EXPLICIT SHIMURA LIFTINGS OF CERTAIN CLASS OF FORMS AND SOME PROBLEMS INVOLVING MODULAR *L*-FUNCTIONS

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

1. "On simultaneous non-vanishing of twisted *L*-functions associated to newforms on $\Gamma_0(N)$", A. K. Jha, A. Juyal, and M. K. Pandey, J. Ramanujan Math. Soc., **2019**, *34*, 245-252.

2. "Signs of Fourier coefficients of cusp form at sum of two squares", S. Banerjee and M. K. Pandey, Proc. Indian Acad. Sci. Math. Sci., Accepted for publication.

Others

1. "Shimura lifts of certain class of modular forms of half-integral weight", M. K. Pandey and B. Ramakrishnan.

2. "Determining modular forms of half-integral weight by central values of convolution *L*-functions", M. K. Pandey and B. Ramakrishnan.

Manish Kumar Pandey

To my family

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Contents

Summary

This thesis studies four problems in the theory of modular forms. The first one is about getting explicit Shimura lifting of a certain class of modular forms of half-integral weight, which generalises the works of B. Cipra, D. Hansen and Y. Naqvi.

In the second problem, simultaneous non-vanishing of twisted L-functions associated to modular forms of integral weight is considered. It is a conjecture that there exists infinitely many twists of L-functions which do not vanish simultaneously. In this direction, following a method of R. Munshi, it is shown that if a product of L-functions associated to two modular forms is non-zero for one character twist, then there are infinitely many character twists of these products of L-functions which are non-zero.

The problem of determining modular forms using the central values of L-functions is well studied in the case of forms of integral weight. In the third problem, we generalise the method adopted by S. Ganguly, J. Hoffstein and J. Sengupta to the case of modular forms of half-integral weight. It is proved that in the case of forms of half-integral weight also the central values of convolution L-functions determine the forms.

The final problem of this thesis is about sign changes of Fourier coefficients of cusp forms. It is well-known that the Fourier coefficients of cusp forms change its sign infinitely often. In this direction there are many works which give quantitative results. Sign changes of Fourier coefficients of cusp forms at some special sequences have also been studied by many authors. The final result in this thesis deals with the sign change problem for the sequence of integers which are sums of two (integer) squares. It is established that on this sparse sequence, the Fourier coefficients of cusp forms of integral weight change sign infinitely often.

CHAPTER 1

Introduction and Preliminaries

1.1 Introduction

In this thesis, we have considered four problems. First problem is about finding explicit Shimura image of certain class of modular forms of half-integral weight and thus giving a generalisation of the work of Cipra [3] and Hansen- Naqvi [10]. Second problem is about simultaneous non-vanishing of the twisted L-function, this problem is a generalisation of the work of Munshi [27]. In the third problem, we have considered the problem of determining modular forms of half-integral weight and thus giving a result analogous to the work of Ganguly-Hoffstein-Sengupta [5]. Finally, we have considered the problem of sign change of the Fourier coefficients of cusp form at integers, which can be written as sum of two squares.

1.2 Preliminaries

Now, in this chapter we give some basic definitions and properties of modular forms of integral and half-integral weights.

1.3 Notations

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. For $z \in \mathbb{C}$, $Re(z)$ denotes the real part of z and $\mathrm{Im}(z)$ denotes the imaginary part of z. For any complex number z and a non-zero real number c, we denote by $e_c(z) = e^{2\pi iz/c}$. If $c = 1$, we simply write $e(z)$ instead of $e_1(z)$. Let $\mathscr{H} = \{\tau \in \mathbb{C} : Im\ \tau > 0\}$ be the complex upper half-plane. We denote by $q = e(\tau)$, for $\tau \in \mathscr{H}$. For a complex number z, the square root is defined as follows:

$$\sqrt{z} = | z |^{\frac{1}{2}} e^{\frac{i}{2}\arg z}, \text{ with } -\pi < \arg z \leq \pi.$$

We set $z^{\frac{k}{2}} = (\sqrt{z})^k$ for any $k \in \mathbb{Z}$. The full modular group $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

For a positive integer N, we denote the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ as follows:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

1.4 Modular forms on $\Gamma_0(N)$

The group $\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc>0 \right\}$ acts on $\mathscr{H}$ via fractional linear transformations, i.e., for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $\tau \in \mathscr{H}$

$$\gamma\tau := \frac{a\tau+b}{c\tau+d}.$$

Let $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. For a complex valued function f define the slash operator as follows:

$$(f\,|_k\,\gamma)(\tau) := (\det\gamma)^{\frac{k}{2}}\,(c\tau+d)^{-k} f(\gamma\tau).$$

Definition 1.4.1 *Let k be a positive integer and χ a Dirichlet character modulo N. A holomorphic function $f : \mathscr{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight k, level N and character χ, if*

1. $(f\,|_k\,\gamma)(\tau) = \chi(d)f(\tau),\ \forall\,\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, *i.e.,*

   $$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau),\ \forall\,\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

2. *f is holomorphic at all the cusps of $\Gamma_0(N)$.*

   Further, we say that f is a cusp form if f vanishes at all the cusps of $\Gamma_0(N)$.

We denote the space of all modular forms and the subspace of all cusp forms of weight k, level N with character χ on $\Gamma_0(N)$ by $M_k(\Gamma_0(N),\chi)$ and $S_k(\Gamma_0(N),\chi)$, respectively.

If χ is the trivial character, then we denote these spaces as $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$, respectively. When $N = 1$, we simply denote these spaces by M_k and S_k respectively.

For $f,\ g \in M_k(\Gamma_0(N), \chi)$ such that fg is a cusp form, the Petersson scalar product of f and g is defined as:

$$\langle f, g\rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int\limits_{\Gamma_0(N)\backslash\mathscr{H}} f(\tau)\overline{g(\tau)}(\mathrm{Im}(\tau))^k d^*\tau,$$

where $\Gamma_0(N)\backslash\mathscr{H}$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathscr{H}$ and $d^*\tau$ is the invariant Haar measure. If $\tau = x + iy$, then the invariant Haar measure is given by $d^*\tau = \frac{dxdy}{y^2}$.

Example 1. Let k be an even integer greater than 2. The normalized Eisenstein series E_k of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is defined as:

$$\begin{aligned} E_k(\tau) :&= \sum_{\gamma\in\Gamma\infty\backslash\Gamma} 1|_k\gamma \\ &= \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2\backslash(0,0) \\ (m,n)=1}} \frac{1}{(m\tau+n)^k}. \end{aligned}$$

Then E_k is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where $\sigma_{k-1}(n) = \sum\limits_{d|n} d^{k-1}$ and B_k's are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{k=o}^{\infty} B_k \frac{x^k}{k!}.$$

We have

$$E_4(\tau) = 1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n,$$
$$E_6(\tau) = 1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n,$$

Example 2. The Ramanujan delta function is defined as

$$\Delta(\tau) := \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2),$$

which is a cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

where $\tau(n)$ is called the Ramanujan tau function.

Example 3. Let $\theta^2(\tau) = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^2 = \sum_{n\geq 0} r_2(n)q^n$, where $r_2(n)$ denotes number of ways n can be written as a sum of two squares. Then

$$\theta^2(z) \in M_1(\Gamma_0(4),\chi),$$

where $\chi(d) = (-1)^{(\frac{d-1}{2})}$.

Example 4. Let m be a be a positive integer. The m-th Poincaré series of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$P_{k,m}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} e^{2\pi i m\tau}|_k\gamma, \tag{1.1}$$

where $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{Z} \right\}$. The m^{th} Poincaré series $P_{k,m}$ is a cusp form of weight $k > 2$ for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$P_{k,m}(\tau) = \sum_{n=1}^{\infty} g_m(n) q^n,$$

where

$$g_m(n) = \delta_{m,n} + (-1)^{\frac{k}{2}+1} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \pi \sum_{c=1}^{\infty} K_c(m,n) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

and $K_c(m,n)$ is the Kloosterman sum defined by

$$\frac{1}{c} \sum_{\substack{d \pmod{c} \\ dd^{-1} \equiv 1 \pmod{c}}} e_c(nd + md^{-1}),$$

and $J_{k-1}(x)$ is the Bessel function of order $k-1$. The Poincaré series has the following property: If $f \in S_k$ has the Fourier expansion $f(\tau) = \sum_{m=1}^{\infty} a(m) q^m$, then

$$\langle f, P_{k,n} \rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a(n). \tag{1.2}$$

We now define Hecke operators which send modular forms to modular forms. Let n be a positive integer such that $(n,N) = 1$. For $f(\tau) = \sum_{m=0}^{\infty} a(m) q^m \in M_k(\Gamma_0(N), \chi)$, the n-th Hecke operator T_n is given in terms of the Fourier expansion by

$$(T_n f)(\tau) = \sum_{m=0}^{\infty} a_n(m) q^m,$$

where $a_n(m) = \sum_{d|(m,n)} \chi(d) d^{k-1} a(\frac{mn}{d^2})$. If $f \in M_k$ (or S_k), then $T_n f \in M_k$ (or S_k). The family $\{T_n : n \in \mathbb{N}\}$ of Hecke operators is commuting. The Hecke operators T_n acting on $S_k(\Gamma_0(N),\chi)$ are self-adjoint with respect to the Petersson inner product.

Definition 1.4.2 *A cusp form is said to be an eigenform if it is a simultaneous eigenfunction for all the Hecke operators.*

The space $S_k(\Gamma_0(N),\chi)$ has a basis of eigenforms. For each prime $p|N$ the Hecke operator $U(p)$ is defined as follows. If $f(\tau)=\sum\limits_{m\geq 1} a(m)q^m$, then $f(\tau)|U(p):=\sum\limits_{m\geq 1} a(pm)q^m$. Also, if d is a positive integer, then the duplicating operator $B(d)$ is defined as

$$f(\tau)|B(d):=f(d\tau).$$

Suppose that $f(\tau)\in S_k(\Gamma_0(N))$, then $f(\tau)\in S_k(\Gamma_0(dN))$ for any positive integer $d\geq 1$ and $f(d\tau)\in S_k(\Gamma_0(dN))$. Therefore, $f(\tau)$ sits in $S_k(\Gamma_0(dN))$ in atleast two different ways. The theory of newforms distinguishes these types of forms, called the oldforms, which are generated by forms of level $<N$.
We define the subspace $S_k^{\text{old}}(\Gamma_0(N))$ by

$$S_k^{\text{old}}(\Gamma_0(N)):=\oplus_{\substack{dM|N\\ M\neq N}} S_k(\Gamma_0(M))|B(d),$$

We now define the subspace of newforms denoted as $S_k^{\text{new}}(\Gamma_0(N))$ as the orthogonal complement of $S_k^{\text{old}}(\Gamma_0(N))$ in $S_k(\Gamma_0(N))$ with respect to the Petersson inner product. One has $S_k(\Gamma_0(N))=S_k^{\text{old}}(\Gamma_0(N))\oplus S_k^{\text{new}}(\Gamma_0(N))$. Further the space $S_k^{\text{new}}(\Gamma_0(N))$ has an orthogonal basis of eigenforms. The following lemma tells us about the growth of the Fourier coefficients of a modular form.

Lemma 1.4.3 *[11] If $f\in M_k(\Gamma_0(N),\chi)$ with Fourier coefficients $a(n)$, then*

$$a(n)\ll n^{k-1+\varepsilon},$$

and moreover, if f is a cusp form, then

$$a(n) \ll n^{\frac{k}{2}-\frac{1}{4}+\varepsilon}.$$

Note that for a Hecke eigen cusp form f, Deligne's bound is $a(n) \ll n^{\frac{k-1}{2}+\varepsilon}$. For more details on the theory of modular forms of integral weight, we refer to the books of Iwaniec [11], Ono [32], Koblitz [15].

We associate an L-function to $f(\tau) = \sum_{m\geq 1} a(m)q^m \in S_k(\Gamma_0(N),\chi))$ defined by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

for $\mathrm{Re(s)} > \frac{\mathrm{k}}{2}+1$.

Theorem 1.4.1 *Let f be as above and put $g = f \mid W_N$, where $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.*

Let

$$L(s,f) = \sum_{n\geq 1} \frac{a(n)}{n^s},$$

$$L(s,g) = \sum_{n\geq 1} \frac{b(n)}{n^s}.$$

Then $L(s,f)$ and $L(s,g)$ can be extended to entire functions and satisfy the functional equation

$$\Lambda(s,f) = i^k \Lambda(s,g),$$

where

$$\Lambda(s,f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s,f)$$

and

$$\Lambda(s,g) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s,g).$$

The following theorem gives modular property of the twist of a modular form by a Dirichlet character.

Theorem 1.4.2 *Let $f \in M_k(\Gamma_0(N), \chi)$, where χ is a Dirichlet character of conductor N^* with $N^*|N$. Let ψ be a primitive Dirichlet character modulo r. If*

$$f(\tau) = \sum_{n \geq 0} a(n) q^n,$$

then the "twisted series"

$$f_\psi(\tau) = \sum_{n \geq 0} a(n)\psi(n) q^n$$

is an element of $M_k(\Gamma_0(M), \chi\psi^2)$, where M is the least common multiple of N, N^*r, and r^2. Moreover if f is a cusp form, then so is f_ψ.

Given $f \in S_k(\Gamma_0(N), \chi)$ and ψ a primitive Dirichlet character modulo r, we associate the twisted L-function by

$$L(s, f, \psi) = \sum_{n \geq 1} \frac{a(n)\psi(n)}{n^s}$$

where $a(n)$'s are the Fourier coefficients of f. Note that this L-function is nothing but $L(s, f_\psi)$, the L-function of the twisted modular form f_ψ.

Theorem 1.4.3 ([11]) *Let $f \in S_k(\Gamma_0(N), \chi)$, ψ a primitive Dirichlet character modulo r with $(N, r) = 1$ and $M = Nr^2$. Put*

$$\Lambda(s, f, \psi) = \left(\frac{\sqrt{M}}{2\pi}\right)^s \Gamma(s) L(s, f, \psi).$$

Then $\Lambda(s, f, \psi)$ is an entire function, bounded in vertical strips and satisfying the functional equation

$$\Lambda(s, f, \psi) = i^k w(\psi) \Lambda(k - s, g, \overline{\psi}),$$

where

$$w(\psi) = \chi(r)\psi(q)\tau(\psi)^2/r$$

and $g = f \mid W_N$.

In the above $\tau(\psi)$ is the Gauss sum given by $\tau(\psi) = \sum_{u \pmod r} \psi(u)e(u/r)$ and W_N is the Fricke involution as defined in Theorem 1.4.1.

1.5 Modular forms of half-integral weight

For a non-negative integer k and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$ we define the slash operator as follows:

$$f \mid_{k+1/2} \gamma(\tau) := \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(c\tau+d)^{-k-1/2}f(\gamma\tau),$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol and $\left(\frac{-4}{d}\right) = \pm 1$ or 0 according as $d \equiv \pm 1 \pmod 4$ or d is even.

Definition 1.5.1 *Let k be a non-negative integer and χ be a Dirichlet character modulo $4N$. A holomorphic function $f : \mathscr{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight $k+1/2$ on $\Gamma_0(4N)$ with character χ, if*

1. $f \mid_{k+1/2} \gamma(\tau) \; = \; \chi(d)f(\tau), \; \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N).$
2. *f is holomorphic at all the cusps of $\Gamma_0(4N)$.*

   Further, we say that f is a cusp form if f vanishes at all the cusps of $\Gamma_0(4N)$.

We denote the space of all modular forms of weight $k+1/2$ with character χ on $\Gamma_0(4N)$ by $M_{k+1/2}(4N,\chi)$ and when $k \geq 1$ the subspace of all cusp forms is denoted by $S_{k+\frac{1}{2}}(4N,\chi)$.

Example Let ψ be a primitive Dirichlet character (mod r), let $\nu = 0$ or 1 according as $\psi(-1) = (-1)^{\nu}$. The theta function associated to the character ψ is defined by

$$h_{\psi}(\tau) = \sum_{-\infty}^{\infty} \psi(m) m^{\nu} e(m^2\tau). \tag{1.3}$$

Then $h_{\psi}(\tau)$ is a modular form of half-integral weight and $h_{\psi} \in M_{\nu+1/2}(4r^2, \psi^{(\nu)})$, where $\psi^{(\nu)}(m) = \psi(m)\left(\frac{-1}{m}\right)^{\nu}$.

When $r = 1$, the above theta series is the classical theta series $\theta(\tau)$ given by

$$\theta(\tau) = \sum_{m\in\mathbb{Z}} e(m^2\tau) \in M_{1/2}(4). \tag{1.4}$$

The Petersson scalar product on $S_{k+1/2}(4N,\chi)$ is defined as follows:

$$\langle f, g\rangle \;=\; \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]} \int_{\Gamma_0(4N)\backslash\mathscr{H}} f(\tau)\overline{g(\tau)}(\mathrm{Im}(\tau))^{k+1/2} d^*\tau,$$

here $d^*\tau$ is the invariant Haar measure as defined in §1.4. The space $S_{k+1/2}(4N,\chi)$ is a finite dimensional Hilbert space.

Definition 1.5.2 *For positive integer n, the n-th Poincaré series of weight $k+1/2$, (k is a positive integer) is defined by*

$$P_{k+1/2,n}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4N)} e^{2\pi i n\tau} |_{k+1/2}\, \gamma. \tag{1.5}$$

It is known that $P_{k+1/2,n} \in S_{k+1/2}(4N)$ (for this fact we refer to the proof of [18, Proposition 4]. The Poincaré series has the following property:

Lemma 1.5.3 *Let $f \in S_{k+1/2}(4N)$ with Fourier expansion*

$$f(\tau) = \sum_{m=1}^{\infty} a(m)q^m.$$

Then

$$\langle f, P_{k+1/2,n} \rangle = \frac{\Gamma(k-\frac{1}{2})}{i_{4N}(4\pi n)^{k-\frac{1}{2}}} a(n). \tag{1.6}$$

Here i_{4N} denotes the index of $\Gamma_0(4N)$ in $\mathrm{SL}_2(\mathbb{Z})$. The following lemma gives the growth of the Fourier coefficients of a half-integral weight modular form.

Lemma 1.5.4 *If $f \in M_{k+1/2}(4N,\chi)$ with Fourier coefficients $a(n)$, then*

$$a(n) \ll n^{k-\frac{1}{2}+\varepsilon},$$

and moreover, if $f \in S_{k+1/2}(4N,\chi)$ is a cusp form, then

$$a(n) \ll n^{\frac{k}{2}+\frac{1}{4}+\varepsilon}.$$

In his 1973 paper [37], G. Shimura introduced the theory of modular forms of half-integral weight. He obtained the following map from the space of cusp forms of half-integral weight to cusp forms of integral weight.

