MAXIMAL SURFACES AND THEIR APPLICATIONS

By

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I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

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Rahul Kumar Singh

Dedicated to

Rummy di, Jija ji, Rohit bhai, Nani

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Contents

Synopsis

This thesis comprise of two parts. In the first part of the thesis we discuss the existence of special kind of maximal surfaces which contain a given curve and having a special singularity. In the second part we discuss some of the applications of maximal surfaces. One of the application which we have discussed in this work shows how the study of solutions of maximal surface equation is related to solutions of Born-Infeld equation (a nonlinear pde which makes it appearance in the context of nonlinear electrodynamics and string theory). Another application of maximal surfaces which we have discussed in this thesis shows there is a beautiful connection between the geometry of maximal surfaces to analytic number theory through certain Ramanujan's identities.

0.1 Existence of interpolating maximal surfaces

The classical Björling problem in Euclidean space, denoted by $\mathbb{E}^3$, was first proposed by Björling in the year 1844 [8]. Given an analytic strip (a real analytic curve and a real analytic normal vector field along the curve) in $\mathbb{E}^3 := (\mathbb{R}^3, dx^2 + dy^2 + dz^2)$, the Björling's problem is to find a minimal surface containing this strip [18]. H.A. Schwarz in the year 1890 gave an explicit formula for such a minimal surface in terms of the prescribed strip [47]. In the year 2003, a similar

Björling problem for maximal surfaces in Lorentz-Minkowski space, denoted by $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$, was formulated and solved by Alías, Chaves and Mira [3]. They obtained a complex representation formula and then used it to solve the Björling problem. Their solution to the Björling problem gives a way to construct new examples of maximal surfaces having interesting geometric properties. For more details on this see [3]. In past, the Björling problem has been studied for different kinds of surfaces which admit a representation formula (see [2, 3, 5, 11, 19, 24, 25, 40]).

In the year 2007, Kim and Yang [32] introduced the singular Björling problem and proved it for the case of real analytic null curves defined on an open interval by obtaining a representation formula. An immediate extension of the singular Björling problem and solution for the case of closed null curve was discussed by the same authors in [32]. As a consequence of this representation formula they got a general method for the construction of maximal surfaces with prescribed singularities. They also applied this singular Björling formula to obtain various interesting properties of maximal surfaces with singularities (we will refer to this as a generalised maximal surface). For more details see [32].

In this work we have revisited the singular Björling problem for the case of closed real analytic null curves and given a different proof for this. We hope that the technique used in our proof helps to know more about generalised maximal surfaces. By using the singular Björling formula in our case, we show the existence of maximal surfaces which contain a given closed spacelike curve and has a special singularity. More generally, we characterize the set of closed spacelike curves such that there exists a generalised maximal surface parametrized by a single chart which contains a given closed spacelike curve and having a special singularity. Now we explain briefly what motivated us to ask such a question.

Question: Does there exists a generalised maximal surface containing a given curve and having a special singularity ?

Let $\alpha(\theta) = (-\frac{3}{4}\cos\theta, -\frac{3}{4}\sin\theta, \ln\frac{1}{2})$ be a closed spacelike real analytic curve. This curve lies on an elliptic catenoid, a maximal surface, given by map

$$F(x,y) = \left(\frac{x(x^2+y^2-1)}{2(x^2+y^2)}, \frac{y(x^2+y^2-1)}{2(x^2+y^2)}, \ln\sqrt{x^2+y^2}\right).$$

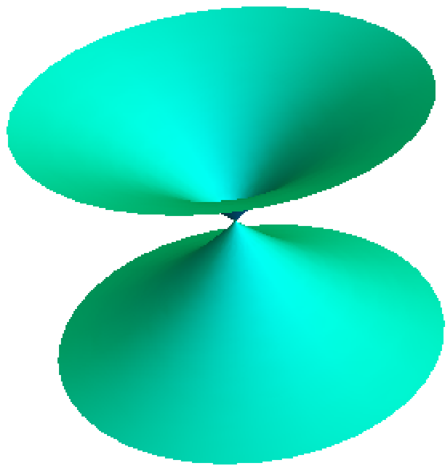

Figure 1: Elliptic Catenoid

We see that

1. the map F is defined for all $z = x+iy \neq 0$ and has a special singularity on $|z| = 1$.

2. there is a positive real r_0, namely $r_0 = \frac{1}{2}$ such that $F(|z| = r_0) = \gamma(\frac{1}{2}e^{i\theta}) := \alpha(\theta)$.

On the other hand if we take $\beta(\theta) = (e^{i\theta}, 1)$, which is a closed spacelike curve, we show that there does not exist any maximal surface F (parametrised by a single chart F defined for all $z \neq 0$) and any $r_0 \neq 1$ such that $F(r_0 e^{i\theta}) = \tilde{\beta}(r_0 e^{i\theta}) := \beta(\theta)$ and has special singularity at $|z| = 1$.

