\Big(6t^5+13ut^4+13u^2t^3+13u^3 t^2+13u^4t+6u^5\Big)s^3$$

$$+ 6(t+u)^2\Big(14t^4+19ut^3+22u^2t^2+19 u^3t+14u^4\Big)s^2+(t+u)^3\Big(65t^4+8ut^3-18u^2t^2+8u^3t+65 u^4\Big)s+44(t+u)^4\Big(t^2+ut$$

$$+ u^2\Big)^2\Big)H(1,z)H(3,2,y) (s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+2ts^3+9t^2s^2+2 \Big(t^2+ut+u^2\Big)^2\Big)H(0,1,z)H(3,2,y)(s+u)^6$$

$$- 216s^2t^2 (s+t)^6u^2(t+u)^6\Big(2s^4+4(t+2u)s^3-3\Big(t^2-u^2\Big)s^2+2 \Big(t^3+3u^3\Big)s+2\Big(t^2+ut+u^2\Big)^2\Big)H(1,0,z) H(3,2,y)(s+u)^6$$

$$+ 18s^2t^2(s+t)^6u^2(t+u)^2\Big(\Big(58t^4+340u t^3+540u^2t^2+340u^3t+58u^4\Big)s^4+2\Big(21t^5+265ut^4+658u^2 t^3+658u^3t^2+265u^4t$$

$$+ 21u^5\Big)s^3+6(t+u)^2\Big(5t^4+86u t^3+138u^2t^2+86u^3t+5u^4\Big)s^2+2(t+u)^3\Big(t^4+97ut^3+96 u^2t^2+97u^3t+u^4\Big)s+3(t+u)^4\Big(t^4$$

$$+ 30ut^3+36u^2t^2+30u^3 t+u^4\Big)\Big)H(1,z)H(3,3,y)(s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6 \Big(4s^4+(6t+4u)s^3+3\Big(t^2+2u^2\Big)s^2+4\Big(2 t^3$$

$$+ u^3\Big)s+4\Big(t^2+ut+u^2\Big)^2\Big)H(0,1,z)H(3,3,y) (s+u)^6-216s^3t^2(s+t)^6u^3(t+u)^6\Big(2s^2-3us+4u^2\Big) H(1,0,z)H(3,3,y)(s+u)^6$$

$$+ 216s^2t^2(s+t)^6u^2(t+u)^6\Big(4s^4+2 (t+u)s^3+9\Big(t^2+u^2\Big)s^2+4\Big(t^2+ut+u^2\Big)^2\Big) H(1,1,z)H(3,3,y)(s+u)^6$$

$$= + 216s^2t^2(s+t)^6u^2(t+u)^6\Big(s^4+2(t+u) s^3+3\Big(t^2+u^2\Big)s^2+2\Big(t^3+u^3\Big)s+t^4+u^4+6t^2u^2-2 t^3u\Big)H(0,y)H(0,0,1,z)(s+u)^6$$

$$+ 216s^2t^2(s+t)^6u^3(t+u)^6 \Big(2s^3-3us^2+4u^2s+6t\Big(t^2-ut+u^2\Big)\Big)H(1,y) H(0,0,1,z)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6\Big(6s^4$$

$$+ 2(4t+u) s^3+15u^2s^2+2\Big(6t^3-u^3\Big)s+6\Big(t^2+u t+u^2\Big)^2\Big)H(2,y)H(0,0,1,z)(s+u)^6+432s^2t^2(s+t)^6u^2 (t+u)^6\Big(2s^4+2(t+u)s^3$$

$$+ 3\Big(t^2+u^2\Big)s^2+2 \Big(t^3+u^3\Big)s+2\Big(t^2+ut+u^2\Big)^2\Big)H(3,y)H(0,0,1,z) (s+u)^6-36s^2t^2(s+t)^2u^2(t+u)^6\Big(22s^8+3(44t+7u) s^7$$

$$+ \Big(374t^2+63ut+18u^2\Big)s^6+\Big(660t^3+54ut^2+36u^2t+4 u^3\Big)s^5+\Big(792t^4+2ut^3+84u^2t^2-11u^3t+18u^4\Big)s^4+t \Big(660t^4+41ut^3$$

$$+ 246u^2t^2+9u^3t+54u^4\Big)s^3+t^2\Big(374 t^4+147ut^3+360u^2t^2+95u^3t+84u^4\Big)s^2+t^3\Big(132t^4+140u t^3+246u^2t^2+115u^3t+64u^4\Big)s$$

$$+ 22t^4\Big(t^2+u t+u^2\Big)^2\Big)H(0,0,2,y)(s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6 \Big(s^4+2ts^3+3\Big(t^2+2u^2\Big)s^2+2\Big(t^3-u^3\Big) s+t^4+u^4$$

$$- 2tu^3+6t^2u^2\Big)H(0,z)H(0,0,2,y)(s+u)^6+108s^2t^2 (s+t)^6u^2(t+u)^6\Big(2s^4+(4t-2u)s^3+\Big(6t^2-3u^2\Big) s^2+\Big(4t^3-2u^3\Big)s$$

$$+ 2\Big(t^2+ut+u^2\Big)^2\Big)H(1,z) H(0,0,2,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6\Big(4s^4+2(4t+u) s^3+3\Big(4t^2+5u^2\Big)s^2+\Big(8t^3-2u^3\Big)s$$

$$+ 4\Big(t^4-u t^3+6u^2t^2-u^3t+u^4\Big)\Big)H(1,z)H(0,0,3,y)(s+u)^6+108s^2t^2 (s+t)^6u^2(t+u)^6\Big(2s^4+2(2t+u)s^3+3\Big(2t^2+u^2\Big)s^2$$

$$+ 2 \Big(2t^3+u^3\Big)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u\Big)H(0,z) H(0,1,0,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+2(2t+u) s^3$$

$$+ 3\Big(2t^2+u^2\Big)s^2+2\Big(2t^3+u^3\Big)s+2t^4+2u^4+2t u^3+3t^2u^2+2t^3u\Big)H(1,z)H(0,1,0,y)(s+u)^6+108s^2t^2(s+t)^6 u^2(t+u)^6\Big(2s^4$$

$$= + (4t-6u)s^3+\Big(6t^2+9u^2\Big)s^2+\Big(4 t^3-2u^3\Big)s+2\Big(t^4+u^4\Big)\Big)H(0,y)H(0,1,0,z) (s+u)^6-216s^2t^2(s+t)^6u^3(t+u)^6\Big(6s^3$$

$$- 6us^2+6u^2s+t \Big(2t^2-3ut+4u^2\Big)\Big)H(1,y)H(0,1,0,z)(s+u)^6-108s^2t^2 (s+t)^6u^2(t+u)^6\Big(2s^4-2(2t+5u)s^3+3\Big(4t^2$$

$$+ 5u^2\Big) s^2-10u^3s+2\Big(t^2+ut+u^2\Big)^2\Big)H(2,y)H(0,1,0,z) (s+u)^6+216s^3t^2(s+t)^6u^2(t+u)^6\Big(4(t+u)s^2-3 \Big(t^2+u^2\Big)s+2\Big(t^3$$

$$+ u^3\Big)\Big)H(3,y)H(0,1,0,z) (s+u)^6-36s^2t^2(s+t)^6u^2(t+u)^2\Big(2\Big(11t^4+32ut^3+42u^2 t^2+27u^3t+9u^4\Big)s^4+\Big(44t^5+115ut^4+95u^2t^3$$

$$+ 9u^3t^2-11 u^4t+4u^5\Big)s^3+6(t+u)^2\Big(11t^4+19ut^3+11u^2t^2+3 u^4\Big)s^2+(t+u)^3\Big(44t^4+8ut^3-9u^2t^2+21u^4\Big)s+22 (t+u)^4\Big(t^2$$

$$+ ut+u^2\Big)^2\Big)H(0,1,1,z)(s+u)^6-108s^2t^2 (s+t)^6u^2(t+u)^6\Big(2s^4+(4t-2u)s^3+\Big(6t^2-3u^2\Big) s^2+\Big(4t^3-2u^3\Big)s+2\Big(t^2+ut$$

$$+ u^2\Big)^2\Big)H(0,y) H(0,1,1,z)(s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+2ts^3+9t^2 s^2+2\Big(t^2+ut+u^2\Big)^2\Big)H(3,y)H(0,1,1,z)(s+u)^6$$

$$= - 36s^2t^2 (s+t)^2u^2(t+u)^6\Big(22s^8+3(44t+7u)s^7+\Big(374t^2+63ut+18 u^2\Big)s^6+\Big(660t^3+54ut^2+36u^2t+4u^3\Big)s^5+\Big(792 t^4+2ut^3$$

$$+ 84u^2t^2-11u^3t+18u^4\Big)s^4+t\Big(660t^4+41u t^3+246u^2t^2+9u^3t+54u^4\Big)s^3+t^2\Big(374t^4+147ut^3+360 u^2t^2+95u^3t+84u^4\Big)s^2$$

$$+ t^3\Big(132t^4+140ut^3+246u^2 t^2+115u^3t+64u^4\Big)s+22t^4\Big(t^2+ut+u^2\Big)^2\Big) H(0,2,0,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4$$

$$+ 12us^3+8 u^3s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u\Big)H(0,z)H(0,2,0,y) (s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6\Big(2s^4+(4t-2u)s^3$$

$$+ \Big(6 t^2-3u^2\Big)s^2+\Big(4t^3-2u^3\Big)s+2\Big(t^2+u t+u^2\Big)^2\Big)H(1,z)H(0,2,0,y)(s+u)^6+36s^2t^2(s+t)^6u^2 (t+u)^2\Big(2\Big(11t^4$$

$$+ 32ut^3+42u^2t^2+27u^3t+9u^4\Big) s^4+\Big(44t^5+115ut^4+95u^2t^3+9u^3t^2-11u^4t+4u^5\Big)s^3+6 (t+u)^2\Big(11t^4+19ut^3$$

$$+ 11u^2t^2+3u^4\Big)s^2+(t+u)^3\Big(44 t^4+8ut^3-9u^2t^2+21u^4\Big)s+22(t+u)^4\Big(t^2+u t+u^2\Big)^2\Big)H(0,2,2,y)(s+u)^6$$

$$+ 108s^2t^2(s+t)^6u^2(t+u)^6 \Big(2s^4+2ts^3+3\Big(t^2+4u^2\Big)s^2+2\Big(t^3-2u^3\Big) s+2\Big(t^2+ut+u^2\Big)^2\Big)H(0,z)H(0,2,2,y)(s+u)^6$$

$$=-216s^2t^2(s+t)^6u^2(t+u)^6(s^4+2(t-u)s^3+3(t^2+u^2)s^2+2(t^3-2u^3)s+t^4+u^4-4tu^3+9t^2u^2-4t^3u)H(1,z)H(0,2,3,y)(s+u)^6$$
$$-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(t+u)s^3+3(t^2+u^2)s^2+2(t^3+u^3)s+2(t^2+ut+u^2)^2)H(1,z)H(0,3,0,y)(s+u)^6$$
$$+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(t+3u)s^3+3(t^2-u^2)s^2+2(t^3+5u^3)s+2(t^2+ut+u^2)^2)H(0,z)H(0,3,2,y)(s+u)^6$$
$$-108s^2t^2(s+t)^6u^2(t+u)^6(4s^4+(6t-2u)s^3+9(t^2+u^2)s^2+6(t^3-u^3)s+4(t^2+ut+u^2)^2)H(1,z)H(0,3,2,y)(s+u)^6$$
$$-108s^3t^2(s+t)^6u^2(t+u)^6(2(t+u)s^2+3(t^2-3u^2)s+2(t^3+3u^3))H(1,z)H(0,3,3,y)(s+u)^6+36s^2t^2(s+t)^2u^2(t+u)^6(44s^8$$
$$+(264t+65u)s^7+(748t^2+203ut+84u^2)s^6+3(440t^3+67ut^2+94u^2t+16u^3)s^5+(1584t^4+43ut^3+444u^2t^2+104u^3t$$
$$+40u^4)s^4+t(1320t^4+43ut^3+492u^2t^2+104u^3t+118u^4)s^3+t^2(748t^4+201ut^3+444u^2t^2+104u^3t+168u^4)s^2$$
$$+t^3(264t^4+203ut^3+282u^2t^2+104u^3t+118u^4)s+t^4(44t^4+65ut^3+84u^2t^2+48u^3t+40u^4))H(1,0,0,y)(s+u)^6$$
$$=+216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4ts^3+(6t^2+9u^2)s^2+2(2t^3+u^3)s+2(t^4+u^4))H(0,z)H(1,0,0,y)(s+u)^6$$
$$-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4ts^3+6t^2s^2+4t^3s+2t^4+2u^4+2tu^3+9t^2u^2)H(1,z)H(1,0,0,y)(s+u)^6$$
$$+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4(t+u)s^3+6(t^2+u^2)s^2+4(t^3+u^3)s+2t^4+2u^4-2tu^3-3t^2u^2$$
$$-2t^3u)H(0,y)H(1,0,0,z)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4-4(t-u)s^3+6(2t^2+u^2)s^2+4u^3s+2t^4+2u^4$$
$$+2tu^3+3t^2u^2+2t^3u)H(2,y)H(1,0,0,z)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4ts^3+6t^2s^2+4t^3s+2t^4+2u^4$$
$$-2tu^3+9t^2u^2-6t^3u)H(0,y)H(1,0,1,z)(s+u)^6+216s^2t^2(s+t)^6u^3(t+u)^6(2s^3-3us^2+4u^2s+6t(t^2-ut$$
$$+u^2))H(1,y)H(1,0,1,z)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(2(t+2u)s^3-3t^2s^2+(4t^3+6u^3)s-tu(2t^2+3ut$$
$$+2u^2))H(2,y)H(1,0,1,z)(s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2ts^3+9t^2s^2+2(t^2+ut+u^2)^2)H(3,y)H(1,0,1,z)(s+u)^6$$
$$=-18s^2t^2(s+t)^3u^2(t+u)^6(8s^7+(28t-6u)s^6+3(11t^2-8ut-5u^2)s^5-(7t^3+42ut^2+81u^2t+12u^3)s^4-2(28t^4+18ut^3$$
$$+105u^2t^2+12u^3t+7u^4)s^3-6t(8t^4+ut^3+37u^2t^2+4u^3t+5u^4)s^2-3(3t^6-4ut^5+21u^2t^4+8u^4t^2)s+3t^4(t^3+2ut^2$$
$$+5u^2t+4u^3))H(1,0,2,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+(4t+6u)s^3+(6t^2-3u^2)s^2+2(2t^3+5u^3)s+2t^4$$
$$+2u^4+10tu^3-3t^2u^2+6t^3u)H(0,z)H(1,0,2,y)(s+u)^6-216s^2t^2(s+t)^6u^3(t+u)^6(2s^3-3us^2+4u^2s+6t(t^2-ut$$
$$+u^2))H(1,z)H(1,0,3,y)(s+u)^6-18s^2t^2(s+t)^2u^2(t+u)^6(3s^8+2(51t+u)s^7+(486t^2+200ut+30u^2)s^6+6(179t^3+130ut^2$$
$$+96u^2t+7u^3)s^5+2(687t^4+677ut^3+945u^2t^2+265u^3t+29u^4)s^4+2t(537t^4+677ut^3+1344u^2t^2+658u^3t+170u^4)s^3$$
$$+2t^2(243t^4+390ut^3+945u^2t^2+658u^3t+270u^4)s^2+2t^3(51t^4+100ut^3+288u^2t^2+265u^3t+170u^4)s+t^4(3t^4+2ut^3$$
$$+30u^2t^2+42u^3t+58u^4))H(1,1,0,y)(s+u)^6+216s^3t^2(s+t)^6u^3(t+u)^6(4s^2-3us+2u^2)H(0,z)H(1,1,0,y)(s+u)^6$$
$$=-216s^2t^3(s+t)^6u^3(t+u)^6(4t^2-3ut+2u^2)H(1,z)H(1,1,0,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(s^4+2(t-u)s^3+3(t^2$$
$$+2u^2)s^2+2t^3s+(t^2+ut+u^2)^2)H(0,y)H(1,1,0,z)(s+u)^6-216s^2t^2(s+t)^6u^3(t+u)^6(6s^3-6us^2+6u^2s+t(2t^2-3ut$$
$$+4u^2))H(1,y)H(1,1,0,z)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(8s^4+4(2t+u)s^3+12(t^2+3u^2)s^2+4(2t^3+u^3)s+8t^4+8u^4$$
$$+14tu^3+21t^2u^2+14t^3u)H(2,y)H(1,1,0,z)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4(t+2u)s^3-3(t^2-u^2)s^2+2(t^3+3u^3)s$$
$$+2(t^2+ut+u^2)^2)H(3,y)H(1,1,0,z)(s+u)^6-18s^2t^2(s+t)^3u^2(t+u)^6(8s^7+(28t-6u)s^6+3(11t^2-8ut-5u^2)s^5-(7t^3$$
$$+42ut^2+81u^2t+12u^3)s^4-2(28t^4+18ut^3+105u^2t^2+12u^3t+7u^4)s^3-6t(8t^4+ut^3+37u^2t^2+4u^3t+5u^4)s^2-3(3t^6$$
$$-4ut^5+21u^2t^4+8u^4t^2)s+3t^4(t^3+2ut^2+5u^2t+4u^3))H(1,2,0,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+(4t-6u)s^3$$
$$+(6t^2+9u^2)s^2+(4t^3-2u^3)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u)H(0,z)H(1,2,0,y)(s+u)^6+216s^2t^2(s+t)^6u^3(t+u)^6(2s^3$$
$$=-3us^2+4u^2s+6t(t^2-ut+u^2))H(1,z)H(1,2,3,y)(s+u)^6-36s^2t^2(s+t)^2u^2(t+u)^6(22s^8+3(44t+7u)s^7+(374t^2$$
$$+63ut+18u^2)s^6+(660t^3+54ut^2+36u^2t+4u^3)s^5+(792t^4+2ut^3+84u^2t^2-11u^3t+18u^4)s^4+t(660t^4+41ut^3$$
$$+246u^2t^2+9u^3t+54u^4)s^3+t^2(374t^4+147ut^3+360u^2t^2+95u^3t+84u^4)s^2+t^3(132t^4+140ut^3+246u^2t^2+115u^3t$$
$$+64u^4)s+22t^4(t^2+ut+u^2)^2)H(2,0,0,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4(t-u)s^3+6(t^2+2u^2)s^2+4t^3s$$
$$+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u)H(0,z)H(2,0,0,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+(4t-2u)s^3+(6t^2-3u^2)s^2$$
$$+(4t^3-2u^3)s+2(t^2+ut+u^2)^2)H(1,z)H(2,0,0,y)(s+u)^6+36s^2t^2(s+t)^6u^2(t+u)^2(2(11t^4+32ut^3+42u^2t^2+27u^3t$$
$$+9u^4)s^4+(44t^5+115ut^4+95u^2t^3+9u^3t^2-11u^4t+4u^5)s^3+6(t+u)^2(11t^4+19ut^3+11u^2t^2+3u^4)s^2+(t+u)^3(44t^4$$
$$+8ut^3-9u^2t^2+21u^4)s+22(t+u)^4(t^2+ut+u^2)^2)H(2,0,2,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(t+4u)s^3$$
$$+3t^2s^2+2(t^3+6u^3)s+2(t^4+u^4))H(0,z)H(2,0,2,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(s^4+(4t-2u)s^3+3u^2s^2$$
$$+(6t^3-4u^3)s+(t^2+ut+u^2)^2)H(1,z)H(2,0,3,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(2t+5u)s^3+(6t^2$$
$$-3u^2)s^2+(4t^3+6u^3)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u)H(0,z)H(2,1,0,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4$$
$$+2(2t+u)s^3+3(2t^2+u^2)s^2+2(2t^3+u^3)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u)H(1,z)H(2,1,0,y)(s+u)^6$$