Theorem 1.5.1 *([37], [31]) Let t be a square-free positive integer and suppose that $f(\tau) = \sum_{n=1}^{\infty} b(n)e(n\tau) \in S_{k+1/2}(4N,\chi)$, where k is a positive integer. If $A_t(n)$ are given by*

$$\sum_{n=1}^{\infty} A_t(n)n^{-s} = L(s-k+1, \chi\chi_{-4}^{k}\chi_t) \sum_{n=1}^{\infty} b(tn^2)n^{-s}, \tag{1.7}$$

where $\mathrm{Re}(s)$ is large and $\chi_t = \left(\frac{t}{\cdot}\right)$ is the real non-principal character modulo t (modulo $4t$ if $t \equiv 2,3 \pmod 4$), $\chi_{-4} = \left(\frac{-4}{\cdot}\right)$ is the odd character modulo 4, define the map

$\mathscr{S}_t$ as follows.

$$\mathscr{S}_t(f)(\tau) = \sum_{n=1}^{\infty}\left(\sum_{d|n}\chi(d)\chi_{-4}^k(d)\chi_t(d)d^{k-1}b(tn^2/d^2)\right)e(n\tau) = \sum_{n=1}^{\infty}A_t(n)e(n\tau). \tag{1.8}$$

Then $\mathscr{S}_t(f)(\tau) \in M_{2k}(2N,\chi^2)$. Moreover, if $k \geq 2$, then $\mathscr{S}_t(f)$ is a cusp form.

When N is odd, we define a canonical subspace of the space of cusp forms of half-integral weight introduced by Kohnen and denoted by $S^+_{k+1/2}(4N)$. This is the subspace of $S_{k+1/2}(4N)$, consisting of cusp forms $\sum_{n\geq 1} c(n)q^n$ with the property that $c(n) = 0$ unless $(-1)^k n \equiv 0,1 \pmod 4$. Later corresponding to each fundamental discriminant D (i.e. D is 1 or the discriminant of a quadratic field) with $(-1)^k D > 0$, Kohnen in [17],[18] defined the D-th Shimura-Kohnen map, given by

$$\mathscr{S}_D(g)(\tau) = \sum_{n\geq 1}\left(\sum_{\substack{d|n\\(d,N)=1}}\left(\frac{D}{d}\right)d^{k-1}c(n^2|D|/d^2)\right)q^n. \tag{1.9}$$

Also when N is squarefree, Kohnen has given a newform theory for $S^+_{k+1/2}(4N)$. Suppose that $S^{+,\text{new}}_{k+1/2}(4N)$ denotes the subspace of newforms of $S^+_{k+1/2}(4N)$. Kohnen has proved that $S^{+,\text{new}}_{k+1/2}(4N)$ and $S^{\text{new}}_{2k}(N)$ [subspace of newforms in $S_{2k}(N)$] are isomorphic as modules over the Hecke algebra.

Theorem 1.5.2 *If $f(\tau) = \sum_{n\geq 1} a(n)q^n \in S^{\text{new}}_{2k}(N)$ is a normalized Hecke eigenform $(a(1) = 1)$ and $g(\tau) = \sum_{n\geq 1} c(n)q^n \in S^{+,\text{new}}_{k+1/2}(4N)$ be the corresponding form of half-integral weight under the $\mathscr{S}_D$ map with $c(|D|) \neq 0$, for a fundamental discriminant D with $(-1)^k D > 0$. Then the Fourier coefficients of f and g are related (via the map $\mathscr{S}_D$) by*

$$c(n^2|D|) = c(|D|)\sum_{\substack{d|n\\(d,N)=1}}\mu(d)\left(\frac{D}{d}\right)d^{k-1}a(n/d).$$

For each prime l dividing N (odd squarefree), let W_l be the Atkin-Lehner involution on $S_{2k}(N)$ associated to l and defined by $W_l = \begin{pmatrix} l & \alpha \\ N & l\beta \end{pmatrix}$, with $l^2\beta - N\alpha = l$, and $\alpha, \beta \in \mathbb{Z}$. Then $f\,|_{2k}\,W_\ell = w_\ell f$, where $w_\ell = \pm 1$. For a fundamental discriminat D with $(D,N) = 1$, $(-1)^k D > 0$ and $\mathrm{Re(s)} \gg 0$ we denote by $L(f,D,s) = \sum_{n\geq 1} \left(\frac{D}{n}\right) a(n) n^{-s}$, the L-function of f twisted by the quadratic character $\left(\frac{D}{\cdot}\right)$. Then $L(f,D,s)$ has a holomorphic continuation to $\mathbb{C}$ and the completed L-function

$$L^*(f,D,s) = (2\pi)^{-s}(ND^2)^{s/2}\Gamma(s)L(f,D,s)$$

satisfies the functional equation

$$L^*(f,D,s) = (-1)^k \left(\frac{D}{-N}\right) w_N L^*(f,D,2k-s),$$

where $w_N = \prod_{l|N} w_l \in \pm 1$. Note that $L(f,D,k) = 0$ for $(-1)^k \left(\frac{D}{-N}\right) = -w_N$.
Below, we state the explicit Waldspurger formula obtained by Kohnen [18].

Theorem 1.5.3 *Let D be a fundamental discriminant with $(-1)^k D > 0$ and N be an odd squarefree natural number. Suppose that for all prime divisors l of N, we have $\left(\frac{D}{l}\right) = w_l$. Then*

$$\frac{|c(|D|)|^2}{\langle g,g\rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f,D,k)}{\langle f,f\rangle}, \tag{1.10}$$

where $\nu(N)$ denotes the number of distinct prime divisors of N.

For more details on the theory of modular forms of half-integral weight, we refer to the book of Koblitz [15], Ono [32], and the work of Shimura [37].

CHAPTER 2

Shimura image of certain modular forms of half-integral weight

2.1 Introduction

In his 1973 paper [37], G. Shimura introduced the theory of modular forms of half-integral weight. In that paper, he obtained a correspondence between the space of modular forms of half-integral weight and modular forms of integral weight. In particular, corresponding to each squarefree positive integer t, he obtained a map $\mathscr{S}_t$ from modular forms of half-integral weight to modular forms of integral weight.

In this connection, in an unpublished work the following property of the first Shimura map was observed by A. Selberg (see [3, p. 58]). Let $f \in S_k(1)$ be a Hecke eigenform, then the image of the function $f(4z)\theta(z)$ under $\mathscr{S}_1$ is $f^2(z) - 2^{k-1} f^2(2z)$, where $\theta(z)$ is the classical theta function given by $\sum_{-\infty}^{\infty} e(n^2 z)$. B. A. Cipra in [3], generalized the observation made by Selberg by considering the general theta function

h_ψ corresponding to a primitive Dirichlet character ψ modulo a prime power. We write $\psi(-1) = (-1)^\nu$, where $\nu = 0,1$ accordingly if ψ is even or odd.

Then, Cipra proved the following theorem.

Theorem 2.1.1 *([3]) Let $f(z)$ be a normalized newform in $S_k(N,\chi)$. For a primitive Dirichlet character ψ modulo $r = p^m$, where p is a prime, and $\mu \geq m$, let $F(z) = f(4p^\mu z)h_\psi(z)$, where $h_\psi(z)$ is the generalized theta function as defined in (1.4). Let $g(z) = \sum_{n\geq 1} c(n)e(nz) = f(z)f(p^\mu z)$, if $\nu = 0$ and $g(z) = \frac{1}{2\pi i}\left(f'(z)f(p^\mu z) - p^\mu f(z)f'(p^\mu z)\right)$, if $\nu = 1$. Here $f'(z)$ is the derivative of f w.r.t z. Let $G(z) = \sum_{n\geq 1}\psi(n)c(n)e(nz) = g_\psi(z)$ be the ψ-twist of g. Then $F \in S_{k+\nu+1/2}(4N_1, \chi\psi\chi_{-4}^{k+\nu})$, $G \in S_{2(k+\nu)}(N_1, \chi^2\psi^2)$, where $N_1 = \mathrm{lcm}(Np^\mu, r^2)$. Moreover,*

$$\mathscr{S}_1(F)(z) = G(z) - 2^{k+\nu-1}\chi(2)\psi(2)G(2z) \in S_{2(k+\nu)}(2N_1, \chi^2\psi^2).$$

In 2008, D. Hansen and Y. Naqvi [10] generalized the work of Cipra by considering ψ to be a Dirichlet character modulo any positive integer. Below we give the result of Hansen-Naqvi.

Theorem 2.1.2 *([10]) Let ψ be a Dirichlet character modulo $r = \prod_{i=1}^{\ell} p_i^{\alpha_i}$, and write $\psi = \prod_{i=1}^{\ell}\psi_i$, where ψ_i is a Dirichlet character modulo $p_i^{\alpha_i}$. Further, let $\psi(-1) = (-1)^\nu$, $\nu = 0$ or 1 according as ψ is even or odd. For a normalized Hecke eigenform $f \in S_k(N,\chi)$ set $F(z) = f(4rz)h_\psi(z) \in S_{k+\nu+1/2}(4N'r^2, \chi\psi\chi_{-4}^{k+\nu})$, where $N' = N/\gcd(N,r)$. Define the function $g(z)$ as*

$$g(z) = \begin{cases} \displaystyle\sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1)f(dz)f(rz/d) & \text{if } \nu = 0, \\ \displaystyle\frac{1}{\pi i}\sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1)df'(dz)f(rz/d) & \text{if } \nu = 1. \end{cases} \tag{2.1}$$

Then we have

$$\mathscr{S}_1(F)(z) = g_\psi(z) - 2^{k+\nu-1}\psi(2)\chi(2)g_\psi(2z),$$

which belongs to the space $S_{2(k+\nu)}(2N'r^2, \chi^2\psi^2)$*, where* $g_\psi(z)$ *is the* ψ*-twist of* $g(z)$*.*

In this chapter, we generalize the result of Hansen-Naqvi to get similar result in the case of general t-th Shimura map $\mathscr{S}_t$. The chapter is organised as follows. First we state our theorems and give a proof of them. At the end, we provide some examples to our result.

Statement of main results

Let $r = \prod_{i=1}^{\ell} p_i^{\alpha_i}$ and ψ be a primitive Dirichlet character modulo r with $\psi(-1) = (-1)^\nu$, where $\nu = 0$ or 1 according as the character ψ is an even or odd character modulo r. As indicated earlier, we write $\psi = \prod_{i=1}^{\ell} \psi_i$, where ψ_i is a Dirichlet character modulo $p_i^{\alpha_i}$ and for any positive divisor $d|r$, we let $\psi_d = \prod_{p_j^{\alpha_j}\|d} \psi_j$. For a modular form $f \in S_k(N,\chi)$ with Fourier expansion $f(z) = \sum_{n\geq 1} a(n)e(nz)$, let us define the function $g_t(z)$ by

$$g_t(z) = a(t)g(z), \tag{2.2}$$

where $a(t)$ is the t-th Fourier coefficient of f and $g(z)$ is the function defined by (2.1). Corresponding to the character ψ modulo r, let $h_\psi(z)$ be the theta function as defined in (1.4).

Theorem 2.1.3 *Let* $f(z)$ *be a normalized Hecke eigenform in* $S_k(N,\chi)$ *and* t *be a square-free positive integer such that* $t|N$, $\gcd(t, 2N/t) = 1$*. For the existence of a square-free* $t \neq 1$*, we assume that* N *has the property that there exists an odd prime* $p|N$ *with* $p^2 \nmid N$*. Let* ψ *be a primitive Dirichlet character modulo* r *such that* $\gcd(r,t) = 1$*. Set* $F_t(z) = f(4rz)h_\psi(tz)$*. Then,* $F_t \in S_{k+\nu+1/2}(4N'r^2, \chi\psi\chi_t\chi_{-4}^{k+\nu})$*,* *where $N' = N/\gcd(N,r)$. For the Hecke eigenform f, let $g_t(z)$ be the function defined by (2.2) with $g(z)$ as in (2.1). Let $G_t(z) = g_{t\psi}(z) = \sum_{n\geq 1} \psi(n)c(n)e(nz)$ ($c(n)$ is the n-th Fourier coefficient of g_t) be the ψ-twist of g_t, which is a modular form in $S_{2(k+\nu)}(N'r^2, \chi^2\psi^2)$. Then we have*

$$\mathscr{S}_t(F_t)(z) = G_t(z) - 2^{k+\nu-1}\psi(2)\chi(2)G_t(2z) \tag{2.3}$$

and it belongs to the space $S_{2(k+\nu)}(2N'r^2, \chi^2\psi^2)$.

Due to our assumption that $\gcd(t, N/t) = 1$, the t-th Fourier coefficient $a(t)$ of f is non-zero, from which it follows that the function $g_t(z)$ defined by (2.2) is non-zero. The second main theorem is the following (which has no conditions on the square-free integer t).

Theorem 2.1.4 *Let $f(z)$ be a normalized Hecke eigenform in $S_k(N,\chi)$ and ψ be a primitive Dirichlet character modulo r as in Theorem 2.1.3. For a square-free positive integer t, set $H_t(z) = f(4rtz)h_\psi(tz)$. Then, $H_t \in S_{k+\nu+\frac{1}{2}}(4N'r^2t, \chi\psi\chi_t\chi_{-4}^{k+\nu})$, where $N' = N/\gcd(N,r)$. Corresponding to the Hecke eigenform f, let $g(z)$ be the function defined by (2.1) and let $G(z)$ be the ψ-twist of g, given by $G(z) = g_\psi(z) = \sum_{n\geq 1} \psi(n)c(n)e(nz)$, where $c(n)$ is the n-th Fourier coefficient of g. Then $G(z) \in S_{2(k+\nu)}(N'r^2, \chi^2\psi^2)$ and the image of the function H_t under the t-th Shimura map is given by*

$$\mathscr{S}_t(H_t)(z) = G(z)|\prod_{p|2t}\left(1 - \chi(p)\psi(p)p^{k+\nu-1}B(p)\right) \tag{2.4}$$

and it belongs to the space $S_{2(k+\nu)}(2N'r^2t, \chi^2\psi^2)$, where $B(d)$ is the operator which *is defined by* $f|B(d)(z) = f(dz)$*. If* t *is such that* $\gcd(N,t) > 1$*, then*

$$\begin{aligned}\mathscr{S}_t(H_t)(z) &= G(z)|\prod_{p|2t/\gcd(N,t)}\left(1-\chi(p)\psi(p)p^{k+\nu-1}B(p)\right)\\ &\in S_{2(k+\nu)}(2N'r^2t/\gcd(N,t),\chi^2\psi^2).\end{aligned} \tag{2.5}$$

More precisely, we have $\mathscr{S}_t(H_t)(z) \in S_{2(k+\nu)}(2N'r^2t/\gcd(Nr,t),\chi^2\psi^2)$*.*

Remark 2.1.1 Though we didn't have any condition on t, the level of the half-integral weight cusp form in Theorem 2.1.4 is divisible by t. So, effectively both the theorems are concerning the t-th Shimura image of certain modular forms whose level is divisible by t. The method of proving these results suggests that these type of constructions won't give information for the t-th Shimura maps on forms of half-integral weight whose level is relatively prime to t.

2.2 Proof of Theorem 2.1.3

First we state an inversion formula needed for the proof of the theorem. As $f \in S_k(N,\chi)$ is a normalized Hecke eigenform, its Fourier coefficients $a(n)$ satisfy the following multiplicative property:

$$a(m)a(n) = \sum_{d|(m,n)} \chi(d)d^{k-1}a(mn/d^2). \tag{2.6}$$

The inversion formula is the inverse of the above property.

Proposition 2.2.1 *([3], [10, Proposition 2.2]) If* $f(z) = \sum_{n\geq 1} a(n)e(nz) \in S_k(N,\chi)$ *is a Hecke eigenform with* $a(1) = 1$*, then we have*

$$a(mn) = \sum_{d|(m,n)} \mu(d)\chi(d)d^{k-1}a(m/d)a(n/d), \tag{2.7}$$

for any positive integers m,n, where μ is the Möbius function.

Let $h_\psi(z)$ be the theta function as defined in (1.4), where ψ is a primitive Dirichlet character modulo a positive integer r. We assume the prime factorization of r as given before. Since $f \in S_k(N,\chi)$, $f(4rz) \in S_k(4Nr,\chi)$ and $h_\psi(tz)$ belongs to $M_{\nu+1/2}(4r^2t, \psi\chi_{-4}^{\nu}\chi_t)$, where t is a square-free positive integer. So, when $t|N$, the function $F(z)$ defined by $F(z) = f(4rz)h_\psi(tz)$ is a cusp form in the space $S_{k+\nu+1/2}(4N'r^2, \chi\psi\chi_t\chi_{-4}^{k+\nu})$, where $N' = N/\gcd(N,r)$. Since $f \in S_k(N,\chi)$, using the definition of the function g_t given by (2.2), we see that the function $G_t(z)$, which is the ψ-twist of g_t, is a cusp form in $S_{2(k+\nu)}(N'r^3, \psi^2\chi^2)$. Below we give a proof of the fact that $G_t \in S_{2(k+\nu)}(N'r^2, \psi^2\chi^2)$.