Next by identifying the vector space structure of $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$, we can define a generalised maximal surface as follows:

Definition 0.1.1 A generalised maximal surface is a map $F = (h := u+iv, w) : \Omega \subset \mathbb{C} \to \mathbb{C} \times \mathbb{R}$, such that

- $h_{z\bar{z}} = 0$ and $w_{z\bar{z}} = 0$ (harmonicity)
- $h_z \overline{h_{\bar{z}}} - w_z^2 = 0$ (conformality)
- $|h_z|$ is not identically equal to $|h_{\bar{z}}|$.

Now we state the singular Björling problem for which we have given a different proof in [14].

Theorem 0.1.2 *Given a real analytic null closed curve $\gamma : S^1 \to \mathbb{L}^3$ and a null vector field $L : S^1 \to \mathbb{L}^3$ such that $\langle \gamma', L \rangle = 0$; atleast one of γ' and L do not vanish identically. If $|g(z)|$ ($g(z)$ is analytic extension of $g(e^{i\theta})$) is not identically equal to 1, then there exists a unique generalised maximal surface $F := (h, w)$ defined on some annulus $A(r, R) := \{z : 0 < r < |z| < R\}; r < 1 < R$, such that*

1. $F(e^{i\theta}) = (h(e^{i\theta}), w(e^{i\theta})) = \gamma(e^{i\theta})$,
2. $\left.\dfrac{\partial F}{\partial \rho}\right|_{e^{i\theta}} = (h_\rho(e^{i\theta}), w_\rho(e^{i\theta})) = L(e^{i\theta})$,

with singular set atleast $\{|z| = 1\}$.

In the above theorem, the function $g(z)$ is the Gauss map of a generalised maximal surface. It is defined in the Section 2.3 of Chapter 2. Next we state a result which we have proved in [14] using singular Björling formula, which roughly says if the given curve γ satisfies some conditions then there exists a generalised maximal surface which has property (1) and (2) as above (mentioned in **Question**).

Theorem 0.1.3 *Let $\tilde{\gamma}(\theta)$ be a nonconstant real analytic closed spacelike curve. Then there exists $s_0 \neq 1$ and a generalised maximal surface $F : \mathbb{C} - \{0\} \to \mathbb{L}^3$ such that $F(s_0 e^{i\theta}) := \tilde{\gamma}(\theta)$ and having a special singularity at $(0,0,0) \in \mathbb{L}^3$ if and only if there exists $r_0 \neq 1$ and constants $c, c_n' s, d, d_n' s$ for the curve $\gamma(r_0 e^{i\theta}) := \tilde{\gamma}(\theta) = (f(r_0 e^{i\theta}), g(r_0 e^{i\theta}))$, which satisfy the relations:*

$$\forall\ k \neq 0; \quad \sum_{n=-\infty}^{\infty} 4n(n-k)(c_n \bar{c}_{n-k} - d_n d_{n-k}) + 2k(c_k \bar{c} - c\bar{c}_{-k} - 2d_k d) = 0$$

$$\text{and} \quad \sum_{n=-\infty}^{\infty} 4n^2(c_n \bar{c}_n - d_n^2) + c\bar{c} - d^2 = 0$$

where the constants are given by

$$c = \frac{1}{2\pi \log r_0} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) d\theta; \quad d = \frac{1}{2\pi \log r_0} \int_{-\pi}^{\pi} g(r_0 e^{i\theta}) d\theta,$$

for $n \neq 0$;

$$c_n = \frac{r_0^n}{2\pi(r_0^{2n} - 1)} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) e^{-in\theta} d\theta; \quad d_n = \frac{r_0^n}{2\pi(r_0^{2n} - 1)} \int_{-\pi}^{\pi} g(r_0 e^{i\theta}) e^{-in\theta} d\theta.$$

0.2 Weierstrass-Enneper representation for maximal surfaces using hodographic coordinates

Weierstrass-Enneper representation formula is a well known complex representation formula (which is expressed in terms of certain holomorphic function and a meromorphic function) for maximal surfaces in Lorentz-Minkowski space $\mathbb{L}^3$ [34].

In [48], we have rederived the Weierstrass-Enneper representation for maximal graphs (assuming the Gauss map for such graphs is one-one). For this we used the method of Barbishov and Chernikov, which was used by them to find the solutions of Born-Infeld equation in hodographic coordinates [6]. Earlier, Dey [13] also used their method to obtain the Weierstrass-Enneper representation of minimal surfaces (Gauss map is one-one) in Euclidean space $\mathbb{E}^3$.

Theorem 0.2.1 *Any maximal surface whose Gauss map is one-one will have a local Weierstrass-Enneper type representation of the following form*

$$x(\zeta) = x_0 + \Re(\int^{\zeta} M(\omega)(1+\omega^2)d\omega),$$

$$y(\zeta) = y_0 + \Re(\int^{\zeta} iM(\omega)(1-\omega^2)d\omega),$$

$$\varphi(\zeta) = \varphi_0 + \Re(\int^{\zeta} 2M(\omega)\omega d\omega)$$

where $M(\zeta)$ is a meromorphic function known as the Weierstrass data.