$$= + 36s^2t^2 (s+t)^6u^2(t+u)^2(2(11t^4+32ut^3+42u^2t^2+27u^3t+9u^4)s^4+(44t^5+115ut^4+95u^2t^3+9u^3t^2-11u^4t$$
$$+4u^5)s^3+6(t+u)^2(11t^4+19ut^3+11u^2t^2+3u^4) s^2+(t+u)^3(44t^4+8ut^3-9u^2t^2+21u^4)s+22(t+u)^4 (t^2+ut$$
$$+u^2)^2)H(2,2,0,y)(s+u)^6-216s^2t^2(s+t)^6u^2 (t+u)^6(s^4-2(t+u)s^3+6(t^2+u^2)s^2+(t^2+ut+u^2)^2)H(0,z)H(2,2,0,y)(s+u)^6$$
$$+432s^2t^2(s+t)^6u^2 (t+u)^6(s^4+3(t^2+u^2)s^2-(t^3+u^3) s+(t^2+ut+u^2)^2)H(1,z)H(2,2,3,y)(s+u)^6$$
$$-108s^2t^2 (s+t)^6u^2(t+u)^6(2s^4+2(t+u)s^3+3(t^2+u^2)s^2+2 (t^3+u^3)s+2(t^2+ut+u^2)^2)H(1,z)H(2,3,0,y) (s+u)^6$$
$$-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(t+u)s^3+3 (t^2+u^2)s^2+2(t^3+u^3)s+2(t^2+ut+u^2)^2)H(0,z)H(2,3,2,y)(s+u)^6$$
$$+432s^2t^2(s+t)^6u^2 (t+u)^6(2s^4+2(t+u)s^3+3(t^2+u^2)s^2+2 (t^3+u^3)s+2(t^2+ut+u^2)^2)H(1,z)H(2,3,3,y) (s+u)^6$$
$$-18s^2t^2(s+t)^6u^2(t+u)^3(2u(12t^2+15ut+7u^2)s^4-12(t^4-2u^2t^2-2u^3t-u^4)s^3-3(t+u)^2 (5t^3-31ut^2-17u^2t$$
$$-5u^3)s^2-6(t+u)^3(t^3-ut^2-u^2 t-u^3)s-(t+u)^3(3t^4-18ut^3-3u^2t^2+4u^3t+8u^4))H(3,0,2,y)(s+u)^6$$
$$=-108s^2t^2(s+t)^6u^2(t+u)^6(2 s^4+2(t-u)s^3+3(t^2+3u^2)s^2+2(t^3-3u^3)s+2 (t^2+ut+u^2)^2)H(0,z)H(3,0,2,y)(s+u)^6$$
$$-216s^2t^2 (s+t)^6u^2(t+u)^6(2s^4+2ts^3+9t^2s^2+2(t^2+ut+u^2)^2)H(1,z)H(3,0,2,y)(s+u)^6-216s^3t^2(s+t)^6u^3 (t+u)^6(2s^2$$
$$-3us+4u^2)H(1,z)H(3,0,3,y)(s+u)^6-18s^2t^2 (s+t)^6u^2(t+u)^3(2u(12t^2+15ut+7u^2)s^4-12 (t^4-2u^2t^2-2u^3t$$
$$-u^4)s^3-3(t+u)^2(5t^3-31ut^2-17u^2t-5u^3)s^2-6(t+u)^3(t^3-ut^2-u^2t-u^3)s-(t+u)^3 (3t^4-18ut^3-3u^2t^2+4u^3t$$
$$+8u^4))H(3,2,0,y) (s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+10(t+u)s^3-3 (t^2+u^2)s^2+6(t^3+u^3)s+2(t^2+ut$$
$$+u^2)^2)H(0,z)H(3,2,0,y)(s+u)^6-216s^2t^2(s+t)^6u^2 (t+u)^6(2s^4+2ts^3+9t^2s^2+2(t^2+ut+u^2)^2) H(1,z)H(3,2,0,y)(s+u)^6$$
$$-36s^2t^2(s+t)^6u^2(t+u)^2(2(20t^4+59ut^3+84u^2t^2+59u^3t+20u^4)s^4+8(6t^5+13ut^4+13u^2t^3+13u^3t^2+13u^4t+6u^5)s^3$$
$$+6(t+u)^2(14t^4+19ut^3+22u^2t^2+19u^3t+14u^4)s^2+(t+u)^3(65t^4+8ut^3-18u^2t^2+8u^3t+65u^4)s+44(t+u)^4(t^2+ut$$
$$+u^2)^2) H(3,2,2,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2us^3+9u^2 s^2+2(t^2+ut+u^2)^2)H(0,z)H(3,2,2,y)(s+u)^6$$
$$=+216s^2 t^2(s+t)^6u^2(t+u)^6(4s^4+2(t+u)s^3+9(t^2+u^2)s^2+4 (t^2+ut+u^2)^2)H(1,z)H(3,2,3,y)(s+u)^6-216s^3t^3 (s+t)^6(2s^2$$
$$-3ts+4t^2)u^2(t+u)^6H(1,z)H(3,3,0,y) (s+u)^6+18s^2t^2(s+t)^6u^2(t+u)^2((58t^4+340ut^3+540u^2t^2+340u^3t+58u^4)s^4+2(21t^5$$
$$+265ut^4+658u^2t^3+658u^3t^2+265u^4t+21u^5)s^3+6(t+u)^2(5t^4+86ut^3+138u^2t^2+86u^3t+5u^4)s^2+2(t+u)^3(t^4+97ut^3$$
$$+96u^2t^2+97u^3t+u^4)s+3(t+u)^4(t^4+30ut^3+36u^2t^2+30u^3t+u^4))H(3,3,2,y)(s+u)^6-216s^3t^2(s+t)^6u^3(t+u)^6 (2s^2$$
$$-3us+4u^2)H(0,z)H(3,3,2,y)(s+u)^6+216s^2t^2(s+t)^6 u^2(t+u)^6(4s^4+2(t+u)s^3+9(t^2+u^2)s^2+4(t^2+ut$$
$$+u^2)^2)H(1,z)H(3,3,2,y)(s+u)^6+216s^2t^2(s+t)^6u^2 (t+u)^6(4s^4+6(t+u)s^3+3(t^2+u^2)s^2+8 (t^3+u^3)s+4(t^2+ut$$
$$+u^2)^2)H(1,z)H(3,3,3,y) (s+u)^6-324s^2t^2(s+t)^6u^3(t+u)^6(2s^3+3us^2+2u^2s-4t (t^2-ut+u^2))H(0,0,0,1,z)(s+u)^6$$
$$=+432s^2t^2(s+t)^6u^2 (t+u)^6(s^4+(2t-u)s^3+3(t^2+u^2)s^2+2t^3s+t^4+u^4+3t^2u^2-t^3u)H(0,0,1,0,y)(s+u)^6$$
$$+432s^2t^2(s+t)^6u^3(t+u)^6 (3s^3+2u^2s-2t^3-3tu^2)H(0,0,1,0,z)(s+u)^6+108s^2t^2 (s+t)^6u^2(t+u)^6(2s^4+(4t-2u)s^3+(6t^2$$
$$-3u^2) s^2+(4t^3-2u^3)s+2(t^2+ut+u^2)^2) H(0,0,1,1,z)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+(4t-2u) s^3+(6t^2-3u^2)s^2$$
$$+(4t^3-2u^3)s+2(t^2+ut+u^2)^2)H(0,0,2,2,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6 (4s^4+2(4t+u)s^3+3(4t^2+5u^2)s^2+(8t^3$$
$$-2u^3)s+4(t^4-ut^3+6u^2t^2-u^3t+u^4)) H(0,0,3,2,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4(t-u) s^3+6(t^2-u^2)s^2+4(t^3$$
$$-u^3)s+2t^4+2u^4+18tu^3-3t^2u^2+18t^3u)H(0,1,0,1,z)(s+u)^6-108s^2t^2(s+t)^6u^2 (t+u)^6(2s^4+2(2t+u)s^3+3(2t^2+u^2)s^2$$
$$+2(2t^3+u^3)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u)H(0,1,0,2,y) (s+u)^6+432s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(2t+u)s^3+3(2t^2$$
$$+u^2)s^2+2(2t^3+u^3)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u)H(0,1,1,0,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6 (2s^4+4(t-u)s^3$$
$$+6(t^2+2u^2)s^2+4t^3s+2t^4+2u^4-10tu^3+3t^2u^2-6t^3u)H(0,1,1,0,z)(s+u)^6-108s^2t^2(s+t)^6u^2 (t+u)^6(2s^4+2(2t+u)s^3$$
$$+3(2t^2+u^2)s^2+2(2t^3+u^3)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u)H(0,1,2,0,y) (s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4$$
$$+(4t-2u)s^3+(6t^2-3u^2)s^2+(4t^3-2u^3)s+2(t^2+ut+u^2)^2)H(0,2,0,2,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6 (s^4$$
$$+2(t-2u)s^3+3(t^2+u^2)s^2+2(t^3-u^3) s+t^4+u^4+4tu^3+6t^3u)H(0,2,1,0,y)(s+u)^6+108s^2t^2(s+t)^6 u^2(t+u)^6(2s^4$$
$$=+(4t-2u)s^3+(6t^2-3u^2)s^2+(4t^3-2u^3)s+2(t^2+ut+u^2)^2)H(0,2,2,0,y) (s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(s^4$$
$$+2(t-u)s^3+3 (t^2+u^2)s^2+2(t^3-2u^3)s+t^4+u^4-4tu^3+9t^2u^2-4t^3u)H(0,2,3,2,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6 (2s^4$$
$$+2(t+u)s^3+3(t^2+u^2)s^2+2(t^3+u^3) s+2(t^2+ut+u^2)^2)H(0,3,0,2,y)(s+u)^6-216s^2t^2 (s+t)^6u^2(t+u)^6(2s^4+2(t+u)s^3$$
$$+3(t^2+u^2)s^2+2 (t^3+u^3)s+2(t^2+ut+u^2)^2)H(0,3,2,0,y) (s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(4s^4+(6t-2u)s^3+9 (t^2$$
$$+u^2)s^2+6(t^3-u^3)s+4(t^2+ut+u^2)^2)H(0,3,2,2,y)(s+u)^6-108s^3t^2(s+t)^6u^2(t+u)^6 (2(t+u)s^2+3(t^2-3u^2)s+2(t^3$$
$$+3u^3)) H(0,3,3,2,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+(4t-6u) s^3+3(2t^2+u^2)s^2+2(2t^3-5u^3)s+2(t^4-2ut^3+6u^2t^2$$
$$+u^4))H(1,0,0,1,z)(s+u)^6-216s^2t^2(s+t)^6 u^2(t+u)^6(2s^4+4ts^3+6t^2s^2+4t^3s+2t^4+2u^4+2tu^3+9t^2u^2)H(1,0,0,2,y)(s+u)^6$$
$$+216s^2t^2(s+t)^6u^2(t+u)^6(4s^4+8ts^3+3(4t^2+3u^2)s^2+2(4t^3+u^3)s+4t^4+4u^4+2tu^3+9t^2u^2)H(1,0,1,0,y)(s+u)^6$$

$$= + 108s^2t^2(s+t)^6 u^2(t+u)^6 (2s^4+2(2t+9u)s^3+(6t^2-3u^2)s^2+2(2t^3+9u^3)s+2(t^4-2ut^3-3u^2t^2-2u^3 t$$
$$+u^4)) H(1,0,1,0,z)(s+u)^6 - 216s^2t^2(s+t)^6u^2(t+u)^6 (2s^4+4ts^3+6t^2s^2+4t^3s+2t^4+2u^4+2tu^3+9t^2u^2) H(1,0,2,0,y)(s+u)^6$$
$$-216s^2t^2(s+t)^6u^3(t+u)^6(2s^3-3us^2+4u^2s+6t(t^2-ut+u^2))H(1,0,3,2,y)(s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6(4s^4+8ts^3$$
$$+3(4t^2+3u^2)s^2+2(4t^3+u^3)s+4t^4+4u^4+2tu^3+9t^2u^2)H(1,1,0,0,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4(t+u)s^3$$
$$+6(t^2+u^2)s^2+4(t^3+u^3)s+2t^4+2u^4-2tu^3-3t^2u^2-2t^3u)H(1,1,0,0,z)(s+u)^6-432s^2t^2(s+t)^6u^3(t+u)^6(2s^3$$
$$+3u^2s-3t^3-2tu^2)H(1,1,0,1,z)(s+u)^6-216s^2t^3(s+t)^6u^3(t+u)^6(4t^2-3ut+2u^2)H(1,1,0,2,y)(s+u)^6$$
$$+216s^2t^2(s+t)^6u^2(t+u)^6(4s^4+8(t+u)s^3+3(4t^2+u^2)s^2+(8t^3+6u^3)s+4t^4+4u^4+6tu^3+3t^2u^2+8t^3u)H(1,1,1,0,y)(s+u)^6$$
$$+324s^2t^2(s+t)^6u^3(t+u)^6(4s^3-4us^2+4u^2s-t(2t^2+3ut+2u^2))H(1,1,1,0,z)(s+u)^6-216s^2t^3(s+t)^6u^3(t+u)^6(4t^2-3ut$$
$$+2u^2)H(1,1,2,0,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+4ts^3+6t^2s^2+4t^3s+2t^4+2u^4+2tu^3+9t^2u^2)H(1,2,0,0,y)(s+u)^6$$
$$= -216s^3t^2(s+t)^6u^3(t+u)^6(4s^2-3us+2u^2)H(1,2,1,0,y)(s+u)^6+216s^2t^2(s+t)^6u^3(t+u)^6(2s^3-3us^2+4u^2s+6t(t^2$$
$$-ut+u^2))H(1,2,3,2,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+(4t-2u)s^3+(6t^2-3u^2)s^2+(4t^3-2u^3)s+2(t^2$$
$$+ut+u^2)^2)H(2,0,0,2,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(s^4-4(t+u)s^3+3(3t^2+u^2)s^2-2(2t^3+u^3)s+(t^2$$
$$+ut+u^2)^2)H(2,0,1,0,y)(s+u)^6+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+(4t-2u)s^3+(6t^2-3u^2)s^2+(4t^3-2u^3)s+2(t^2$$
$$+ut+u^2)^2)H(2,0,2,0,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(s^4+(4t-2u)s^3+3u^2s^2+(6t^3-4u^3)s+(t^2+ut$$
$$+u^2)^2)H(2,0,3,2,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(4s^4+(8t-6u)s^3+3(4t^2+3u^2)s^2+(8t^3-2u^3)s+4t^4+4u^4+6tu^3$$
$$+9t^2u^2+6t^3u)H(2,1,0,0,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(2t+u)s^3+3(2t^2+u^2)s^2+2(2t^3+u^3)s+2t^4+2u^4+2tu^3$$
$$+3t^2u^2+2t^3u)H(2,1,0,2,y)(s+u)^6-108s^2t^2(s+t)^6u^3(t+u)^6(6s^3-9us^2+2u^2s+t(2t^2+3ut+2u^2))H(2,1,1,0,y)(s+u)^6$$
$$-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(2t+u)s^3+3(2t^2+u^2)s^2+2(2t^3+u^3)s+2t^4+2u^4+2tu^3+3t^2u^2+2t^3u)H(2,1,2,0,y)(s+u)^6$$
$$+108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+(4t-2u)s^3+(6t^2-3u^2)s^2+(4t^3-2u^3)s+2(t^2+ut+u^2)^2)H(2,2,0,0,y)(s+u)^6$$
$$= -108s^2t^2(s+t)^6u^2(t+u)^6(4s^4-2(2t+u)s^3+3(8t^2+5u^2)s^2+(2u^3-4t^3)s+4(t^2+ut+u^2)^2)H(2,2,1,0,y)(s+u)^6$$
$$+432s^2t^2(s+t)^6u^2(t+u)^6(s^4+3(t^2+u^2)s^2-(t^3+u^3)s+(t^2+ut+u^2)^2)H(2,2,3,2,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4$$
$$+2(t+u)s^3+3(t^2+u^2)s^2+2(t^3+u^3)s+2(t^2+ut+u^2)^2)H(2,3,0,2,y)(s+u)^6-108s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(t+u)s^3$$
$$+3(t^2+u^2)s^2+2(t^3+u^3)s+2(t^2+ut+u^2)^2)H(2,3,2,0,y)(s+u)^6+432s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2(t+u)s^3+3(t^2$$
$$+u^2)s^2+2(t^3+u^3)s+2(t^2+ut+u^2)^2)H(2,3,3,2,y)(s+u)^6+216s^3t^2(s+t)^6u^2(t+u)^6((6t+4u)s^2-3(2t^2+u^2)s$$
$$+2(3t^3+u^3))H(3,0,1,0,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2ts^3+9t^2s^2+2(t^2+ut+u^2)^2)H(3,0,2,2,y)(s+u)^6$$
$$= -216s^3t^2(s+t)^6u^3(t+u)^6(2s^2-3us+4u^2)H(3,0,3,2,y)(s+u)^6-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2ts^3+9t^2s^2+2(t^2+ut$$
$$+u^2)^2)H(3,2,0,2,y)(s+u)^6-216s^3t^2(s+t)^6u^2(t+u)^6((6t+4u)s^2-3(2t^2+u^2)s+2(3t^3+u^3))H(3,2,1,0,y)(s+u)^6$$
$$-216s^2t^2(s+t)^6u^2(t+u)^6(2s^4+2ts^3+9t^2s^2+2(t^2+ut+u^2)^2)H(3,2,2,0,y)(s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6(4s^4+2(t+u)s^3$$
$$+9(t^2+u^2)s^2+4(t^2+ut+u^2)^2)H(3,2,3,2,y)(s+u)^6-216s^3t^3(s+t)^6(2s^2-3ts+4t^2)u^2(t+u)^6H(3,3,0,2,y)(s+u)^6$$
$$-216s^3t^3(s+t)^6(2s^2-3ts+4t^2)u^2(t+u)^6H(3,3,2,0,y)(s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6(4s^4+2(t+u)s^3+9(t^2+u^2)s^2$$
$$+4(t^2+ut+u^2)^2)H(3,3,2,2,y)(s+u)^6+216s^2t^2(s+t)^6u^2(t+u)^6(4s^4+6(t+u)s^3+3(t^2+u^2)s^2+8(t^3+u^3)s+4(t^2$$
$$+ut+u^2)^2)H(3,3,3,2,y)(s+u)^6-s^2t^2(s+t)^3u^2(t+u)^3(7948(t+u)^3s^{10}+(39359t^4+155744ut^3+236082u^2t^2+155744u^3t$$
$$+39359u^4)s^9+(93651t^5+434331ut^4+850587u^2t^3+850587u^3t^2+434331u^4t+93651u^5)s^8+3(46598t^6+240304ut^5+550627u^2t^4$$
$$+717154u^3t^3+550627u^4t^2+240304u^5t+46598u^6)s^7+(139794t^7+836962ut^6+2146599u^2t^5+3271156u^3t^4+3271156u^4t^3$$
$$+2146599u^5t^2+836962u^6t+139794u^7)s^6+3(31217t^8+240304ut^7+715533u^2t^6+1216094u^3t^5+1422608u^4t^4+1216094u^5t^3$$
$$+715533u^6t^2+240304u^7t+31217u^8)s^5+(39359t^9+434331ut^8+1651881u^2t^7+3271156u^3t^6+4267824u^4t^5+4267824u^5t^4$$
$$= +3271156u^6t^3+1651881u^7t^2+434331u^8t+39359u^9)s^4+(7948t^{10}+155744ut^9+850587u^2t^8+2151462u^3t^7+3271156u^4t^6$$
$$+3648282u^5t^5+3271156u^6t^4+2151462u^7t^3+850587u^8t^2+155744u^9t+7948u^{10})s^3+3tu(7948t^9+78694ut^8+283529u^2t^7$$
$$+550627u^3t^6+715533u^4t^5+715533u^5t^4+550627u^6t^3+283529u^7t^2+78694u^8t+7948u^9)s^2+t^2u^2(t+u)^2(23844t^6+108056ut^5$$
$$+194375u^2t^4+224106u^3t^3+194375u^4t^2+108056u^5t+23844u^6)s+t^3u^3(t+u)^3(7948t^4+15515ut^3+23262u^2t^2+15515u^3t$$
$$+7948u^4))(s+u)^3+18s^2t^2u^2((11t^6+162ut^5+153u^2t^4+884u^3t^3+255u^4t^2+246u^5t+33u^6)s^{13}+(62t^7+1339ut^6$$
$$+2160u^2t^5+8525u^3t^4+7390u^4t^3+3807u^5t^2+1728u^6t+189u^7)s^{12}+6(26t^8+774ut^7+1639u^2t^6+5548u^3t^5+7678u^4t^4$$
$$+4900u^5t^3+2621u^6t^2+826u^7t+84u^8)s^{11}+2(140t^9+4728ut^8+12480u^2t^7+37199u^3t^6+69840u^4t^5+60090u^5t^4+36224u^6t^3$$
$$+17091u^7t^2+4188u^8t+404u^9)s^{10}+(534t^{10}+13578ut^9+45012u^2t^8+119424u^3t^7+270061u^4t^6+311178u^5t^5+220301u^6t^4$$
$$+125964u^7t^3+47997u^8t^2+9560u^9t+839u^{10})s^9+3(340t^{11}+5526ut^{10}+22370u^2t^9+55138u^3t^8+129254u^4t^7+190253u^5t^6$$

$$= + 166896u^6t^5 + 109793u^7t^4 + 53466u^8t^3 + 15573u^9 t^2 + 2466u^{10}t + 189u^{11}\Big)s^8 + 2\Big(736t^{12} + 9471u t^{11} + 44292u^2t^{10} + 110988u^3t^9$$

$$+ 235059u^4t^8 + 387831u^5t^7 + 413404 u^6t^6 + 317445u^7t^5 + 187035u^8t^4 + 72214u^9t^3 + 15774u^{10}t^2 + 1827 u^{11}t + 116u^{12}\Big)s^7$$

$$+ 2\Big(708t^{13} + 9054ut^{12} + 47823u^2 t^{11} + 134712u^3t^{10} + 269098u^4t^9 + 435561u^5t^8 + 508839u^6t^7 + 430834 u^7t^6 + 288621u^8t^5$$

$$+ 141397u^9t^4 + 43186u^{10}t^3 + 7236u^{11}t^2 + 541 u^{12}t + 22u^{13}\Big)s^6 + 3t\Big(285t^{13} + 4132ut^{12} + 24795u^2 t^{11} + 81510u^3t^{10} + 175113u^4t^9$$

$$+ 286188u^5t^8 + 351323u^6t^7 + 311558 u^7t^6 + 211822u^8t^5 + 112744u^9t^4 + 42788u^{10}t^3 + 10552u^{11}t^2 + 1454 u^{12}t + 56u^{13}\Big)s^5$$

$$+ t^2\Big(294t^{13} + 5499ut^{12} + 38730u^2 t^{11} + 148723u^3t^{10} + 362736u^4t^9 + 642597u^5t^8 + 864762u^6t^7 + 853029 u^7t^6 + 614934u^8t^5$$

$$+ 328490u^9t^4 + 124788u^{10}t^3 + 32430u^{11} t^2 + 6072u^{12}t + 708u^{13}\Big)s^4 + 2t^3\Big(22t^{13} + 707u t^{12} + 6399u^2t^{11} + 29412u^3t^{10} + 82842u^4t^9$$

$$+ 162087u^5t^8 + 239183 u^6t^7 + 267073u^7t^6 + 222465u^8t^5 + 137118u^9t^4 + 59592u^{10}t^3 + 16485 u^{11}t^2 + 2553u^{12}t + 174u^{13}\Big)s^3$$

$$+ 6t^4u(t+u)^2\Big(26 t^{10} + 335ut^9 + 1643u^2t^8 + 4383u^3t^7 + 7465u^4t^6 + 9430u^5t^5 + 8350 u^6t^4 + 5555u^7t^3 + 2649u^8t^2 + 809u^9t + 127u^{10}\Big)s^2$$

$$= + t^5u^2 (t+u)^3\Big(162t^8 + 1164ut^7 + 3573u^2t^6 + 6255u^3t^5 + 7589u^4t^4 + 5865 u^5t^3 + 3192u^6t^2 + 1112u^7t + 216u^8\Big)s + t^6u^3(t+u)^6\Big(58 t^4$$

$$+ 100ut^3 + 153u^2t^2 + 100u^3t + 58u^4\Big)\Big)H(2,y)H(1,0,z) (s+u)^3 + st(s+t)^2u^2(t+u)^2\Big(1531t(t+u)^4s^{13} + 3\Big(3029 t^6 + 15191ut^5$$

$$+ 28746u^2t^4 + 30440u^3t^3 + 14567u^4t^2 + 3351u^5t + 120 u^6\Big)s^{12} + \Big(25433t^7 + 151345ut^6 + 347338u^2t^5 + 471508u^3 t^4 + 364546u^4t^3$$

$$+ 144937u^5t^2 + 32273u^6t + 1800u^7\Big) s^{11} + \Big(44445t^8 + 300713ut^7 + 814816u^2t^6 + 1300511u^3t^5 + 1362455 u^4t^4 + 833113u^5t^3$$

$$+ 303956u^6t^2 + 65055u^7t + 3960u^8\Big) s^{10} + \Big(53136t^9 + 409309ut^8 + 1281415u^2t^7 + 2291071u^3t^6 + 2894942 u^4t^5 + 2471851u^5t^4$$

$$+ 1320295u^6t^3 + 459769u^7t^2 + 87252u^8t + 5040 u^9\Big)s^9 + \Big(44445t^{10} + 409309ut^9 + 1473770u^2t^8 + 2915618u^3 t^7 + 4102638u^4t^6$$

$$+ 4464012u^5t^5 + 3318386u^6t^4 + 1630106u^7t^3 + 508285 u^8t^2 + 77799u^9t + 3960u^{10}\Big)s^8 + \Big(25433t^{11} + 300713u t^{10} + 1281415u^2t^9$$