As $[\Gamma_0(N'r^2) : \Gamma_0(N'r^3)] = r$, we let the coset representatives be given by

$$\alpha_j := \begin{pmatrix} 1 & 0 \\ jN'r^2 & 1 \end{pmatrix}$$

for $j = 0, 1, \ldots, r-1$. To prove our claim, we need to show that $G_t(z) |_k \alpha_j = G_t(z)$ for all $j = 0, 1, \ldots, r-1$. We consider the Gauss sum $\tau(\psi) = \sum_{m=0}^{r-1} \psi(m) e^{2\pi i m/r}$. By using the Gauss sum (refer to [15], page 128), we can write

$$G_t(z) = \frac{\tau(\psi)}{r} \sum_{\nu=0}^{r-1} \overline{\psi(\nu)} g_t(z - \nu/r) = \frac{\tau(\psi)}{r} \sum_{\nu=0}^{r-1} \overline{\psi(\nu)} g_t(z) |_k \gamma_\nu,$$

where $\gamma_\nu = \begin{pmatrix} 1 & -\nu/r \\ 0 & 1 \end{pmatrix}$. By observing that

$$\begin{pmatrix} 1 & -\nu/r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ jN'r^2 & 1 \end{pmatrix} = \begin{pmatrix} 1 - j\nu N'r & -jN'\nu^2 \\ jN'r^2 & j\nu N'r + 1 \end{pmatrix} \begin{pmatrix} 1 & -\nu/r \\ 0 & 1 \end{pmatrix},$$

we finally conclude that $G_t(z)\mid_k \alpha_j = G_t(z)$ for all $j = 0,1,\ldots,r-1$. For a detailed proof we refer to [[15], Page 128]. We write the Fourier expansion of the function $g_t(z)$ by $g_t(z) = \sum_{n\geq 1} c(n)e(nz)$ (2.2). Then, if ψ is an even character (i.e., when $\nu = 0$), the Fourier coefficients $c(n)$ are given by

$$c(n) = a(t) \sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1) \sum_{m=-\infty}^{\infty} a(m)a(\frac{n-dm}{r/d}).$$

Note that $a(\ell) = 0$ whenever ℓ is not an integer or $\ell \leq 0$. Since f is a normalized Hecke eigenform, and $t|N$, we have $a(t)a(m) = a(tm)$, $m \geq 1$. Note that for $t|N$, the newform f is an eigenfunction under the Hecke operator $U(t)$ and since $a(1) = 1$, we have the relation $a(tm) = a(t)a(m) \forall m \geq 1$. Therefore, we have

$$c(n) = \sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1) \sum_{m=-\infty}^{\infty} a(tm)a(\frac{n-dm}{r/d}). \tag{2.8}$$

When ψ is an odd character (i.e., when $\nu = 1$), the Fourier coefficients $c(n)$ are given by

$$c(n) = \sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1) \sum_{m=-\infty}^{\infty} (n-2dm)a(tm)a(\frac{n-dm}{r/d}). \tag{2.9}$$

Let $F_t(z) = \sum_{n\geq 1} b(n)e(nz)$. Then as $f(z) = \sum_{n\geq 1} a(n)e(nz)$, by using the definition of the theta function $h_\psi(z)$ given by (1.4), we get

$$F_t(z) = \sum_{n=1}^{\infty} a(n)e(4rnz) \sum_{m=-\infty}^{\infty} m^\nu \psi(m)e(tm^2 z). \tag{2.10}$$

This gives the Fourier coefficient $b(n)$ as

$$b(n) = \sum_{m=-\infty}^{\infty} m^{\nu}\psi(m)a\left(\frac{n-tm^2}{4r}\right). \tag{2.11}$$

So, we have

$$b(tn^2) = \sum_{m=-\infty}^{\infty} m^{\nu}\psi(m)a\left(\frac{t(n^2-m^2)}{4r}\right). \tag{2.12}$$

By our assumption, $(t,2r)=1$. Also $\psi(m)=0$ for any m divisible by d, where d is a divisor of r. When n and m are integers with $4|(n^2-m^2)$ both the factors $n-m$ and $n+m$ have to be even. So, let $\left(\frac{n-m}{2},r\right)=d$, then $m\equiv n \pmod{2d}$ and $m\equiv -n \pmod{2r/d}$. If $(d,r/d)=d'\neq 1$, then $m\equiv n\equiv -n\equiv 0 \pmod{2d'}$. This implies that $\psi(m)=0$. So, it is enough to assume that $(d,r/d)=1$ and we substitute $m=n+2dm'$ for some $m'\in\mathbb{Z}$ and $m+n=2n+2dm'$. Since $(d,r/d)=1$, we have

$$\psi(m) = \psi_d(m)\psi_{r/d}(m).$$

As $m\equiv n \pmod{d}$ and $m\equiv -n \pmod{r/d}$, we have

$$\psi(m) = \psi_d(n)\psi_{r/d}(-n) = \psi_d(n)\psi_{r/d}(n)\psi_{r/d}(-1) = \psi_{r/d}(-1)\psi(n). \tag{2.13}$$

Using this in (2.12), we get

$$b(tn^2) = \psi(n)\sum_{m=-\infty}^{\infty}(n-2m)^{\nu}\sum_{\substack{d|r\\(d,r/d)=1}}\psi_{r/d}(-1)a\left(\frac{tm(n-dm)}{r/d}\right). \tag{2.14}$$

Case (i): $\psi(-1) = 1$, i.e., $\nu = 0$. In this case $\psi_d(-1)\psi_{r/d}(-1) = 1$ gives the fact that $\psi_d(-1) = \psi_{r/d}(-1) = \pm 1$. Therefore, we get

$$b(tn^2) = \psi(n) \sum_{\substack{d|r \\ (d,r/d)=1}} \psi_d(-1) \sum_{m\in\mathscr{Z}} a\left(\frac{tm(n-dm)}{r/d}\right). \tag{2.15}$$

Now by applying Selberg inversion (Proposition 2.2.1) we have,

$$a\left(\frac{tm(n-dm)}{r/d}\right) = \sum_{\substack{\delta|tm \\ \delta|\left(\frac{n-dm}{r/d}\right)}} \mu(\delta)\chi(\delta)\delta^{k-1}a(tm/\delta)a\left(\frac{n-dm}{\delta r/d}\right).$$

Substituting this in (2.15), we get

$$b(tn^2) = \sum_{m=-\infty}^{\infty} \sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1)\left(\sum_{\substack{\delta|tm \\ (\delta,t)=1 \\ \delta|\left(\frac{n-dm}{r/d}\right)}} \mu(\delta)\chi(\delta)\delta^{k-1}a(tm/\delta)a\left(\frac{n-dm}{\delta r/d}\right)\right)\psi(n) \tag{2.16}$$

As t divides N, we have an additional condition $(\delta,t) = 1$ (in the sum over δ) and so we have

$$b(tn^2) = \sum_{m=-\infty}^{\infty} \sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1)\left(\sum_{\substack{\delta|m \\ \delta|n}} \mu(\delta)\chi(\delta)\delta^{k-1}a(tm/\delta)a\left(\frac{n-dm}{\delta r/d}\right)\right)\psi(n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1)\left(\sum_{\substack{\delta|m \\ \delta|n}} \mu(\delta)\chi(\delta)\delta^{k-1}\psi(\delta)\psi(n/\delta)a(tm/\delta)a\left(\frac{n/\delta - dm/\delta}{r/d}\right)\right)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1)\left(\sum_{\delta|n} \mu(\delta)\chi(\delta)\delta^{k-1}\psi(\delta)\psi(n/\delta)a(tm)a\left(\frac{n/\delta - dm}{r/d}\right)\right).$$

Now, using the Fourier expansion of g given by (2.8), it follows that

$$\begin{aligned} b(tn^2) &= \sum_{\delta|n} \mu(\delta)\chi(\delta)\delta^{k-1}\psi(\delta)\psi(n/\delta)c(n/\delta) \\ &= \sum_{\delta|n} \mu(\delta)\chi(\delta)\delta^{k+\nu-1}\psi(\delta)\psi(n/\delta)c(n/\delta). \end{aligned} \tag{2.17}$$

Case (ii): $\psi(-1) = -1$, i.e., $\nu = 1$. In this case $\psi_d(-1)\psi_{r/d}(-1) = -1$. So, equation (2.12) now becomes

$$b(tn^2) = \psi(n) \sum_{\substack{d|r \\ (d,r/d)=1}} \psi_d(-1) \sum_{m\in\mathscr{Z}} (n-2dm)a\left(\frac{tm(n-dm)}{r/d}\right). \tag{2.18}$$

Now, proceeding as in the previous case, we get

$$b(tn^2) = \sum_{m=-\infty}^{\infty} \sum_{\substack{d|r \\ \gcd(d,r/d)=1}} \psi_d(-1) \left(\sum_{\delta|n} \mu(\delta)\chi(\delta)\delta^{k+\nu-1}\psi(\delta)\psi(n/\delta)(n/\delta-2dm)a(tm)a\left(\frac{n/\delta-dm}{r/d}\right)\right).$$

Using the Fourier expansion of g given by (2.9) in the above expression, we get the same expression as in (2.17). Thus, in both the cases (ψ even or odd) we have the same expression for $b(tn^2)$ given by (2.17). Now we evaluate the Dirichlet series associated to the coefficients $b(tn^2)$ in the following.

$$\begin{aligned} \sum_{n=1}^{\infty} b(tn^2)n^{-s} &= \sum_{n=1}^{\infty}\sum_{\delta|n} \mu(\delta)\chi(\delta)\delta^{k+\nu-1}\psi(\delta)\psi(n/\delta)c(n/\delta)n^{-s} \\ &= \sum_{n=1}^{\infty} \mu(n)\chi(n)\psi(n)n^{k+\nu-1-s} \sum_{n=1}^{\infty} \psi(n)c(n)n^{-s} \\ &= L(s+1-k-\nu,\chi\psi)^{-1} \sum_{n=1}^{\infty} \psi(n)c(n)n^{-s} \end{aligned}$$

Since $F_t \in S_{k+\nu+1/2}(4N'r^2, \chi\psi\chi_t\chi_{-4}^{k+\nu})$, we have to multiply the left-hand side of the above equation by $L(s-k-\nu+1, \chi\chi_t\psi\chi_{-4}^{k+\nu}\chi_{-4}^{k+\nu}\chi_t)$ to get the Dirichlet series corresponding to the image of F_t under Shimura map $\mathscr{S}_t$ (see equation (1.7)). However, $L(s-k-\nu+1, \chi\chi_t\psi\chi_{-4}^{k+\nu}\chi_{-4}^{k+\nu}\chi_t) = L(s-k-\nu+1, \chi\psi\chi_4)$. So, multiplying both the sides by $L(s-k-\nu+1, \chi\psi\chi_4)$, we get

$$L(s-k-\nu+1, \chi\psi\chi_4)\sum_{n=1}^{\infty} b(tn^2)n^{-s} = \frac{L(s+1-k-\nu, \chi\psi\chi_{-4}^2)}{L(s+1-k-\nu, \chi\psi)}\sum_{n=1}^{\infty}\psi(n)c(n)n^{-s}. \tag{2.19}$$

All Dirichlet series that appears above converge absolutely for $\mathrm{Re}(s)$ sufficiently large, and so by an easy consideration of Euler products, the quotient of the L-functions simplifies as follows:

$$\frac{L(s+1-k-\nu, \chi\psi\chi_{-4}^2)}{L(s+1-k-\nu, \chi\psi)} = \frac{\prod_{p\nmid 2Nr}(1-\chi(p)\psi(p)p^{k+\nu-1-s})^{-1}}{\prod_{p\nmid Nr}(1-\chi(p)\psi(p)p^{k+\nu-1-s})^{-1}} \tag{2.20}$$

$$= \left(1-\chi(2)\psi(2)2^{k+\nu-1-s}\right). \tag{2.21}$$

Therefore,

$$\sum_{n=1}^{\infty} A_t(n)n^{-s} = \left(1-\chi(2)\psi(2)2^{k+\nu-1-s}\right)\sum_{n=1}^{\infty}\psi(n)c(n)n^{-s}$$

and so,

$$\mathscr{S}_t(F_t) = \sum_{n=1}^{\infty}\psi(n)c(n)e(nz) - \chi(2)\psi(2)2^{k+\nu-1}\sum_{n=1}^{\infty}\psi(n)c(n)e(2nz).$$

In other words, $\mathscr{S}_t(F_t)(z) = G_t(z) - \chi(2)\psi(2)2^{k+\nu-1}G_t(2z)$. This completes the proof.

2.3 Proof of Theorem 2.1.4

The proof is similar and we indicate only the relevant changes to be carried out. Let the Fourier expansion of H be given by $H_t(z) = \sum_{n\geq 1} h(n)e(nz)$. Taking the n-th Fourier coefficient of f as $a(n)$ as before, $h(n)$ is given by (compare this with (2.11)):

$$h(n) = \sum_{m=-\infty}^{\infty} m^{\nu}\psi(m)a\left(\frac{n-tm^2}{4rt}\right). \tag{2.22}$$

So, we have

$$h(tn^2) = \sum_{m=-\infty}^{\infty} m^{\nu}\psi(m)a\left(\frac{(n^2-m^2)}{4r}\right).$$

Here we note that the above expansion is the same as in Eq. (3.5) of [10] (in the case $\nu = 0$). Again giving the same argument as presented after (2.12) , we arrive at the following in the general case ($\nu = 0$ or 1).

$$h(tn^2) = \psi(n) \sum_{\substack{d|r,\\ (d,r/d)=1}} (-1)^{\nu}\psi_d(-1) \sum_{m\in\mathbb{Z}} a\left(\frac{m(n-dm)}{r/d}\right)(n-2dm)^{\nu}$$

and so we finally get

$$\begin{aligned}\sum_{n=1}^{\infty} h(tn^2)n^{-s} &= \sum_{n=1}^{\infty}\sum_{\delta|n}\mu(\delta)\chi(\delta)\delta^{k+\nu-1}\psi(\delta)\psi(n/\delta)c(n/\delta)n^{-s}\\ &= L(s+1-k-\nu,\chi\psi)^{-1}\sum_{n=1}^{\infty}\psi(n)c(n)n^{-s}\end{aligned}$$

To get the Dirichlet series corresponding to the image of H_t under $\mathscr{S}_t$ we have to multiply both sides of the above equation by $L(s-k-\nu+1,\chi\psi\chi_4\chi_t^2)$. In the present case, the ratio of the L-functions that appear on the right-hand side has the following simplification.

$$\frac{L(s+1-k-\nu,\chi\psi\chi_4\chi_t^2)}{L(s+1-k-\nu,\chi\psi)} = \frac{\prod_{p\nmid 2Nrt}(1-\chi(p)\psi(p)p^{k+\nu-1-s})^{-1}}{\prod_{p\nmid Nr}(1-\chi(p)\psi(p)p^{k+\nu-1-s})^{-1}}$$
$$= \prod_{p|2t}\left(1-\chi(p)\psi(p)p^{k+\nu-1-s}\right).$$

Therefore,

$$\mathscr{S}_t(H_t) = \left(\sum_{n=1}^{\infty}\psi(n)c(n)e(nz)\right)|\prod_{p|2t}\left(1-\chi(p)\psi(p)p^{k+\nu-1}B(p)\right)$$

In other words, $\mathscr{S}_t(H_t)(z) = G(z)|\prod_{p|2t}\left(1-\chi(p)\psi(p)p^{k+\nu-1}B(p)\right)$. This completes the proof.

2.4 Examples

In this section, we shall give some examples to illustrate our results. Examples 1 to 3 are for Theorem 2.1.3 and Examples 4 and 5 are for Theorem 2.1.4. Also in the examples $q = e(z)$.

Example 1: The case $r = 1$. Let

$$f(z) = \eta^3(2z)\eta^3(6z) = q - 3q^3 + 2q^7 + 9q^9 - 22q^{13} + 26q^{19} + \mathbf{O}(q^{20})$$

be the newform in $S_3(12, (\frac{\cdot}{3}))$. Let $\theta(z) = \sum_{n=-\infty}^{\infty} e(n^2 z)$ be the theta function. Set

$$F_3(z) = f(4z)\theta(3z) = q^4 + 2q^7 - 3q^{12} - 6q^{15} + 2q^{16} + \mathbf{O}(q^{17}),$$

which belongs to $S_{3+1/2}(48)$. Since $N = 12$, let $t = 3$ and so let $g_t(z) = a(t)f^2(z) =$

$-3f^2(z) \in S_6(12)$. Since $r=1$, we have

$$G_3(z) = g_3(z) = -3f^2(z) = -3q^2 + 18q^4 - 27q^6 - 12q^8 + 36q^{10} + \mathbf{O}(q^{11}).$$

As $N=12$, $\chi(2)=0$ and so by Theorem 2.1.3

$$\mathscr{S}_3(F_3)(z) = G_3(z) - 2^{k-1}\chi(2)\psi(2)G_3(2z) = -3f^2(z) \in S_6(12).$$

Example 2: In this example, we will take ψ to be the Legendre symbol $\left(\frac{\cdot}{5}\right)$ modulo 5 which is an even character modulo 5. So, $r=5$. Let f be the newform in the space $S_4(6)$ given by the eta product $\eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z)$. Then

$$f(z) = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 - 8q^8 + 9q^9 + \mathbf{O}(q^{10}).$$

Since $N=6$, we take $t=3$. Also, $N'=N=6$, $a(3)=-3$. The functions F and g are defined as follows.

$$\begin{aligned} F_3(z) &= f(20z)h_\psi(3z) \ \in S_{4+1/2}(600, (\frac{15}{\cdot})), \\ g_3(z) = 2a(3)f(z)f(5z) &= -6f(z)f(5z) = \sum_{n\geq 6} c(n)e(nz) \in S_8(30), \end{aligned}$$

where $h_\psi(z) = \sum_{n\in\mathbb{Z}} \left(\frac{n}{5}\right) e(n^2 z)$. The function $G_3(z) \in S_8(150)$ is the ψ-twist of $g_3(z)$ and is given by

$$G_3(z) = \sum_{n\geq 1} \left(\frac{n}{5}\right) c(n)e(nz).$$

Since N is even, $\chi(2)=0$ and so by Theorem 2.1.3, we get

$$\mathscr{S}_3(F_3)(z) = G_3(z) \in S_8(150).$$

Example 3: In this case, we shall take ψ to be the odd Dirichlet character modulo 3 given by the Legendre symbol $\left(\frac{\cdot}{3}\right)$. Let $f(z) = \eta^3(z)\eta^3(7z) = q - 3q^2 + 5q^4 - 7q^7 - 3q^8 + 9q^9 + \mathbf{O}(q^{10})$ be the newform in $S_3(7, \left(\frac{\cdot}{7}\right))$. We have $N = 7, t = 7, r = 3$ and $F_7(z) = f(12z)h_\psi(7z) \in S_{4+1/2}(252, \left(\frac{3}{\cdot}\right))$, where $h_\psi(z) = \sum_{n\in\mathbb{Z}} n \left(\frac{n}{3}\right) e(n^2z)$. The 7-th Fourier coefficient $a(7)$ of f is -7. The function g_7 is defined by

$$g_7(z) = \frac{-7}{\pi i}\left[f'(z)f(3z) - 3f'(3z)f(z)\right] = \sum_{n\geq 1} c(n)e(nz) \in S_8(21)$$

and $G_7(z)$ is defined by

$$G_7(z) = \sum_{n\geq 1} \left(\frac{n}{3}\right) c(n)e(nz).$$

By Theorem 2.1.3 we have

$$\mathscr{S}_7(F_7)(z) = G_7(z) - 2^3 \left(\frac{2}{7}\right)\left(\frac{2}{3}\right) G_7(2z) = G_7(z) + 8G_7(2z) \in S_8(126).$$

Example 4: As in Example 2, let ψ be the Legendre symbol $\left(\frac{\cdot}{5}\right)$ modulo $r = 5$. Let f be the newform in the space $S_4(6)$ given by the eta product $\eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z)$. Then

$$f(z) = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 - 8q^8 + 9q^9 + \mathbf{O}(q^{10}).$$

Let us take $t = 7$. In this case, $N' = N$, $g(z) = 2f(z)f(5z) = \sum_{n\geq 6} c(n)e(nz) \in S_8(30)$ and $G(z) = \sum_{n\geq 1} \left(\frac{n}{5}\right) c(n)e(nz) \in S_8(150)$. The function $F_7(z)$ is defined by $F_7(z) = f(140z)h_\psi(7z) \in S_{4+1/2}(4200, \left(\frac{35}{\cdot}\right))$, where $h_\psi(z) = \sum_{n\in\mathbb{Z}} \left(\frac{n}{5}\right) e(n^2z)$. Then by Theorem 2.1.4, we have the following.

$$\begin{aligned}\mathscr{S}_7(F_7)(z) &= G(z)\big|\left(1 + 2^3B(2)\right)\left(1 + 7^3B(7)\right)\\ &= G(z) + 8G(2z) + 343G(7z) + 2744G(14z) \in S_8(2100).\end{aligned}$$

Example 5: In Example 4, we take $t = 5$. Then, $N' = N$ and the functions f, g, G are the same. The function $F_5(z)$ is defined by $F_5(z) = f(100z)h_\psi(5z) \in S_{4+1/2}(3000)$, where $h_\psi(z)$ is as defined in Example 4. In this case, Theorem 2.1.4, gives the following.

$$\mathscr{S}_5(F_5)(z) = G(z)\big|\left(1 + 2^3 B(2)\right) = G(z) + 8G(2z) \in S_8(300).$$

Note that the level of the integral weight form is 300, where it should have been 1500. Since t divides r, the level of the integral weight form is further divided by t.