Next we have obtained Weierstrass-Enneper representation using hodographic coordinates [48]

$$x(\rho) = \frac{\rho+\bar{\rho}}{2} + \frac{1}{2}\left(\int (F^{-1}(\rho))^2 d\rho + \int (H^{-1}(\bar{\rho}))^2 d\bar{\rho}\right),$$

$$y(\rho) = \frac{\bar{\rho}-\rho}{2i} + \frac{1}{2i}\left(\int (F^{-1}(\rho))^2 d\rho - \int (H^{-1}(\bar{\rho}))^2 d\bar{\rho}\right),$$

$$\varphi(\rho) = \int F^{-1}(\rho)d\rho + \int H^{-1}(\bar{\rho})d\bar{\rho}$$

where $M(\zeta) = F'(\zeta) \neq 0$ and $F(\zeta) = \rho$, $H(\bar{\zeta}) = \overline{F(\zeta)} = \bar{\rho}$.

0.3 Relationship with Born-Infeld equation

M. Born and L. Infeld [9] introduced in the year 1934 a geometric (nonlinear) theory of electromagnetism (known as Born-Infeld model) in order to overcome the infinity problem associated with a point charge source in the original Maxwell theory. The corresponding nonlinear PDE which describes this model is what is known as Born-Infeld equation (see the PDE in next paragraph). Born-Infeld equation also arises in the context of string theory [52].

Any smooth function $\varphi(x,t)$ which is a solution to Born-Infeld equation (see [50])

$$(1+\varphi_x^2)\varphi_{tt} - 2\varphi_x\varphi_t\varphi_{xt} + (\varphi_t^2 - 1)\varphi_{xx} = 0 \tag{1}$$

is known as a Born-Infeld soliton.

A graph $(x, t, f(x,t))$ in Lorentz-Minkowski space $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dt^2 - dz^2)$ is maximal if it satisfies

$$(1 - f_x^2)f_{tt} + 2f_xf_tf_{xt} + (1 - f_t^2)f_{xx} = 0, \tag{2}$$

for some smooth function $f(x,t)$ satisfying $f_x^2 + f_t^2 < 1$, see [35]. This equation is known as the maximal surface equation.

Remark 0.3.1 Note that the Born-Infeld equation is related to the maximal surface equation by a Wick rotation in the variable x, i.e., if we replace x by ix and define $f(x,t) := \varphi(ix,t)$, in (1), we get back the maximal surface equation (2) and vice-versa [48].

It was known that the Born-Infeld equation is related to the minimal surface equation in $\mathbb{R}^3$ by a Wick rotation in the variable t i.e., if we replace t by it in (1), we get back the minimal surface equation and vice-versa [13]. This fact has been used by Mallory and others in [38] to obtain some exact solutions of the Born-Infeld equation. Also, Dey and Kumar in [16], using the idea of Wick rotation constructed a one parameter family of Born-Infeld solitons from a given one parameter family of minimal surfaces. In this work, we explored the interrelation between Born-Infeld equation and maximal surface equation and have obtained some analogous results. In this regard, the very first result which we obtain, gives us a way to see Born-Infeld soliton as a minimal graph over timelike plane $\{x = 0\}$ in Lorentz-Minkowski space $\mathbb{L}^3$ [15].

Theorem 0.3.2 *The solutions of (1), i.e. Born-Infeld solitons, can be represented as a spacelike minimal graph or timelike minimal graph over a domain in timelike plane or a combination of both away from singular points (those points where tangent plane degenerates), i.e., points where the determinant of the coefficients of first fundamental form vanishes.*

Next, we discuss a method which explains how to construct a one parameter family of complex solitons from a given one parameter family of maximal surfaces [15].

Let $X_1(\tau,\bar{\tau})$=$(x_1(\tau,\bar{\tau}), t_1(\tau,\bar{\tau}), f_1(\tau,\bar{\tau}))$ and $X_2(\tau,\bar{\tau})$=$(x_2(\tau,\bar{\tau}), t_2(\tau,\bar{\tau}), f_2(\tau,\bar{\tau}))$ be isothermal paramerizations of two maximal surfaces, where $X_j(\tau,\bar{\tau}) : \Omega \subseteq \mathbb{C} \to \mathbb{L}^3, \tau = \tilde{u} + i\tilde{v} \in \Omega \;\; ; \;\; j = 1, 2$ such that

$$X := X_1 + iX_2 : \Omega \subseteq \mathbb{C} \to \mathbb{C}^3$$

is a holomorphic mapping. Then we say that X_1 and X_2 are *conjugate maximal surfaces.*

It should be remarked that if the Gauss map of a given maximal surface in $\mathbb{L}^3$ is one-one, then its conjugate maximal surface exist.