$$+ 2915618u^3t^8 + 4495118u^4t^7 + 5664674u^5 t^6 + 5440250u^6t^5 + 3552410u^7t^4 + 1539427u^8t^3 + 391013u^9t^2 + 44585 u^{10}t + 1800u^{11}\Big)s^7$$

$$+ \Big(9087t^{12} + 151345ut^{11} + 814816u^2 t^{10} + 2291071u^3t^9 + 4102638u^4t^8 + 5664674u^5t^7 + 6338504u^6 t^6 + 5147724u^7t^5 + 2865079u^8t^4$$

$$+ 1027309u^9t^3 + 196900u^{10}t^2 + 15021 u^{11}t + 360u^{12}\Big)s^6 + t\Big(1531t^{12} + 45573ut^{11} + 347338u^2 t^{10} + 1300511u^3t^9 + 2894942u^4t^8$$

$$+ 4464012u^5t^7 + 5440250u^6 t^6 + 5147724u^7t^5 + 3498698u^8t^4 + 1616591u^9t^3 + 448090u^{10}t^2 + 58725 u^{11}t + 2287u^{12}\Big)s^5 + t^2u\Big(6124t^{11}$$

$$+ 86238u t^{10} + 471508u^2t^9 + 1362455u^3t^8 + 2471851u^4t^7 + 3318386u^5 t^6 + 3552410u^6t^5 + 2865079u^7t^4 + 1616591u^8t^3 + 581902u^9t^2$$

$$= + 110862 u^{10}t + 7726u^{11}\Big)s^4 + t^3u^2\Big(9186t^{10} + 91320ut^9 + 364546 u^2t^8 + 833113u^3t^7 + 1320295u^4t^6 + 1630106u^5t^5 + 1539427u^6 t^4$$

$$+ 1027309u^7t^3 + 448090u^8t^2 + 110862u^9t + 11166u^{10}\Big)s^3 + t^4 u^3(t+u)^2\Big(6124t^7 + 31453ut^6 + 75907u^2t^5 + 120689u^3t^4 + 142484 u^4t^3$$

$$+ 102628u^5t^2 + 43273u^6t + 7726u^7\Big)s^2 + t^5u^4(t+u)^4 \Big(1531t^4 + 3929ut^3 + 7371u^2t^2 + 5873u^3t + 2287u^4\Big)s + 360t^6 u^6(t+u)^4\Big(t^2$$

$$+ ut + u^2\Big)\Big)H(0,y)(s+u)^2 + st^2(s+t)^2u (t+u)^2\Big(1531u(t+u)^4s^{13} + 3\Big(120t^6 + 3351ut^5 + 14567u^2 t^4 + 30440u^3t^3 + 28746u^4t^2$$

$$+ 15191u^5t + 3029u^6\Big) s^{12} + \Big(1800t^7 + 32273ut^6 + 144937u^2t^5 + 364546u^3t^4 + 471508u^4 t^3 + 347338u^5t^2 + 151345u^6t + 25433u^7\Big)s^{11}$$

$$+ \Big(3960t^8 + 65055 ut^7 + 303956u^2t^6 + 833113u^3t^5 + 1362455u^4t^4 + 1300511u^5t^3 + 814816 u^6t^2 + 300713u^7t + 44445u^8\Big)s^{10}$$

$$+ \Big(5040t^9 + 87252u t^8 + 459769u^2t^7 + 1320295u^3t^6 + 2471851u^4t^5 + 2894942u^5t^4 + 2291071 u^6t^3 + 1281415u^7t^2 + 409309u^8t$$

$$+ 53136u^9\Big)s^9 + \Big(3960 t^{10} + 77799ut^9 + 508285u^2t^8 + 1630106u^3t^7 + 3318386u^4t^6 + 4464012 u^5t^5 + 4102638u^6t^4 + 2915618u^7t^3$$

$$= + 1473770u^8t^2 + 409309u^9t + 44445 u^{10}\Big)s^8 + \Big(1800t^{11} + 44585ut^{10} + 391013u^2t^9 + 1539427 u^3t^8 + 3552410u^4t^7 + 5440250u^5t^6$$

$$+ 5664674u^6t^5 + 4495118u^7 t^4 + 2915618u^8t^3 + 1281415u^9t^2 + 300713u^{10}t + 25433u^{11}\Big) s^7 + \Big(360t^{12} + 15021ut^{11} + 196900u^2t^{10}$$

$$+ 1027309u^3 t^9 + 2865079u^4t^8 + 5147724u^5t^7 + 6338504u^6t^6 + 5664674u^7t^5 + 4102638 u^8t^4 + 2291071u^9t^3 + 814816u^{10}t^2$$

$$+ 151345u^{11}t + 9087u^{12}\Big) s^6 + u\Big(2287t^{12} + 58725ut^{11} + 448090u^2t^{10} + 1616591u^3 t^9 + 3498698u^4t^8 + 5147724u^5t^7 + 5440250u^6t^6$$

$$+ 4464012u^7t^5 + 2894942 u^8t^4 + 1300511u^9t^3 + 347338u^{10}t^2 + 45573u^{11}t + 1531u^{12}\Big) s^5 + tu^2\Big(7726t^{11} + 110862ut^{10} + 581902u^2t^9$$

$$+ 1616591u^3 t^8 + 2865079u^4t^7 + 3552410u^5t^6 + 3318386u^6t^5 + 2471851u^7t^4 + 1362455 u^8t^3 + 471508u^9t^2 + 86238u^{10}t + 6124u^{11}\Big)s^4$$

$$+ t^2u^3 \Big(11166t^{10} + 110862ut^9 + 448090u^2t^8 + 1027309u^3t^7 + 1539427u^4 t^6 + 1630106u^5t^5 + 1320295u^6t^4 + 833113u^7t^3 + 364546u^8t^2$$

$$+ 91320u^9 t + 9186u^{10}\Big)s^3 + t^3u^4(t+u)^2\Big(7726t^7 + 43273ut^6 + 102628 u^2t^5 + 142484u^3t^4 + 120689u^4t^3 + 75907u^5t^2 + 31453u^6t$$

$$+ 6124 u^7\Big)s^2 + t^4u^5(t+u)^4\Big(2287t^4 + 5873ut^3 + 7371u^2t^2 + 3929 u^3t + 1531u^4\Big)s + 360t^6u^6(t+u)^4\Big(t^2 + ut + u^2\Big)\Big) H(0,z)(s+u)^2$$

$$= + 3t^2(s+t)^2u^2(t+u)\Big(259(t+u)^5s^{14} + 36\Big(43 t^6 + 242ut^5 + 651u^2t^4 + 764u^3t^3 + 651u^4t^2 + 242u^5t + 43u^6\Big) s^{13} + \Big(4367t^7 + 26163ut^6$$

$$+ 84905u^2t^5 + 124733u^3t^4 + 124733u^4 t^3 + 84905u^5t^2 + 26163u^6t + 4367u^7\Big)s^{12} + 4\Big(1920t^8 + 12056 ut^7 + 43065u^2t^6 + 78684u^3t^5$$

$$+ 86470u^4t^4 + 78684u^5t^3 + 43065u^6 t^2 + 12056u^7t + 1920u^8\Big)s^{11} + 6\Big(1534t^9 + 10415ut^8 + 38108 u^2t^7 + 83108u^3t^6 + 101419u^4t^5$$

$$+ 101419u^5t^4 + 83108u^6t^3 + 38108u^7 t^2 + 10415u^8t + 1534u^9\Big)s^{10} + 12\Big(640t^{10} + 4992ut^9 + 18355 u^2t^8 + 44658u^3t^7 + 62385u^4t^6$$

$$+ 60940u^5t^5 + 62385u^6t^4 + 44658u^7 t^3 + 18355u^8t^2 + 4992u^9t + 640u^{10}\Big)s^9 + \Big(4367t^{11} + 41875u t^{10} + 163877u^2t^9 + 417437u^3t^8$$

$$+ 677572u^4t^7 + 645896u^5t^6 + 645896 u^6t^5 + 677572u^7t^4 + 417437u^8t^3 + 163877u^9t^2 + 41875u^{10}t + 4367 u^{11}\Big)s^8 + 2\Big(774t^{12}$$

$$+ 9924ut^{11} + 46320u^2t^{10} + 124333 u^3t^9 + 232260u^4t^8 + 249279u^5t^7 + 198284u^6t^6 + 249279u^7t^5 + 232260 u^8t^4 + 124333u^9t^3$$

$$= + 46320u^{10}t^2 + 9924u^{11}t + 774u^{12}\Big)\ s^7 + \Big(259t^{13} + 5481ut^{12} + 35278u^2t^{11} + 110676u^3t^{10} + 240564\ u^4t^9 + 349066u^5t^8 + 300068u^6t^7$$

$$+ 300068u^7t^6 + 349066u^8t^5 + 240564\ u^9t^4 + 110676u^{10}t^3 + 35278u^{11}t^2 + 5481u^{12}t + 259u^{13}\Big)\ s^6 + 2tu\Big(308t^{12} + 3564ut^{11} + 15365u^2t^{10}$$

$$+ 42591u^3\ t^9 + 94706u^4t^8 + 136033u^5t^7 + 141098u^6t^6 + 136033u^7t^5 + 94706u^8\ t^4 + 42591u^9t^3 + 15365u^{10}t^2 + 3564u^{11}t + 308u^{12}\Big)s^5$$

$$+ t^2\ u^2\Big(426t^{11} + 3380ut^{10} + 15835u^2t^9 + 65921u^3t^8 + 171209u^4\ t^7 + 259557u^5t^6 + 259557u^6t^5 + 171209u^7t^4 + 65921u^8t^3 + 15835u^9\ t^2$$

$$+ 3380u^{10}t + 426u^{11}\Big)s^4 - 2t^3u^3(t + u)^2\Big(34t^8 - 527u\ t^7 - 6013u^2t^6 - 19815u^3t^5 - 26374u^4t^4 - 19815u^5t^3 - 6013u^6t^2 - 527\ u^7t$$

$$+ 34u^8\Big)s^3 + t^4u^4(t + u)^3\Big(109t^6 + 2242ut^5 + 8867u^2\ t^4 + 13216u^3t^3 + 8867u^4t^2 + 2242u^5t + 109u^6\Big)s^2 + 30t^5u^5\ (t + u)^6\Big(10t^2$$

$$+ 23ut + 10u^2\Big)s + 120t^6u^6(t + u)^5\Big(t^2 + u\ t + u^2\Big)\Big)H(0, y)H(0, z)(s + u)^2 - s^2t(s + t)^2u(t + u)^2\ \Big(\Big(360t^6 + 2287ut^5 + 7726u^2t^4$$

$$+ 11166u^3t^3 + 7726u^4t^2 + 2287\ u^5t + 360u^6\Big)s^{12} + 9\Big(200t^7 + 1669ut^6 + 6525u^2t^5 + 12318\ u^3t^4 + 12318u^4t^3 + 6525u^5t^2 + 1669u^6t$$

$$+ 200u^7\Big)\ s^{11} + \Big(3960t^8 + 44585ut^7 + 196900u^2t^6 + 448090u^3t^5 + 581902u^4\ t^4 + 448090u^5t^3 + 196900u^6t^2 + 44585u^7t + 3960u^8\Big)\ s^{10}$$

$$= + \Big(5040t^9 + 77799ut^8 + 391013u^2t^7 + 1027309u^3t^6 + 1616591u^4\ t^5 + 1616591u^5t^4 + 1027309u^6t^3 + 391013u^7t^2 + 77799u^8t$$

$$+ 5040\ u^9\Big)s^9 + \Big(3960t^{10} + 87252ut^9 + 508285u^2t^8 + 1539427u^3\ t^7 + 2865079u^4t^6 + 3498698u^5t^5 + 2865079u^6t^4 + 1539427u^7t^3$$

$$+ 508285\ u^8t^2 + 87252u^9t + 3960u^{10}\Big)s^8 + \Big(1800t^{11} + 65055u\ t^{10} + 459769u^2t^9 + 1630106u^3t^8 + 3552410u^4t^7 + 5147724u^5\ t^6$$

$$+ 5147724u^6t^5 + 3552410u^7t^4 + 1630106u^8t^3 + 459769u^9t^2 + 65055\ u^{10}t + 1800u^{11}\Big)s^7 + \Big(360t^{12} + 32273ut^{11} + 303956u^2\ t^{10}$$

$$+ 1320295u^3t^9 + 3318386u^4t^8 + 5440250u^5t^7 + 6338504u^6\ t^6 + 5440250u^7t^5 + 3318386u^8t^4 + 1320295u^9t^3 + 303956u^{10}t^2 + 32273\ u^{11}t$$

$$+ 360u^{12}\Big)s^6 + tu\Big(10053t^{11} + 144937ut^{10} + 833113\ u^2t^9 + 2471851u^3t^8 + 4464012u^4t^7 + 5664674u^5t^6 + 5664674u^6\ t^5 + 4464012u^7t^4$$

$$+ 2471851u^8t^3 + 833113u^9t^2 + 144937u^{10}t + 10053\ u^{11}\Big)s^5 + tu\Big(1531t^{12} + 43701ut^{11} + 364546u^2\ t^{10} + 1362455u^3t^9 + 2894942u^4t^8$$

$$+ 4102638u^5t^7 + 4495118u^6\ t^6 + 4102638u^7t^5 + 2894942u^8t^4 + 1362455u^9t^3 + 364546u^{10}t^2 + 43701\ u^{11}t + 1531u^{12}\Big)s^4 + t^2u^2\Big(6124t^{11}$$

$$+ 91320u\ t^{10} + 471508u^2t^9 + 1300511u^3t^8 + 2291071u^4t^7 + 2915618u^5\ t^6 + 2915618u^6t^5 + 2291071u^7t^4 + 1300511u^8t^3 + 471508u^9t^2$$

$$+ 91320\ u^{10}t + 6124u^{11}\Big)s^3 + t^3u^3(t + u)^2\Big(9186t^8 + 67866u\ t^7 + 202420u^2t^6 + 342110u^3t^5 + 394775u^4t^4 + 342110u^5t^3 + 202420u^6\ t^2$$

$$+ 67866u^7t + 9186u^8\Big)s^2 + t^4u^4(t + u)^3\Big(6124t^6 + 27201u\ t^5 + 51370u^2t^4 + 58876u^3t^3 + 51370u^4t^2 + 27201u^5t + 6124u^6\Big)\ s$$

$$+ t^5u^5(t + u)^4\Big(1531t^4 + 2963ut^3 + 4395u^2t^2 + 2963u^3t + 1531\ u^4\Big)\Big)H(1, z)(s + u)^2 - s^2t(s + t)^2u(t + u)^2\Big(\Big(360\ t^6 + 2287ut^5$$

$$= + 7726u^2t^4 + 11166u^3t^3 + 7726u^4t^2 + 2287u^5t + 360\ u^6\Big)s^{12} + 9\Big(200t^7 + 1669ut^6 + 6525u^2t^5 + 12318u^3\ t^4 + 12318u^4t^3 + 6525u^5t^2$$

$$+ 1669u^6t + 200u^7\Big)s^{11} + \Big(3960\ t^8 + 44585ut^7 + 196900u^2t^6 + 448090u^3t^5 + 581902u^4t^4 + 448090u^5\ t^3 + 196900u^6t^2 + 44585u^7t$$

$$+ 3960u^8\Big)s^{10} + \Big(5040t^9 + 77799u\ t^8 + 391013u^2t^7 + 1027309u^3t^6 + 1616591u^4t^5 + 1616591u^5t^4 + 1027309\ u^6t^3 + 391013u^7t^2$$

$$+ 77799u^8t + 5040u^9\Big)s^9 + \Big(3960\ t^{10} + 87252ut^9 + 508285u^2t^8 + 1539427u^3t^7 + 2865079u^4t^6 + 3498698\ u^5t^5 + 2865079u^6t^4$$

$$+ 1539427u^7t^3 + 508285u^8t^2 + 87252u^9t + 3960\ u^{10}\Big)s^8 + \Big(1800t^{11} + 65055ut^{10} + 459769u^2t^9 + 1630106\ u^3t^8 + 3552410u^4t^7$$

$$+ 5147724u^5t^6 + 5147724u^6t^5 + 3552410u^7\ t^4 + 1630106u^8t^3 + 459769u^9t^2 + 65055u^{10}t + 1800u^{11}\Big)\ s^7 + \Big(360t^{12} + 32273ut^{11}$$

$$+ 303956u^2t^{10} + 1320295u^3\ t^9 + 3318386u^4t^8 + 5440250u^5t^7 + 6338504u^6t^6 + 5440250u^7t^5 + 3318386\ u^8t^4 + 1320295u^9t^3$$

$$+ 303956u^{10}t^2 + 32273u^{11}t + 360u^{12}\Big)\ s^6 + tu\Big(10053t^{11} + 144937ut^{10} + 833113u^2t^9 + 2471851u^3\ t^8 + 4464012u^4t^7 + 5664674u^5t^6$$

$$+ 5664674u^6t^5 + 4464012u^7t^4 + 2471851\ u^8t^3 + 833113u^9t^2 + 144937u^{10}t + 10053u^{11}\Big)s^5 + tu\ \Big(1531t^{12} + 43701ut^{11} + 364546u^2t^{10}$$

$$+ 1362455u^3t^9 + 2894942\ u^4t^8 + 4102638u^5t^7 + 4495118u^6t^6 + 4102638u^7t^5 + 2894942u^8\ t^4 + 1362455u^9t^3 + 364546u^{10}t^2$$

$$+ 43701u^{11}t + 1531u^{12}\Big)\ s^4 + t^2u^2\Big(6124t^{11} + 91320ut^{10} + 471508u^2t^9 + 1300511u^3\ t^8 + 2291071u^4t^7 + 2915618u^5t^6 + 2915618u^6t^5$$

$$+ 2291071u^7t^4 + 1300511\ u^8t^3 + 471508u^9t^2 + 91320u^{10}t + 6124u^{11}\Big)s^3 + t^3u^3(t + u)^2\ \Big(9186t^8 + 67866ut^7 + 202420u^2t^6$$

$$+ 342110u^3t^5 + 394775u^4\ t^4 + 342110u^5t^3 + 202420u^6t^2 + 67866u^7t + 9186u^8\Big)s^2 + t^4u^4\ (t + u)^3\Big(6124t^6 + 27201ut^5 + 51370u^2t^4$$

$$+ 58876u^3t^3 + 51370u^4\ t^2 + 27201u^5t + 6124u^6\Big)s + t^5u^5(t + u)^4\Big(1531t^4 + 2963u\ t^3 + 4395u^2t^2 + 2963u^3t + 1531u^4\Big)\Big)H(2, y)(s + u)^2$$

$$= - 3s^2t^2\ (t + u)^2\Big(\Big(120t^6 + 300ut^5 + 109u^2t^4 - 68u^3t^3 + 426u^4t^2 + 616\ u^5t + 259u^6\Big)s^{14} + 2\Big(420t^7 + 1395ut^6 + 1339u^2t^5 + 425u^3\ t^4$$

$$+ 1903u^4t^3 + 3872u^5t^2 + 2870u^6t + 774u^7\Big)s^{13} + \Big(2640\ t^8 + 11430ut^7 + 18489u^2t^6 + 14984u^3t^5 + 19215u^4t^4 + 37858u^5\ t^3 + 40759u^6t^2$$

$$+ 21396u^7t + 4367u^8\Big)s^{12} + 2\Big(2460t^9 + 13545\ ut^8 + 31286u^2t^7 + 39401u^3t^6 + 40878u^4t^5 + 57956u^5t^4 + 72977u^6\ t^3 + 56244u^7t^2$$

$$+ 23121u^8t + 3840u^9\Big)s^{11} + 2\Big(3000\ t^{10} + 20475ut^9 + 62005u^2t^8 + 104385u^3t^7 + 118565u^4t^6 + 137297u^5\ t^5 + 175620u^6t^4 + 170653u^7t^3$$

$$+ 102876u^8t^2 + 33792u^9t + 4602\ u^{10}\Big)s^{10} + 2\Big(2460t^{11} + 20475ut^{10} + 77358u^2t^9 + 164395\ u^3t^8 + 215383u^4t^7 + 230739u^5t^6$$

$$+ 294815u^6t^5 + 356593u^7t^4 + 290657\ u^8t^3 + 140082u^9t^2 + 35847u^{10}t + 3840u^{11}\Big)s^9 + \Big(2640\ t^{12} + 27090ut^{11} + 124010u^2t^{10}$$

$$+ 328790u^3t^9 + 519114u^4t^8 + 554262\ u^5t^7 + 649134u^6t^6 + 963078u^7t^5 + 1095009u^8t^4 + 756156u^9t^3 + 291138\ u^{10}t^2 + 55904u^{11}t$$

$$+ 4367u^{12}\Big)s^8 + 2\Big(420t^{13} + 5715u\ t^{12} + 31286u^2t^{11} + 104385u^3t^{10} + 215383u^4t^9 + 277131u^5\ t^8 + 300068u^6t^7 + 447563u^7t^6$$

$$+ 661734u^8t^5 + 642258u^9t^4 + 363648\ u^{10}t^3 + 110242u^{11}t^2 + 15265u^{12}t + 774u^{13}\Big)s^7 + \Big(120\ t^{14} + 2790ut^{13} + 18489u^2t^{12} + 78802u^3t^{11}$$

$$+ 237130u^4\ t^{10} + 461478u^5t^9 + 649134u^6t^8 + 895126u^7t^7 + 1291792u^8t^6 + 1479900\ u^9t^5 + 1107162u^{10}t^4 + 486996u^{11}t^3 + 111068u^{12}t^2$$