CHAPTER 3

On simultaneous non-vanishing of the twisted L-functions of newforms on $\Gamma_0(N)$

3.1 Introduction

Let f_1 and f_2 be two normalized Hecke eigenforms of weight $2k_1$ and $2k_2$ respectively and of level N and assume that $k_1 \equiv k_2 \pmod 2$. Let $\mathscr{D}$ denotes the set of fundamental discriminants, i.e. for a $d \in \mathscr{D}$ either $d = 1$ or d is the discriminant of a quadratic field. Define the set

$$\Delta(f_1, f_2) := \{d \in \mathscr{D} : L(f_1 \otimes \chi_d, 1/2)L(f_2 \otimes \chi_d, 1/2) \neq 0\},$$

where $L(f_i \otimes \chi_d, s)$, is the twisted L-function associated with f_i $(i = 1, 2)$ by quadratic character χ_d for $d \in \mathscr{D}$, here χ_d is defined by $\chi_d(n) = \left(\frac{d}{n}\right)$, where $\left(\frac{d}{n}\right)$ is the generalised Jacobi symbol. We prove that if the set $\Delta(f_1, f_2)$ is non-empty, then the cardinality of the set $\Delta(f_1, f_2)$ is infinite. This result is a generalisation of the work of R. Munshi [27], who first obtained similar result when $N = 1$. Our method is very similar to the one adopted by Munshi. The idea is to use the connection between half-integral weight modular forms and integral weight modular forms as developed by Kohnen [16]. Especially the Waldspurger formula relating the special values of the twisted L-function associated to a modular form of integral weight and square of the Fourier coefficients of the corresponding half-integral weight modular form. Before we proceed to state our result we fix the notation about the L-functions studied in this chapter. For a cusp form $f(\tau) = \sum_{n=1}^{\infty} a(n)e(n\tau)$ and for a Dirichlet character χ, the two L-functions associated to f, denoted by $L(f,s)$ and $L(f,\chi,s)$ are defined as follows

$$L(f,s) = \sum_{n=1}^{\infty} a(n)n^{-s}$$
$$L(f,\chi,s) = \sum_{n=1}^{\infty} a(n)\chi(n)n^{-s}.$$

However, we consider the normalized L-function associated to a cusp form of weight $2k$ defined by

$$L(f\otimes\chi,s) = \sum_{n\geq 1} \frac{a(n)}{n^{k-1/2}}\chi(n)n^{-s}. \tag{3.1}$$

We give below the precise statement of our result.

Theorem 3.1.1 *Let f_1 and f_2 be two normalized Hecke eigenforms of weight $2k_1$ and $2k_2$ respectively and of level N, where N is odd square-free and also assume that k_1 and k_2 are of same parity. Suppose that there exists a fundamental discriminant d such that $(d,N) = 1$ and*

$$L(f_1\otimes\chi_d, 1/2)L(f_2\otimes\chi_d, 1/2) \neq 0,$$

then there are infinitely many such fundamental discriminants d with the above property. In other words the cardinality of the set $\Delta(f_1, f_2)$ is either zero or infinite.

3.2 Proof of Theorem 3.1.1

Let N be a fixed positive odd square-free integer. Suppose that

$$f_i(\tau) = \sum_{n=1}^{\infty} a_i(n)e(n\tau) \in S_{2k_i}(N), \quad i = 1,2$$

be two Hecke eigenforms of weight $2k_i$ and level N with $a_i(1) = 1$. Let

$$F_i(\tau) = \sum_{n=1}^{\infty} A_i(n)e(n\tau) \in S^+_{k+1/2}(4N)$$

be the half-integral weight modular form corresponding to f_i, under the Shimura-Kohnen correspondence given by (1.9).

We assume that the set $\Delta(f_1, f_2)$ is nonempty and cardinality of $\Delta(f_1, f_2)$ is finite. We show that it leads to a contradiction.

Define $H(\tau) := \overline{F_1(\tau)}F_2(\tau)$. Then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, the transformation of $F_1(\tau)$ and $F_2(\tau)$ w.r.t γ implies that

$$H(\gamma\tau) = \overline{(c\tau + d)}^{k_1+\frac{1}{2}}(c\tau + d)^{k_2+\frac{1}{2}}H(\tau). \tag{3.2}$$

Note that the condition $k_1 \equiv k_2 \pmod 2$ implies that the $\left(\frac{-4}{d}\right)$ factor in the transformation cancel with each other. Let $\kappa_1 = \infty, \kappa_2, \ldots, \kappa_h$ be the inequivalent cusps of $\Gamma_0(4N)$, Then for each cusp κ_i there exists $g_i \in \mathrm{SL}_2(\mathbb{Q})$ such that $g_i\infty = \kappa_i$ and $\Gamma_i := g_i\Gamma_\infty g_i^{-1}$, where Γ_i and Γ_∞ are the stabilizer group of the cusps κ_i $(i > 1)$ and ∞ respectively.

We have an Eisenstein series corresponding to each cusp κ_i as follows;

$$E_i(\tau,s,k) := \sum_{\gamma\in\Gamma_i\backslash\Gamma_0(4N)} j(g_i^{-1}\gamma,\tau)^k \mathrm{Im}(g_i^{-1}\gamma\tau)^s, \tag{3.3}$$

where $j(\gamma,\tau) = \overline{(c\tau+d)}(c\tau+d)^{-1}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

These Eisenstein series converge absolutely for Re(s) > 1 and have analytic continuation to the whole complex plane. If we write

$$\overrightarrow{E}(\tau,s,k) = \begin{bmatrix} E_1(\tau,s,k) \\ E_2(\tau,s,k) \\ \vdots \\ E_h(\tau,s,k) \end{bmatrix}$$

Then these Eisenstein series together satisfy a functional equation given by

$$\overrightarrow{E}(\tau,s,k) = \Phi(s)\overrightarrow{E}(\tau,1-s,k), \tag{3.4}$$

here $\Phi(s)$ is a $h\times h$ matrix called scattering matrix [11, p. 238] and satisfies the condition

$$\Phi(s)\Phi(1-s) = I_{h\times h},$$

where $I_{h\times h}$ is the $h\times h$ identity matrix.

Observing that for any $\alpha,\beta\in \mathrm{SL}_2(\mathbb{Z})$, $j(\alpha\beta,\tau) = j(\alpha,\beta\tau)j(\beta,\tau)$. Consequently we have,

$$E_i(\eta\tau,s,k) = \overline{(c\tau+d)}^{-k}(c\tau+d)^k E_i(\tau,s,k), \quad \eta\in\Gamma_0(4N). \tag{3.5}$$

From the transformation of $H(\tau)$ and $E_i(\tau,s,k)$ equations (3.2) and (3.5) it follows that the function

$$H(\tau)E_i\left(\tau, s, \frac{k_1-k_2}{2}\right) y^{\frac{k_1+k_2+1}{2}}$$

is invariant under $\Gamma_0(4N)$. Therefore, we can consider the Rankin-Selberg integral

$$R_i(s) = \int_{\Gamma_0(4N)\backslash\mathbb{H}} H(\tau)E_i\left(\tau, s, \frac{k_1-k_2}{2}\right) \mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}} \frac{dudv}{v^2}, \quad (\tau = u+iv \in \mathbb{H}). \quad (3.6)$$

We use the functional equation satisfied by $R_i(s)$, which was obtained by Shamita Dutta Gupta [7], which we present below.

Theorem 3.2.1 ([7]) *Let $R_i(s)$ be as defined in (3.6). Then $R_i(s)$ has a meromorphic continuation to all s, the only possible poles being at $s = 0, 1, \alpha_{ij}, 1-\alpha_{ij}$ and $\rho/2$, where ρ's are the nontrivial zeros of the Riemann zeta function. Further we have the following functional equation*

$$\overrightarrow{R}(s) = \begin{bmatrix} R_1(s) \\ R_2(s) \\ \vdots \\ R_h(s) \end{bmatrix} = \Phi(s) \begin{bmatrix} R_1(1-s) \\ R_2(1-s) \\ \vdots \\ R_h(1-s) \end{bmatrix} = \Phi(s)\overrightarrow{R}(1-s),$$

where $\phi(s)$ is as in equation (3.4).

To prove our theorem, we adopt the following strategy. We use the Rankin unfolding argument and the Fourier series expansion of $F_i(\tau)$, to express Rankin-Selbebrg integrals $R_i(s)$ for the cusp $\kappa_i = \infty$, in terms of the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{\overline{A_1(n)}A_2(n)}{n^{s+k*}},$$

where $k^* = \frac{k_1+k_2-1}{2}$. Since $A_i(n)$ are the n^{th} Fourier coefficient of the newform $F_i(\tau)$

of weight $k_i+1/2$ in the Kohnen plus space of level $4N$, we use the Shimura-Kohnen map obtained by Kohnen (1.9). Using this map, we are able to write the Dirichlet Series $D(s)$ in terms of (upto certain Dirichlet series $\Xi(s)$) the Rankin-Selberg L-function corresponding to the integral weight newforms $f_i(\tau)$. Finally, by computing the order of zeros of $\Xi(s)$ in a rectangle and using the known result about the zeros of Rankin-Selberg L-function, we arrive at a contradiction. We now give a detailed argument.

Consider the integral

$$\begin{aligned} R_i(s) &= \int\limits_{\Gamma_0(4N)\backslash\mathbb{H}} H(\tau)E_i\left(\tau,s,\frac{(k_1-k_2)}{2}\right)\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}}\frac{dudv}{v^2}, \\ &= \int\limits_{\Gamma_0(4N)\backslash\mathbb{H}} H(\tau)\sum_{\gamma\in\Gamma_i\backslash\Gamma_0(4N)} j(g_i^{-1}\gamma,\tau)^{\frac{k_1-k_2}{2}}\mathrm{Im}(g_i^{-1}\gamma\tau)^s\,\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}}\frac{dudv}{v^2}, \\ &= \int\limits_{\Gamma_0(4N)\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_i\backslash\Gamma_0(4N)} H(\tau)j(g_i^{-1}\gamma,\tau)^{\frac{k_1-k_2}{2}}\mathrm{Im}(g_i^{-1}\gamma\tau)^s\,\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}}\frac{dudv}{v^2}. \end{aligned}$$

We can interchange the sum and integration in the above equation to get

$$R_i(s) = \sum_{\gamma\in\Gamma_i\backslash\Gamma_0(4N)}\int\limits_{\Gamma_0(4N)\backslash\mathbb{H}} H(\tau)j(g_i^{-1}\gamma,\tau)^{\frac{k_1-k_2}{2}}\mathrm{Im}(g_i^{-1}\gamma\tau)^s\,\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}}\frac{dudv}{v^2}.$$

Using the change of variable τ to $g_i\tau$, we get

$$R_i(s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4N)}\int\limits_{g_i^{-1}\Gamma_0(4N)\backslash\mathbb{H}} H(g_i\tau)j(g_i^{-1}\gamma,g_i\tau)^{\frac{k_1-k_2}{2}}\mathrm{Im}(g_i^{-1}\gamma g_i\tau)^s\,\mathrm{Im}(g_i\tau)^{\frac{k_1+k_2+1}{2}}\frac{dudv}{v^2}.$$

Now using the Rankin unfolding argument, we have

$$R_i(s) = \int_{\Gamma_\infty\backslash\mathbb{H}} j(g_i,\tau)^{\frac{-(k_1-k_2)}{2}}H(g_i\tau)\mathrm{Im}(g_i\tau)^{\frac{k_1+k_2+1}{2}}\mathrm{Im}(\tau)^s\frac{dudv}{v^2}.$$

Therefore,

$$\begin{aligned} R_\infty(s) &= \int\limits_{\Gamma_\infty\backslash\mathbb{H}} H(\tau)\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}}\mathrm{Im}(\tau)^s\frac{dudv}{v^2}, \\ &= \int\limits_{\Gamma_\infty\backslash\mathbb{H}} \overline{F_1(\tau)}F_2(\tau)\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}}y^s\frac{dudv}{v^2}. \end{aligned}$$

Now replacing F_1 and F_2 by their Fourier series expansions, we have

$$R_\infty(s) = \int\limits_{\Gamma_\infty\backslash\mathbb{H}} \left(\sum_{n=1}^{\infty}\overline{A_1(n)}e(-n\overline{\tau})\right)\left(\sum_{m=1}^{\infty}A_2(m)e(m\tau)\right)\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}+s}\frac{dudv}{v^2}. \quad (3.7)$$

A fundamental domain for the action of Γ_∞ on $\mathbb{H}$ is given by $[0,1]\times[0,\infty]$. Integrating (3.7) over this region, we get

$$\begin{aligned} R_\infty(s) &= \int_0^1\int_0^\infty \left(\sum_{n=1}^{\infty}\overline{A_1(n)}e(-n\overline{\tau})\right)\left(\sum_{m=1}^{\infty}A_2(m)e(m\tau)\right)\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}+s}\frac{dudv}{v^2} \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\overline{A_1(n)}A_2(m)\int_0^1\int_0^\infty e(-n\overline{\tau})e(m\tau)\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}+s}\frac{dudv}{v^2} \\ &= \sum_{n=1}^{\infty}\overline{A_1(n)}A_2(n)\int_0^\infty e(-n\overline{\tau})e(n\tau)\mathrm{Im}(\tau)^{\frac{k_1+k_2+1}{2}+s}\frac{dudv}{v^2} \\ &= \frac{\Gamma(s+k^*)}{(4\pi)^{(s+k^*)}}\sum_{n=1}^{\infty}\frac{\overline{A_1(n)}A_2(n)}{n^{s+k^*}}, \quad k^*=\frac{k_1+k_2-1}{2}. \end{aligned} \quad (3.8)$$

We now consider the Dirichlet series defined by

$$D(s) := \sum_{n=1}^{\infty}\frac{\overline{A_1(n)}A_2(n)}{n^{s+k^*}},$$

which appeared in the the Rankin-Selberg integral R_∞ (3.8). Now we relate $D(s)$ with the normalized Rankin-Selberg L- function $L(f_1\times f_2,s)$, where $L(f_1\times f_2,s) =$

$\sum_{n=1}^{\infty} \frac{a_1(n)a_2(n)}{n^{k_1+k_2+s-1}}$. Now by writing $n = |d|m^2$ and using the explicit Shimura-Kohnen map Theorem 1.5.2, we have

$$\overline{A_1(n)}A_2(n) = \overline{A_1(|d|)}A_2(|d|) \prod_{i=1,2} \sum_{\substack{\delta_i|m \\ (\delta_i,N)=1}} \mu(\delta_i)\chi_d(\delta_i)\delta_i^{k_i-1}a_i(m/\delta_i)$$

Therefore,

$$D(s) = \sum_{d\in\Delta(f_1,f_2)} \frac{\overline{A_1(|d|)}A_2(|d|)}{|d|^{s+k^*}} \sum_{m=1}^{\infty} \frac{1}{m^{2(s+k^*)}} \prod_{i=1,2} \sum_{\substack{\delta_i|m \\ (\delta_i,N)=1}} \mu(\delta_i)\chi_d(\delta_i)\delta_i^{k_i-1}a_i(m/\delta_i).$$

Since f_i, $i = 1,2$ are normalized Hecke eigenforms, the functions

$$\sum_{\substack{\delta_i|m \\ (\delta_i,N)=1}} \mu(\delta_i)\chi_d(\delta_i)\delta_i^{k_i-1}a_i(m/\delta_i)$$

are multiplicative in m. Hence, we get

$$\sum_{m=1}^{\infty} \frac{1}{m^{2\omega}} \prod_{i=1,2} \sum_{\substack{\delta_i|m \\ (\delta_i,N)=1}} \mu(\delta_i)\chi_d(\delta_i)\delta_i^{k_i-1}a_i(m/\delta_i) = \prod_p L_p(w,d),$$

where

$$L_p(w,d) = 1 + \sum_{l=1}^{\infty} \frac{1}{p^{2lw}} \prod_{i=1,2} \sum_{\substack{\delta_i|p^l \\ (\delta_i,N)=1}} \mu(\delta_i)\chi_d(\delta_i)\delta_i^{k_i-1}a_i(p^l/\delta_i).$$

As

$$\sum_{\delta|p^l} \mu(\delta)\chi_d(\delta)\delta^{k-1}a(p^l/\delta) = a(p^l) - \chi_d(p)p^{k-1}a(p^{l-1}),$$

we have

$$L_p(w,d) = 1 + \sum_{l=1}^{\infty} \frac{1}{p^{2lw}} \left[\left(a_1(p^l) - \chi_d(p)p^{k_1-1}a_1(p^{l-1}) \right) \left(a_2(p^l) - \chi_d(p)p^{k_2-1}a(p^{l-1}) \right) \right].$$

Now by observing that the local Euler factor of $L(f_1 \times f_2, s)$ is given by $1+\sum_{l=1}^{\infty} \frac{a_1(p^l)a_2(p^l)}{p^{ls}}$ and the local Euler factor of $L(f_i \otimes \chi_d, s)$ is given by

$$\frac{1}{(1-\chi_d(p)a_i(p)p^{-s}+\chi_N(p)p^{2k_i-1-2s})}$$

where, $\chi_N(p) = 1$ if $p \nmid N$ and $\chi_N(p) = 0$ if $p \mid N$.