Theorem 0.3.3 *Let* $X_1 = (x_1, t_1, f_1)$ *and* $X_2 = (x_2, t_2, f_2)$ *be two conjugate maximal surfaces and let* $X_\theta = (x_1 \cos\theta + x_2 \sin\theta, t_1 \cos\theta + t_2 \sin\theta, f_1 \cos\theta + f_2 \sin\theta) = (x_\theta, t_\theta, f_\theta)$ *denotes the one parameter family of maximal surfaces corresponding to* X_1 *and* X_2. *Then* $X_\theta^s = (i(x_1 \cos\theta + x_2 \sin\theta), (t_1 \cos\theta + t_2 \sin\theta), (\varphi_1 \cos\theta + \varphi_2 \sin\theta)) = (x_\theta^s, t_\theta^s, \varphi_\theta^s)$, *where* $\varphi_j(x_j, t_j) := f_j(ix_j, t_j)$, $j = 1, 2$ *will give us a one parameter family of complex solitons, i.e., for each* θ *we will have a complex solution to the Born-Infeld equation* (1).

0.4 Some identities

Dey in [17] had obtained some nontrivial identities using certain Ramanujan's identities and Weierstrass-Enneper representation for minimal surfaces. In this work we have also used these identities in order to arrive at further nontrivial identities using Weierstrass-Enneper representation for maximal surfaces[15].

Let X and A be complex, where A is not an odd multiple of $\frac{\pi}{2}$. Then

$$\frac{\cos(X + A)}{\cos(A)} = \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{X}{(k - \frac{1}{2})\pi - A}\right) \left(1 + \frac{X}{(k - \frac{1}{2})\pi + A}\right) \right\}.[1]$$

If X and A are real, then

$$\tan^{-1}(\tanh X \cot A) = \tan^{-1}\left(\frac{X}{A}\right) + \sum_{k=1}^{\infty} \left(\tan^{-1}\left(\frac{X}{k\pi + A}\right) - \tan^{-1}\left(\frac{X}{k\pi - A}\right) \right).$$

The first identity was known to Jacques Hadamard and Karl Weierstrass. The above identities were also obtained by Srinivasa Ramanujan [45].

The Weierestrass-Enneper representation for a maximal surface (x, y, z) in Lorentz-Minkowski space $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$, whose Gauss map is one-one is

[1] This identity can also be obtained from Hadamard-Weierstrass factorization theorem [12]

given by [35],

$$x(\zeta) = Re\left(\int^{\zeta} M(\omega)(1+\omega^2)d\omega\right) \; ; \; y(\zeta) = Re\left(\int^{\zeta} iM(\omega)(1-\omega^2)d\omega\right)$$

$$z(\zeta) = Re\left(\int^{\zeta} -2M(\omega)\omega d\omega\right),$$

where $\zeta = u + iv$.

Now we state the identities corresponding to different maximal surfaces which we have proved as propositions in [15].

Proposition 0.4.1 (Identity corresponding to Scherk's surface of first kind) *For $\zeta \in \Omega \subset \mathbb{C} - \{\pm 1, \pm i\}$, we have the following identity*

$$\ln|\frac{\zeta^2-1}{\zeta^2+1}| = \sum_{k=1}^{\infty} \ln\left(\frac{(k-\frac{1}{2})\pi - i\ln|\frac{\zeta-i}{\zeta+i}|}{(k-\frac{1}{2})\pi - i\ln|\frac{\zeta+1}{\zeta-1}|}\right)\left(\frac{(k-\frac{1}{2})\pi + i\ln|\frac{\zeta-i}{\zeta+i}|}{(k-\frac{1}{2})\pi + i\ln|\frac{\zeta+1}{\zeta-1}|}\right).$$

Proposition 0.4.2 (Identity corresponding to helicoid of second kind) *For $\zeta \in \Omega \subset \mathbb{C} - \{0\}$, we have the following identity*

$$\frac{Im\left(\zeta+\frac{1}{\zeta}\right)}{Im\left(\zeta-\frac{1}{\zeta}\right)} = \frac{1}{i}\prod_{k=1}^{\infty}\left\{\left(\frac{(k-1)\pi + i\ln|\zeta|}{(k-\frac{1}{2})\pi + i\ln|\zeta|}\right)\left(\frac{k\pi - i\ln|\zeta|}{(k-\frac{1}{2})\pi - i\ln|\zeta|}\right)\right\}.$$

Proposition 0.4.3 (Identity corresponding to Lorentzian helicoid) *For $\zeta = u + iv$, such that $\zeta \in \Omega \subset \mathbb{C} - \{0\}$, we have the following identity*

$$Im(\ln(\zeta)) - \tan^{-1}\left(\tanh\left(-\frac{1}{2}Re\left(\zeta+\frac{1}{\zeta}\right)\right)\cot\left(\frac{1}{2}Im\left(\zeta-\frac{1}{\zeta}\right)\right)\right)$$

$$= \pm\frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\tan^{-1}\left(\frac{Re(\zeta+\frac{1}{\zeta})}{Im(\zeta-\frac{1}{\zeta})-2k\pi}\right) + \tan^{-1}\left(\frac{Re(\zeta+\frac{1}{\zeta})}{Im(\zeta-\frac{1}{\zeta})+2k\pi}\right)\right),$$

where the constant term is $\frac{\pi}{2}$, *when either* $u > 0$ *and* $v > 0$ *or* $u < 0$ *and* $v < 0$ *or* $u = 0$ *or* $v = 0$ *and the constant term is* $-\frac{\pi}{2}$ *otherwise.*

Remark 0.4.4 Maximal surface equation can be obtained from the minimal surface equation by Wick rotation in both the variables and vice-versa, but in general we get complex surfaces this way. The identities use Weierstrass-Enneper representation of real maximal surfaces and hence they cannot be obtained from Weierstrass-Enneper representation of real minimal surfaces.