$$=+10260u^{13}t+259u^{14}\Big)s^6+2tu\Big(150t^{13}+1339ut^{12}+7492u^2t^{11}+40878u^3t^{10}+137297u^4t^9+294815u^5t^8+481539u^6t^7+661734u^7t^6$$
$$+739950u^8t^5+608514u^9t^4+330308u^{10}t^3+104819u^{11}t^2+16074u^{12}t+777u^{13}\Big)s^5+t^2u^2\Big(109t^{12}+850ut^{11}+19215u^2t^{10}$$
$$+115912u^3t^9+351240u^4t^8+713186u^5t^7+1095009u^6t^6+1284516u^7t^5+1107162u^8t^4+660616u^9t^3+249466u^{10}t^2+50940u^{11}t$$
$$+3885u^{12}\Big)s^4+2t^3u^3\Big(-34t^{11}+1903ut^{10}+18929u^2t^9+72977u^3t^8+170653u^4t^7+290657u^5t^6+378078u^6t^5+363648u^7t^4$$
$$+243498u^8t^3+104819u^9t^2+25470u^{10}t+2590u^{11}\Big)s^3+t^4u^4(t+u)^2\Big(426t^8+6892ut^7+26549u^2t^6+52498u^3t^5+74207u^4t^4$$
$$+79252u^5t^3+58427u^6t^2+24378u^7t+3885u^8\Big)s^2+2t^5u^5(t+u)^3\Big(308t^6+1946ut^5+3936u^2t^4+5167u^3t^3+4537u^4t^2+2799u^5t$$
$$+777u^6\Big)s+t^6u^6(t+u)^4\Big(259t^4+512ut^3+765u^2t^2+512u^3t+259u^4\Big)\Big)H(0,z)H(2,y)\,(s+u)^2+3s^2t^2(s+t)^6u^2(t+u)^6\Big(363s^8$$
$$+726(t+3u)s^7+3\Big(363t^2+796ut+2057u^2\Big)s^6+6\Big(121t^3+348ut^2+480u^2t+1815u^3\Big)s^5+\Big(363t^4+958ut^3-297u^2t^2+1782u^3t$$
$$+13068u^4\Big)s^4+2u\Big(246t^4-58ut^3-1296u^2t^2+891u^3t+5445u^4\Big)s^3+u^2\Big(504t^4-116ut^3-297u^2t^2+2880u^3t+6171u^4\Big)s^2$$
$$=+2u^3\Big(246t^4+479ut^3+1044u^2t^2+1194u^3t+1089u^4\Big)s+363u^4\Big(t^2+ut+u^2\Big)^2\Big)H(0,0,z)(s+u)^2+36s^2t^2(s+t)^6u^2(t+u)^6\Big(22s^8$$
$$+44(t+3u)s^7+2\Big(33t^2+70ut+187u^2\Big)s^6+\Big(44t^3+246ut^2+147u^2t+660u^3\Big)s^5+\Big(22t^4+115ut^3+360u^2t^2+41u^3t+792u^4\Big)s^4$$
$$+u\Big(64t^4+95ut^3+246u^2t^2+2u^3t+660u^4\Big)s^3+u^2\Big(84t^4+9ut^3+84u^2t^2+54u^3t+374u^4\Big)s^2+u^3\Big(54t^4-11ut^3+36u^2t^2+63u^3t$$
$$+132u^4\Big)s+u^4\Big(18t^4+4ut^3+18u^2t^2+21u^3t+22u^4\Big)\Big)H(0,y)H(0,0,z)(s+u)^2-36s^2t^2(s+t)^6u^2(t+u)^6\Big(22s^8+3(7t+44u)s^7$$
$$+\Big(18t^2+63ut+374u^2\Big)s^6+\Big(4t^3+36ut^2+54u^2t+660u^3\Big)s^5+\Big(18t^4-11ut^3+84u^2t^2+2u^3t+792u^4\Big)s^4+u\Big(54t^4+9ut^3$$
$$+246u^2t^2+41u^3t+660u^4\Big)s^3+u^2\Big(84t^4+95ut^3+360u^2t^2+147u^3t+374u^4\Big)s^2+u^3\Big(64t^4+115ut^3+246u^2t^2+140u^3t+132u^4\Big)s$$
$$+22u^4\Big(t^2+ut+u^2\Big)^2\Big)H(2,y)H(0,0,z)\,(s+u)^2+18s^2t^2u^2(t+u)^2\Big(\Big(25t^4+64ut^3+78u^2t^2+44u^3t+11u^4\Big)s^{14}+2\Big(92t^5+278ut^4$$
$$+393u^2t^3+307u^3t^2+133u^4t+27u^5\Big)s^{13}+\Big(633t^6+2332ut^5+3918u^2t^4+3896u^3t^3+2351u^4t^2+816u^5t+124u^6\Big)s^{12}+2\Big(684t^7$$
$$+2970ut^6+5715u^2t^5+6827u^3t^4+5495u^4t^3+2871u^5t^2+846u^6t+96u^7\Big)s^{11}+\Big(2094t^8+10548ut^7+21762u^2t^6+27160u^3t^5$$
$$+25897u^4t^4+19800u^5t^3+10356u^6t^2+2820u^7t+231u^8\Big)s^{10}+2\Big(1200t^9+6984ut^8+15768u^2t^7+19041u^3t^6+16319u^4t^5$$
$$=+14613u^5t^4+13072u^6t^3+7593u^7t^2+2013u^8t+105u^9\Big)s^9+\Big(2094t^{10}+13968ut^9+35988u^2t^8+47158u^3t^7+37273u^4t^6$$
$$+27528u^5t^5+31140u^6t^4+31554u^7t^3+17931u^8t^2+4336u^9t+130u^{10}\Big)s^8+2\Big(684t^{11}+5274ut^{10}+15768u^2t^9+23579u^3t^8$$
$$+22058u^4t^7+21537u^5t^6+26874u^6t^5+27399u^7t^4+18504u^8t^3+7571u^9t^2+1472u^{10}t+24u^{11}\Big)s^7+\Big(633t^{12}+5940ut^{11}$$
$$+21762u^2t^{10}+38082u^3t^9+37273u^4t^8+43074u^5t^7+83288u^6t^6+111834u^7t^5+83094u^8t^4+33834u^9t^3+7744u^{10}t^2+1056u^{11}t$$
$$+8u^{12}\Big)s^6+2t\Big(92t^{12}+1166ut^{11}+5715u^2t^{10}+13580u^3t^9+16319u^4t^8+13764u^5t^7+26874u^6t^6+55917u^7t^5+62742u^8t^4$$
$$+36199u^9t^3+9748u^{10}t^2+930u^{11}t+72u^{12}\Big)s^5+t^2\Big(25t^{12}+556ut^{11}+3918u^2t^{10}+13654u^3t^9+25897u^4t^8+29226u^5t^7$$
$$+31140u^6t^6+54798u^7t^5+83094u^8t^4+72398u^9t^3+32492u^{10}t^2+6012u^{11}t+84u^{12}\Big)s^4+2t^3u\Big(32t^{11}+393ut^{10}+1948u^2t^9$$
$$+5495u^3t^8+9900u^4t^7+13072u^5t^6+15777u^6t^5+18504u^7t^4+16917u^8t^3+9748u^9t^2+3006u^{10}t+368u^{11}\Big)s^3+t^4u^2(t+u)^2\Big(78t^8$$
$$+458ut^7+1357u^2t^6+2570u^3t^5+3859u^4t^4+4898u^5t^3+4276u^6t^2+1692u^7t+84u^8\Big)s^2+2t^5u^3(t+u)^3\Big(22t^6+67ut^5+141u^2t^4$$
$$+200u^3t^3+320u^4t^2+312u^5t+72u^6\Big)s+t^6u^4(t+u)^4\Big(11t^4+10ut^3+18u^2t^2+16u^3t+8u^4\Big)\Big)H(0,z)H(0,2,y)(s+u)^2$$
$$=+3t^2(s+t)u^2\Big(\Big(17t^6+1338ut^5+1209u^2t^4+3688u^3t^3+4269u^4t^2+762u^5t+293u^6\Big)s^{15}+(t+u)^2\Big(125t^5+9282ut^4-1574u^2t^3$$
$$+24402u^3t^2+2031u^4t+1740u^5\Big)s^{14}+\Big(421t^8+28668ut^7+79874u^2t^6+122578u^3t^5+210974u^4t^4+216624u^5t^3+90402u^6t^2$$
$$+18674u^7t+4873u^8\Big)s^{13}+(t+u)^2\Big(853t^7+47050ut^6+99979u^2t^5+64212u^3t^4+244839u^4t^3+99258u^5t^2+22313u^6t+8520u^7\Big)s^{12}$$
$$+4\Big(288t^{10}+12551ut^9+71057u^2t^8+139212u^3t^7+179702u^4t^6+266064u^5t^5+276429u^6t^4+139626u^7t^3+41347u^8t^2+13917u^9t$$
$$+2547u^{10}\Big)s^{11}+2(t+u)^2\Big(546t^9+13510ut^8+92097u^2t^7+124154u^3t^6+79909u^4t^5+282798u^5t^4+136349u^6t^3+25806u^7t^2$$
$$+17949u^8t+4260u^9\Big)s^{10}+\Big(745t^{12}+5476ut^{11}+88790u^2t^{10}+380158u^3t^9+559919u^4t^8+624124u^5t^7+1280316u^6t^6+1630684u^7t^5$$
$$+908135u^8t^4+264280u^9t^3+94614u^{10}t^2+32670u^{11}t+4873u^{12}\Big)s^9+(t+u)^2\Big(361t^{11}-5942ut^{10}-13109u^2t^9+45310u^3t^8$$
$$-83270u^4t^7-225424u^5t^6+359776u^6t^5+341850u^7t^4+131133u^8t^3+49098u^9t^2+8565u^{10}t+1740u^{11}\Big)s^8+\Big(113t^{14}-5422ut^{13}$$
$$-49927u^2t^{12}-155092u^3t^{11}-318038u^4t^{10}-778482u^5t^9-1327656u^6t^8-974758u^7t^7-19040u^8t^6+432306u^9t^5+380872u^{10}t^4$$
$$=+185882u^{11}t^3+39939u^{12}t^2+2122u^{13}t+293u^{14}\Big)s^7+t(t+u)^2\Big(17t^{12}-2466ut^{11}-25660u^2t^{10}-88158u^3t^9-136236u^4t^8$$
$$-237708u^5t^7-370210u^6t^6-209898u^7t^5-117978u^8t^4-9304u^9t^3+43600u^{10}t^2+14262u^{11}t+49u^{12}\Big)s^6-t^2u\Big(424t^{13}+9046ut^{12}$$
$$+64296u^2t^{11}+209881u^3t^{10}+368916u^4t^9+405336u^5t^8+306108u^6t^7+185900u^7t^6+201510u^8t^5+255266u^9t^4+158896u^{10}t^3$$
$$+31479u^{11}t^2-5494u^{12}t-2252u^{13}\Big)s^5-t^3u^2(t+u)^2\Big(1038t^{10}+11826ut^9+41953u^2t^8+50370u^3t^7-10747u^4t^6-92766u^5t^5$$
$$-113853u^6t^4-35658u^7t^3+15137u^8t^2+11676u^9t+2344u^{10}\Big)s^4-t^4u^3(t+u)^3\Big(1108t^8+5303ut^7+2515u^2t^6-23685u^3t^5$$
$$-48783u^4t^4-46581u^5t^3-17785u^6t^2-2753u^7t-67u^8\Big)s^3-t^5u^4(t+u)^4\Big(133t^6-1574ut^5-8533u^2t^4-14102u^3t^3-10237u^4t^2$$
$$-2876u^5t-167u^6\Big)s^2+30t^6u^5(t+u)^6\Big(10t^3+37ut^2+37u^2t+14u^3\Big)s+120t^7u^6(t+u)^6\Big(t^2+ut+u^2\Big)\Big)H(1,0,z)(s+u)^2$$

$$= + 18s^2t^2(s+t)^2u^2 \ \Big(\Big(11t^6 + 162ut^5 + 129u^2t^4 + 796u^3t^3 + 129u^4t^2 + 162u^5 \ t + 11u^6\Big)s^{12} + 6\Big(11t^7 + 193ut^6 + 349u^2t^5 + 1047u^3 \ t^4$$

$$+ 1027u^4t^3 + 313u^5t^2 + 173u^6t + 7u^7\Big)s^{11} + \Big(187 \ t^8 + 3280ut^7 + 8506u^2t^6 + 20204u^3t^5 + 32372u^4t^4 + 18608u^5t^3 + 6910 \ u^6t^2$$

$$+ 2596u^7t + 73u^8\Big)s^{10} + 2\Big(165t^9 + 2522ut^8 + 8324u^2 \ t^7 + 17309u^3t^6 + 35002u^4t^5 + 33469u^5t^4 + 14236u^6t^3 + 6119u^7 \ t^2 + 1745u^8t$$

$$+ 53u^9\Big)s^9 + \Big(396t^{10} + 4920ut^9 + 19227u^2 \ t^8 + 34792u^3t^7 + 71793u^4t^6 + 105726u^5t^5 + 63533u^6t^4 + 25164u^7 \ t^3 + 13635u^8t^2$$

$$+ 3190u^9t + 168u^{10}\Big)s^8 + 2\Big(165t^{11} + 1725u \ t^{10} + 7356u^2t^9 + 10992u^3t^8 + 12447u^4t^7 + 31129u^5t^6 + 30585u^6 \ t^5 + 10947u^7t^4$$

$$+ 9212u^8t^3 + 6186u^9t^2 + 1323u^{10}t + 109u^{11}\Big) \ s^7 + \Big(187t^{12} + 1872ut^{11} + 8220u^2t^{10} + 10632u^3t^9 - 15915u^4 \ t^8 - 29212u^5t^7 - 9538u^6t^6$$

$$- 13152u^7t^5 + 1543u^8t^4 + 19604u^9 \ t^3 + 10518u^{10}t^2 + 2096u^{11}t + 185u^{12}\Big)s^6 + 2\Big(33 \ t^{13} + 372ut^{12} + 1782u^2t^{11} + 2908u^3t^{10}$$

$$- 8202u^4t^9 - 31332u^5 \ t^8 - 35999u^6t^7 - 26337u^7t^6 - 11010u^8t^5 + 8183u^9t^4 + 10293u^{10} \ t^3 + 3681u^{11}t^2 + 623u^{12}t + 45u^{13}\Big)s^5$$

$$+ \Big(11t^{14} + 182u \ t^{13} + 1142u^2t^{12} + 3266u^3t^{11} - 2041u^4t^{10} - 30594u^5t^9 - 56568 \ u^6t^8 - 48084u^7t^7 - 20480u^8t^6 + 13148u^9t^5$$

$$= + 25835u^{10}t^4 + 13918 \ u^{11}t^3 + 3530u^{12}t^2 + 452u^{13}t + 19u^{14}\Big)s^4 + 2tu\Big(10 \ t^{13} + 115ut^{12} + 604u^2t^{11} + 1366u^3t^{10} - 633u^4t^9 - 5650u^5 \ t^8$$

$$- 5310u^6t^7 + 1442u^7t^6 + 9166u^8t^5 + 12427u^9t^4 + 8168u^{10} \ t^3 + 2766u^{11}t^2 + 499u^{12}t + 38u^{13}\Big)s^3 + t^2u^2(t+u)^2 \ \Big(24t^{10} + 190ut^9$$

$$+ 841u^2t^8 + 1482u^3t^7 + 1601u^4t^6 + 4172u^5 \ t^5 + 4881u^6t^4 + 5362u^7t^3 + 3151u^8t^2 + 930u^9t + 126u^{10}\Big)s^2 + 2 \ t^3u^3(t+u)^3\Big(10t^8$$

$$+ 70ut^7 + 248u^2t^6 + 399u^3t^5 + 669u^4 \ t^4 + 647u^5t^3 + 545u^6t^2 + 236u^7t + 48u^8\Big)s + t^4u^4(t+u)^6 \ \Big(11t^4 + 22ut^3 + 39u^2t^2 + 28u^3t$$

$$+ 33u^4\Big)\Big)H(0,y) \ H(1,0,z)(s+u)^2 + 18s^2t^2(s+t)^2u^2\Big(4\Big(2t^6 + 36ut^5 + 21u^2 \ t^4 + 184u^3t^3 + 21u^4t^2 + 36u^5t + 2u^6\Big)s^{12} + 12\Big(4t^7$$

$$+ 88u \ t^6 + 155u^2t^5 + 501u^3t^4 + 501u^4t^3 + 155u^5t^2 + 88u^6t + 4u^7\Big) \ s^{11} + 2\Big(65t^8 + 1472ut^7 + 3872u^2t^6 + 9748u^3t^5 + 16246u^4 \ t^4$$

$$+ 9748u^5t^3 + 3872u^6t^2 + 1472u^7t + 65u^8\Big)s^{10} + 2\Big(105 \ t^9 + 2168ut^8 + 7571u^2t^7 + 16917u^3t^6 + 36199u^4t^5 + 36199u^5t^4 + 16917 \ u^6t^3$$

$$+ 7571u^7t^2 + 2168u^8t + 105u^9\Big)s^9 + 3\Big(77t^{10} + 1342u \ t^9 + 5977u^2t^8 + 12336u^3t^7 + 27698u^4t^6 + 41828u^5t^5 + 27698u^6 \ t^4 + 12336u^7t^3$$

$$+ 5977u^8t^2 + 1342u^9t + 77u^{10}\Big)s^8 + 6\Big(32 \ t^{11} + 470ut^{10} + 2531u^2t^9 + 5259u^3t^8 + 9133u^4t^7 + 18639u^5 \ t^6 + 18639u^6t^5 + 9133u^7t^4$$

$$= + 5259u^8t^3 + 2531u^9t^2 + 470u^{10}t + 32 \ u^{11}\Big)s^7 + 4\Big(31t^{12} + 423ut^{11} + 2589u^2t^{10} + 6536u^3 \ t^9 + 7785u^4t^8 + 13437u^5t^7 + 20822u^6t^6$$

$$+ 13437u^7t^5 + 7785u^8t^4 + 6536 \ u^9t^3 + 2589u^{10}t^2 + 423u^{11}t + 31u^{12}\Big)s^6 + 6\Big(9 \ t^{13} + 136ut^{12} + 957u^2t^{11} + 3300u^3t^{10} + 4871u^4t^9$$

$$+ 4588u^5 \ t^8 + 7179u^6t^7 + 7179u^7t^6 + 4588u^8t^5 + 4871u^9t^4 + 3300u^{10}t^3 + 957 \ u^{11}t^2 + 136u^{12}t + 9u^{13}\Big)s^5 + \Big(11t^{14} + 266u \ t^{13}$$

$$+ 2351u^2t^{12} + 10990u^3t^{11} + 25897u^4t^{10} + 32638u^5t^9 + 37273 \ u^6t^8 + 44116u^7t^7 + 37273u^8t^6 + 32638u^9t^5 + 25897u^{10}t^4 + 10990 \ u^{11}t^3$$

$$+ 2351u^{12}t^2 + 266u^{13}t + 11u^{14}\Big)s^4 + 2tu\Big(22 \ t^{13} + 307ut^{12} + 1948u^2t^{11} + 6827u^3t^{10} + 13580u^4t^9 + 19041u^5 \ t^8 + 23579u^6t^7 + 23579u^7t^6$$

$$+ 19041u^8t^5 + 13580u^9t^4 + 6827u^{10} \ t^3 + 1948u^{11}t^2 + 307u^{12}t + 22u^{13}\Big)s^3 + 6t^2u^2(t+u)^2 \ \Big(13t^{10} + 105ut^9 + 430u^2t^8 + 940u^3t^7$$

$$+ 1317u^4t^6 + 1682u^5 \ t^5 + 1317u^6t^4 + 940u^7t^3 + 430u^8t^2 + 105u^9t + 13u^{10}\Big)s^2 + 4 \ t^3u^3(t+u)^3\Big(16t^8 + 91ut^7 + 262u^2t^6 + 410u^3t^5$$

$$+ 530u^4 \ t^4 + 410u^5t^3 + 262u^6t^2 + 91u^7t + 16u^8\Big)s + t^4u^4(t+u)^6 \ \Big(25t^4 + 34ut^3 + 54u^2t^2 + 34u^3t + 25u^4\Big)\Big)H(0,z) \ H(2,0,y)(s+u)^2$$

$$= + 18s^2t^2u^2\Big(6\Big(11t^6 + 46ut^5 + 156u^2 \ t^4 + 102u^3t^3 + 156u^4t^2 + 46u^5t + 11u^6\Big)s^{14} + 2\Big(270 \ t^7 + 1259ut^6 + 4389u^2t^5 + 4796u^3t^4$$

$$+ 5120u^4t^3 + 3729u^5t^2 + 1049u^6 \ t + 204u^7\Big)s^{13} + \Big(2055t^8 + 10770ut^7 + 38032u^2t^6 + 57306u^3 \ t^5 + 58498u^4t^4 + 54214u^5t^3 + 27450u^6t^2$$

$$+ 7450u^7t + 1185u^8\Big) \ s^{12} + 2\Big(2410t^9 + 14457ut^8 + 51810u^2t^7 + 97505u^3t^6 + 109719u^4 \ t^5 + 108252u^5t^4 + 77806u^6t^3 + 32031u^7t^2$$

$$+ 8199u^8t + 1059u^9\Big) \ s^{11} + \Big(7744t^{10} + 53588ut^9 + 198480u^2t^8 + 439066u^3t^7 + 579501 \ u^4t^6 + 598332u^5t^5 + 517783u^6t^4 + 294682u^7t^3$$

$$+ 105495u^8t^2 + 24196 \ u^9t + 2541u^{10}\Big)s^{10} + 2\Big(4458t^{11} + 35621ut^{10} + 138966 \ u^2t^9 + 344982u^3t^8 + 538700u^4t^7 + 609874u^5t^6$$

$$+ 590760u^6t^5 + 426236 \ u^7t^4 + 200386u^8t^3 + 62613u^9t^2 + 12146u^{10}t + 1050u^{11}\Big) \ s^9 + \Big(7450t^{12} + 68928ut^{11} + 290076u^2t^{10}$$

$$+ 782276u^3 \ t^9 + 1406634u^4t^8 + 1787400u^5t^7 + 1901521u^6t^6 + 1654302u^7t^5 + 999621 \ u^8t^4 + 402156u^9t^3 + 106839u^{10}t^2 + 16578u^{11}t$$

$$+ 1179u^{12}\Big) \ s^8 + 2\Big(2230t^{13} + 24137ut^{12} + 113268u^2t^{11} + 330723u^3 \ t^{10} + 666867u^4t^9 + 948111u^5t^8 + 1082558u^6t^7 + 1072690u^7t^6$$

$$= + 809304 \ u^8t^5 + 421377u^9t^4 + 148590u^{10}t^3 + 32739u^{11}t^2 + 3807u^{12}t + 207 \ u^{13}\Big)s^7 + \Big(1830t^{14} + 23864ut^{13} + 129744u^2t^{12}$$

$$+ 419622 \ u^3t^{11} + 946629u^4t^{10} + 1526944u^5t^9 + 1879479u^6t^8 + 1986148u^7 \ t^7 + 1729402u^8t^6 + 1108500u^9t^5 + 506934u^{10}t^4$$

$$+ 157722u^{11} \ t^3 + 28413u^{12}t^2 + 2220u^{13}t + 69u^{14}\Big)s^6 + 2t\Big(232 \ t^{14} + 3972ut^{13} + 26409u^2t^{12} + 97770u^3t^{11} + 247866u^4 \ t^{10}$$

$$+ 463269u^5t^9 + 652628u^6t^8 + 749862u^7t^7 + 712998u^8t^6 + 514895 \ u^9t^5 + 272466u^{10}t^4 + 106176u^{11}t^3 + 28022u^{12}t^2 + 3996u^{13} \ t$$

$$+ 159u^{14}\Big)s^5 + t^2\Big(55t^{14} + 1614ut^{13} + 14748u^2 \ t^{12} + 66434u^3t^{11} + 190555u^4t^{10} + 406308u^5t^9 + 671989u^6 \ t^8 + 896290u^7t^7$$