By executing the sum over l and using the Hecke relation for the Fourier coefficients $a_1(p^l)$ and $a_2(p^l)$. We obtain

$$L_p(w;d) = \frac{L_p(f_1 \times f_2, 2s)}{L_p(f_1 \otimes \chi_d, 2s+1/2)L_p(f_2 \otimes \chi_d, 2s+1/2)} E_p(s;d),$$

where the first factor on the right-hand side is the local Euler factor for the normalized Rankin-Selberg $L-$function $L(f_1 \times f_2, 2s)$, and the last factor is such that the Euler product $\prod_p E_p(s;d)$ is absolutely convergent for $\sigma > 1/8$. Since $L(f_1 \times f_2, 2s) = \prod_p L_p(f_1 \times f_2, 2s)$
and $L(f_i \otimes \chi_d, 2s+1/2) = \prod_p L_p(f_i \otimes \chi_d, 2s+1/2)$ for $i = 1,2$. It follows that

$$D(s) = L(f_1 \times f_2, 2s)\Xi(s), \tag{3.9}$$

where

$$\Xi(s) = \sum_{d \in \Delta(f_1, f_2)} \frac{\overline{A_1(|d|)}A_2(|d|)}{|d|^{s+k^*}} l(s;d)$$

with

$$l(s;d) = \frac{E(s;d)}{L(f_1 \otimes \chi_d, 2s+1/2)L(f_2 \otimes \chi_d, 2s+1/2)}$$

and $E(s;d)$ is an Euler product which converges absolutely in the half-plane $\sigma >$

$1/8$ and the center of $L(f_1 \times f_2, s)$ is at $s = 1/2$.

Now order the elements of $\Delta(f_1, f_2)$ as follows.

$$\Delta(f_1, f_2) = \{d_1, ..., d_m\}, \text{with } |d_1| < |d_2| < ... < |d_m|.$$

Then $\Xi(s)$ is meromorphic in the half plane $\sigma > \frac{1}{8}$, and it is holomorphic in the half-plane $\sigma \geq \frac{1}{4}$. Now we consider the rectangle

$$R = \{s = \sigma + it : 1/3 \leq \sigma \leq \alpha, T \leq t \leq T + H\}$$

and let

$$f(s) = \frac{|d_1|^{s+k^*}\Xi(s)}{A_1(|d_1|)A_2(|d_1|)l(s;d_1)} = 1 + \sum_{i=2}^{m} \alpha(i) \frac{l(s;d_i)}{l(s;d_1)} \left(\frac{d_1}{d_i}\right)^{s+k^*},$$

where αi depends on $A_j(|d_i|)$, $j = 1, 2$ and $i = 1, 2, ..., m$. Then by applying the Littlewood lemma which states that:

If $\nu(\sigma) =$ the number of zeros $-$ number of poles, of f in the region having $\mathrm{Re(s)} \geq \sigma$, where $1/3 < \sigma < \alpha$. Zeros and poles being counted with multiplicity and given weight $1/2$ if occuring on the boundary, Then

$$\int_{1/3}^{\alpha} \nu(\sigma) d\sigma = \frac{-1}{2\pi i} \int_{\partial R} \log \mathrm{f(s)ds},$$

here ∂R denotes the boundary of R oriented counterclockwise. we get that

$$\begin{aligned}\sum_{\rho=\beta+i\gamma\in R, f(\rho)=0} \beta - 1/3 = &\frac{1}{2\pi} \int_{T}^{T+H} \log|\mathrm{f}(1/3 + \mathrm{it})|\mathrm{dt} - \frac{1}{2\pi} \int_{\mathrm{T}}^{\mathrm{T+H}} \log|\mathrm{f}(\alpha + \mathrm{it})|\mathrm{dt} \\ &+ \frac{1}{2\pi} \int_{1/3}^{\alpha} \arg(\mathrm{f}(\sigma + \mathrm{i(T+H)}))\mathrm{d}\sigma - \frac{1}{2\pi} \int_{1/3}^{\alpha} \arg(\mathrm{f}(\sigma + \mathrm{iT}))\mathrm{d}\sigma.\end{aligned}$$

Now by observing that for large α, $\log|f(\alpha+it)| = O_\Delta(1)$ and as $\sigma\to\infty$ we have,

$$f(\sigma+it) = 1+O_\Delta(e^{-\sigma(\log|d_2|-\log|d_1|)}),$$

we get

$$\sum_{\rho=\beta+i\gamma\in R, f(\rho)=0} \beta-1/3 = O_\Delta(H).$$

If we denote by $N(T,\Xi)$ the number of zeros of $\Xi(s)$ (counted with multiplicity) in the region $\{s:\sigma\geq 1/2, |t|<T\}$, from the above observation we obtain

$$N(T,\Xi) = O_\Delta(T).$$

Now by using the functional equation $D(s) = G(s)D(1-s)$, where $G(s)$ is determined by the scattering matrix $\Phi(s)$ Theorem 3.2.1, and it involves Γ-functions. Using (3.9), we get

$$L(f_1\otimes f_2,s) = G(s)L(f_1\otimes f_2, 2-s)\frac{\Xi(1-s/2)}{\Xi(s/2)} \tag{3.10}$$

now we look at the number of zeros of $L(f_1\otimes f_2,s)$ in the ractangle

$$R = \{S=\sigma+it : 1/2\leq\sigma\leq 1, |t|\leq T\}.$$

as it is well known that (see for example [12]) number of zeros $N(T,f_1\otimes f_2)$ of $L(f_1\otimes f_2,s)$ is of order $cT\log T$, where c is a non zero constant. Now we look at the number of zeros of right hand side of the functional equation (3.10). $G(s)$ will have atmost $O(1)$ possible zeros in R, $L(f_1\otimes f_2, 2-s)$ will not have any zeros in R as $2-\mathrm{Re}(s)\geq 1$. Also $\Xi(s/2)$ has no poles in the region and so the major contribution to zeros of right hand side is coming from $\Xi(1-s/2)$, which is of order $O_\Delta(T)$. As the zeros of LHS in (3.10) has order $O(T\mathrm{logT})$ and RHS has order $O(T)$ we get a contradiction.

CHAPTER 4

Determining modular forms of half-integral weight by central values of convolution L-function

4.1 Introduction and Statement of the main theorem

The problem of determining Hecke eigenforms is one of the central problems in the theory of modular forms. In this chapter we consider the problem of determining half-integral weight Hecke eigenforms by central values of the convolution L-functions. There are many works related to the problem of determination of a Hecke eigenform of integral weight by the special values of its twisted L-functions. Here we mention some of these results. In [23] W. Luo and D. Ramakrishnan showed that if two normalized newforms f and f' are such that $L(1/2, f, \chi_D) = L(1/2, f', \chi_D)$ for all quadratic characters $\chi_D = \left(\frac{D}{\cdot}\right)$, then f and f' are equal. As an application, they proved that if g_1 and g_2 are newforms in the Kohnen plus space of weight $k+1/2$ on $\Gamma_0(4N)$ (N is odd and square-free) with Fourier coefficients $b_1(n)$ and $b_2(n)$ with the property that $b_1^2(|D|) = b_2^2(|D|)$ for almost all fundamental discriminants D with $(-1)^k D > 0$, then $g_1 = \pm g_2$. This question was posed by W. Kohnen in [18]. In 1999, Luo [22] proved the following result. Let $H_k(N)$ denote the orthogonal basis of normalized Hecke eigenforms of weight k on $\Gamma_0(N)$. Suppose that f and g are two normalized newforms of weight $2k$ (resp. $2k'$) on $\Gamma_0(N)$ (resp. $\Gamma_0(N')$). If there exists a positive integer ℓ and infinitely many primes p such that for all forms $h \in H_{2\ell}(p)$, the central values of the Rankin-Selberg L-functions are equal (i.e., $L(1/2, f \otimes h) = L(1/2, f' \otimes h)$), then $k = k'$, $N = N'$ and $f = f'$. This result of Luo can be viewed as the $GL(2)$ analog of the result of Luo and Ramakrishnan which is mentioned above. As a variant of Luo's result, in [5], S. Ganguly, J. Hoffstein and J. Sengupta considered twists by Hecke eigenforms of fixed level and varying weight. More precisely, if $g \in H_\ell(1)$ and $g' \in H_{\ell'}(1)$ are such that $L(1/2, f \otimes g) = L(1/2, f \otimes g')$, $f \in H_k(1)$ for infinitely many k, then $\ell = \ell'$ and $g = g'$. (Here k, ℓ, ℓ' are all even positive integers.) There are other generalizations in the case of eigenforms of integral weight (see for example [35, 38]). In this chapter, we generalise the work of Ganguly et. al to the case of forms of half-integral weight. We consider Hecke eigenforms of half-integral weight on $\Gamma_0(4)$ which lie in the Kohnen plus space and prove the following result.

Theorem 4.1.1 *Let g, g' be two Hecke eigenforms belonging to the Kohnen plus space on $\Gamma_0(4)$ of weights $\ell + 1/2$ and $\ell' + 1/2$ respectively, such that $\ell \equiv \ell' \pmod 2$. Suppose that*

$$L(1/2, f \otimes g) = L(1/2, f \otimes g') \tag{4.1}$$

for any Hecke eigenform f of weight $k+1/2$ on $\Gamma_0(4)$ belonging to the Kohnen plus space, for sufficiently large weights k. Then we have, $\ell = \ell'$ and $g = g'$.

The method of proof has several steps, which we explain in brief below before we proceed to the details of the proof. For Hecke eigenforms $f \in S^+_{k+1/2}(4)$ and $g \in S^+_{\ell+1/2}(4)$, we consider their Rankin-Selberg convolution $L(s, f \otimes g)$ and use its approximate functional equation, we obtain an asymptotic expression for the following average:

$$\sum_{f \in \mathscr{F}_{k+1/2}(4)} \omega_f L(1/2, f \otimes g) \frac{a_f(|D|)}{|D|^{k/2-1/4}},$$

where $\mathscr{F}^+_{k+1/2}(4)$ is an orthogonal basis of Hecke eigenforms in the Kohnen plus space $S^+_{k+1/2}(4)$, $a_f(|D|)$ is the $|D|$-th Fourier coefficient of f and ω_f is a constant defined in the next section. Considering these averages for both g and g' over $\mathscr{F}^+_{k+1/2}(4)$, we deduce (using the explicit main and error terms of the above average) that

$$\frac{a_g(|D|)}{|D|^{\ell/2-1/4}} = \frac{a_{g'}(|D|)}{|D|^{\ell'/2-1/4}}, \tag{4.2}$$

for all fundamental discriminants D with $(-1)^k D > 0$. By the Shimura-Kohnen correspondence, the functions g and g' correspond to Hecke eigenforms F and F' in $S_{2\ell}$ and $S_{2\ell'}$ resp., and the explicit Waldspurger theorem connecting the special values of the L-functions corresponding to F and F' and the square of the $|D|$-th Fourier coefficients of g and g'. Therefore, using (4.2), it follows that the special values of the L-functions corresponding to F and F' are equal. At this stage, we apply the result of Luo and Ramakrishnan [23] to conclude that $\ell = \ell'$ and $g = g'$, proving our main theorem.

The necessary auxiliary results are obtained in the following subsections.

In §4.2, we establish the Petersson formula and in §4.3, we consider the Rankin-Selberg L-function of half-integral weight. In §4.4 we obtain an approximate functional equation for the Rankin-Selberg L-function and apply this to get an auxiliary theorem in §4.5. In §4.6 and §4.7 we get estimates for the main term and error term respectively. Finally in §4.8 we present a proof of the main theorem.

4.2 Petersson Formula

For $k \geq 2$ a natural number. Let $S^+_{k+1/2}(4)$, denote the Kohnen plus space. Let $m \in \mathbb{N}$ be such that $(-1)^k m \equiv 0, 1 \pmod 4$. Then the m-th Poincaré series in $S^+_{k+1/2}(4)$ is characterised by

$$\langle f, P^+_{k+1/2,m} \rangle = \frac{\Gamma(k-1/2)}{6(4\pi m)^{k-1/2}} a_f(m),$$

for all $f \in S^+_{k+1/2}(4)$. The factor 6 in the denominator is exactly the index of $\Gamma_0(4)$ in $\mathrm{SL}_2(\mathbb{Z})$. The Fourier expansion of the Poincaré series $P^+_{k+1/2,m}(z)$ in $S^+_{k+1/2}(\Gamma_0(4))$ is obtained in [18, Proposition 4], which is given by

$$P^+_{k+1/2,m}(z) = \sum_{\substack{n \geq 1 \\ (-1)^k n \equiv 0,1 \pmod 4}} g_{k+1/2,m}(n) q^n, \tag{4.3}$$

where

$$g_{k+1/2,m}(n) = \frac{2}{3}\left[\delta_{m,n} + (-1)^{[\frac{k+1}{2}]} \pi\sqrt{2}(n/m)^{(k/2-1/4)} \sum_{c \geq 1} H_c(m,n) J_{k-1/2}(\frac{\pi\sqrt{mn}}{c})\right], \tag{4.4}$$

and $q = e^{2\pi i z}$, $z \in \mathscr{H}$, the complex upper half-plane. In the above,

$$H_c(m,n) = (1-(-1)^k i)\big(1+\left(\frac{4}{c}\right)\big)\frac{1}{4c}\sum_{\delta(4c)^*}\left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k+1/2} e_{4c}(m\delta + n\delta^{-1}),$$

where δ^{-1} is an integer such that $\delta\delta^{-1} \equiv 1 \pmod{4c}$, $J_{k-1/2}(x)$ is the Bessel function of order $k-1/2$ and $\delta_{m,n}$ is the Kronecker delta function. We also recall the notation $e_c(x) = e^{2\pi ix/c}$ for a complex number x and an integer c. The symbol $\left(\frac{c}{d}\right)$ denotes the generalised quadratic residue symbol as described in [37, 15]. Now, as $P^+_{k+1/2,m}(z) \in S^+_{k+1/2}(4)$, writing it in terms of an orthogonal basis and using the characteristic property of Poincaré series we have,

$$P^+_{k+1/2,m}(z) = \sum_{f \in \mathscr{F}^+_{k+1/2}(4)} \omega_f \frac{\overline{a_f(m)}}{m^{k-1/2}} \, f, \tag{4.5}$$

where $\mathscr{F}^+_{k+1/2}(4)$ denotes an orthogonal basis for the plus space $S^+_{k+1/2}(4)$ and $\omega_f = \dfrac{\Gamma(k-1/2)}{6(4\pi)^{k-1/2}} \dfrac{1}{\langle f, f\rangle}$. Now by comparing the n-th Fourier coefficients of both the sides of the above equation and using (4.4), we get

$$\sum_{f \in \mathscr{F}^+_{k+1/2}(4)} \omega_f \frac{\overline{a_f(m)} a_f(n)}{(mn)^{k/2-1/4}} = \frac{2}{3}\left[\delta_{m,n} + (-1)^{[\frac{k+1}{2}]}\pi\sqrt{2}\sum_{c\geq 1} H_c(m,n) J_{k-1/2}\left(\pi\sqrt{mn}/c\right)\right], \tag{4.6}$$

where ω_f is defined as above.

4.3 Rankin-Selberg L-functions

In this section, we shall obtain a functional equation satisfied by the Rankin-Selberg L-function associated to forms of half-integral weight. Let $f_i \in S^+_{k_i+1/2}(\Gamma_0(4))$, $i = 1,2$, and $f_i(z) = \sum_{n\geq 1} a_{f_i}(n) e(nz)$ be their Fourier expansions. We also assume that k_1 and k_2 have the same parity, i.e., $k_1 \equiv k_2 \pmod 2$. Set $H(z) = \overline{f_1(z)} f_2(z)$. Then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we get

$$H(\gamma(z)) = \overline{(cz+d)}^{k_1+1/2} (cz+d)^{k_2+1/2} \overline{f_1(z)} f_2(z),$$

since $\overline{\left(\frac{-4}{d}\right)}^{-k_1-1/2}\left(\frac{-4}{d}\right)^{-k_2-1/2} = 1$, as $k_1 \equiv k_2 \pmod 2$.