CHAPTER 1

Introduction

Maximal surfaces arises at the junction of different fields of Mathematics and Physics. From the point of view of geometry, to understand the structure of spacetime it is important to study maximal surfaces, as they reflect the structure of the ambient spacetime [7]. From a physical standpoint, a maximal surface, more generally, constant mean curvature spacelike surfaces, play an important role in the theory of General Relativity, as they represent the natural initial conditions for Einstein equations, for instance, the issue of existence of maximal hypersurfaces appears in the Schoen-Yau proof of the *positive mass theorem* [46]. Also, the existence of constant mean curvature (in particular maximal) hypersurfaces is necessary for the study of the structure of singularities in the space of solutions to the Einstein equations (see [44, 4, 39], and references therein).

It is interesting to note that maximal surfaces can also be seen as a model for conical singularities, which occurs in various natural phenomena (e.g., certain solitons in fluid dynamics, cosmology, electromagnetism) [30]. It is again interesting to note that there is a connection between maximal surfaces in $\mathbb{L}^3$ (Lorentz-Minkowski space) and two-dimensional barotropic steady flows. In fact, the graphs of maximal surfaces can be interpreted as stream functions of a virtual gas (for details see [23]).

A maximal surface in Lorentz-Minkowski space $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ is a spacelike surface such that the mean curvature function vanishes at all its points. In general, a maximal surface can have singularities (i.e., those points where the metric degenerates), then we refer to it as a generalised maximal surface [20].

Maximal surfaces can be seen from various different point of views, from the point of view of calculus of variation, they can be seen as a critical point of certain area functional in Lorentz-Minkowski space $\mathbb{L}^3$. In fact locally (i.e. there exists neighbourhood of points on the surface such that it has maximum area for all possible spacelike variation which fixes the boundary of the neighbourhood) they are the maxima for this area functional [10].

From the PDE point of view, they can be described as solutions of a certain second order quasilinear elliptic PDE

$$(1-\varphi_x^2)\varphi_{yy} + 2\varphi_x\varphi_y\varphi_{xy} + (1-\varphi_y^2)\varphi_{xx} = 0$$

where $\varphi_x^2 + \varphi_y^2 < 1$ [35]. The condition $\varphi_x^2 + \varphi_y^2 < 1$ is to ensure the spacelike nature of a maximal surface. From the point of view of complex analysis, a spacelike surface $M \subset \mathbb{L}^3$ is said to be maximal if its stereographically projected Gauss map $g : M \to \mathbb{C} \setminus \{|z| = 1\} \cup \{\infty\}$ is meromorphic with respect to the Riemann surface structure on M. For a spacelike surface in $\mathbb{L}^3$, the Gauss map N is defined as a map which assigns to a point of the surface M, the unit normal vector at that point. Since the unit normal vector field to a spacelike surface is timelike, it is natural to consider the image of Gauss map lying inside $\mathbb{H}^2 = \{(x, y, z) \in \mathbb{L}^3 | x^2 + y^2 - z^2 = -1\}$. Then there exists a stereographic mapping $\sigma : \mathbb{C} \setminus \{|z| = 1\} \cup \{\infty\} \to \mathbb{H}^2$ such that $\sigma^{-1} \circ N = g$ (see [34]).

Maximal surfaces in $\mathbb{L}^3$, in a sense, are analogous to minimal surfaces in Euclidean space $\mathbb{E}^3$, as both of them share some common geometric features. For instance, as we have discussed above, like maximal surfaces, minimal surfaces in $\mathbb{E}^3$ also arises as a critical (minimal) point of a suitable area functional in $\mathbb{E}^3$. Similar to the case of minimal surfaces in $\mathbb{E}^3$, there exists a representation formula (involving two complex functions one of which is holomorphic and other is meromorphic over a domain in $\mathbb{C}$) for maximal surfaces known as Weierstrass-Enneper representation.

The classical Björling problem in Euclidean space $\mathbb{E}^3$ was proposed by Björling in the year 1844 [8]. Given a real analytic strip in $\mathbb{E}^3$, Björling's problem is to find a minimal surface containing this strip (for details see [18]). In the year 1890, H.A. Schwarz gave an explicit formula for such a minimal surface in terms of the prescribed strip (a real analytic curve and a real analytic normal vector field along the curve) [47]. A similar Björling problem for maximal surfaces in

$\mathbb{L}^3$ was solved by Alías, Chaves and Mira [3]. In order to solve this problem they obtained a complex representation formula and then used it to solve the Björling problem. Their solution to the Björling problem gives a way to construct new examples of maximal surfaces having interesting geometric properties. For more details see the reference [3].