$$+ 978039u^8t^6 + 811640u^9t^5 + 480198u^{10}t^4 + 201792 \ u^{11}t^3 + 59561u^{12}t^2 + 11310u^{13}t + 1083u^{14}\Big)s^4 + 2t^3u \ \Big(74t^{13} + 1262ut^{12}$$

$$+ 7944u^2t^{11} + 27682u^3t^{10} + 67049u^4 \ t^9 + 125496u^5t^8 + 189647u^6t^7 + 237429u^7t^6 + 234672u^8t^5 + 169330u^9 \ t^4 + 85057u^{10}t^3$$

$$+ 28209u^{11}t^2 + 5389u^{12}t + 424u^{13}\Big) \ s^3 + t^4u^2(t+u)^2\Big(192t^{10} + 1868ut^9 + 7244u^2t^8 + 17304u^3 \ t^7 + 30738u^4t^6 + 40806u^5t^5$$

$$+ 45573u^6t^4 + 36880u^7t^3 + 20764u^8 \ t^2 + 7278u^9t + 1137u^{10}\Big)s^2 + 2t^5u^3(t+u)^3\Big(64t^8 + 420u \ t^7 + 1164u^2t^6 + 2189u^3t^5 + 2787u^4t^4$$

$$+ 2772u^5t^3 + 1870u^6t^2 + 855u^7 \ t + 183u^8\Big)s + t^6u^4(t+u)^6\Big(47t^4 + 64ut^3 + 117u^2t^2 + 92u^3 \ t + 83u^4\Big)\Big)H(0,1,0,z)(s+u)^2$$

$$= + 36s^2t^2(s+t)^6u^2(t+u)^6 \ (22s^8 + 44(t+3u)s^7 + 2(33t^2 + 70ut + 187u^2) \ s^6 + (44t^3 + 246ut^2 + 147u^2t + 660u^3)s^5 + (22t^4$$
$$+ 115u \ t^3 + 360u^2t^2 + 41u^3t + 792u^4)s^4 + u(64t^4 + 95ut^3 + 246u^2 \ t^2 + 2u^3t + 660u^4)s^3 + u^2(84t^4 + 9ut^3 + 84u^2t^2 + 54u^3 \ t$$
$$+ 374u^4)s^2 + u^3(54t^4 - 11ut^3 + 36u^2t^2 + 63u^3t + 132 \ u^4)s + u^4(18t^4 + 4ut^3 + 18u^2t^2 + 21u^3t + 22 \ u^4))H(1,0,0,z)(s+u)^2$$
$$+ 36s^2t^2u^3((48t^5 - 6u \ t^4 + 332u^2t^3 + 45u^3t^2 + 90u^4t + 11u^5)s^{14} + (420t^6 + 282u \ t^5 + 2684u^2t^4 + 2292u^3t^3 + 810u^4t^2$$
$$+ 542u^5t + 42u^6) \ s^{13} + (1386t^7 + 1149ut^6 + 7038u^2t^5 + 12044u^3t^4 + 4972u^4 \ t^3 + 2592u^5t^2 + 1202u^6t + 73u^7)s^{12} + (2181t^8$$
$$- 6ut^7 + 3 \ u^2t^6 + 15414u^3t^5 + 7299u^4t^4 - 896u^5t^3 + 2931u^6t^2 + 1368u^7t + 106 \ u^8)s^{11} + (985t^9 - 9867ut^8 - 44480u^2t^7$$
$$- 48559u^3t^6 - 45186 \ u^4t^5 - 50331u^5t^4 - 19580u^6t^3 + 585u^7t^2 + 1097u^8t + 168u^9) \ s^{10} - (2469t^{10} + 31352ut^9 + 135204u^2t^8$$
$$+ 241022u^3t^7 + 253354 \ u^4t^6 + 222072u^5t^5 + 127806u^6t^4 + 30846u^7t^3 + 657u^8t^2 - 980u^9 \ t - 218u^{10})s^9 - (5517t^{11}$$
$$+ 52233ut^{10} + 228083u^2 \ t^9 + 506619u^3t^8 + 645063u^4t^7 + 575952u^5t^6 + 368997u^6t^5 + 131304u^7 \ t^4 + 16832u^8t^3 - 1527u^9t^2$$
$$- 936u^{10}t - 185u^{11})s^8 - (5797 \ t^{12} + 55314ut^{11} + 250929u^2t^{10} + 646444u^3t^9 + 990366u^4 \ t^8 + 1001980u^5t^7 + 717884u^6t^6$$
$$+ 319380u^7t^5 + 58843u^8t^4 - 6082u^9 \ t^3 - 4017u^{10}t^2 - 684u^{11}t - 90u^{12})s^7 - (3849 \ t^{13} + 40039ut^{12} + 191778u^2t^{11}$$
$$= + 546879u^3t^{10} + 969791u^4 \ t^9 + 1126182u^5t^8 + 922078u^6t^7 + 512534u^7t^6 + 148569u^8t^5 - 1403u^9 \ t^4 - 14004u^{10}t^3 - 3726u^{11}t^2$$
$$- 315u^{12}t - 19u^{13})s^6 - t \ (1647t^{13} + 19818ut^{12} + 104248u^2t^{11} + 323862u^3t^{10} + 638973 \ u^4t^9 + 819288u^5t^8 + 723606u^6t^7 + 455700u^7t^6$$
$$+ 180305u^8t^5 + 24144 \ u^9t^4 - 13260u^{10}t^3 - 8070u^{11}t^2 - 1683u^{12}t - 66u^{13}) \ s^5 - t^2(413t^{13} + 6240ut^{12} + 38435u^2t^{11} + 133574u^3 \ t^{10}$$
$$+ 291906u^4t^9 + 405133u^5t^8 + 357472u^6t^7 + 200055u^7t^6 + 58195u^8 \ t^5 - 7313u^9t^4 - 13779u^{10}t^3 - 6238u^{11}t^2 - 1920u^{12}t - 309 \ u^{13})s^4$$
$$- t^3(48t^{13} + 1112ut^{12} + 8730u^2t^{11} + 35276 \ u^3t^{10} + 86245u^4t^9 + 130602u^5t^8 + 113599u^6t^7 + 38248u^7t^6 - 30777 \ u^8t^5 - 50450u^9t^4$$
$$- 31999u^{10}t^3 - 10602u^{11}t^2 - 1774u^{12}t - 114 \ u^{13})s^3 - t^4u(t+u)^2(84t^{10} + 898ut^9 + 3400u^2t^8 + 6309 \ u^3t^7 + 5217u^4t^6 - 2505u^5t^5$$
$$- 8514u^6t^4 - 9343u^7t^3 - 5743u^8 \ t^2 - 2007u^9t - 336u^{10})s^2 - t^5u^2(t+u)^3(48t^8 + 200u \ t^7 + 135u^2t^6 - 571u^3t^5 - 1726u^4t^4 - 1854u^5t^3$$
$$- 1219u^6t^2 - 459u^7 \ t - 90u^8)s + t^6u^4(t+u)^6(23t^3 + 54ut^2 + 46u^2t + 26 \ u^3))H(1,1,0,z)(s+u)^2 + 9(s+t)^2u^2(t+u)(2(92 \ t^7 + 224ut^6$$
$$+ 807u^2t^5 + 41u^3t^4 + 227u^4t^3 + 71u^5t^2 + 70u^6t + 20 \ u^7)s^{15} + 2(548t^8 + 1800ut^7 + 6507u^2t^6 + 5556u^3t^5 + 2001 \ u^4t^4 + 2149u^5t^3$$
$$+ 864u^6t^2 + 535u^7t + 120u^8)s^{14} + (3080 \ t^9 + 12672ut^8 + 43909u^2t^7 + 68677u^3t^6 + 39172u^4t^5 + 27176u^5 \ t^4 + 19411u^6t^3 + 8427u^7t^2$$
$$+ 3620u^8t + 640u^9)s^{13} + (5400 \ t^{10} + 26796ut^9 + 85362u^2t^8 + 182471u^3t^7 + 168303u^4t^6 + 105098u^5 \ t^5 + 92562u^6t^4 + 53097u^7t^3$$
$$+ 21885u^8t^2 + 7050u^9t + 1000u^{10}) \ s^{12} + (6464t^{11} + 38196ut^{10} + 114563u^2t^9 + 283713u^3t^8 + 385450 \ u^4t^7 + 275704u^5t^6 + 227431u^6t^5$$
$$= + 183473u^7t^4 + 90148u^8t^3 + 33770u^9 \ t^2 + 8600u^{10}t + 1000u^{11})s^{11} + (5400t^{12} + 38196u \ t^{11} + 123420u^2t^{10} + 320669u^3t^9 + 573417u^4t^8$$
$$+ 513000u^5t^7 + 343086 \ u^6t^6 + 321375u^7t^5 + 218509u^8t^4 + 96742u^9t^3 + 32200u^{10}t^2 + 6690 \ u^{11}t + 640u^{12})s^{10} + (3080t^{13} + 26796ut^{12}$$
$$+ 114563 \ u^2t^{11} + 320669u^3t^{10} + 641380u^4t^9 + 698714u^5t^8 + 386310u^6 \ t^7 + 282028u^7t^6 + 279284u^8t^5 + 162550u^9t^4 + 65139u^{10}t^3$$
$$+ 18779 \ u^{11}t^2 + 3220u^{12}t + 240u^{13})s^9 + (1096t^{14} + 12672u \ t^{13} + 85362u^2t^{12} + 283713u^3t^{11} + 573417u^4t^{10} + 698714u^5 \ t^9$$
$$+ 392272u^6t^8 + 129314u^7t^7 + 178058u^8t^6 + 158492u^9t^5 + 72298u^{10} \ t^4 + 25025u^{11}t^3 + 6257u^{12}t^2 + 870u^{13}t + 40u^{14})s^8 + t \ (184t^{14}$$
$$+ 3600ut^{13} + 43909u^2t^{12} + 182471u^3t^{11} + 385450u^4 \ t^{10} + 513000u^5t^9 + 386310u^6t^8 + 129314u^7t^7 + 92160u^8t^6 + 110380u^9 \ t^5$$
$$+ 48211u^{10}t^4 + 11865u^{11}t^3 + 3700u^{12}t^2 + 994u^{13}t + 100 \ u^{14})s^7 + t^2u(448t^{13} + 13014ut^{12} + 68677u^2 \ t^{11} + 168303u^3t^{10}$$
$$+ 275704u^4t^9 + 343086u^5t^8 + 282028u^6t^7 + 178058 \ u^7t^6 + 110380u^8t^5 + 33784u^9t^4 - 4267u^{10}t^3 - 3791u^{11}t^2 - 474 \ u^{12}t + 42u^{13})s^6$$
$$+ t^3u^2(1614t^{12} + 11112ut^{11} + 39172 \ u^2t^{10} + 105098u^3t^9 + 227431u^4t^8 + 321375u^5t^7 + 279284u^6 \ t^6 + 158492u^7t^5 + 48211u^8t^4$$
$$- 4267u^9t^3 - 7188u^{10}t^2 - 1770u^{11} \ t - 164u^{12})s^5 + t^4u^3(82t^{11} + 4002ut^{10} + 27176u^2 \ t^9 + 92562u^3t^8 + 183473u^4t^7 + 218509u^5t^6$$
$$+ 162550u^6t^5 + 72298u^7 \ t^4 + 11865u^8t^3 - 3791u^9t^2 - 1770u^{10}t - 204u^{11})s^4 + t^5u^4 \ (t+u)^2(454t^8 + 3390ut^7 + 12177u^2t^6$$
$$+ 25353u^3t^5 + 27265u^4 \ t^4 + 16859u^5t^3 + 4156u^6t^2 - 146u^7t - 164u^8)s^3 + t^6u^5(t+u)^3 \ (142t^6 + 1302ut^5 + 4095u^2t^4 + 5552u^3t^3$$
$$+ 3527u^4t^2 + 868u^5t + 42 \ u^6)s^2 + 10t^7u^6(t+u)^5(14t^3 + 37ut^2 + 37u^2t + 10 \ u^3)s + 40t^8u^7(t+u)^5(t^2 + ut + u^2))H(1,0,y) \ (s+u)$$
$$= - 3s^2(s+t)^2u^2(t+u)^2((259t^6 + 616ut^5 + 426u^2t^4 - 68 \ u^3t^3 + 109u^4t^2 + 300u^5t + 120u^6)s^{14} + 2(774t^7 + 2870u \ t^6 + 3872u^2t^5$$
$$+ 1903u^3t^4 + 425u^4t^3 + 1339u^5t^2 + 1395u^6t + 420 \ u^7)s^{13} + (4367t^8 + 21396ut^7 + 40759u^2t^6 + 37858u^3 \ t^5 + 19215u^4t^4 + 14984u^5t^3$$
$$+ 18489u^6t^2 + 11430u^7t + 2640u^8) \ s^{12} + 2(3840t^9 + 23121ut^8 + 56244u^2t^7 + 72977u^3t^6 + 57956u^4 \ t^5 + 40878u^5t^4 + 39401u^6t^3$$
$$+ 31286u^7t^2 + 13545u^8t + 2460u^9) \ s^{11} + 2(4602t^{10} + 33792ut^9 + 102876u^2t^8 + 170653u^3t^7 + 175620 \ u^4t^6 + 137297u^5t^5 + 118565u^6t^4$$
$$+ 104385u^7t^3 + 62005u^8t^2 + 20475u^9 \ t + 3000u^{10})s^{10} + 2(3840t^{11} + 35847ut^{10} + 140082u^2 \ t^9 + 290657u^3t^8 + 356593u^4t^7$$
$$+ 294815u^5t^6 + 230739u^6t^5 + 215383u^7 \ t^4 + 164395u^8t^3 + 77358u^9t^2 + 20475u^{10}t + 2460u^{11}) \ s^9 + (4367t^{12} + 55904ut^{11}$$
$$+ 291138u^2t^{10} + 756156u^3 \ t^9 + 1095009u^4t^8 + 963078u^5t^7 + 649134u^6t^6 + 554262u^7t^5 + 519114u^8 \ t^4 + 328790u^9t^3 + 124010u^{10}t^2$$
$$+ 27090u^{11}t + 2640u^{12})s^8 + 2 \ (774t^{13} + 15265ut^{12} + 110242u^2t^{11} + 363648u^3t^{10} + 642258 \ u^4t^9 + 661734u^5t^8 + 447563u^6t^7$$
$$+ 300068u^7t^6 + 277131u^8t^5 + 215383 \ u^9t^4 + 104385u^{10}t^3 + 31286u^{11}t^2 + 5715u^{12}t + 420u^{13}) \ s^7 + (259t^{14} + 10260ut^{13} + 111068u^2t^{12}$$

$$= + 486996u^3 t^{11} + 1107162u^4t^{10} + 1479900u^5t^9 + 1291792u^6t^8 + 895126u^7 t^7 + 649134u^8t^6 + 461478u^9t^5 + 237130u^{10}t^4 + 78802u^{11}t^3$$
$$+ 18489 u^{12}t^2 + 2790u^{13}t + 120u^{14}\Big) s^6 + 2tu\Big(777t^{13} + 16074 ut^{12} + 104819u^2t^{11} + 330308u^3t^{10} + 608514u^4t^9 + 739950u^5 t^8$$
$$+ 661734u^6t^7 + 481539u^7t^6 + 294815u^8t^5 + 137297u^9t^4 + 40878u^{10} t^3 + 7492u^{11}t^2 + 1339u^{12}t + 150u^{13}\Big)s^5 + t^2u^2\Big(3885 t^{12}$$
$$+ 50940ut^{11} + 249466u^2t^{10} + 660616u^3t^9 + 1107162u^4 t^8 + 1284516u^5t^7 + 1095009u^6t^6 + 713186u^7t^5 + 351240u^8t^4 + 115912u^9 t^3$$
$$+ 19215u^{10}t^2 + 850u^{11}t + 109u^{12}\Big)s^4 + 2t^3u^3\Big(2590 t^{11} + 25470ut^{10} + 104819u^2t^9 + 243498u^3t^8 + 363648u^4t^7 + 378078 u^5t^6$$
$$+ 290657u^6t^5 + 170653u^7t^4 + 72977u^8t^3 + 18929u^9t^2 + 1903 u^{10}t - 34u^{11}\Big)s^3 + t^4u^4(t+u)^2\Big(3885t^8 + 24378u t^7 + 58427u^2t^6$$
$$+ 79252u^3t^5 + 74207u^4t^4 + 52498u^5t^3 + 26549u^6 t^2 + 6892u^7t + 426u^8\Big)s^2 + 2t^5u^5(t+u)^3\Big(777t^6 + 2799u t^5 + 4537u^2t^4 + 5167u^3t^3$$
$$+ 3936u^4t^2 + 1946u^5t + 308u^6\Big)s + t^6 u^6(t+u)^4\Big(259t^4 + 512ut^3 + 765u^2t^2 + 512u^3t + 259 u^4\Big)\Big)H(0,y)H(1,z) + 9s^2(t+u)^2\Big(2\Big(20t^8$$
$$+ 50u t^7 + 21u^2t^6 - 82u^3t^5 - 102u^4t^4 - 82u^5t^3 + 21u^6t^2 + 50u^7t + 20 u^8\Big)s^{16} + 2\Big(140t^9 + 505ut^8 + 568u^2t^7 - 298u^3t^6 - 1069u^4 t^5$$
$$- 1069u^5t^4 - 298u^6t^3 + 568u^7t^2 + 505u^8t + 140u^9\Big) s^{15} + \Big(880t^{10} + 4370ut^9 + 8221u^2t^8 + 4262u^3t^7 - 6199u^4 t^6 - 10932u^5t^5$$
$$- 6199u^6t^4 + 4262u^7t^3 + 8221u^8t^2 + 4370u^9t + 880 u^{10}\Big)s^{14} + 2\Big(820t^{11} + 5395ut^{10} + 14563u^2t^9 + 17988 u^3t^8 + 5650u^4t^7 - 8508u^5t^6$$
$$= - 8508u^6t^5 + 5650u^7t^4 + 17988u^8 t^3 + 14563u^9t^2 + 5395u^{10}t + 820u^{11}\Big)s^{13} + \Big(2000 t^{12} + 16930ut^{11} + 60889u^2t^{10} + 115200u^3t^9$$
$$+ 112888u^4t^8 + 52018 u^5t^7 + 18062u^6t^6 + 52018u^7t^5 + 112888u^8t^4 + 115200u^9t^3 + 60889 u^{10}t^2 + 16930u^{11}t + 2000u^{12}\Big)s^{12} + 2\Big(820t^{13}$$
$$+ 8825 ut^{12} + 40630u^2t^{11} + 106430u^3t^{10} + 162506u^4t^9 + 145433u^5 t^8 + 94054u^6t^7 + 94054u^7t^6 + 145433u^8t^5 + 162506u^9t^4$$
$$+ 106430u^{10} t^3 + 40630u^{11}t^2 + 8825u^{12}t + 820u^{13}\Big)s^{11} + \Big(880 t^{14} + 12310ut^{13} + 71305u^2t^{12} + 252860u^3t^{11} + 542940u^4 t^{10}$$
$$+ 672624u^5t^9 + 495141u^6t^8 + 346704u^7t^7 + 495141u^8t^6 + 672624 u^9t^5 + 542940u^{10}t^4 + 252860u^{11}t^3 + 71305u^{12}t^2 + 12310u^{13} t$$
$$+ 880u^{14}\Big)s^{10} + 2\Big(140t^{15} + 2785ut^{14} + 20491u^2 t^{13} + 99450u^3t^{12} + 294436u^4t^{11} + 490859u^5t^{10} + 448931u^6 t^9 + 254956u^7t^8$$
$$+ 254956u^8t^7 + 448931u^9t^6 + 490859u^{10}t^5 + 294436 u^{11}t^4 + 99450u^{12}t^3 + 20491u^{13}t^2 + 2785u^{14}t + 140u^{15}\Big) s^9 + \Big(40t^{16} + 1490ut^{15}$$
$$+ 14845u^2t^{14} + 102820u^3t^{13} + 419280 u^4t^{12} + 950788u^5t^{11} + 1225773u^6t^{10} + 975710u^7t^9 + 743060u^8 t^8 + 975710u^9t^7 + 1225773u^{10}t^6$$
$$+ 950788u^{11}t^5 + 419280u^{12} t^4 + 102820u^{13}t^3 + 14845u^{14}t^2 + 1490u^{15}t + 40u^{16}\Big)s^8 + 2 tu\Big(90t^{15} + 1540ut^{14} + 16932u^2t^{13} + 96123u^3t^{12}$$
$$+ 304282 u^4t^{11} + 583798u^5t^{10} + 762212u^6t^9 + 803305u^7t^8 + 803305u^8 t^7 + 762212u^9t^6 + 583798u^{10}t^5 + 304282u^{11}t^4 + 96123u^{12} t^3$$
$$+ 16932u^{13}t^2 + 1540u^{14}t + 90u^{15}\Big)s^7 + t^2u^2\Big(282 t^{14} + 6622ut^{13} + 54887u^2t^{12} + 264008u^3t^{11} + 776536u^4 t^{10} + 1517240u^5t^9$$
$$+ 2171441u^6t^8 + 2431080u^7t^7 + 2171441u^8 t^6 + 1517240u^9t^5 + 776536u^{10}t^4 + 264008u^{11}t^3 + 54887u^{12} t^2 + 6622u^{13}t + 282u^{14}\Big)s^6$$
$$= + 2t^3u^3\Big(298t^{13} + 4418u t^{12} + 40731u^2t^{11} + 190625u^3t^{10} + 505964u^4t^9 + 877790u^5 t^8 + 1117090u^6t^7 + 1117090u^7t^6 + 877790u^8t^5$$
$$+ 505964u^9t^4 + 190625 u^{10}t^3 + 40731u^{11}t^2 + 4418u^{12}t + 298u^{13}\Big)s^5 + t^4u^4 \Big(536t^{12} + 16810ut^{11} + 131975u^2t^{10} + 463360u^3t^9$$
$$+ 936996u^4 t^8 + 1292362u^5t^7 + 1406138u^6t^6 + 1292362u^7t^5 + 936996u^8t^4 + 463360 u^9t^3 + 131975u^{10}t^2 + 16810u^{11}t + 536u^{12}\Big)s^4$$
$$+ 2t^5u^5 \Big(848t^{11} + 13094ut^{10} + 64600u^2t^9 + 162207u^3t^8 + 255217u^4 t^7 + 298424u^5t^6 + 298424u^6t^5 + 255217u^7t^4 + 162207u^8t^3$$
$$+ 64600u^9 t^2 + 13094u^{10}t + 848u^{11}\Big)s^3 + t^6u^6(t+u)^2\Big(2062 t^8 + 13122ut^7 + 32971u^2t^6 + 48846u^3t^5 + 54292u^4t^4 + 48846u^5 t^3$$
$$+ 32971u^6t^2 + 13122u^7t + 2062u^8\Big)s^2 + 4t^7u^7(t+u)^3 \Big(158t^6 + 746ut^5 + 1500u^2t^4 + 1923u^3t^3 + 1500u^4t^2 + 746u^5t + 158 u^6\Big)s$$
$$+ 8t^8u^8(t+u)^4\Big(23t^4 + 45ut^3 + 67u^2t^2 + 45u^3t + 23 u^4\Big)\Big)H(1,z)H(3,y) + 3s^2(s+t)u^2(t+u)^2\Big(\Big(17 t^6 - 424ut^5 - 1038u^2t^4 - 1108u^3t^3$$
$$- 133u^4t^2 + 300u^5t + 120u^6\Big) s^{15} + \Big(113t^7 - 2432ut^6 - 9046u^2t^5 - 13902u^3t^4 - 8627u^4t^3 + 1042 u^5t^2 + 2910u^6t + 840u^7\Big)s^{14} + \Big(361t^8$$
$$- 5422ut^7 - 30575u^2 t^6 - 64296u^3t^5 - 66643u^4t^4 - 21748u^5t^3 + 14031u^6t^2 + 12270u^7t + 2640 u^8\Big)s^{13} + \Big(745t^9 - 5220ut^8 - 49927u^2t^7$$
$$- 141944u^3 t^6 - 209881u^4t^5 - 146102u^5t^4 - 877u^6t^3 + 57146u^7t^2 + 29730u^8t + 4920 u^9\Big)s^{12} + 2\Big(546t^{10} + 2738ut^9 - 12316u^2t^8$$
$$- 77546u^3 t^7 - 169106u^4t^6 - 184458u^5t^5 - 65973u^6t^4 + 53495u^7t^3 + 62003u^8 t^2 + 22935u^9t + 3000u^{10}\Big)s^{11} + 2\Big(576t^{11} + 14602u t^{10}$$
$$= + 44395u^2t^9 + 6575u^3t^8 - 159019u^4t^7 - 299169u^5t^6 - 202668u^6 t^5 + 31945u^7t^4 + 130735u^8t^3 + 82071u^9t^2 + 23475u^{10}t + 2460 u^{11}\Big)s^{10}$$
$$+ \Big(853t^{12} + 50204ut^{11} + 239326u^2 t^{10} + 380158u^3t^9 - 5759u^4t^8 - 778482u^5t^7 - 981862u^6t^6 - 306108u^7 t^5 + 310132u^8t^4$$
$$+ 327562u^9t^3 + 138034u^{10}t^2 + 32010u^{11}t + 2640 u^{12}\Big)s^9 + \Big(421t^{13} + 48756ut^{12} + 284228u^2t^{11} + 643716 u^3t^{10} + 559919u^4t^9$$
$$- 346654u^5t^8 - 1327656u^6t^7 - 1188026u^7 t^6 - 185900u^8t^5 + 356130u^9t^4 + 244634u^{10}t^3 + 72974u^{11}t^2 + 14070 u^{12}t + 840u^{13}\Big)s^8$$
$$+ \Big(125t^{14} + 28668ut^{13} + 194932u^2 t^{12} + 556848u^3t^{11} + 840628u^4t^{10} + 624124u^5t^9 - 174342u^6 t^8 - 974758u^7t^7 - 907984u^8t^6$$
$$- 201510u^9t^5 + 170032u^{10}t^4 + 108262 u^{11}t^3 + 22743u^{12}t^2 + 3630u^{13}t + 120u^{14}\Big)s^7 + t\Big(17 t^{14} + 9532ut^{13} + 79874u^2t^{12} + 311220u^3t^{11}$$
$$+ 718808u^4 t^{10} + 1133540u^5t^9 + 1280316u^6t^8 + 835978u^7t^7 - 19040u^8t^6 - 455158 u^9t^5 - 255266u^{10}t^4 - 6292u^{11}t^3 + 26245u^{12}t^2$$
$$+ 3544u^{13}t + 420 u^{14}\Big)s^6 + t^2u\Big(1338t^{13} + 17115ut^{12} + 122578u^2 t^{11} + 473242u^3t^{10} + 1064256u^4t^9 + 1563708u^5t^8 + 1630684u^6 t^7$$
$$+ 1174609u^7t^6 + 432306u^8t^5 - 92986u^9t^4 - 158896u^{10}t^3 - 40833 u^{11}t^2 + 2954u^{12}t + 167u^{13}\Big)s^5 + t^3u^2\Big(1209 t^{12} + 30536ut^{11}$$
$$+ 210974u^2t^{10} + 653148u^3t^9 + 1105716u^4 t^8 + 1162604u^5t^7 + 908135u^6t^6 + 653214u^7t^5 + 380872u^8t^4 + 92158u^9 t^3 - 31479u^{10}t^2$$
$$- 16364u^{11}t + 67u^{12}\Big)s^4 + t^4u^3\Big(3688 t^{11} + 49261ut^{10} + 216624u^2t^9 + 465668u^3t^8 + 558504u^4t^7 + 411820 u^5t^6 + 264280u^6t^5$$