Note that the group $\Gamma_0(4)$ has three cusps $\infty, 0, 1/2$ and the matrices $g_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g_0 = \begin{pmatrix} 0 & \frac{-1}{2} \\ 2 & 0 \end{pmatrix}$, $g_{1/2} = \begin{pmatrix} 1 & \frac{-1}{2} \\ 2 & 0 \end{pmatrix}$ take the cusp ∞ to the corresponding cusps $\omega = \infty, 0, 1/2$ respectively. Corresponding to each cusp $\omega = \infty, 0, 1/2$, there is an integral weight Eisenstein series of level 4 (and with weight ℓ) given by the following.

$$E_\omega(z,s;\ell) = \sum_{\gamma\in\Gamma_\omega\backslash\Gamma_0(4)} j(g_\omega^{-1}\gamma,z)^\ell \mathrm{Im}(g_\omega^{-1}\gamma z)^s. \tag{4.7}$$

In the above, $j(\gamma,z) = \overline{(cz+d)}(cz+d)^{-1}$ and the stabilizer Γ_ω of the cusps ω is given by $\Gamma_\omega = g_\omega \Gamma_\infty g_\omega^{-1}$, $\omega = \infty, 0, 1/2$. It is known that these Eisenstein series converge absolutely for $\mathrm{Re}(s) > 1$, have analytic continuations to the whole of $\mathbb{C}$ and they satisfy a functional equation, we refer to [11], [27] for details. In particular, we have

$$E_\infty(z,1-s;\ell) = \phi_\infty(s)E_\infty(z,s;\ell) + \phi_0(s)E_0(z,s;\ell) + \phi_{1/2}(s)E_{1/2}(z,s;\ell), \tag{4.8}$$

where

$$\phi_\infty(s) = \frac{2^{4s-3}\zeta(2s)\Gamma(s+\ell)\pi^{-s}}{(1-2^{2s-2}\zeta(2-2s)\Gamma(1-s+\ell)\pi^{-(1-s)})} \tag{4.9}$$

and

$$\phi_0(s) = \phi_{1/2}(s) = \left(\frac{1}{2^{2s-1}} - 1\right)\phi_\infty(s). \tag{4.10}$$

Also for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ we have

$$E_\omega(\gamma z,s;\ell) = \overline{(cz+d)}^{-\ell}(cz+d)^\ell E_\omega(z,s;\ell).$$

So, the function $H(z)E_\omega(z,s;\frac{k_1-k_2}{2})y^{\frac{k_1+k_2+1}{2}}$ is invariant under $\Gamma_0(4)$ and therefore, one can consider the integral

$$R_\omega = \int_{\Gamma_0(4)\backslash\mathbb{H}} H(z)E_\omega(z,s;\frac{k_1-k_2}{2})y^{\frac{k_1+k_2+1}{2}}\frac{dxdy}{y^2}. \tag{4.11}$$

Now following the standard unfolding argument, we obtain the Rankin-Selberg L-function as follows:

$$L(s,f_1\times f_2) = \sum_{n=1}^{\infty}\frac{\overline{a_{f_1}(n)}a_{f_2}(n)}{n^{s+(k_1+k_2-1)/2}}. \tag{4.12}$$

In the following, we define the completed Rankin-Selberg L-functions associated to f_1 and f_2 corresponding to each of the three cusps $\infty, 0, 1/2$.

$$\begin{aligned}
\Lambda_\infty(s,f_1\otimes f_2) &= \pi^{-2s-k^*}\Gamma(s+k')\Gamma(s+k^*)\zeta(2s)\sum_{n=1}^{\infty}\frac{\overline{a_{f_1}(n)}a_{f_2}(n)}{n^{s+k^*}},\\
\Lambda_0(s,f_1\otimes f_2) &= \pi^{-2s-k^*}\Gamma(s+k')\Gamma(s+k^*)\zeta(2s)\sum_{\substack{n=1\\ n\equiv 0 \pmod 4}}^{\infty}\frac{\overline{a_{f_1}(n)}a_{f_2}(n)}{n^{s+k^*}},\\
\Lambda_{1/2}(s,f_1\otimes f_2) &= \pi^{-2s-k^*}\Gamma(s+k')\Gamma(s+k^*)\zeta(2s)\sum_{\substack{n=1\\ n\equiv(-1)^{k_1} \pmod 4}}^{\infty}\frac{\overline{a_{f_1}(n)}a_{f_2}(n)}{n^{s+k^*}}.
\end{aligned} \tag{4.13}$$

In the above, we have made the following substitutions, $k' = (k_1-k_2)/2$ and $k^* = (k_1+k_2-1)/2$. We also assume w.l.g that $k_1 > k_2$. Since k_1 and k_2 have the same parity, k' is an integer. These completed Rankin-Selberg L-functions together satisfy a functional equation, which is given below.

$$\Lambda_\infty(1-s,f_1\otimes f_2) = \psi_\infty(s)\Lambda_\infty(s,f_1\otimes f_2)+\psi_0(s)\Lambda_0(s,f_1\otimes f_2)+\psi_{1/2}(s)\Lambda_{1/2}(s,f_1\otimes f_2), \tag{4.14}$$

where

$$\psi_\infty(s) = \frac{1}{2(1-2^{2s-2})}, \ \psi_0(s) = \psi_{1/2}(s) = \frac{(-1)^{k'}(1-2^{2s-1})}{2(1-2^{2s-2})}. \tag{4.15}$$

Note that $\psi_\infty(1/2) = 1$. Since both f_1 and f_2 belong to the Kohnen plus space, we see that

$$\Lambda_\infty(s, f_1 \otimes f_2) = \Lambda_0(s, f_1 \otimes f_2) + \Lambda_{1/2}(s, f_1 \otimes f_2).$$

So, with this observation we have the following.

$$\Lambda_\infty(1-s, f_1 \otimes f_2) = \begin{cases} \Lambda_\infty(s, f_1 \otimes f_2), & \text{if } k' \text{ is even,} \\ (2\psi_\infty(s) - 1)\Lambda_\infty(s, f_1 \otimes f_2), & \text{if } k' \text{ is odd.} \end{cases} \tag{4.16}$$

4.4 Approximate functional equation

In this section, we determine approximate functional equation for the completed Rankin-Selberg L-function and use it to get an expression for the central value of the Rankin-Selberg L-function. We assume that f_i's are modular forms in $S^+_{k_i+1/2}(4)$. Let $G(u)$ be a holomorphic function on an open set containing $|\mathrm{Re}(u)| \le 3/2$ and bounded therein. We also choose the function G such that $G(0) = 1$, $G(-u) = G(u)$ (later we will be taking $G(u) = e^{u^2}$). For $X > 0$, we consider the integral

$$I(X, s) = \frac{1}{2\pi i} \int_{(3/2)} X^u \Lambda_\infty(s+u, f_1 \otimes f_2) \frac{G(u)}{u} du, \tag{4.17}$$

where $\int_{(c)}$ means the integral over the line $\mathrm{Re(s)} = \mathrm{c}$. We now move the line of integration from $3/2$ to $-3/2$, which will pick up the residue at $u = 0$ (which is

$\Lambda_\infty(s, f_1 \otimes f_2)$) and so we get

$$I(X,s) = \frac{1}{2\pi i}\int_{(-3/2)} X^u \Lambda_\infty(s+u, f_1\otimes f_2)\frac{G(u)}{u}du + \Lambda_\infty(s, f_1\otimes f_2).$$

Therefore,

$$\Lambda_\infty(s, f_1\otimes f_2) = I(X,s) - \frac{1}{2\pi i}\int_{(-3/2)} X^u \Lambda_\infty(s+u, f_1\otimes f_2)\frac{G(u)}{u}du.$$

In the above, we use the functional equation given by (4.16) to get

$$\Lambda_\infty(s, f_1\otimes f_2) = \begin{cases} I(X,s) - \frac{1}{2\pi i}\int\limits_{-3/2} X^u \frac{\Lambda_\infty(1-s-u, f_1\otimes f_2)G(u)}{u}du, & \text{if } k' \text{ is even} \\ I(X,s) - \frac{1}{2\pi i}\int\limits_{-3/2} X^u \frac{\Lambda_\infty(1-s-u, f_1\otimes f_2)}{2\psi_\infty(s)-1}\frac{G(u)}{u}du, & \text{if } k' \text{ is odd.} \end{cases} \tag{4.18}$$

Now by making the change of variable u going to $-u$ and using the definition of $I(X,s)$ given by (4.17), we have

$$\Lambda_\infty(s, f_1\otimes f_2) = \begin{cases} I(X,s) + I(X^{-1}, 1-s), & \text{if } k' \text{ is even} \\ I(X,s) + \dfrac{I(X^{-1}, 1-s)}{2\psi_\infty(s)-1}, & \text{if } k' \text{ is odd.} \end{cases} \tag{4.19}$$

(Recall that $k' = (k_1 - k_2)/2$.) We now define

$$L(s, f_1\otimes f_2) := \zeta(2s)L(s, f_1\times f_2), \tag{4.20}$$

where $L(s, f_1\times f_2)$ is defined by (4.12). We write the Dirichlet series corresponding to $L(s, f_1\otimes f_2)$ as follows.

$$L(s, f_1\otimes f_2) = \zeta(2s)L(s, f_1\times f_2) = \sum_{n=1}^{\infty} b_{f_1\otimes f_2}(n)n^{-s}, \tag{4.21}$$

where the coefficients are given by

$$b_{f_1\otimes f_2}(n) = \sum_{n=mt^2} \frac{\overline{a_{f_1}(m)}a_{f_2}(m)}{m^{k^*}}, \tag{4.22}$$

where $k^* = \frac{k_1+k_2-1}{2}$. Substituting the above equation, the integral $I(X,s)$ defined by equation (4.17) becomes

$$I(X,s) = \frac{1}{2\pi i}\int_{(3/2)} X^u \pi^{-2s-2u-k^*}\Gamma(s+u+k^*)\Gamma(s+u+k')\sum_{n=1}^{\infty}\frac{b_{f_1\otimes f_2(n)}}{n^{s+u}}\frac{G(u)}{u}du.$$

Now, interchanging the order of integration and summation, we get

$$I(X,s) = \pi^{-2s-k^*}\Gamma(s+k^*)\Gamma(s+k')\sum_{n=1}^{\infty}\frac{b_{f_1\otimes f_2(n)}}{n^s}V_s(\frac{\pi^2 n}{X}), \tag{4.23}$$

where

$$\begin{aligned} V_s(\frac{\pi^2 n}{X}) &= \frac{1}{2\pi i}\int_{(3/2)}\frac{X^u}{(\pi^2 n)^u}\gamma(s,u)\frac{G(u)}{u}du, \\ \gamma(s,u) &= \frac{\Gamma(s+u+k^*)\Gamma(s+u+k')}{\Gamma(s+k^*)\Gamma(s+k')}. \end{aligned} \tag{4.24}$$

Replacing X by X^{-1} and s by $1-s$ in (4.23), we get

$$I(X^{-1},1-s) = \pi^{-2+2s-k^*}\Gamma(1-s+k^*)\Gamma(1-s+k')\sum_{n=1}^{\infty}\frac{b_{f_1\otimes f_2(n)}}{n^{1-s}}V_{1-s}(\frac{\pi^2 n}{X^{-1}}).$$

Therefore, using (4.19), when k' is even, we get

$$\begin{aligned} \Lambda_\infty(s,f_1\otimes f_2) = {} & \pi^{-2+2s-k^*}\Gamma(1-s+k^*)\Gamma(1-s+k')\sum_{n=1}^{\infty}\frac{b_{f_1\otimes f_2(n)}}{n^{1-s}}V_{1-s}(\frac{\pi^2 n}{X^{-1}}) \\ & + \pi^{-2s-k^*}\Gamma(s+k^*)\Gamma(s+k')\sum_{n=1}^{\infty}\frac{b_{f_1\otimes f_2(n)}}{n^s}V_s(\frac{\pi^2 n}{X}) \end{aligned}$$

and when k' is odd, it follows that

$$\Lambda_\infty(s, f_1 \otimes f_2) = \pi^{-2s-k^*}\Gamma(s+k^*)\Gamma(s+k') \sum_{n=1}^{\infty} \frac{b_{f_1\otimes f_2(n)}}{n^s} V_s(\frac{\pi^2 n}{X})$$
$$+ \frac{\pi^{-2+2s-k^*}\Gamma(1-s+k^*)\Gamma(1-s+k')}{2\psi_\infty(s)-1} \sum_{n=1}^{\infty} \frac{b_{f_1\otimes f_2(n)}}{n^{1-s}} V_{1-s}(\frac{\pi^2 n}{X^{-1}}).$$

Observing that at the point $s = 1/2$, both sides of the above expressions have the same gamma factor and the same power of π, and so after cancellation of these terms, we get

$$L(1/2, f_1 \otimes f_2) = \sum_{n=1}^{\infty} \frac{b_{f_1\otimes f_2(n)}}{n^{1/2}} (V_{1/2}(\frac{\pi^2 n}{X}) + V_{1/2}(\frac{\pi^2 n}{X^{-1}})).$$

Now, substituting $X = 1$ in the above, we have the following expression for the central value:

$$L(1/2, f_1 \otimes f_2) = 2\sum_{n=1}^{\infty} \frac{b_{f_1\otimes f_2(n)}}{n^{1/2}} V_{1/2}(\pi^2 n). \tag{4.25}$$

4.5 An Auxiliary Theorem

Let $g \in S^+_{\ell+1/2}(4)$ be a Hecke eigenform with Fourier coefficients $a_g(n)$ and let $\mathscr{F}^+_{k+1/2}$ denotes an orthogonal basis for the space $S^+_{k+1/2}(4)$. For a fixed fundamental discriminat D with $(-1)^k D > 0$, we are interested in obtaining an asymptotic expression for the following average :

$$\sum_{f\in\mathscr{F}_{k+1/2}(4)} \omega_f L(1/2, f \otimes g) \frac{a_f(|D|)}{|D|^{k/2-1/4}}.$$

Using the equation (4.25), the above average becomes

$$\sum_{f\in\mathscr{F}^+_{k+1/2}(4)} \omega_f L(1/2,f\otimes g)\frac{a_f(|D|)}{|D|^{k/2-1/4}} = 2\sum_{f\in\mathscr{F}^+_{k_1+1/2}(4)}\sum_{n=1}^{\infty}\omega_f\frac{b_{f\otimes g}(n)}{n^{1/2}}\frac{a_f(|D|)}{|D|^{k/2-1/4}}V_{1/2}(\pi^2 n). \tag{4.26}$$

From now onwards, we use the following notation κ and κ^* (instead of k' and k^*): $\kappa = (k-\ell)/2$ and $\kappa^* = (k+\ell-1)/2$. Now substituting for $b_{f\otimes g}(n)$ from Eq.(4.22), we get,

$$\begin{aligned}\sum_{f\in\mathscr{F}^+_{k+1/2}(4)} \omega_f L(1/2,f\otimes g)\frac{a_f(|D|)}{|D|^{k/2-1/4}} &= 2\sum_{f\in\mathscr{F}^+_{k+1/2}(4)}\omega_f\sum_{n=1}^{\infty}\left(\sum_{n=mt^2}\frac{\overline{a_f(m)}a_g(m)}{m^{\kappa^*}}\right)\\ &\quad\times\frac{a_f(|D|)}{|D|^{k/2-1/4}}\frac{V_{1/2}(\pi^2 n)}{n^{1/2}}\\ &= 2\sum_{n=1}^{\infty}\frac{V_{1/2}(\pi^2 n)}{n^{1/2}}\sum_{n=mt^2}\frac{a_g(m)}{m^{\ell/2-1/4}}\\ &\quad\times\sum_{f\in\mathscr{F}^+_{k+1/2}(4)}\omega_f\frac{\overline{a_f(m)}a_f(|D|)}{(m|D|)^{k/2-1/4}}.\end{aligned} \tag{4.27}$$

Using the Petersson formula (Eq.(4.6)), the above becomes

$$\sum_{f\in\mathscr{F}_{k+1/2}(4)} \omega_f L(1/2,f\otimes g)\frac{a_f(|D|)}{|D|^{k/2-1/4}} = \frac{4}{3}\left(\frac{a_g(|D|)}{|D|^{\ell/2-1/4}}M_{|D|}(k,\ell)+E_{g,|D|}(k,\ell)\right), \tag{4.28}$$

where $M_{|D|}(k,\ell)$ is the main term given by

$$M_{|D|}(k,\ell) = |D|^{-1/2}\sum_{t=1}^{\infty}\frac{V_{1/2}(\pi^2|D|t^2)}{t} \tag{4.29}$$

and $E_{g,|D|}(k,\ell)$ is the error term given by

$$E_{g,|D|}(k,\ell)=\sum_{n=1}^{\infty}\frac{V_{1/2}(\pi^2 n)}{n^{1/2}}\sum_{n=mt^2}\frac{a_g(m)}{m^{\ell/2-1/4}}(-1)^{\left[\frac{k+1}{2}\right]}\pi\sqrt{2}$$
$$\times\sum_{c\geq 1}H_c(|D|,m)J_{k-1/2}\left(\frac{\pi\sqrt{m|D|}}{c}\right), \tag{4.30}$$

with $V_{1/2}(\pi^2 x)$ is given by (4.24).

Thus, we obtained the following (auxiliary) theorem to prove our main result.

Theorem A: *Let g be a cusp form in the Kohnen plus space* $S^+_{\ell+1/2}(4)$, $\mathscr{F}^+_{k+1/2}(4)$ *be an orthogonal basis for the space* $S^+_{k+1/2}(4)$ *and D be a fundamental discriminant such that* $(-1)^k D>0$*. Then we have the following formula for the spectral average of the central values of the Rankin-Selberg (convolution) L-functions.*

$$\sum_{f\in\mathscr{F}_{k+1/2}(4)}\omega_f L(1/2,f\otimes g)\frac{a_f(|D|)}{|D|^{k/2-1/4}}=\frac{4}{3}\left(\frac{a_g(|D|)}{|D|^{\ell/2-1/4}}M_{|D|}(k,\ell)+E_{g,|D|}(k,\ell)\right), \tag{4.31}$$

where $M_{|D|}(k,\ell)$ *and* $E_{g,|D|}(k,\ell)$ *are given by Eqs.*(4.32), (4.30) *(*ω_f *and* $L(1/2,f\otimes g)$ *are defined in §4.2 and §4.3 respectively).*

In the next sections, we shall give estimates for these main and error terms in order to get our main result.

4.6 Estimation of the Main Term $M_{|D|}(k,\ell)$

$$M_{|D|}(k,\ell)=|D|^{-1/2}\sum_{t=1}^{\infty}\frac{V_{1/2}(\pi^2|D|t^2)}{t}$$

By using (4.24), we have

$$M_{|D|}k,\ell=|D|^{-1/2}\sum_{t=1}^{\infty}\frac{1}{t}\frac{1}{2\pi i}\int_{(3/2)}\frac{(\pi^2|D|t^2)^{-u}\Gamma(u+a)\Gamma(u+b)G(u)}{\Gamma(a)\Gamma(b)u}du.$$

Here we have put $a=\kappa^*+1/2$ and $b=\kappa+1/2$, where $\kappa=\frac{k-\ell}{2}$, $\kappa^*=\frac{k+\ell-1}{2}$. So we have,

$$M_{|D|}(k,\ell)=\frac{|D|^{-1/2}}{2\pi i}\int_{(3/2)}\frac{(\pi^2|D|)^{-u}\Gamma(u+a)\Gamma(u+b)G(u)}{\Gamma(a)\Gamma(b)u}\zeta(2u+1)du.$$

By moving the line of integration to $\mathrm{Re}(u)=-1/2$, we note that the integrand has a double pole at $u=0$, with the residue at $u=0$ given by

$$\frac{\Gamma'}{\Gamma}(a)+\frac{\Gamma'}{\Gamma}(b)+2\gamma_0-\log(\pi^2|D|),$$

where γ_0 is the Euler's constant. Therefore, we have

$$|D|^{1/2}M_{|D|}(k,\ell)=\frac{\Gamma'}{\Gamma}(a)+\frac{\Gamma'}{\Gamma}(b)+2\gamma_0-\log(\pi^2|D|)+I, \tag{4.32}$$

where I denotes the following integral along the line $(-1/2)$:

$$I=\frac{1}{2\pi i}\int_{(-1/2)}\frac{(\pi^2|D|)^{-u}\Gamma(u+a)\Gamma(u+b)G(u)}{\Gamma(a)\Gamma(b)u}\zeta(2u+1)du.$$

By making the change of variable $u=-1/2+iv$, we get

$$I=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\pi^2|D|)^{1/2-iv}\Gamma(a-1/2+iv)\Gamma(b-1/2+iv)G(-1/2+iv)}{\Gamma(a)\Gamma(b)(-1/2+iv)}\zeta(2iv)dv. \tag{4.33}$$

Now using the estimate for the ratio of Γ-functions

$$\frac{\Gamma(A+c+it)}{\Gamma(A+it)}\ll|A+it|^c, \tag{4.34}$$

where the implied constant depends on c, along with the fact that $|\Gamma(x+iy)|\le|\Gamma(x)|$, We get the following estimate for the integral I along the line $\mathrm{Re}(u) = -1/2$:

$$I \leq \frac{|D|^{1/2}}{2} \int_{-\infty}^{\infty} |a+iv|^{-1/2}|b+iv|^{-1/2} \frac{|G(-1/2+iv)|}{|(-1/2+iv)|} |\zeta(2iv)| dv$$

and this finaly gives us

$$I \leq \frac{|D|^{1/2}}{2\kappa} \int_{-\infty}^{\infty} \frac{|G(-1/2+iv)|}{|(-1/2+iv)|} |\zeta(2iv)| dv.$$

Further, from the fact that $\zeta(it) \ll |t|^{1/2}$ and the exponential decay of the function $G(u)$, it follows that $I \ll \frac{|D|^{1/2}}{\kappa}$. Finally, combining everything in (4.32), we have the following estimate for the main term:

$$M_{|D|}(k,\ell) = |D|^{1/2} \left(\frac{\Gamma'}{\Gamma}(a) + \frac{\Gamma'}{\Gamma}(b) + 2\gamma_0 - \log(\pi^2|D|) \right) + O(1/k). \qquad (4.35)$$

4.7 Estimation of the error term $E_{g,|D|}(k,\ell)$

Before we proceed to estimate the error term, we obtain some preliminary results on certain Dirichlet series in the following subsection.