However, despite these similarities, there are notable differences between the behaviour of maximal and minimal surfaces. For instance, in the case of minimal surfaces in $\mathbb{E}^3$, the Weierstrass-Enneper representation can be applied to study their global geometry (in the study of complete minimal surfaces). On the other hand, a different situation appears when one considers studying the global geometry of maximal surfaces in $\mathbb{L}^3$, as there is so called Calabi-Bernstein theorem which states that the only complete maximal surfaces in $\mathbb{L}^3$ are spacelike planes. There are a plethora of examples of complete maximal surfaces with singularities [31]. But if we are just looking for complete maximal surfaces without singularities then there is only one such example which is spacelike plane (Calabi-Bernstein theorem) [10]. This is the reason Weierstrass-Enneper representation in the case of maximal surfaces is not as useful as it was in the Euclidean case. Nonetheless, in past several authors have studied the global geometry of complete maximal surfaces with singularities in $\mathbb{L}^3$.

Due to the indefinite nature of Lorentzian metric, singularities arises naturally when one consider surfaces in Lorentz-Minkowski space. This makes the study of local geometry of surfaces in $\mathbb{L}^3$ much more complicated than that of surfaces in $\mathbb{E}^3$. Therefore it is good to have a tool for constructing maximal surfaces which contains a given set in $\mathbb{L}^3$ as its singularities. In fact, Kim and Yang [32], investigated the singularities of maximal surfaces in $\mathbb{L}^3$ by solving a singular Björling problem which asks, approximately, whether there exists a generalised maximal surface that contains a given curve in $\mathbb{L}^3$ as a singular set and prescribed null directions on the curve as the normal to the surface.

In the year 1934, M. Born and L. Infeld [9], introduced a geometric theory of electromagnetism in order to overcome the infinity problem associated with a point charge source in the original Maxwell theory. There also exist a connection between the Born-Infeld theory and the Nambu-Goto string theory [52]. The corresponding model which describes this theory is known as Born-Infeld model. The non-linear PDE which describes this model is what is known as Born-Infeld equation (BIE). Any solution to this PDE is called a Born-Infeld soliton. The

Born-Infeld equation [50]

$$(1+\varphi_x^2)\varphi_{tt} - 2\varphi_x\varphi_t\varphi_{xt} + (\varphi_t^2 - 1)\varphi_{xx} = 0 \quad (1.1)$$

looks similar to the minimal surface equation (MSE) [18]

$$(1+\varphi_x^2)\varphi_{yy} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_y^2)\varphi_{xx} = 0. \quad (1.2)$$

It also looks similar to maximal surface equation (MaSE) which is given by

$$(1-\varphi_x^2)\varphi_{yy} + 2\varphi_x\varphi_y\varphi_{xy} + (1-\varphi_y^2)\varphi_{xx} = 0. \quad (1.3)$$

where $\varphi_x^2 + \varphi_y^2 < 1$, upto change of signs.

More generally, the BIE is similar to Zero Mean Curvature (ZMC) equation which is also given by the PDE (1.3) (except the condition $\varphi_x^2 + \varphi_y^2 < 1$) in Lorentz-Minkowski space, i.e., it is not necessary to put a spacelike condition $\varphi_x^2+\varphi_y^2 < 1$. In other words, a ZMC equation (which includes spacelike, timelike and lightlike cases) with spacelike condition is a MaSE (like a ZMC equation in Euclidean space is MSE). Above considerations naturally leads one to expect a connection among the solutions of Born-Infeld equation, solutions of minimal surface equation and solutions of maximal surface equation. In past several authors have discussed about the connection of Born-Infeld equation and minimal surface equation [13, 38, 16]. For instance, Dey in [13], was able to rederive the Weierstrass-Enneper representation of minimal surfaces (she obtained this representation in neighbourhood of points which are not umbilic) using a technique of Barbishov and Chenikov [6] (which they used to construct general solution to Born-Infeld equation). Again using this representation she and her collaborator in [16] constructed a one parameter family of complex Born-Infeld solitons.

There is a link between certain Ramanujan's identities and minimal surfaces. This was explored by R. D. Kamien in [28]. Recently, Dey in [17], connected the Weierstrass-Enneper representations of some well known examples of minimal graphs and some solutions of Born-Infeld equation to certain Ramanujan's identities [45] in order to arrive at further new nontrivial identities (which might find its interest in number theory).

In Chapter 2, we give a different formulation of the singular Björling problem for the case of real analytic closed null curve and prove it in this setting. Next using this formulation of singular Björling problem we show existence of special kind of maximal surfaces (we call them interpolating maximal surfaces) containing a given spacelike closed curve and a point as singularity. More generally, the main theorem of this chapter gives a necessary and sufficient condition for existence of interpolating maximal surfaces for a given real analytic closed spacelike curve (for details see the section "proof of the main theorem" to Chapter 2).

In Chapter 3, 4 and 5, following Dey [13, 16, 17], we prove analogous results for maximal surfaces. For instance, in Chapter 3, we rederive the Weierstrass-Enneper representation for maximal surfaces, construction of one parameter family of solitons. At the base of all these results there lie a fact or an observtion that the MaSE and BIE related to each other by a Wick rotation (for details on this see the introduction to the Chapter 3). In Chapter 4, we also give a geometric interpretation to Born-Infeld solitons (for details see the first section to Chapter 4). Finally, in Chapter 5, we obtain new nontrivial identities using Weierstrass-Enneper representation of maximal surfaces and some Ramanujan's identities. This also concludes the thesis.