$$= + 237894u^7t^4 + 185882u^8t^3 + 72173u^9t^2 + 5494\, u^{10}t - 2344u^{11}\Big)s^3 + t^5u^4(t+u)^2\Big(4269t^8 + 21666u\, t^7 + 42801u^2t^6 + 45136u^3t^5$$

$$+ 32315u^4t^4 + 22162u^5t^3 + 17975u^6\, t^2 + 9856u^7t + 2252u^8\Big)s^2 + t^6u^5(t+u)^3\Big(762t^6 + 3225u\, t^5 + 6713u^2t^4 + 8777u^3t^3 + 5973u^4t^2$$

$$+ 1975u^5t + 49u^6\Big)s + t^7\, u^6(t+u)^4\Big(293t^4 + 568ut^3 + 843u^2t^2 + 568u^3t + 293\, u^4\Big)\Big)H(0,1,z) + 18s^2t^2(s+t)^2u^2(t+u)^2\Big(\Big(11\, t^4 + 20ut^3$$

$$+ 24u^2t^2 + 20u^3t + 11u^4\Big)s^{14} + 2\Big(33t^5 + 91u\, t^4 + 115u^2t^3 + 119u^3t^2 + 100u^4t + 44u^5\Big)s^{13} + \Big(187t^6 + 744\, ut^5 + 1142u^2t^4 + 1208u^3t^3$$

$$+ 1245u^4t^2 + 976u^5t + 336u^6\Big)\, s^{12} + 2\Big(165t^7 + 936ut^6 + 1782u^2t^5 + 1633u^3t^4 + 1366u^4t^3 + 1677\, u^5t^2 + 1363u^6t + 406u^7\Big)s^{11}$$

$$+ \Big(396t^8 + 3450ut^7 + 8220u^2\, t^6 + 5816u^3t^5 - 2041u^4t^4 - 1266u^5t^3 + 5406u^6t^2 + 5360u^7t + 1391\, u^8\Big)s^{10} + 2\Big(165t^9 + 2460ut^8$$

$$+ 7356u^2t^7 + 5316u^3t^6 - 8202\, u^4t^5 - 15297u^5t^4 - 5650u^6t^3 + 4428u^7t^2 + 4099u^8t + 897u^9\Big)\, s^9 + \Big(187t^{10} + 5044ut^9 + 19227u^2t^8$$

$$+ 21984u^3t^7 - 15915u^4\, t^6 - 62664u^5t^5 - 56568u^6t^4 - 10620u^7t^3 + 14826u^8t^2 + 9784u^9t + 1783\, u^{10}\Big)s^8 + 2\Big(33t^{11} + 1640ut^{10}$$

$$+ 8324u^2t^9 + 17396u^3\, t^8 + 12447u^4t^7 - 14606u^5t^6 - 35999u^6t^5 - 24042u^7t^4 + 1442u^8\, t^3 + 9648u^9t^2 + 4481u^{10}t + 668u^{11}\Big)s^7$$

$$+ \Big(11t^{12} + 1158u\, t^{11} + 8506u^2t^{10} + 34618u^3t^9 + 71793u^4t^8 + 62258u^5t^7 - 9538u^6\, t^6 - 52674u^7t^5 - 20480u^8t^4 + 18332u^9t^3$$

$$+ 18756u^{10}t^2 + 6076u^{11}\, t + 702u^{12}\Big)s^6 + 2u\Big(81t^{12} + 1047ut^{11} + 10102u^2\, t^{10} + 35002u^3t^9 + 52863u^4t^8 + 30585u^5t^7 - 6576u^6t^6$$

$$= - 11010u^7\, t^5 + 6574u^8t^4 + 12427u^9t^3 + 6297u^{10}t^2 + 1397u^{11}t + 113\, u^{12}\Big)s^5 + u^2\Big(129t^{12} + 6282ut^{11} + 32372u^2t^{10} + 66938\, u^3t^9$$

$$+ 63533u^4t^8 + 21894u^5t^7 + 1543u^6t^6 + 16366u^7t^5 + 25835u^8\, t^4 + 16336u^9t^3 + 5137u^{10}t^2 + 760u^{11}t + 33u^{12}\Big)s^4 + 2tu^3\, \Big(398t^{11}$$

$$+ 3081ut^{10} + 9304u^2t^9 + 14236u^3t^8 + 12582u^4t^7 + 9212\, u^5t^6 + 9802u^6t^5 + 10293u^7t^4 + 6959u^8t^3 + 2766u^9t^2 + 591u^{10}t + 48\, u^{11}\Big)s^3$$

$$+ t^2u^4(t+u)^2\Big(129t^8 + 1620ut^7 + 3541u^2t^6 + 3536\, u^3t^5 + 3022u^4t^4 + 2792u^5t^3 + 1912u^6t^2 + 746u^7t + 126u^8\Big)\, s^2 + 2t^3u^5(t+u)^3\Big(81t^6$$

$$+ 276ut^5 + 227u^2t^4 + 155u^3t^3 + 173u^4\, t^2 + 112u^5t + 38u^6\Big)s + t^4u^6(t+u)^4\Big(11t^4 - 2ut^3 + 15u^2\, t^2 + 14u^3t + 19u^4\Big)\Big)H(0,y)H(0,1,z)$$

$$- 18s^2t^2u^2(t+u)^2\, \Big(\Big(47t^4 + 116ut^3 + 144u^2t^2 + 96u^3t + 39u^4\Big)s^{16} + 2\, \Big(211t^5 + 697ut^4 + 1032u^2t^3 + 886u^3t^2 + 487u^4t + 147u^5\Big)\, s^{15}$$

$$+ 3\Big(587t^6 + 2466ut^5 + 4371u^2t^4 + 4476u^3t^3 + 3059u^4\, t^2 + 1466u^5t + 355u^6\Big)s^{14} + 2\Big(2246t^7 + 11795ut^6 + 25344u^2\, t^5 + 30555u^3t^4$$

$$+ 24384u^4t^3 + 14649u^5t^2 + 6280u^6t + 1255u^7\Big)\, s^{13} + \Big(7761t^8 + 49924ut^7 + 132409u^2t^6 + 195056u^3t^5 + 184034u^4\, t^4 + 125820u^5t^3$$

$$+ 68009u^6t^2 + 25468u^7t + 4325u^8\Big)s^{12} + 6\, \Big(1587t^9 + 12289ut^8 + 40330u^2t^7 + 74208u^3t^6 + 86445u^4t^5 + 69435\, u^5t^4 + 42172u^6t^3$$

$$+ 20076u^7t^2 + 6418u^8t + 944u^9\Big)\, s^{11} + \Big(8447t^{10} + 78224ut^9 + 319917u^2t^8 + 731786u^3t^7 + 1046888\, u^4t^6 + 1017372u^5t^5 + 727368u^6t^4$$

$$= + 404718u^7t^3 + 167685u^8t^2 + 44164\, u^9t + 5599u^{10}\Big)s^{10} + 2\Big(2688t^{11} + 30091ut^{10} + 156379\, u^2t^9 + 443281u^3t^8 + 763361u^4t^7$$

$$+ 878974u^5t^6 + 745371u^6t^5 + 500624\, u^7t^4 + 259751u^8t^3 + 91571u^9t^2 + 19006u^{10}t + 2015u^{11}\Big)s^9 + 3\, \Big(784t^{12} + 11080ut^{11} + 75305u^2t^{10}$$

$$+ 263492u^3t^9 + 545731u^4\, t^8 + 746656u^5t^7 + 740518u^6t^6 + 577912u^7t^5 + 364335u^8t^4 + 170508u^9\, t^3 + 50345u^{10}t^2 + 7940u^{11}t$$

$$+ 660u^{12}\Big)s^8 + 2\Big(318\, t^{13} + 6360ut^{12} + 59340u^2t^{11} + 252696u^3t^{10} + 631038u^4\, t^9 + 1061526u^5t^8 + 1285210u^6t^7 + 1158562u^7t^6$$

$$+ 806991u^8t^5 + 444563\, u^9t^4 + 181058u^{10}t^3 + 45120u^{11}t^2 + 5167u^{12}t + 295u^{13}\Big)\, s^7 + \Big(80t^{14} + 3066ut^{13} + 43714u^2t^{12} + 225298u^3t^{11}$$

$$+ 680647\, u^4t^{10} + 1441224u^5t^9 + 2222801u^6t^8 + 2456276u^7t^7 + 1932681u^8\, t^6 + 1122364u^9t^5 + 510799u^{10}t^4 + 177102u^{11}t^3$$

$$+ 37210u^{12}\, t^2 + 2790u^{13}t + 80u^{14}\Big)s^6 + 6tu\Big(58t^{13} + 1689u\, t^{12} + 11116u^2t^{11} + 42847u^3t^{10} + 116103u^4t^9 + 225254u^5\, t^8 + 308029u^6t^7$$

$$+ 293932u^7t^6 + 196649u^8t^5 + 94795u^9t^4 + 34537u^{10}\, t^3 + 9491u^{11}t^2 + 1574u^{12}t + 58u^{13}\Big)s^5 + t^2u^2\Big(1092\, t^{12} + 11428ut^{11} + 67756u^2t^{10}$$

$$+ 245052u^3t^9 + 575825u^4t^8 + 929600\, u^5t^7 + 1064520u^6t^6 + 875584u^7t^5 + 515905u^8t^4 + 213096u^9t^3 + 59096\, u^{10}t^2 + 10508u^{11}t$$

$$+ 1092u^{12}\Big)s^4 + 2t^3u^3\Big(394\, t^{11} + 5820ut^{10} + 29696u^2t^9 + 83756u^3t^8 + 156154u^4t^7 + 208169u^5\, t^6 + 204213u^6t^5 + 148024u^7t^4$$

$$+ 77656u^8t^3 + 27425u^9t^2 + 5475u^{10}\, t + 394u^{11}\Big)s^3 + 6t^4u^4(t+u)^2\Big(173t^8 + 987ut^7 + 2643u^2\, t^6 + 4684u^3t^5 + 5690u^4t^4 + 4568u^5t^3$$

$$+ 2519u^6t^2 + 941u^7t + 173\, u^8\Big)s^2 + 2t^5u^5(t+u)^3\Big(150t^6 + 629ut^5 + 1361u^2t^4 + 1851\, u^3t^3 + 1353u^4t^2 + 606u^5t + 150u^6\Big)s$$

$$+ 66t^6u^6(t+u)^4\, \Big(t^2 + ut + u^2\Big)^2\Big)H(2,y)H(0,1,z) - 3s^2(s+t)^2u^2(t+u)^2\, \Big(\Big(259t^6 + 616ut^5 + 426u^2t^4 - 68u^3t^3 + 109u^4t^2 + 300u^5\, t$$

$$+ 120u^6\Big)s^{14} + 2\Big(774t^7 + 2870ut^6 + 3872u^2t^5 + 1903u^3\, t^4 + 425u^4t^3 + 1339u^5t^2 + 1395u^6t + 420u^7\Big)s^{13} + \Big(4367\, t^8 + 21396ut^7$$

$$= + 40759u^2t^6 + 37858u^3t^5 + 19215u^4t^4 + 14984u^5\, t^3 + 18489u^6t^2 + 11430u^7t + 2640u^8\Big)s^{12} + 2\Big(3840t^9 + 23121\, ut^8 + 56244u^2t^7$$

$$+ 72977u^3t^6 + 57956u^4t^5 + 40878u^5t^4 + 39401u^6\, t^3 + 31286u^7t^2 + 13545u^8t + 2460u^9\Big)s^{11} + 2\Big(4602\, t^{10} + 33792ut^9 + 102876u^2t^8$$

$$+ 170653u^3t^7 + 175620u^4t^6 + 137297u^5\, t^5 + 118565u^6t^4 + 104385u^7t^3 + 62005u^8t^2 + 20475u^9t + 3000\, u^{10}\Big)s^{10} + 2\Big(3840t^{11}$$

$$+ 35847ut^{10} + 140082u^2\, t^9 + 290657u^3t^8 + 356593u^4t^7 + 294815u^5t^6 + 230739u^6t^5 + 215383u^7\, t^4 + 164395u^8t^3 + 77358u^9t^2$$

$$+ 20475u^{10}t + 2460u^{11}\Big)\, s^9 + \Big(4367t^{12} + 55904ut^{11} + 291138u^2t^{10} + 756156u^3\, t^9 + 1095009u^4t^8 + 963078u^5t^7 + 649134u^6t^6$$

$$+ 554262u^7t^5 + 519114u^8\, t^4 + 328790u^9t^3 + 124010u^{10}t^2 + 27090u^{11}t + 2640u^{12}\Big)s^8 + 2\, \Big(774t^{13} + 15265ut^{12} + 110242u^2t^{11}$$

$$+ 363648u^3t^{10} + 642258\, u^4t^9 + 661734u^5t^8 + 447563u^6t^7 + 300068u^7t^6 + 277131u^8t^5 + 215383\, u^9t^4 + 104385u^{10}t^3 + 31286u^{11}t^2$$

$$+ 5715u^{12}t + 420u^{13}\Big)\, s^7 + \Big(259t^{14} + 10260ut^{13} + 111068u^2t^{12} + 486996u^3\, t^{11} + 1107162u^4t^{10} + 1479900u^5t^9 + 1291792u^6t^8$$

$$+ 895126u^7\, t^7 + 649134u^8t^6 + 461478u^9t^5 + 237130u^{10}t^4 + 78802u^{11}t^3 + 18489\, u^{12}t^2 + 2790u^{13}t + 120u^{14}\Big)s^6 + 2tu\Big(777t^{13}$$

$$+ 16074\, ut^{12} + 104819u^2t^{11} + 330308u^3t^{10} + 608514u^4t^9 + 739950u^5\, t^8 + 661734u^6t^7 + 481539u^7t^6 + 294815u^8t^5 + 137297u^9t^4$$