4.7.1 Some facts on certain Dirichlet series

In this section we prove the functional equation of a Dirichlet series associated to modular form of half-integral weight in the Kohnen plus space. Let $g(z) = \sum_{n=1}^{\infty} a_g(n) e^{2\pi i n z} \in S^+_{\ell+1/2}(4)$. We consider the following Dirichlet series associated to g, defined by

$$L_g(s, \frac{\alpha}{\beta}) = \sum_{n=1}^{\infty} \frac{a_g(n) e(\frac{\alpha n}{\beta})}{n^{\ell/2-1/4+s}}, \qquad (4.36)$$

where α,β are positive integers with $(\alpha,\beta)=1$. We derive the functional equation satisfied by the above Dirichlet series. Since g is invariant under the action of $\Gamma_0(4)$, we get

$$g(\gamma z)=\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-\ell-1/2}(cz+d)^{\ell+1/2}g(z),$$

where $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(4)$. Now, taking $z=\frac{-d}{c}+\frac{it}{c}$ with $t\in\mathbb{R}^+$, we get $\gamma z=\frac{a}{c}+\frac{i}{ct}$, so that

$$g(\frac{-d}{c}+\frac{it}{c})=\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\ell+1/2}(it)^{-\ell-1/2}g(\frac{a}{c}+\frac{i}{ct}). \tag{4.37}$$

The Fourier expansions of $g(z)$ and $g(\gamma z)$ (for the above value of z) are given by

$$g(\frac{-d}{c}+\frac{it}{c})=\sum_{n=1}^{\infty}a_g(n)e(\frac{-nd}{c})e^{\frac{-2\pi tn}{c}},$$
$$g(\frac{a}{c}+\frac{i}{ct})=\sum_{n=1}^{\infty}a_g(n)e(\frac{na}{c})e^{\frac{-2\pi n}{ct}}.$$

Using (4.37), the Mellin transform of g becomes

$$\int_0^{\infty}g(\frac{-d}{c}+\frac{it}{c})t^{s+\ell/2-1/4}\frac{dt}{t}=\int_0^{\infty}\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\ell+1/2}(it)^{-\ell-1/2}g(\frac{a}{c}+\frac{i}{ct})t^{s+\ell/2-1/4}\frac{dt}{t}. \tag{4.38}$$

Now substituting the Fourier expansion of $g(z)$ and $g(\gamma z)$ as given above, we get

$$\begin{aligned}LHS&=\int_0^{\infty}\sum_{n=1}^{\infty}a_g(n)e(\frac{-nd}{c})e^{\frac{-2\pi nt}{c}}t^{s+\ell/2-1/4}\frac{dt}{t}\\&=\sum_{n=1}^{\infty}a_g(n)e(\frac{-nd}{c})\int_0^{\infty}t^{s+\ell/2-1/4}e^{\frac{-2\pi nt}{c}}\frac{dt}{t}\\&=(c/2\pi)^{s+\ell/2-1/4}\Gamma(s+\ell/2-1/4)\sum_{n=1}^{\infty}\frac{a_g(n)e(\frac{-nd}{c})}{n^{s+\ell/2-1/4}}\end{aligned}$$

and

$$\begin{aligned}
RHS &= \int_0^\infty \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\ell+1/2} (it)^{-\ell-1/2} \sum_{n=1}^{\infty} a_g(n) e\left(\frac{na}{c}\right) e^{\frac{-2\pi n}{ct}} t^{s+\ell/2-1/4} \frac{dt}{t} \\
&= \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\ell+1/2} (i)^{-\ell-1/2} \sum_{n=1}^{\infty} a_g(n) e\left(\frac{na}{c}\right) \int_0^\infty t^{s-\ell/2-3/4} e^{\frac{-2\pi n}{ct}} \frac{dt}{t} \\
&= (c/2\pi)^{\ell/2+3/4-s} \Gamma(\ell/2+3/4-s) \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\ell+1/2} i^{-\ell-1/2} \sum_{n=1}^{\infty} \frac{a_g(n) e\left(\frac{na}{c}\right)}{n^{\ell/2+3/4-s}}.
\end{aligned}$$

Now by equating the LHS and RHS, we get the following functional equation

$$\begin{aligned}
&(c/2\pi)^{s+\ell/2-1/4} \Gamma(s+\ell/2-1/4) \sum_{n=1}^{\infty} \frac{a_g(n) e\left(\frac{-nd}{c}\right)}{n^{s+\ell/2-1/4}} \\
&\qquad = (c/2\pi)^{\ell/2+3/4-s} \Gamma(\ell/2+3/4-s) \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\ell+1/2} i^{-\ell-1/2} \sum_{n=1}^{\infty} \frac{a_g(n) e\left(\frac{na}{c}\right)}{n^{\ell/2+3/4-s}}
\end{aligned} \tag{4.39}$$

Remark 4.7.1 Note that it is possible to derive a Voronoi type summation formula using the above functional equation (similar to [12, p.83]) for modular forms of half-integral weight.

4.7.2 Error term estimation

For simplicity, we write the error term as E and it is given by

$$E = \sum_{n=1}^{\infty} \frac{V_{1/2}(\pi^2 n)}{n^{1/2}} \sum_{n=mt^2} \frac{a_g(m)}{m^{\ell/2-1/4}} (-1)^{\left[\frac{k+1}{2}\right]} \pi\sqrt{2} \sum_{c\geq 1} H_c(|D|,m) J_{k-1/2}\left(\frac{\pi\sqrt{m|D|}}{c}\right),$$

So, we write it as

$$E = (-1)^{\left[\frac{k+1}{2}\right]} \pi\sqrt{2} \sum_{m=1}^{\infty} \frac{a_g(m)}{m^{\ell/2+1/4}} \sum_{t=1}^{\infty} \frac{V_{1/2}(\pi^2 mt^2)}{t} \sum_{c\geq 1} H_c(|D|,m) J_{k-1/2}\left(\frac{\pi\sqrt{m|D|}}{c}\right).$$

Now by using the inverse Mellin transform of the J-Bessel function (See for example [5]), we can write E as

$$E = \frac{(-1)^{\left[\frac{k+1}{2}\right]}\pi\sqrt{2}}{2(2\pi i)^2} \times \int_{(3/2)}\int_{(\alpha)} \zeta(2u+1)\frac{G(u)}{u}\frac{\Gamma(\frac{k}{2}-\frac{1}{4}+\frac{s}{2})}{\Gamma(\frac{k}{2}+\frac{3}{4}-\frac{s}{2})}\frac{\Gamma(\frac{1}{2}+u+\kappa)}{\Gamma(\frac{1}{2}+\kappa)}\frac{\Gamma(\frac{1}{2}+u+\kappa^*)}{\Gamma(\frac{1}{2}+\kappa^*)} \times 2^s\pi^{-2u-s}|D|^{-s/2}S_u du ds,$$

where

$$S_u = \sum_{m=1}^{\infty}\sum_{c\geq 1}\frac{a_g(m)}{m^{\ell/2+1/4+u+s/2}}\frac{H_c(|D|,m)}{c^{-s}}$$

and κ, κ^* are as in §4.5. Using the Weil bound for the Kloostermann sum $H_c(|D|,m)$ (i.e., $H_c(|D|,m) \ll c^{1/2}$), the series converges absolutely and so we can change the order of summation in S_u. Using the definition of $H_c(|D|,m)$, we get

$$\begin{aligned} S_u &= \sum_{c\geq 1}(1-(-1)^k i)(1+\left(\frac{4}{c}\right))\frac{c^{s-1}}{4}\sum_{a \pmod{4c}^*}\left(\frac{4c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2} \\ &\quad\times \sum_{m=1}^{\infty}\frac{a_g(m)}{m^{\ell/2+1/4+u+s/2}}e\left(\frac{|D|a+md}{4c}\right) \\ &= \sum_{c\geq 1}(1-(-1)^k i)(1+\left(\frac{4}{c}\right))\sum_{a \pmod{4c}^*}\left(\frac{4c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2} \\ &\quad\times \frac{e(\frac{|D|a}{4c})}{c^{1-s}}L_g(1/2+u+s/2,\frac{d}{4c}), \end{aligned}$$

where d is an integer such that $ad \equiv 1 \pmod{4c}$ and we have denoted the sum over m by the Dirichlet series $L_g(s,d/4c)$ as defined in (4.36). Now by applying the functional equation for the Dirichlet series given by (4.39), we get

$$S_u = (1-(-1)^k i)\frac{i^{\ell+1/2}}{4}(\pi/2)^{2u+s}\frac{\Gamma(\frac{\ell}{2}+\frac{1}{4}-u-\frac{s}{2})}{\Gamma(\frac{\ell}{2}+\frac{1}{4}+u+\frac{s}{2})}\sum_{c\geq 1}(1+\left(\frac{4}{c}\right))c^{-1-2u}$$
$$\sum_{a \pmod{4c}^*}\left(\frac{-4}{a}\right)^{k-\ell}\times L_g\big(1/2-u-s/2,\frac{-a}{4c}\big)e(\frac{|D|a}{4c}).$$

Now we move the line of integration in the s variable to $\mathrm{Re}(s)=\alpha=-7$. Since $\mathrm{Re}(-u-s/2+1/2)=5/2$, the Dirichlet series $L_g(1/2-u-s/2,-a/4c)$ is absolutely convergent. Therefore, we can write

$$\sum_{a \pmod{4c}^*}\left(\frac{-4}{a}\right)^{k-\ell}e(\frac{|D|a}{4c})L_g\big(1/2-u-s/2,\frac{-a}{4c}\big)$$
$$=\sum_{m=1}^{\infty}\frac{a_g(m)}{m^{\ell/2+1/4-u-s/2}}\sum_{a \pmod{4c}^*}\left(\frac{-4}{a}\right)^{k-\ell}e(\frac{(|D|-m)a}{4c})$$

Since we are interested in the case where k and ℓ having the same parity, the sum over a reduces to the Ramanujan sum and so we have the following estimate for S_u:

$$S_u \ll (\pi/2)^{2u+s}\frac{\Gamma(\ell/2+1/4-u-s/2)}{\Gamma(\ell/2+1/4+u+s/2)}.$$

Thus, the estimate of the error term E simplifies as

$$E \ll \int_{(3/2)}\int_{(-7)}(\pi/2)^{2u+s}\zeta(2u+1)\frac{G(u)}{u}\frac{\Gamma(k/2-1/4+s/2)}{\Gamma(k/2+3/4-s/2)}\frac{\Gamma(1/2+u+\kappa)}{\Gamma(1/2+\kappa)}$$
$$\times\frac{\Gamma(1/2+u+\kappa^*)}{\Gamma(1/2+\kappa^*)}\frac{\Gamma(\ell/2+1/4-u-s/2)}{\Gamma(\ell/2+1/4+u+s/2)}duds. \tag{4.40}$$

Now writing $u=3/2+iv$ and $s=-7+it$ and integrating with respect to the t variable we get,

$$E \ll \int_{-\infty}^{\infty}k^{-7}(k^2+v^2)^{1/2}\left|\frac{e^{-v^2}\Gamma(2+iv+\kappa^*)\Gamma(2+iv+\kappa)}{(9/4+v^2)^{1/2}\Gamma(1/2+\kappa^*)\Gamma(1/2+\kappa)}\right|dv.$$

Finally, using the bound for the ratio of Γ-functions (4.34) and the estimate $|\Gamma(x+iy)| \leq |\Gamma(x)|$, we have

$$E \ll k^{-7} \int_{-\infty}^{\infty} (v^2+k^2)^{7/2} \frac{e^{-v^2}}{(9/4+v^2)^{1/2}} dv \ll 1.$$

4.8 Proof of the Main Theorem

Using our auxiliary result (Theorem A), we shall prove the main result in this section. By assumption, the functions $g \in S^{+}_{\ell+1/2}(4)$ and $g' \in S^{+}_{\ell'+1/2}(4)$ are Hecke eigenforms such that $L(1/2, f \otimes g) = L(1/2, f \otimes g')$, for all Hecke eigenforms $f \in S^{+}_{k+1/2}(4)$. Therefore, Theorem A implies that for all fundamental discriminants D with $(-1)^k D > 0$,

$$\frac{a_g(|D|)}{|D|^{\ell/2-1/4}} M_{|D|}(k,\ell) + E_{g,|D|}(k,\ell) = \frac{a_{g'}(|D|)}{|D|^{\ell'/2-1/4}} M_{|D|}(k,\ell') + E_{g',|D|}(k,\ell'). \quad (4.41)$$

Using Stirling's formula for the derivatives of $\Gamma(s)$ and equation (4.35) it follows that for k large, $M_{|D|}(k,\ell) = \log k + O(1)$. Also the error terms are bounded for large k. Using these two observations in (4.41) we get

$$\frac{a_g(|D|)}{|D|^{\ell/2-1/4}} = \frac{a_{g'}(|D|)}{|D|^{\ell'/2-1/4}}, \quad (4.42)$$

for all fundamental discriminants D with $(-1)^k D > 0$. Let F and F' be the normalised Hecke eigenforms of weights 2ℓ and $2\ell'$ on $SL_2(\mathbb{Z})$, corresponding to the Hecke eigenforms g and g' (via the Shimura-Kohnen maps) (1.9). Using the corresponding Waldspurger's formula for g and g', obtained by Kohnen which is presented in (1.5.3) and using (4.42), we see that

$$L(F,\chi_D,\ell) = C\,L(F',\chi_D,\ell'), \tag{4.43}$$

for all fundamental discriminants with $(-1)^k D > 0$ and $C > 0$ is a constant. (Here $L(F,\chi_D,\ell)$ denotes the usual L-function associated to the modular form F twisted with the character $\chi_D = \left(\frac{D}{\cdot}\right)$.) Now Theorem B of Luo-Ramakrishnan [23],which states that if $L(F,\chi_D,\ell) = C\,L(F',\chi_D,\ell')$ then $\ell = \ell'$ and $F = CF'$ this implies that $\ell = \ell'$ and $F = F'$. Our main theorem now follows using the 'multiplicity 1' result in $S^+_{\ell+1/2}(\Gamma_0(4))$ (see [16]).

CHAPTER 5

Sign changes of Fourier coefficients of cusp form at sum of two squares

5.1 Introduction

Let S_k be the space of holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Suppose that $f(z)$ is a normalized Hecke-eigenform in S_k. Then the Hecke eigenform $f(z)$ has the following Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

with $a(1) = 1$.

For every $n \in \mathbb{N}$, let

$$\lambda(n) = \frac{a(n)}{n^{\frac{k-1}{2}}}$$

denote the normalized Fourier coefficients. Here $\lambda(n)$ are real and satisfy the multiplicative property that

$$\lambda(m)\lambda(n) = \sum_{d|(m,n)} \lambda\left(\frac{mn}{d^2}\right), \tag{5.1}$$

where m and n are positive integers. It also satisfies the celebrated Deligne's bound

$$\lambda(n) \leq d(n) \ll_{\varepsilon} n^{\varepsilon}, \tag{5.2}$$

where $d(n)$ is the number of divisors of n and ε is any arbitrary small positive constant.

In this chapter, we are interested in studying the sign changes of the subsequence $\{\lambda(n_k)\}_{n_k \geq 1}$, where n_k is a sum of two squares i.e, $n_k = c^2 + d^2$ for some integers c and d. The proof depends on the observation that the function $r_2(n)$, the number of ways n can be written as sum of two squares, can be used as the weighted characteristic function for those n which can be written as sum of two squares. First we state the result of this chapter and then give a proof.

5.2 Main theorem

Let $f \in S_k$ be a normalized Hecke eigenform of even integral weight k for the full modular group and $\lambda(n)$ denotes its n-th normalized Fourier coefficient as described above. We state our main result.