CHAPTER 2

Existence of interpolating maximal surfaces

2.1 Introduction

In this chapter we discuss the singular Björling problem [32] for the case of real analytic null closed curves and give a different proof for the same. We make use of the solution of the singular Björling problem to show the existence of maximal surfaces which contain a given real analytic spacelike closed curve and has a special singularity at a point in $\mathbb{L}^3$. But it turns out that this is not always the case, i.e., for a given spacelike closed curve we cannot always demand a maximal surface containing it and having a special singularity. We illustrate this fact through an example in the subsection. This observation naturally raises a question as to why for certain spacelike closed curve we get a maximal surface having special singularity and for others we do not. Here we give a characterisation (in terms of certain series conditions) of such spacelike closed curves for which we get a maximal surface which contains it and have special singularity. More generally, we characterize the set of spacelike closed curves such that there exists a generalised maximal surface parametrized by a single chart which contains a given spacelike closed curve and having a special singularity.

2.1.1 Motivating example

We start with a spacelike closed real analytic curve $\alpha(\theta)=(-\frac{3}{4}\cos\theta, -\frac{3}{4}\sin\theta, \ln\frac{1}{2})$. This curve lies on elliptic catenoid, a maximal surface, given by a map $F(x, y) =$

$\left(\frac{x(x^2+y^2-1)}{2(x^2+y^2)}, \frac{y(x^2+y^2-1)}{2(x^2+y^2)}, \ln\sqrt{x^2+y^2}\right)$. Here, we see that

1. the map F is defined for all $z = x + iy \neq 0$ and has a special singularity on $|z| = 1$,
2. there is a positive real r_0, namely $r_0 = \frac{1}{2}$, such that $F(|z| = r_0) = \gamma(\frac{1}{2}e^{i\theta}) := \alpha(\theta)$.

On the other hand if we take $\beta(\theta) = (e^{i\theta}, 1)$, which is a real analytic spacelike closed curve, we see in Section 4 that there does not exist any maximal surface F (parametrised by a single chart F defined for all $z \neq 0$) and any $r_0 \neq 1$ such that $F(r_0 e^{i\theta}) = \tilde{\beta}(r_0 e^{i\theta}) := \beta(\theta)$ and has special singularity at $|z| = 1$.

2.2 Maximal surface

The vector space $\mathbb{R}^3$ with the metric $dx^2 + dy^2 - dt^2$, denoted by $\mathbb{L}^3$, is known as 3-dimensional Lorentz-Minkowski space. We identify the vector space structure of $\mathbb{L}^3$ with $\mathbb{C} \times \mathbb{R}$, by $(x, y, t) \to (x + iy, t)$. Then the metric can be represented as $(dx + idy)(dx - idy) - dt^2$.

Definition 2.2.1 Let $\Omega \subset \mathbb{C}$ be a domain and $F = (u, v, w) : \Omega \subset \mathbb{C} \to \mathbb{L}^3$ be a nonconstant, smooth harmonic map such that the coordinate functions u, v, w satisfy the conformality relations (with $z = x + iy$),

$$\begin{aligned} u_x^2 + v_x^2 - w_x^2 &= u_y^2 + v_y^2 - w_y^2 \\ u_x u_y + v_x v_y - w_x w_y &= 0 \end{aligned} \tag{2.1}$$

and on Ω, $|u_z|^2 + |v_z|^2 - |w_z|^2$ does not vanish identically. Then F is said to be a *generalised maximal surface.*

Let $F = (h := u + iv, w)$, where h is the complex coordinate of F, the conformality relations (2.1) is equivalent to (see appendix **A.3**)

$$h_z \overline{h_{\bar{z}}} - w_z^2 = 0.$$

On Ω, nonvanishing of $|u_z|^2 + |v_z|^2 - |w_z|^2$ is equivalent to $|h_z|$ is not identically equal to $|h_{\bar{z}}|$ (see appendix **A.3**). In view of the above complex representation, we have an equivalent definition of the generalised maximal surface.

Definition 2.2.2 Let $F = (h, w) : \Omega \to \mathbb{C} \times \mathbb{R}$ be a smooth map such that $h_{z\bar{z}} = 0$ and $w_{z\bar{z}} = 0$ (harmonic) with $h_z\overline{h_{\bar{z}}} - w_z^2 = 0$ (conformal) and $|h_z|$ is not identically equal to $|h_{\bar{z}}|$. A *generalised maximal surface* is the equivalence class of map F, where equivalence relation is change of the conformal parameter.

The above definition is motivated by the analogous definition of minimal surfaces [27].