$$= + 40878u^{10} t^3 + 7492u^{11}t^2 + 1339u^{12}t + 150u^{13}\Big)s^5 + t^2u^2\Big(3885 t^{12} + 50940ut^{11} + 249466u^2t^{10} + 660616u^3t^9 + 1107162u^4 t^8$$
$$+ 1284516u^5t^7 + 1095009u^6t^6 + 713186u^7t^5 + 351240u^8t^4 + 115912u^9 t^3 + 19215u^{10}t^2 + 850u^{11}t + 109u^{12}\Big)s^4 + 2t^3u^3\Big(2590 t^{11}$$
$$+ 25470ut^{10} + 104819u^2t^9 + 243498u^3t^8 + 363648u^4t^7 + 378078 u^5t^6 + 290657u^6t^5 + 170653u^7t^4 + 72977u^8t^3 + 18929u^9t^2 + 1903 u^{10}t$$
$$- 34u^{11}\Big)s^3 + t^4u^4(t+u)^2\Big(3885t^8 + 24378u t^7 + 58427u^2t^6 + 79252u^3t^5 + 74207u^4t^4 + 52498u^5t^3 + 26549u^6 t^2 + 6892u^7t + 426u^8\Big)s^2$$
$$+ 2t^5u^5(t+u)^3\Big(777t^6 + 2799u t^5 + 4537u^2t^4 + 5167u^3t^3 + 3936u^4t^2 + 1946u^5t + 308u^6\Big)s + t^6 u^6(t+u)^4\Big(259t^4 + 512ut^3 + 765u^2t^2$$
$$+ 512u^3t + 259 u^4\Big)\Big)H(0,2,y) - 18s^2t^2u^2(t+u)^3\Big(2t\Big(7t^2 + 15u t + 12u^2\Big)s^{16} + 12\Big(8t^4 + 24ut^3 + 29u^2t^2 + 12u^3 t - u^4\Big)s^{15} + 3\Big(101t^5$$
$$+ 423ut^4 + 720u^2t^3 + 556u^3t^2 + 111 u^4t - 31u^5\Big)s^{14} + \Big(598t^6 + 3066ut^5 + 7134u^2t^4 + 8992u^3 t^3 + 5154u^4t^2 + 510u^5t - 318u^6\Big)s^{13}$$
$$+ \Big(851t^7 + 4495u t^6 + 13290u^2t^5 + 25506u^3t^4 + 26949u^4t^3 + 12309u^5t^2 + 688u^6t - 652 u^7\Big)s^{12} + 6\Big(158t^8 + 586ut^7 + 2628u^2t^6$$
$$+ 7708u^3t^5 + 11649 u^4t^4 + 9092u^5t^3 + 3324u^6t^2 + 218u^7t - 155u^8\Big)s^{11} + \Big(851 t^9 - 417ut^8 + 9564u^2t^7 + 62150u^3t^6 + 123813u^4t^5$$
$$+ 121269u^5t^4 + 67982 u^6t^3 + 23208u^7t^2 + 2838u^8t - 1002u^9\Big)s^{10} + 2\Big(299 t^{10} - 2075ut^9 - 1896u^2t^8 + 23751u^3t^7 + 72903u^4t^6$$
$$+ 97515u^5 t^5 + 71423u^6t^4 + 31963u^7t^3 + 10800u^8t^2 + 1966u^9t - 409u^{10}\Big) s^9 + 3\Big(101t^{11} - 1463ut^{10} - 3896u^2t^9 + 208u^3t^8 + 20429u^4 t^7$$
$$= + 58669u^5t^6 + 80268u^6t^5 + 55172u^7t^4 + 19400u^8t^3 + 4832u^9 t^2 + 1000u^{10}t - 156u^{11}\Big)s^8 + 2\Big(48t^{12} - 1128ut^{11} - 4410 u^2t^{10}$$
$$- 17090u^3t^9 - 31983u^4t^8 + 6084u^5t^7 + 100874u^6t^6 + 143160 u^7t^5 + 88338u^8t^4 + 23800u^9t^3 + 2622u^{10}t^2 + 582u^{11}t - 81 u^{12}\Big)s^7$$
$$+ \Big(14t^{13} - 572ut^{12} - 3036u^2t^{11} - 35818u^3 t^{10} - 106376u^4t^9 - 101688u^5t^8 + 48822u^6t^7 + 213630u^7t^6 + 239397u^8 t^5 + 134809u^9t^4$$
$$+ 31274u^{10}t^3 - 36u^{11}t^2 + 157u^{12}t - 25 u^{13}\Big)s^6 - 6tu\Big(8t^{12} + 37ut^{11} + 3213u^2t^{10} + 11984u^3 t^9 + 14592u^4t^8 + 1987u^5t^7 - 12914u^6t^6$$
$$- 20563u^7t^5 - 20499u^8 t^4 - 11612u^9t^3 - 2423u^{10}t^2 + 147u^{11}t + 3u^{12}\Big)s^5 + t^2u^2 \Big(102t^{11} - 5234ut^{10} - 25530u^2t^9 - 39126u^3t^8$$
$$- 12408u^4 t^7 + 24648u^5t^6 + 35505u^6t^5 + 40505u^7t^4 + 40476u^8t^3 + 21240u^9 t^2 + 3629u^{10}t - 267u^{11}\Big)s^4 + 2t^3u^3\Big(-266t^{10} - 1804u t^9$$
$$- 4752u^2t^8 - 4830u^3t^7 + 378u^4t^6 + 5301u^5t^5 + 6806u^6t^4 + 6178u^7 t^3 + 3642u^8t^2 + 1119u^9t + 156u^{10}\Big)s^3 + 3t^4u^4(t+u)^2\Big(16 t^7$$
$$- 382ut^6 - 532u^2t^5 + 8u^3t^4 + 467u^4t^3 + 637u^5t^2 + 265u^6t - 71 u^7\Big)s^2 - 2t^5u^5(t+u)^3\Big(48t^5 + 151ut^4 - 84u^2t^3 + 18u^3 t^2 - 160u^4t$$
$$- 15u^5\Big)s + t^6u^7(t+u)^3\Big(12t^3 + 3ut^2 + 2u^2 t - 11u^3\Big)\Big)H(1,z)H(0,3,y) - 3s^2(s+t)^2u^2(t+u)^2 \Big(\Big(259t^6 + 616ut^5 + 426u^2t^4 - 68u^3t^3$$
$$+ 109u^4t^2 + 300u^5 t + 120u^6\Big)s^{14} + 2\Big(774t^7 + 2870ut^6 + 3872u^2t^5 + 1903u^3 t^4 + 425u^4t^3 + 1339u^5t^2 + 1395u^6t + 420u^7\Big)s^{13} + \Big(4367 t^8$$
$$+ 21396ut^7 + 40759u^2t^6 + 37858u^3t^5 + 19215u^4t^4 + 14984u^5 t^3 + 18489u^6t^2 + 11430u^7t + 2640u^8\Big)s^{12} + 2\Big(3840t^9 + 23121 ut^8$$
$$+ 56244u^2t^7 + 72977u^3t^6 + 57956u^4t^5 + 40878u^5t^4 + 39401u^6 t^3 + 31286u^7t^2 + 13545u^8t + 2460u^9\Big)s^{11} + 2\Big(4602 t^{10} + 33792ut^9$$
$$+ 102876u^2t^8 + 170653u^3t^7 + 175620u^4t^6 + 137297u^5 t^5 + 118565u^6t^4 + 104385u^7t^3 + 62005u^8t^2 + 20475u^9t + 3000 u^{10}\Big)s^{10}$$
$$= + 2\Big(3840t^{11} + 35847ut^{10} + 140082u^2 t^9 + 290657u^3t^8 + 356593u^4t^7 + 294815u^5t^6 + 230739u^6t^5 + 215383u^7 t^4 + 164395u^8t^3$$
$$+ 77358u^9t^2 + 20475u^{10}t + 2460u^{11}\Big) s^9 + \Big(4367t^{12} + 55904ut^{11} + 291138u^2t^{10} + 756156u^3 t^9 + 1095009u^4t^8 + 963078u^5t^7$$
$$+ 649134u^6t^6 + 554262u^7t^5 + 519114u^8 t^4 + 328790u^9t^3 + 124010u^{10}t^2 + 27090u^{11}t + 2640u^{12}\Big)s^8 + 2 \Big(774t^{13} + 15265ut^{12}$$
$$+ 110242u^2t^{11} + 363648u^3t^{10} + 642258 u^4t^9 + 661734u^5t^8 + 447563u^6t^7 + 300068u^7t^6 + 277131u^8t^5 + 215383 u^9t^4 + 104385u^{10}t^3$$
$$+ 31286u^{11}t^2 + 5715u^{12}t + 420u^{13}\Big) s^7 + \Big(259t^{14} + 10260ut^{13} + 111068u^2t^{12} + 486996u^3 t^{11} + 1107162u^4t^{10} + 1479900u^5t^9$$
$$+ 1291792u^6t^8 + 895126u^7 t^7 + 649134u^8t^6 + 461478u^9t^5 + 237130u^{10}t^4 + 78802u^{11}t^3 + 18489 u^{12}t^2 + 2790u^{13}t + 120u^{14}\Big)s^6$$
$$+ 2tu\Big(777t^{13} + 16074 ut^{12} + 104819u^2t^{11} + 330308u^3t^{10} + 608514u^4t^9 + 739950u^5 t^8 + 661734u^6t^7 + 481539u^7t^6 + 294815u^8t^5$$
$$+ 137297u^9t^4 + 40878u^{10} t^3 + 7492u^{11}t^2 + 1339u^{12}t + 150u^{13}\Big)s^5 + t^2u^2\Big(3885 t^{12} + 50940ut^{11} + 249466u^2t^{10} + 660616u^3t^9$$
$$+ 1107162u^4 t^8 + 1284516u^5t^7 + 1095009u^6t^6 + 713186u^7t^5 + 351240u^8t^4 + 115912u^9 t^3 + 19215u^{10}t^2 + 850u^{11}t + 109u^{12}\Big)s^4$$
$$+ 2t^3u^3\Big(2590 t^{11} + 25470ut^{10} + 104819u^2t^9 + 243498u^3t^8 + 363648u^4t^7 + 378078 u^5t^6 + 290657u^6t^5 + 170653u^7t^4 + 72977u^8t^3$$
$$= + 18929u^9t^2 + 1903 u^{10}t - 34u^{11}\Big)s^3 + t^4u^4(t+u)^2\Big(3885t^8 + 24378u t^7 + 58427u^2t^6 + 79252u^3t^5 + 74207u^4t^4 + 52498u^5t^3$$
$$+ 26549u^6 t^2 + 6892u^7t + 426u^8\Big)s^2 + 2t^5u^5(t+u)^3\Big(777t^6 + 2799u t^5 + 4537u^2t^4 + 5167u^3t^3 + 3936u^4t^2 + 1946u^5t + 308u^6\Big)s$$
$$+ t^6 u^6(t+u)^4\Big(259t^4 + 512ut^3 + 765u^2t^2 + 512u^3t + 259 u^4\Big)\Big)H(2,0,y) - 18s^2t^2u^2(t+u)^2\Big(\Big(83t^4 + 224u t^3 + 312u^2t^2 + 224u^3t$$
$$+ 83u^4\Big)s^{16} + \Big(646t^5 + 2236ut^4 + 3738 u^2t^3 + 3738u^3t^2 + 2236u^4t + 646u^5\Big)s^{15} + 3\Big(795t^6 + 3434u t^5 + 6735u^2t^4 + 8252u^3t^3 + 6735u^4t^2$$
$$+ 3434u^5t + 795u^6\Big) s^{14} + \Big(5590t^7 + 29722ut^6 + 68154u^2t^5 + 96362u^3t^4 + 96362u^4 t^3 + 68154u^5t^2 + 29722u^6t + 5590u^7\Big)s^{13} + \Big(9297t^8$$
$$+ 59792u t^7 + 162893u^2t^6 + 265856u^3t^5 + 307366u^4t^4 + 265856u^5t^3 + 162893u^6 t^2 + 59792u^7t + 9297u^8\Big)s^{12} + 6\Big(1912t^9 + 14697ut^8$$
$$+ 48146 u^2t^7 + 92627u^3t^6 + 123322u^4t^5 + 123322u^5t^4 + 92627u^6t^3 + 48146u^7 t^2 + 14697u^8t + 1912u^9\Big)s^{11} + \Big(10571t^{10} + 97460ut^9$$
$$+ 390525 u^2t^8 + 895950u^3t^7 + 1372200u^4t^6 + 1560372u^5t^5 + 1372200u^6 t^4 + 895950u^7t^3 + 390525u^8t^2 + 97460u^9t + 10571u^{10}\Big)s^{10}$$
$$+ 2 \Big(3555t^{11} + 40253ut^{10} + 202415u^2t^9 + 561210u^3t^8 + 989305u^4 t^7 + 1257298u^5t^6 + 1257298u^6t^5 + 989305u^7t^4 + 561210u^8t^3$$
$$+ 202415u^9 t^2 + 40253u^{10}t + 3555u^{11}\Big)s^9 + 3\Big(1100t^{12} + 16144u t^{11} + 105097u^2t^{10} + 356156u^3t^9 + 736699u^4t^8 + 1061804u^5 t^7$$
$$+ 1182566u^6t^6 + 1061804u^7t^5 + 736699u^8t^4 + 356156u^9t^3 + 105097 u^{10}t^2 + 16144u^{11}t + 1100u^{12}\Big)s^8 + 2\Big(471t^{13} + 10059u t^{12}$$

$$=+89007u^2t^{11}+370856u^3t^{10}+918809u^4t^9+1568172u^5t^8+2010976u^6t^7+2010976u^7t^6+1568172u^8t^5+918809u^9t^4+370856u^{10}t^3$$
$$+89007u^{11}t^2+10059u^{12}t+471u^{13}\Big)s^7+\Big(124t^{14}+5166ut^{13}+68942u^2t^{12}+360694u^3t^{11}+1093775u^4t^{10}+2281660u^5t^9$$
$$+3515021u^6t^8+4060864u^7t^7+3515021u^8t^6+2281660u^9t^5+1093775u^{10}t^4+360694u^{11}t^3+68942u^{12}t^2+5166u^{13}t+124u^{14}\Big)s^6$$
$$+6tu\Big(102t^{13}+2718ut^{12}+19299u^2t^{11}+75952u^3t^{10}+198691u^4t^9+370135u^5t^8+503605u^6t^7+503605u^7t^6+370135u^8t^5$$
$$+198691u^9t^4+75952u^{10}t^3+19299u^{11}t^2+2718u^{12}t+102u^{13}\Big)s^5+t^2u^2\Big(1752t^{12}+21508ut^{11}+127396u^2t^{10}+438996u^3t^9$$
$$+986209u^4t^8+1559032u^5t^7+1808388u^6t^6+1559032u^7t^5+986209u^8t^4+438996u^9t^3+127396u^{10}t^2+21508u^{11}t+1752u^{12}\Big)s^4$$
$$+2t^3u^3\Big(834t^{11}+10755ut^{10}+52845u^2t^9+146657u^3t^8+271322u^4t^7+363015u^5t^6+363015u^6t^5+271322u^7t^4+146657u^8t^3$$
$$+52845u^9t^2+10755u^{10}t+834u^{11}\Big)s^3+6t^4u^4(t+u)^2\Big(283t^8+1733ut^7+4779u^2t^6+8360u^3t^5+10092u^4t^4+8360u^5t^3+4779u^6t^2$$
$$+1733u^7t+283u^8\Big)s^2+2t^5u^5(t+u)^3\Big(282t^6+1178ut^5+2417u^2t^4+3162u^3t^3+2417u^4t^2+1178u^5t+282u^6\Big)s+110t^6u^6(t+u)^4\Big(t^2$$
$$+ut+u^2\Big)^2\Big)H(1,z)H(2,3,y)+9s^2(t+u)^2\Big(2\Big(20t^8+50ut^7+21u^2t^6-82u^3t^5-102u^4t^4-82u^5t^3+21u^6t^2+50u^7t+20u^8\Big)s^{16}$$
$$+2\Big(140t^9+505ut^8+568u^2t^7-298u^3t^6-1069u^4t^5-1069u^5t^4-298u^6t^3+568u^7t^2+505u^8t+140u^9\Big)s^{15}+\Big(880t^{10}+4370ut^9$$
$$=+8221u^2t^8+4262u^3t^7-6199u^4t^6-10932u^5t^5-6199u^6t^4+4262u^7t^3+8221u^8t^2+4370u^9t+880u^{10}\Big)s^{14}+2\Big(820t^{11}+5395ut^{10}$$
$$+14563u^2t^9+17988u^3t^8+5650u^4t^7-8508u^5t^6-8508u^6t^5+5650u^7t^4+17988u^8t^3+14563u^9t^2+5395u^{10}t+820u^{11}\Big)s^{13}$$
$$+\Big(2000t^{12}+16930ut^{11}+60889u^2t^{10}+115200u^3t^9+112888u^4t^8+52018u^5t^7+18062u^6t^6+52018u^7t^5+112888u^8t^4+115200u^9t^3$$
$$+60889u^{10}t^2+16930u^{11}t+2000u^{12}\Big)s^{12}+2\Big(820t^{13}+8825ut^{12}+40630u^2t^{11}+106430u^3t^{10}+162506u^4t^9+145433u^5t^8$$
$$+94054u^6t^7+94054u^7t^6+145433u^8t^5+162506u^9t^4+106430u^{10}t^3+40630u^{11}t^2+8825u^{12}t+820u^{13}\Big)s^{11}+\Big(880t^{14}+12310ut^{13}$$
$$+71305u^2t^{12}+252860u^3t^{11}+542940u^4t^{10}+672624u^5t^9+495141u^6t^8+346704u^7t^7+495141u^8t^6+672624u^9t^5+542940u^{10}t^4$$
$$+252860u^{11}t^3+71305u^{12}t^2+12310u^{13}t+880u^{14}\Big)s^{10}+2\Big(140t^{15}+2785ut^{14}+20491u^2t^{13}+99450u^3t^{12}+294436u^4t^{11}$$
$$+490859u^5t^{10}+448931u^6t^9+254956u^7t^8+254956u^8t^7+448931u^9t^6+490859u^{10}t^5+294436u^{11}t^4+99450u^{12}t^3+20491u^{13}t^2$$
$$+2785u^{14}t+140u^{15}\Big)s^9+\Big(40t^{16}+1490ut^{15}+14845u^2t^{14}+102820u^3t^{13}+419280u^4t^{12}+950788u^5t^{11}+1225773u^6t^{10}$$
$$+975710u^7t^9+743060u^8t^8+975710u^9t^7+1225773u^{10}t^6+950788u^{11}t^5+419280u^{12}t^4+102820u^{13}t^3+14845u^{14}t^2+1490u^{15}t$$
$$+40u^{16}\Big)s^8+2tu\Big(90t^{15}+1540ut^{14}+16932u^2t^{13}+96123u^3t^{12}+304282u^4t^{11}+583798u^5t^{10}+762212u^6t^9+803305u^7t^8$$
$$+803305u^8t^7+762212u^9t^6+583798u^{10}t^5+304282u^{11}t^4+96123u^{12}t^3+16932u^{13}t^2+1540u^{14}t+90u^{15}\Big)s^7+t^2u^2\Big(282t^{14}$$
$$=+6622ut^{13}+54887u^2t^{12}+264008u^3t^{11}+776536u^4t^{10}+1517240u^5t^9+2171441u^6t^8+2431080u^7t^7+2171441u^8t^6+1517240u^9t^5$$
$$+776536u^{10}t^4+264008u^{11}t^3+54887u^{12}t^2+6622u^{13}t+282u^{14}\Big)s^6+2t^3u^3\Big(298t^{13}+4418ut^{12}+40731u^2t^{11}+190625u^3t^{10}$$
$$+505964u^4t^9+877790u^5t^8+1117090u^6t^7+1117090u^7t^6+877790u^8t^5+505964u^9t^4+190625u^{10}t^3+40731u^{11}t^2+4418u^{12}t$$
$$+298u^{13}\Big)s^5+t^4u^4\Big(536t^{12}+16810ut^{11}+131975u^2t^{10}+463360u^3t^9+936996u^4t^8+1292362u^5t^7+1406138u^6t^6+1292362u^7t^5$$
$$+936996u^8t^4+463360u^9t^3+131975u^{10}t^2+16810u^{11}t+536u^{12}\Big)s^4+2t^5u^5\Big(848t^{11}+13094ut^{10}+64600u^2t^9+162207u^3t^8$$
$$+255217u^4t^7+298424u^5t^6+298424u^6t^5+255217u^7t^4+162207u^8t^3+64600u^9t^2+13094u^{10}t+848u^{11}\Big)s^3+t^6u^6(t+u)^2\Big(2062t^8$$
$$+13122ut^7+32971u^2t^6+48846u^3t^5+54292u^4t^4+48846u^5t^3+32971u^6t^2+13122u^7t+2062u^8\Big)s^2+4t^7u^7(t+u)^3\Big(158t^6$$
$$+746ut^5+1500u^2t^4+1923u^3t^3+1500u^4t^2+746u^5t+158u^6\Big)s+8t^8u^8(t+u)^4\Big(23t^4+45ut^3+67u^2t^2+45u^3t+23u^4\Big)\Big)H(3,2,y)$$
$$=-36s^2t^2u^3(t+u)^2\Big(-12t\Big(4t^2+7ut+4u^2\Big)s^{16}-\Big(413t^4+1112ut^3+1066u^2t^2+344u^3t-23u^4\Big)s^{15}-3\Big(549t^5+2080ut^4+2910u^2t^3$$
$$+1760u^3t^2+293u^4t-64u^5\Big)s^{14}-\Big(3849t^6+19818ut^5+38435u^2t^4+35276u^3t^3+14007u^4t^2+482u^5t-715u^6\Big)s^{13}-\Big(5797t^7$$
$$+40039ut^6+104248u^2t^5+133574u^3t^4+86245u^4t^3+21235u^5t^2-2834u^6t-1572u^7\Big)s^{12}-3\Big(1839t^8+18438ut^7+63926u^2t^6$$
$$+107954u^3t^5+97302u^4t^4+43534u^5t^3+4746u^6t^2-2870u^7t-757u^8\Big)s^{11}-\Big(2469t^9+52233ut^8+250929u^2t^7+546879u^3t^6$$
$$+638973u^4t^5+405133u^5t^4+113599u^6t^3-8307u^7t^2-12530u^8t-2258u^9\Big)s^{10}+\Big(985t^{10}-31352ut^9-228083u^2t^8-646444u^3t^7$$
$$-969791u^4t^6-819288u^5t^5-357472u^6t^4-38248u^7t^3+28876u^8t^2+11404u^9t+1557u^{10}\Big)s^9+3\Big(727t^{11}-3289ut^{10}-45068u^2t^9$$
$$-168873u^3t^8-330122u^4t^7-375394u^5t^6-241202u^6t^5-66685u^7t^4+10259u^8t^3+10981u^9t^2+2326u^{10}t+240u^{11}\Big)s^8+\Big(1386t^{12}$$
$$-6ut^{11}-44480u^2t^{10}-241022u^3t^9-645063u^4t^8-1001980u^5t^7-922078u^6t^6-455700u^7t^5-58195u^8t^4+50450u^9t^3+22836u^{10}t^2$$
$$+2866u^{11}t+202u^{12}\Big)s^7+\Big(420t^{13}+1149ut^{12}+3u^2t^{11}-48559u^3t^{10}-253354u^4t^9-575952u^5t^8-717884u^6t^7-512534u^7t^6$$
$$-180305u^8t^5+7313u^9t^4+31999u^{10}t^3+10093u^{11}t^2+729u^{12}t+26u^{13}\Big)s^6+3t\Big(16t^{13}+94ut^{12}+2346u^2t^{11}+5138u^3t^{10}-15062u^4t^9$$
$$-74024u^5t^8-122999u^6t^7-106460u^7t^6-49523u^8t^5-8048u^9t^4+4593u^{10}t^3+3534u^{11}t^2+893u^{12}t+30u^{13}\Big)s^5+t^2u\Big(-6t^{12}$$
$$+2684ut^{11}+12044u^2t^{10}+7299u^3t^9-50331u^4t^8-127806u^5t^7-131304u^6t^6-58843u^7t^5+1403u^8t^4+13260u^9t^3+6238u^{10}t^2$$
$$+1774u^{11}t+336u^{12}\Big)s^4+2t^3u^2\Big(166t^{11}+1146ut^{10}+2486u^2t^9-448u^3t^8-9790u^4t^7-15423u^5t^6-8416u^6t^5+3041u^7t^4$$

$$= + 7002u^8t^3 + 4035u^9t^2 + 960u^{10}t + 57u^{11}\Big)s^3 + 3t^4\ u^3(t+u)^2\Big(15t^8 + 240ut^7 + 369u^2t^6 - u^3t^5 - 172u^4t^4 + 126u^5\ t^3 + 429u^6t^2$$

$$+ 355u^7t + 103u^8\Big)s^2 + t^5u^4(t+u)^3\Big(90\ t^6 + 272ut^5 + 116u^2t^4 + 114u^3t^3 + 135u^4t^2 + 117u^5t + 66u^6\Big)\ s + t^6u^5(t+u)^4\Big(11t^4 - 2ut^3$$

$$+ 15u^2t^2 + 14u^3t + 19\ u^4\Big)\Big)H(0,0,1,z) + 18s^2t^2(s+t)^2u^2\Big(2\Big(55t^6 + 282\ ut^5 + 849u^2t^4 + 834u^3t^3 + 876u^4t^2 + 306u^5t + 62u^6\Big)s^{14}$$

$$+ 2\ \Big(330t^7 + 2024ut^6 + 6897u^2t^5 + 10755u^3t^4 + 10754u^4t^3 + 8154u^5\ t^2 + 2583u^6t + 471u^7\Big)s^{13} + 2\Big(935t^8 + 6797ut^7 + 25584u^2\ t^6$$

$$+ 52845u^3t^5 + 63698u^4t^4 + 57897u^5t^3 + 34471u^6t^2 + 10059u^7t + 1650\ u^8\Big)s^{12} + 2\Big(1650t^9 + 14229ut^8 + 58953u^2t^7 + 146657u^3\ t^6$$

$$+ 219498u^4t^5 + 227856u^5t^4 + 180347u^6t^3 + 89007u^7t^2 + 24216u^8\ t + 3555u^9\Big)s^{11} + \Big(3960t^{10} + 40664ut^9 + 189546u^2t^8 + 542644\ u^3t^7$$

$$+ 986209u^4t^6 + 1192146u^5t^5 + 1093775u^6t^4 + 741712u^7t^3 + 315291\ u^8t^2 + 80506u^9t + 10571u^{10}\Big)s^{10} + 2\Big(1650t^{11} + 20332u\ t^{10}$$

$$+ 110712u^2t^9 + 363015u^3t^8 + 779516u^4t^7 + 1110405u^5t^6 + 1140830\ u^6t^5 + 918809u^7t^4 + 534234u^8t^3 + 202415u^9t^2 + 48730u^{10}t$$