Theorem 5.2.1 *The sequence $\{\lambda(c^2+d^2)\}_{c,d \geq 1}$ has infinitely many sign changes. Moreover, the sequence changes its sign at least $x^{1/8-2\varepsilon}$ times in the interval $(x, 2x]$ for sufficiently large x, where ε is an arbitrarily small positive constant.*

5.3 Background set up

To get the sign change results at sum of two squares, one needs to consider the partial sums

$$S_1(x) = \sum_{n=c^2+d^2\leq x} \lambda(c^2+d^2),$$

$$S_2(x) := \sum_{n=c^2+d^2\leq x} \lambda^2(c^2+d^2),$$

for $x \geq 1$ and some $(c,d) \in \mathbb{Z}^2$. Also, one needs to find the upper bound of $S_1(x)$ and the approximate behaviour of $S_2(x)$. As we are just interested in the sign change of the coefficients at sum of two squares, one can use $r_2(n)$ as the weighted characteristic function of sum of two squares. Also, $r_2(n)$ is always non-negative. So to consider the sign change of the Fourier coefficients at sum of two squares, it is enough to consider the following sums.

$$S(x) = \sum_{n\leq x} \lambda(n) r_2(n),$$

$$S_f(x) = \sum_{n\leq x} \lambda^2(n) r_2(n).$$

In number theory the function $r_2(n)$ has received much attention. It is well known that [11], $r_2(n) = 4\sum_{d|n}\chi_{-4}(d)$. We set $r(n) := \frac{1}{4}r_2(n) = \sum_{d|n}\chi_{-4}(d)$. So for any prime p we have,

$$r(p) = 1+\chi_{-4}(p), \quad r(p^2) = 1+\chi_{-4}(p)+\chi_{-4}(p^2) \tag{5.3}$$

and so on. We define

$$L(s) := \sum_{n=1}^{\infty} \frac{\lambda^2(n) r(n)}{n^s} \tag{5.4}$$

for $\mathrm{Re(s)} > 1$. We use the following L-functions associated to f defined by

$$L(s, f \times f) := \sum_{n=1}^{\infty} \frac{\lambda^2(n)}{n^s} \tag{5.5}$$

and

$$L(s, f \times f \times \chi_{-4}) := \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)\lambda^2(n)}{n^s}, \tag{5.6}$$

where $\mathrm{Re(s)} > 1$. Note that one can think of $L(s, f \times f \times \chi_{-4})$ as the convolution L- function associated to f and $g = f \otimes \chi_{-4} (= \sum_{n \geq 1} a(n)\chi_{-4}(n) e^{2\pi i n z})$. Then the Rankin-Selberg L-function associated to $f \otimes f$ and $f \otimes g$ are given by

$$L(s, f \otimes f) := \zeta(2s) L(s, f \times f) \tag{5.7}$$

and

$$L(s, f \otimes f \otimes \chi_{-4}) := \zeta(2s) L(s, f \times f \times \chi_{-4}). \tag{5.8}$$

These L-functions are well studied. They have analytic continuation and satisfy a functional equation (see for instance [2]).

The following lemmas are important to study the average behaviour of $S(x)$ and $S_f(x)$.

Lemma 5.3.1 *For* $\mathrm{Re(s)} > 1$, *we have*

$$L(s) = L(s, f \times f) L(s, f \times f \times \chi_{-4}) \mathscr{U}(s) \tag{5.9}$$

where $\mathscr{U}(s)$ *converges absolutely and uniformly in the half plane* $\mathrm{Re(s)} \geq 1/2 + \varepsilon$ *for any* $\varepsilon > 0$ *and* $L(s, f \times f)$ *and* $L(s, f \times f \times \chi_{-4})$ *are defined as in* (5.5) *and* (5.6) *respectively.*

Proof. The Euler product representations of $L(s, f\times f)$ and $L(s, f\times f\times\chi_{-4})$ are given by

$$L(s, f\times f) = \prod_{p,prime}\left(1+\frac{\lambda^2(p)}{p^s}+\frac{\lambda^2(p^2)}{p^{2s}}+\cdots\right), \tag{5.10}$$

and

$$L(s, f\times f\times\chi_{-4}) = \prod_{p,prime}\left(1+\frac{\lambda^2(p)\chi_{-4}(p)}{p^s}+\frac{\chi_{-4}(p^2)\lambda^2(p^2)}{p^{2s}}+\cdots\right) \tag{5.11}$$

respectively, for Re(s) >1. Since the Dirichlet coefficients ($\lambda^2(n)$ and $r(n)$) in (5.4) are multiplicative, using the multiplicative relation in (5.1) and (5.3), we have the following Euler product representation for $L(s)$:

$$\begin{aligned} L(s) &= \sum_{n=1}^{\infty}\frac{\lambda^2(n)r(n)}{n^s} \\ &= \prod_{p,prime}\left(1+\frac{\lambda^2(p)r(p)}{p^s}+\frac{\lambda^2(p^2)r(p^2)}{p^{2s}}+\cdots\right) \\ &= \prod_{p,prime}\left(1+\frac{\lambda^2(p)(1+\chi_{-4}(p))}{p^s}+\frac{\lambda^2(p^2)(1+\chi_{-4}(p)+\chi_{-4}(p^2))}{p^{2s}}+\cdots\right), \end{aligned} \tag{5.12}$$

for Re(s) >1. Now for $\mathrm{Re}(s)>1$, we write

$$\begin{aligned} L(s) &= \prod_{p,prime}\left(\sum_{\ell=0}^{\infty}\lambda^2(p^\ell)r(p^\ell)p^{-\ell s}\right) \\ L(s, f\times f) &= \prod_{p,prime}\left(\sum_{\ell=0}^{\infty}\lambda^2(p^\ell)p^{-\ell s}\right) = \prod_{p,prime}\sum_{\ell=0}^{\infty}a_\ell\, p^{-\ell s} \quad (say), \\ L(s, f\times f\times\chi_{-4}) &= \prod_{p,prime}\left(\sum_{\ell=0}^{\infty}\lambda^2(p^\ell)\chi_{-4}(p^\ell)p^{-\ell s}\right) = \prod_{p,prime}\sum_{\ell=0}^{\infty}b_\ell\, p^{-\ell s} \quad (say), \end{aligned} \tag{5.13}$$

where $a_\ell=\lambda^2(p^\ell)$ and $b_\ell=\lambda^2(p^\ell)\chi_{-4}(p^\ell)$, with $a_0=b_0=1$.

Let $\mathscr{U}(s) = \prod_{p,prime} \left(\sum_{\ell=0}^{\infty} \alpha_\ell \, p^{\ell s} \right)$ be the Dirichlet series such that

$$L(s) = L(s, f \times f) L(s, f \times f \times \chi_{-4}) \mathscr{U}(s),$$

for Re(s) > 1 where $\alpha_0 = 1$. We see that the Dirichlet series coefficients α_ℓ of $\mathscr{U}(s)$ can be determined recursively as follows. It is easy to see that $\alpha_1 = 0$. By comparing the powers of p^{-s} both the sides, we get

$$\lambda^2(p) r(p) = \alpha_1 + \lambda^2(p) + \lambda^2(p)\chi_{-4}(p).$$

Since $r(p) = 1 + \chi_{-4}(p)$, the above relation implies that $\alpha_1 = 0$. Next, we compare the coefficients of $p^{-\ell s}$, $\ell > 1$ both the sides to get

$$\lambda^2(p^\ell) r(p^\ell) = \sum_{j=0}^{\ell} \alpha_j \left(a_{\ell-j} + a_{\ell-j-1} b_1 + \ldots + a_1 b_{\ell-j-1} + b_{\ell-j} \right). \tag{5.14}$$

In the above equation, we assume that $a_j = 0 = b_j$ if $j < 0$. With $\alpha_0 = 1$, $\alpha_1 = 0$, the other coefficients α_ℓ, $\ell \geq 2$ can be computed recursively from the above equation. Thus, the Dirichlet series $\mathscr{U}(s)$ is determined completely by the above relation. In particular one gets that

$$\mathscr{U}(s) = \prod_{p,prime} \Big(1 + \frac{\chi_{-4}(p)(\lambda^2(p^2) - \lambda^4(p))}{p^{2s}} + \frac{(\lambda^2(p^3) - 2\lambda^2(p)\lambda^2(p^2) + \lambda^6(p))(1 + \chi_{-4}(p))}{p^{3s}} + \cdots \Big).$$

We now use the multiplicative relation (5.1) satisfied by $\lambda(n)$ to conclude that

$$\mathscr{U}(s) = \prod_{p,prime} \left(1 + \frac{\chi_{-4}(p)(1 - 2\lambda^2(p))}{p^{2s}} + \frac{2\lambda^2(p)(1 + \chi_{-4}(p))}{p^{3s}} + \cdots \right).$$

Since $\lambda(n) \ll n^{\varepsilon}$ (by (5.2)) and since the term corresponding to p^{-s} is zero, the Dirichlet series $\mathscr{U}(s)$ given by the above Euler product converges absolutely and uniformly for $\mathrm{Re(s)} \geq 1/2+\varepsilon$. This completes the proof.

Lemma 5.3.2 *For any $s=\sigma+it$ with $0 \leq \sigma \leq 1$ and $\varepsilon > 0$, we have*

$$\frac{s-1}{s+1}L(\sigma+it, f\otimes f) \ll_{f,\varepsilon} (1+|t|)^{2(1-\sigma)+\varepsilon}. \tag{5.15}$$

Proof. The proof involves standard arguments using the Stirling formula for the Gamma function in the functional equation of $L(\sigma+it, f\otimes f)$ and the Phargmen-Lindelöf principle. For details, we refer to Chapter 5 page 100 of [12].

Lemma 5.3.3 *For $1/2 \leq \sigma \leq 3/4$, we have*

$$\text{(i)} \qquad \int_0^T |L(\sigma+it, f\otimes f)|^2 \mathrm{d}t \ll T^{4-4\sigma}(\log T)^{1+\varepsilon}, \tag{5.16}$$

and

$$\text{(ii)} \qquad \int_T^{2T} |L(\sigma+it, f\otimes f)|^2 \mathrm{d}t \ll T^{4-4\sigma}(\log T)^{1+\varepsilon}. \tag{5.17}$$

Proof. We refer [24] for a proof. We also mention that the second inequality is valid for Rankin-Selberg convolution of two different forms f and g. The same proof works in the case $f=g$.

Now we state the main proposition, which provides the asymptotic behaviour of $S_f(x)$ and an upper bound for $S(x)$.

Proposition 5.3.1 *We have*

$$S(x) \ll x^{3/4+\varepsilon} \tag{5.18}$$

and

$$S_f(x) = Cx + O_{f,\varepsilon}(x^{3/4+\varepsilon}) \tag{5.19}$$

where C is a constant and $\varepsilon > 0$ is arbitrarily small.

Proof. Define

$$L_j(s) = \begin{cases} L(s, f \otimes \theta^2) & \text{if } j = 1 \\ L(s) & \text{if } j = 2, \end{cases} \tag{5.20}$$

here θ is as in 1.4. Now by using the truncated Perron's formula (cf. [30, Exercise 4.4.16, page 67]), we have

$$S(x) = \sum_{n \le x} \lambda(n) r_2(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L_1(s) \frac{x^s}{s} \mathrm{d}s + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{5.21}$$

and

$$S_f(x) = 4 \sum_{n \le x} \lambda^2(n) r(n) = \frac{4}{2\pi i} \int_{b-iT}^{b+iT} L_2(s) \frac{x^s}{s} \mathrm{d}s + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{5.22}$$

where $b = 1 + \varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later. We observe that $L_1(s)$ has an analytic continuation to the whole complex plane. Using Lemma 5.3.1 and using the analytic continuation of Rankin-Selberg L- function (cf. [2]) we see that $L_2(s)$ can be meromorphically continued to the half plane $\mathrm{Re(s)} > 1/2$. In this region, $L_2(s)$ has a simple pole at $s = 1$. Next we move the line of integration to $\mathrm{Re(s)} = 1/2 + \varepsilon$ and apply the Cauchy residue theorem to obtain

$$\begin{aligned} \sum_{n \le x} \lambda(n) r_2(n) &= \frac{1}{2\pi i} \left\{ \int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} + \int_{1/2+\varepsilon+iT}^{1+\varepsilon+iT} + \int_{1+\varepsilon-iT}^{1/2+\varepsilon-iT} \right\} L_1(s) \frac{x^s}{s} \mathrm{d}s \\ &\quad + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &= I_1 + I_2 + I_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right) \end{aligned} \tag{5.23}$$

and

$$\sum_{n\leq x}\lambda^2(n)r(n) = 4\underset{s=1}{\mathrm{Res}}L(s)\,x + \frac{4}{2\pi i}\left\{\int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} + \int_{1/2+\varepsilon+iT}^{1+\varepsilon+iT} + \int_{1+\varepsilon-iT}^{1/2+\varepsilon-iT}\right\}L(s)\frac{x^s}{s}\mathrm{d}s$$
$$+O\left(\frac{x^{1+\varepsilon}}{T}\right)$$
$$= Cx + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right). \tag{5.24}$$

Here $C = 4\underset{s=1}{\mathrm{Res}}L(s)$ is a constant.

Now we evaluate the integrals in equation (5.23). By using the convexity bound for the Rankin-Selberg L- function Lemma 5.3.2, we have

$$I_1 \ll x^{1/2+\varepsilon}\left(1+\int_1^T \frac{|L_1(1/2+\varepsilon+it)|}{t}dt\right)$$
$$\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}T^{1-\varepsilon}. \tag{5.25}$$

To evaluate the horizontal integral, we write $s=\sigma+it$ so we have

$$I_2+I_3 = \int_{1/2+\varepsilon}^{1+\varepsilon}\frac{L_1(\sigma+iT)x^{\sigma+iT}}{\sigma+iT}\mathrm{d}\sigma - \int_{1/2+\varepsilon}^{1+\varepsilon}\frac{L_1(\sigma-iT)x^{\sigma-iT}}{\sigma-iT}\mathrm{d}\sigma$$
$$\ll \int_{1/2+\varepsilon}^{1+\varepsilon}|L_1(\sigma+iT)|\frac{x^{\sigma}}{T}\mathrm{d}\sigma$$
$$\ll \int_{1/2+\varepsilon}^{1+\varepsilon}\frac{\max\limits_{1/2+\varepsilon<\sigma\leq 1+\varepsilon}|L_1(\sigma+iT)x^{\sigma}|}{T}d\sigma \tag{5.26}$$
$$\ll \frac{x^{1/2+\varepsilon}}{T^{\varepsilon}} + \frac{x^{1+\varepsilon}}{T^{1+\varepsilon}} \tag{5.27}$$

Now by using (5.25),(5.26), and (5.23), we have

$$S(x) = O(x^{1/2+\varepsilon}T^{1-\varepsilon}) + O(\frac{x^{1+\varepsilon}}{T^{1+\varepsilon}}). \tag{5.28}$$

Now we evaluate the integrals in equation (5.24). By applying Lemma 5.3.1 and

Cauchy-Schwartz inequality on the first integral J_1, we obtain

$$\begin{aligned} J_1 \ll x^{1/2+\varepsilon}\Bigg[& \left(\int_0^T |L(1/2+\varepsilon+it, f\otimes f)|^2 \mathrm{d}t\right)^{1/2} \\ & \times \left(\int_0^T \frac{|L(1/2+\varepsilon+it,\chi_{-4}\otimes f\otimes f)|^2}{|1/2+\varepsilon+it|^2}\mathrm{d}t\right)^{1/2}\Bigg] \\ \ll x^{1/2+\varepsilon}\Bigg[& \left(\int_0^T |L(1/2+\varepsilon+it, f\otimes f)|^2 \mathrm{d}t\right)^{1/2} \\ & \times \left(1+\int_1^T \frac{|L(1/2+\varepsilon+it,\chi_{-4}\otimes f\otimes f)|^2}{t^2}\mathrm{d}t\right)^{1/2}\Bigg]. \end{aligned} \tag{5.29}$$

Here, we use standard argument and Lemma 5.3.3 (ii) in the second integral of (5.29) and obtain

$$\begin{aligned} & \left(\int_1^T \frac{|L(1/2+\varepsilon+it,\chi_{-4}\otimes f\otimes f)|^2}{t^2}\mathrm{d}t\right)^{1/2} \\ & \ll \log T \max_{1<T_1\le T} \frac{1}{T_1^2}\int_{T_1/2}^{T_1} |L(1/2+\varepsilon+it,\chi_{-4}\otimes f\otimes f)|^2 \mathrm{d}t \\ & \ll \log T \end{aligned} \tag{5.30}$$

We insert (5.30) in (5.29) and apply Lemma 5.3.3 (i) to obtain

$$J_1 \ll x^{1/2+\varepsilon}T^{1-2\varepsilon}(\log T)^{3/2+\varepsilon}. \tag{5.31}$$

Now, we will concentrate on the horizontal integrals J_2 and J_3. Consider $s=\sigma+it$. After applying Lemma 5.3.2 we get

$$\begin{aligned} J_2+J_3 & \ll \int_{1/2+\varepsilon}^{1+\varepsilon} |L(\sigma+iT, f\otimes f)||L(\sigma+iT,\chi_{-4}\otimes f\otimes f)|\frac{x^\sigma}{T}\mathrm{d}\sigma \\ & \ll \max_{1/2+\varepsilon<\sigma\le 1+\varepsilon} x^\sigma T^{4(1-\sigma)+2\varepsilon}T^{-1} = \max_{1/2+\varepsilon<\sigma\le 1+\varepsilon}\left(\frac{x}{T^4}\right)^\sigma T^{3+2\varepsilon} \\ & \ll \frac{x^{1+\varepsilon}}{T^{1+2\varepsilon}} + x^{1/2+\varepsilon}T^{1-2\varepsilon}. \end{aligned} \tag{5.32}$$

Finally from (5.24), (5.31) and (5.32) we get

$$S_f(x) = Cx + O\left(\frac{x^{1+\varepsilon}}{T^{1+2\varepsilon}}\right) + O\left(x^{1/2+\varepsilon}T^{1-2\varepsilon}(\log T)^{3/2+\varepsilon}\right) \tag{5.33}$$

Now, we choose $T = x^{1/4}$ in both the cases (5.28), (5.33) and we obtain

$$S(x) \ll x^{3/4+\varepsilon}$$

and

$$S_f(x) = Cx + O(x^{3/4+\varepsilon}),$$

which completes the proof of the proposition.

5.4 Proof of Theorem 5.2

Now consider $h = h(x) = x^{7/8}$. The proof is by contradiction, so assume that the sequence $\{\lambda(n) : n = c^2 + d^2)\}_{n\geq 1}$ has constant sign, say positive for all $n \in (x, x+h)$. Now we apply (5.2) and Proposition 5.3.1 respectively to obtain

$$\begin{aligned} \sum_{x<n\leq x+h} \lambda^2(n)r_2(n) &= \sum_{x<n\leq x+h} \lambda(n)\lambda(n)r_2(n) \ll x^{\varepsilon} \sum_{x<n\leq x+h} \lambda(n)r_2(n) \\ &\ll x^{2\varepsilon}[(x+h)^{3/4+\varepsilon} + x^{3/4+\varepsilon}] \ll x^{3/4+2\varepsilon}. \end{aligned} \tag{5.34}$$

On the other hand, from Proposition 5.3.1, we get

$$\sum_{x<n\leq x+h} \lambda^2(n)r_2(n) = 4Ch + O_{f,\varepsilon}(x^{3/4+\varepsilon}) \gg x^{7/8}. \tag{5.35}$$

Here one notes that each time ε may have different value. Now we compare the bounds in (5.34) and (5.35) and arrive at a contradiction. Therefore, the sequence

$\{\lambda(n)r_2(n)\}_{n\geq 1}$ has at least one sign change in the interval $(x, x+h]$. This in particular implies that the sequence $\{\lambda(c^2+d^2)\}_{c,d\geq 1}$ has infinitely many sign changes. In fact, there are at least $x^{1/8-2\varepsilon}$ many sign changes in the interval $(x, 2x]$, for sufficiently large x, where ε is arbitrarily small.

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