$$+ 5736\ u^{11}\Big)s^9 + \Big(1870t^{12} + 28458ut^{11} + 189546u^2t^{10} + 726030\ u^3t^9 + 1808388u^4t^8 + 3021630u^5t^7 + 3515021u^6t^6 + 3136344u^7\ t^5$$

$$+ 2210097u^8t^4 + 1122420u^9t^3 + 390525u^{10}t^2 + 88182u^{11}t + 9297\ u^{12}\Big)s^8 + 2\Big(330t^{13} + 6797ut^{12} + 58953u^2t^{11} + 271322\ u^3t^{10}$$

$$+ 779516u^4t^9 + 1510815u^5t^8 + 2030432u^6t^7 + 2010976u^7\ t^6 + 1592706u^8t^5 + 989305u^9t^4 + 447975u^{10}t^3 + 144438u^{11}t^2 + 29896\ u^{12}t$$

$$+ 2795u^{13}\Big)s^7 + \Big(110t^{14} + 4048ut^{13} + 51168u^2\ t^{12} + 293314u^3t^{11} + 986209u^4t^{10} + 2220810u^5t^9 + 3515021u^6\ t^8 + 4021952u^7t^7$$

$$= + 3547698u^8t^6 + 2514596u^9t^5 + 1372200u^{10}\ t^4 + 555762u^{11}t^3 + 162893u^{12}t^2 + 29722u^{13}t + 2385u^{14}\Big)\ s^6 + 2u\Big(282t^{14} + 6897ut^{13}$$

$$+ 52845u^2t^{12} + 219498u^3\ t^{11} + 596073u^4t^{10} + 1140830u^5t^9 + 1568172u^6t^8 + 1592706u^7\ t^7 + 1257298u^8t^6 + 780186u^9t^5 + 369966u^{10}t^4$$

$$+ 132928u^{11}t^3 + 34077\ u^{12}t^2 + 5151u^{13}t + 323u^{14}\Big)s^5 + u^2\Big(1698t^{14} + 21510u\ t^{13} + 127396u^2t^{12} + 455712u^3t^{11} + 1093775u^4t^{10}$$

$$+ 1837618u^5\ t^9 + 2210097u^6t^8 + 1978610u^7t^7 + 1372200u^8t^6 + 739932u^9t^5 + 307366\ u^{10}t^4 + 96362u^{11}t^3 + 20205u^{12}t^2 + 2236u^{13}t$$

$$+ 83u^{14}\Big)\ s^4 + 2tu^3\Big(834t^{13} + 10754ut^{12} + 57897u^2t^{11} + 180347u^3\ t^{10} + 370856u^4t^9 + 534234u^5t^8 + 561210u^6t^7 + 447975u^7t^6$$

$$+ 277881\ u^8t^5 + 132928u^9t^4 + 48181u^{10}t^3 + 12378u^{11}t^2 + 1869u^{12}t + 112\ u^{13}\Big)s^3 + t^2u^4(t+u)^2\Big(1752t^{10} + 12804ut^9 + 41582u^2\ t^8$$

$$+ 82046u^3t^7 + 109617u^4t^6 + 103550u^5t^5 + 73808u^6t^4 + 37710u^7\ t^3 + 13665u^8t^2 + 3114u^9t + 312u^{10}\Big)s^2 + 2t^3u^5(t+u)^3\ \Big(306t^8$$

$$+ 1665ut^7 + 4146u^2t^6 + 6477u^3t^5 + 6719u^4t^4 + 4996u^5\ t^3 + 2469u^6t^2 + 782u^7t + 112u^8\Big)s + t^4u^6(t+u)^6\Big(124\ t^4 + 198ut^3 + 252u^2t^2$$

$$+ 148u^3t + 83u^4\Big)\Big)H(0,1,0,y) - 18s^2\ t^2u^2(t+u)^3\Big(2t\Big(7t^2 + 15ut + 12u^2\Big)s^{16} + 12\Big(8\ t^4 + 24ut^3 + 29u^2t^2 + 12u^3t - u^4\Big)s^{15}$$

$$+ 3\Big(101t^5 + 423u\ t^4 + 720u^2t^3 + 556u^3t^2 + 111u^4t - 31u^5\Big)s^{14} + \Big(598\ t^6 + 3066ut^5 + 7134u^2t^4 + 8992u^3t^3 + 5154u^4t^2 + 510u^5t$$

$$- 318\ u^6\Big)s^{13} + \Big(851t^7 + 4495ut^6 + 13290u^2t^5 + 25506u^3t^4 + 26949\ u^4t^3 + 12309u^5t^2 + 688u^6t - 652u^7\Big)s^{12} + 6\Big(158t^8 + 586u\ t^7$$

$$+ 2628u^2t^6 + 7708u^3t^5 + 11649u^4t^4 + 9092u^5t^3 + 3324u^6t^2 + 218\ u^7t - 155u^8\Big)s^{11} + \Big(851t^9 - 417ut^8 + 9564u^2t^7 + 62150u^3\ t^6$$

$$= + 123813u^4t^5 + 121269u^5t^4 + 67982u^6t^3 + 23208u^7t^2 + 2838u^8\ t - 1002u^9\Big)s^{10} + 2\Big(299t^{10} - 2075ut^9 - 1896u^2t^8 + 23751\ u^3t^7$$

$$+ 72903u^4t^6 + 97515u^5t^5 + 71423u^6t^4 + 31963u^7t^3 + 10800u^8\ t^2 + 1966u^9t - 409u^{10}\Big)s^9 + 3\Big(101t^{11} - 1463ut^{10} - 3896\ u^2t^9$$

$$+ 208u^3t^8 + 20429u^4t^7 + 58669u^5t^6 + 80268u^6t^5 + 55172u^7\ t^4 + 19400u^8t^3 + 4832u^9t^2 + 1000u^{10}t - 156u^{11}\Big)s^8 + 2\ \Big(48t^{12} - 1128ut^{11}$$

$$- 4410u^2t^{10} - 17090u^3t^9 - 31983u^4\ t^8 + 6084u^5t^7 + 100874u^6t^6 + 143160u^7t^5 + 88338u^8t^4 + 23800u^9\ t^3 + 2622u^{10}t^2 + 582u^{11}t$$

$$- 81u^{12}\Big)s^7 + \Big(14t^{13} - 572u\ t^{12} - 3036u^2t^{11} - 35818u^3t^{10} - 106376u^4t^9 - 101688u^5t^8 + 48822\ u^6t^7 + 213630u^7t^6 + 239397u^8t^5$$

$$+ 134809u^9t^4 + 31274u^{10}t^3 - 36\ u^{11}t^2 + 157u^{12}t - 25u^{13}\Big)s^6 - 6tu\Big(8t^{12} + 37u\ t^{11} + 3213u^2t^{10} + 11984u^3t^9 + 14592u^4t^8 + 1987u^5t^7$$

$$- 12914u^6\ t^6 - 20563u^7t^5 - 20499u^8t^4 - 11612u^9t^3 - 2423u^{10}t^2 + 147u^{11}t + 3\ u^{12}\Big)s^5 + t^2u^2\Big(102t^{11} - 5234ut^{10} - 25530u^2t^9$$

$$- 39126\ u^3t^8 - 12408u^4t^7 + 24648u^5t^6 + 35505u^6t^5 + 40505u^7t^4 + 40476u^8\ t^3 + 21240u^9t^2 + 3629u^{10}t - 267u^{11}\Big)s^4 + 2t^3u^3\Big($$

$$- 266\ t^{10} - 1804ut^9 - 4752u^2t^8 - 4830u^3t^7 + 378u^4t^6 + 5301u^5t^5 + 6806\ u^6t^4 + 6178u^7t^3 + 3642u^8t^2 + 1119u^9t + 156u^{10}\Big)s^3$$

$$+ 3t^4u^4\ (t+u)^2\Big(16t^7 - 382ut^6 - 532u^2t^5 + 8u^3t^4 + 467u^4t^3 + 637u^5\ t^2 + 265u^6t - 71u^7\Big)s^2 - 2t^5u^5(t+u)^3\Big(48t^5 + 151ut^4$$

$$- 84\ u^2t^3 + 18u^3t^2 - 160u^4t - 15u^5\Big)s + t^6u^7(t+u)^3\Big(12t^3 + 3\ ut^2 + 2u^2t - 11u^3\Big)\Big)H(0,3,2,y) - 18s^2t^2u^2(t+u)^2\ \Big(\Big(55t^4$$

$$= + 148ut^3 + 192u^2t^2 + 128u^3t + 47u^4\Big)s^{16} + 2\ \Big(232t^5 + 807ut^4 + 1262u^2t^3 + 1126u^3t^2 + 612u^4t + 173u^5\Big)\ s^{15} + 6\Big(305t^6 + 1324ut^5$$

$$+ 2458u^2t^4 + 2648u^3t^3 + 1862u^4t^2 + 872\ u^5t + 201u^6\Big)s^{14} + 2\Big(2230t^7 + 11932ut^6 + 26409u^2t^5 + 33217\ u^3t^4 + 27682u^4t^3 + 16830u^5t^2$$

$$+ 7005u^6t + 1347u^7\Big)\ s^{13} + \Big(7450t^8 + 48274ut^7 + 129744u^2t^6 + 195540u^3t^5 + 190555u^4\ t^4 + 134098u^5t^3 + 72590u^6t^2 + 26532u^7t$$

$$+ 4375u^8\Big)s^{12} + 6\ \Big(1486t^9 + 11488ut^8 + 37756u^2t^7 + 69937u^3t^6 + 82622u^4t^5 + 67718\ u^5t^4 + 41832u^6t^3 + 19931u^7t^2 + 6288u^8t$$

$$+ 910u^9\Big)\ s^{11} + \Big(7744t^{10} + 71242ut^9 + 290076u^2t^8 + 661446u^3t^7 + 946629\ u^4t^6 + 926538u^5t^5 + 671989u^6t^4 + 379294u^7t^3$$

$$+ 157923u^8t^2 + 41472\ u^9t + 5271u^{10}\Big)s^{10} + 2\Big(2410t^{11} + 26794ut^{10} + 138966\ u^2t^9 + 391138u^3t^8 + 666867u^4t^7 + 763472u^5t^6$$

$$+ 652628u^6t^5 + 448145\ u^7t^4 + 237429u^8t^3 + 84416u^9t^2 + 17568u^{10}t + 1903u^{11}\Big)s^9 + 3\ \Big(685t^{12} + 9638ut^{11} + 66160u^2t^{10} + 229988u^3t^9$$

$$+ 468878u^4\ t^8 + 632074u^5t^7 + 626493u^6t^6 + 499908u^7t^5 + 326013u^8t^4 + 156448u^9\ t^3 + 46699u^{10}t^2 + 7420u^{11}t + 638u^{12}\Big)s^8$$

$$+ 2\Big(270\ t^{13} + 5385ut^{12} + 51810u^2t^{11} + 219533u^3t^{10} + 538700u^4\ t^9 + 893700u^5t^8 + 1082558u^6t^7 + 993074u^7t^6 + 712998u^8t^5$$

$$+ 405820u^9\ t^4 + 169330u^{10}t^3 + 42843u^{11}t^2 + 4984u^{12}t + 295u^{13}\Big)\ s^7 + \Big(66t^{14} + 2518ut^{13} + 38032u^2t^{12} + 195010u^3t^{11} + 579501\ u^4t^{10}$$

$$= + 1219748u^5t^9 + 1901521u^6t^8 + 2145380u^7t^7 + 1729402u^8t^6 + 1029790u^9t^5 + 480198u^{10}t^4 + 170114u^{11}t^3 + 36457u^{12}t^2 + 2808u^{13}t$$

$$+ 83u^{14}\Big)s^6 + 6tu\Big(46t^{13} + 1463ut^{12} + 9551u^2t^{11} + 36573u^3t^{10} + 99722u^4t^9 + 196920u^5t^8 + 275717u^6t^7 + 269768u^7t^6 + 184750u^8t^5$$

$$+ 90822u^9t^4 + 33632u^{10}t^3 + 9403u^{11}t^2 + 1592u^{12}t + 61u^{13}\Big)s^5 + t^2u^2\Big(936t^{12} + 9592ut^{11} + 58498u^2t^{10} + 216504u^3t^9 + 517783u^4t^8$$

$$+ 852472u^5t^7 + 999621u^6t^6 + 842754u^7t^5 + 506934u^8t^4 + 212352u^9t^3 + 59561u^{10}t^2 + 10778u^{11}t + 1137u^{12}\Big)s^4 + 2t^3u^3\Big(306t^{11}$$

$$+ 5120ut^{10} + 27107u^2t^9 + 77806u^3t^8 + 147341u^4t^7 + 200386u^5t^6 + 201078u^6t^5 + 148590u^7t^4 + 78861u^8t^3 + 28022u^9t^2 + 5655u^{10}t$$

$$+ 424u^{11}\Big)s^3 + 3t^4u^4(t+u)^2\Big(312t^8 + 1862ut^7 + 5114u^2t^6 + 9264u^3t^5 + 11523u^4t^4 + 9432u^5t^3 + 5226u^6t^2 + 1942u^7t + 361u^8\Big)s^2$$

$$+ 2t^5u^5(t+u)^3\Big(138t^6 + 635ut^5 + 1406u^2t^4 + 1938u^3t^3 + 1431u^4t^2 + 633u^5t + 159u^6\Big)s + 3t^6u^6(t+u)^4\Big(22t^4 + 48ut^3 + 71u^2t^2$$

$$+ 46u^3t + 23u^4\Big)\Big)H(1,0,1,z) - 18s^2t^2(s+t)^3u^2\Big(\Big(11t^6 - 30ut^5 + 213u^2t^4 - 312u^3t^3 + 267u^4t^2 + 18u^5t + 25u^6\Big)s^{13} + \Big(31t^7 - 410ut^6$$

$$- 369u^2t^5 - 2238u^3t^4 - 3629u^4t^3 + 882u^5t^2 - 157u^6t + 162u^7\Big)s^{12} + 6\Big(4t^8 - 169ut^7 - 548u^2t^6 - 1214u^3t^5 - 3540u^4t^4 - 2423u^5t^3$$

$$+ 6u^6t^2 - 194u^7t + 78u^8\Big)s^{11} - 2\Big(8t^9 + 525ut^8 + 3009u^2t^7 + 6178u^3t^6 + 20238u^4t^5 + 34836u^5t^4 + 15637u^6t^3 + 2622u^7t^2 + 1500u^8t$$

$$- 409u^9\Big)s^{10} - \Big(47t^{10} + 414ut^9 + 4737u^2t^8 + 13612u^3t^7 + 40505u^4t^6 + 122994u^5t^5 + 134809u^6t^4 + 47600u^7t^3 + 14496u^8t^2 + 3932u^9t$$

$$- 1002u^{10}\Big)s^9 - 3\Big(13t^{11} - 178ut^{10} - 49u^2t^9 + 3534u^3t^8 + 11835u^4t^7 + 41126u^5t^6 + 79799u^6t^5 + 58892u^7t^4 + 19400u^8t^3 + 7200u^9t^2$$

$$= + 946u^{10}t - 310u^{11}\Big)s^8 - 2\Big(6t^{12} - 513ut^{11} - 2157u^2t^{10} + 378u^3t^9 + 12324u^4t^8 + 38742u^5t^7 + 106815u^6t^6 + 143160u^7t^5$$

$$+ 82758u^8t^4 + 31963u^9t^3 + 11604u^{10}t^2 + 654u^{11}t - 326u^{12}\Big)s^7 + 2u\Big(295t^{12} + 1920ut^{11} + 4830u^2t^{10} + 6204u^3t^9 + 5961u^4t^8$$

$$- 24411u^5t^7 - 100874u^6t^6 - 120402u^7t^5 - 71423u^8t^4 - 33991u^9t^3 - 9972u^{10}t^2 - 344u^{11}t + 159u^{12}\Big)s^6 + 3u\Big(32t^{13} + 350ut^{12}$$

$$+ 3168u^2t^{11} + 13042u^3t^{10} + 29184u^4t^9 + 33896u^5t^8 - 4056u^6t^7 - 58669u^7t^6 - 65010u^8t^5 - 40423u^9t^4 - 18184u^{10}t^3$$

$$- 4103u^{11}t^2 - 170u^{12}t + 31u^{13}\Big)s^5 + u^2\Big(- 48t^{13} + 3608ut^{12} + 25530u^2t^{11} + 71904u^3t^{10} + 106376u^4t^9 + 63966u^5t^8 - 61287u^6t^7$$

$$- 145806u^7t^6 - 123813u^8t^5 - 69894u^9t^4 - 26949u^{10}t^3 - 5154u^{11}t^2 - 333u^{12}t + 12u^{13}\Big)s^4 + 2tu^3\Big(266t^{12} + 2617ut^{11} + 9639u^2t^{10}$$

$$+ 17909u^3t^9 + 17090u^4t^8 - 312u^5t^7 - 23751u^6t^6 - 31075u^7t^5 - 23124u^8t^4 - 12753u^9t^3 - 4496u^{10}t^2 - 834u^{11}t - 72u^{12}\Big)s^3$$

$$- 6tu^4(t+u)^2\Big(17t^{10} - 71ut^9 - 381u^2t^8 - 637u^3t^7 - 293u^4t^6 + 591u^5t^5 + 705u^6t^4 + 627u^7t^3 + 256u^8t^2 + 50u^9t + 4u^{10}\Big)s^2$$

$$+ t^2u^5(t+u)^3\Big(48t^8 + 428ut^7 + 828u^2t^6 + 573u^3t^5 - 481u^4t^4 - 687u^5t^3 - 585u^6t^2 - 198u^7t - 30u^8\Big)s - t^3u^6(t+u)^6\Big(14t^4$$

$$+ 12ut^3 + 21u^2t^2 + 12u^3t + 14u^4\Big)\Big)\ H(2,1,0,y) - 18s^2t^2u^2(t+u)^2\Big(\Big(83t^4 + 224ut^3 + 312u^2t^2 + 224u^3t + 83u^4\Big)s^{16} + \Big(646t^5$$

$$+ 2236ut^4 + 3738u^2t^3 + 3738u^3t^2 + 2236u^4t + 646u^5\Big)s^{15} + 3\Big(795t^6 + 3434ut^5 + 6735u^2t^4 + 8252u^3t^3 + 6735u^4t^2 + 3434u^5t$$

$$+ 795u^6\Big)\ s^{14} + \Big(5590t^7 + 29722ut^6 + 68154u^2t^5 + 96362u^3t^4 + 96362u^4t^3 + 68154u^5t^2 + 29722u^6t + 5590u^7\Big)s^{13} + \Big(9297t^8$$

$$= + 59792ut^7 + 162893u^2t^6 + 265856u^3t^5 + 307366u^4t^4 + 265856u^5t^3 + 162893u^6t^2 + 59792u^7t + 9297u^8\Big)s^{12} + 6\Big(1912t^9$$

$$+ 14697ut^8 + 48146u^2t^7 + 92627u^3t^6 + 123322u^4t^5 + 123322u^5t^4 + 92627u^6t^3 + 48146u^7t^2 + 14697u^8t + 1912u^9\Big)s^{11}$$

$$+ \Big(10571t^{10} + 97460ut^9 + 390525u^2t^8 + 895950u^3t^7 + 1372200u^4t^6 + 1560372u^5t^5 + 1372200u^6t^4 + 895950u^7t^3 + 390525u^8t^2$$

$$+ 97460u^9t + 10571u^{10}\Big)s^{10} + 2\ \Big(3555t^{11} + 40253ut^{10} + 202415u^2t^9 + 561210u^3t^8 + 989305u^4t^7 + 1257298u^5t^6 + 1257298u^6t^5$$

$$+ 989305u^7t^4 + 561210u^8t^3 + 202415u^9t^2 + 40253u^{10}t + 3555u^{11}\Big)s^9 + 3\Big(1100t^{12} + 16144ut^{11} + 105097u^2t^{10} + 356156u^3t^9$$

$$+ 736699u^4t^8 + 1061804u^5t^7 + 1182566u^6t^6 + 1061804u^7t^5 + 736699u^8t^4 + 356156u^9t^3 + 105097u^{10}t^2 + 16144u^{11}t + 1100u^{12}\Big)s^8$$

$$+ 2\Big(471t^{13} + 10059ut^{12} + 89007u^2t^{11} + 370856u^3t^{10} + 918809u^4t^9 + 1568172u^5t^8 + 2010976u^6t^7 + 2010976u^7t^6 + 1568172u^8t^5$$

$$+ 918809u^9t^4 + 370856u^{10}t^3 + 89007u^{11}t^2 + 10059u^{12}t + 471u^{13}\Big)s^7 + \Big(124t^{14} + 5166ut^{13} + 68942u^2t^{12} + 360694u^3t^{11}$$

$$+ 1093775u^4t^{10} + 2281660u^5t^9 + 3515021u^6t^8 + 4060864u^7t^7 + 3515021u^8t^6 + 2281660u^9t^5 + 1093775u^{10}t^4 + 360694u^{11}t^3$$

$$= + 68942u^{12}t^2 + 5166u^{13}t + 124u^{14}\Big)s^6 + 6tu\Big(102t^{13} + 2718ut^{12} + 19299u^2t^{11} + 75952u^3t^{10} + 198691u^4t^9 + 370135u^5t^8$$

$$+ 503605u^6t^7 + 503605u^7t^6 + 370135u^8t^5 + 198691u^9t^4 + 75952u^{10}t^3 + 19299u^{11}t^2 + 2718u^{12}t + 102u^{13}\Big)s^5 + t^2u^2\Big(1752t^{12}$$

$$+ 21508ut^{11} + 127396u^2t^{10} + 438996u^3t^9 + 986209u^4t^8 + 1559032u^5t^7 + 1808388u^6t^6 + 1559032u^7t^5 + 986209u^8t^4 + 438996u^9t^3$$

$$+ 127396u^{10}t^2 + 21508u^{11}t + 1752u^{12}\Big)s^4 + 2t^3u^3\ \Big(834t^{11} + 10755ut^{10} + 52845u^2t^9 + 146657u^3t^8 + 271322u^4t^7 + 363015u^5t^6$$

$$+ 363015u^6t^5 + 271322u^7t^4 + 146657u^8t^3 + 52845u^9t^2 + 10755u^{10}t + 834u^{11}\Big)s^3 + 6t^4u^4(t+u)^2\Big(283t^8 + 1733ut^7 + 4779u^2t^6$$

$$+ 8360u^3t^5 + 10092u^4t^4 + 8360u^5t^3 + 4779u^6t^2 + 1733u^7t + 283u^8\Big)s^2 + 2t^5u^5(t+u)^3\Big(282t^6 + 1178ut^5 + 2417u^2t^4 + 3162u^3t^3$$

$$+ 2417u^4t^2 + 1178u^5t + 282u^6\Big)s + 110t^6u^6(t+u)^4\Big(t^2 + ut + u^2\Big)^2\Big)H(2,3,2,y)\Big\}\Big/\Big(27s^3t^3u^3(s+t)^6(s+u)^6(t+u)^6\Big)$$

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