Example 2.2.1 (Elliptic catenoid) Let $\Omega = \mathbb{C} - \{0\}$ and $h(z) = \frac{1}{2}\left(z - \frac{1}{\bar{z}}\right)$, $w(z) = \frac{1}{2}\log(z\bar{z})$. Then we define $F : \mathbb{C}-\{0\} \to \mathbb{C}\times\mathbb{R}$, $F(z) = (h(z), w(z))$. We have $h_z\overline{h_{\bar{z}}} - w_z^2 = 0$ and $h_{z\bar{z}} = w_{z\bar{z}} = 0$ for all $z \in \Omega$. On $|z| = 1$, $|h_z| = |h_{\bar{z}}| = \frac{1}{2}$. Here $|h_z|$ is not identically equal to $|h_{\bar{z}}|$.

Example 2.2.2 If we take $h : \mathbb{C} \to \mathbb{C}$ defined by $h(z) = \sin z + \sin\bar{z} + i0$ and $w(z) = \sin z + \sin\bar{z}$, then we can easily see $h_z\overline{h_{\bar{z}}} - w_z^2 = 0$ and $h_{z\bar{z}} = w_{z\bar{z}} = 0$ for all $z \in \mathbb{C}$, but $|h_z|$ is identically equal to $|h_{\bar{z}}|$ on whole of $\mathbb{C}$. Therefore it is not a generalised maximal surface.

Since F is a generalised maximal surface in isothermal parameters, we have $\langle F_x(z), F_x(z)\rangle_L = \langle F_y(z), F_y(z)\rangle_L = \eta(z)(\geq 0)$, $\langle F_x(z), F_y(z)\rangle = 0$, we have

$$ds^2 = \eta(z)(dx^2 + dy^2) = \eta(z)|dz|^2, \tag{2.2}$$

where

$$\begin{aligned}\eta(z) &= \langle F_x, F_x\rangle \\ &= \langle (h_z + h_{\bar{z}}, w_z + w_{\bar{z}}), (h_z + h_{\bar{z}}, w_z + w_{\bar{z}})\rangle \\ &= (h_z + h_{\bar{z}})\overline{(h_z + h_{\bar{z}})} - (w_z + w_{\bar{z}})^2,\end{aligned} \tag{2.3}$$

now use conformality relation $h_z\overline{h_{\bar{z}}} - w_z^2 = 0$, to obtain

$$\eta(z) = (|h_z| - |h_{\bar{z}}|)^2. \tag{2.4}$$

Definition 2.2.3 A point of $\Omega \subseteq \mathbb{C}$ on which the equation $|h_z| = |h_{\bar{z}}|$ holds is called a singular point of (F, Ω) and set of all *singular points* is called the *singular set* of the maximal surface (F, Ω).

Authors in [32, 23, 35, 49] have defined and studied different kind of singularities based on image of singularity set, such as shrinking, curvilinear singularity, cuspidal edges, swallowtails etc., Fernández, López, and Souam in [36] discussed two type of isolated singularities namely branch and special singularity. We also use the name special singularity for the singularity defined below.

Definition 2.2.4 A point p in $\mathbb{L}^3$ is such that $F(\{|z| = r\}) = p$ for some $r > 0$, then we say that at p the generalised maximal surface (F, Ω) has *special singularity*, if $|z| = r$ is a subset of the singular set (set of all singular points) of (F, Ω).

If (F, Ω) has a special singularity at a point p for $|z| = r$, we often refer to it as p or $|z| = r$.

Points where $|h_z| \neq |h_{\bar{z}}|$ holds are called regular points of (F, Ω) in the sense that at those points of Ω, F will be an immersion. We have following easy observation that if F is not an immersion, then $u_x v_y - u_y v_x = 0$. In turn $u_x v_y - u_y v_x = |h_z|^2 - |h_{\bar{z}}|^2$. Thus $|h_z| = |h_{\bar{z}}|$.

Conversely, suppose $|h_z| = |h_{\bar{z}}|$, as $F = (h, w)$ is a generalized maximal surface then $|h_z| = |h_{\bar{z}}|$ corresponds to singular set of the surface. Indeed, since we have $h_z \overline{h_{\bar{z}}} - w_z^2 = 0$, this implies $|w_z|^2 = |h_z|^2 = |h_{\bar{z}}|^2$. This also gives

$$2(u_x v_y - u_y v_x) = (u_x^2 + v_x^2 - w_x^2) + (u_y^2 + v_y^2 - w_y^2). \tag{2.5}$$

As we have F maximal, so by definition F is spacelike. The vectors $F_x = (u_x, v_x, w_x)$ and $F_y = (u_y, v_y, w_y)$ are spacelike vectors and hence

$$|F_x|^2 = u_x^2 + v_x^2 - w_x^2 \geq 0,$$
$$|F_y|^2 = u_y^2 + v_y^2 - w_y^2 \geq 0.$$

Therefore at the singular points we get $|F_x|^2 + |F_y|^2 = 0$. This imply $F_x = F_y = 0$. Hence, F is not an immersion. Thus $F = (h, w) : \Omega \to \mathbb{L}^3$ is a generalised maximal surface and F is an immersion at $p \in \Omega$ if and only if at p, $|h_z| \neq |h_{\bar{z}}|$.

With this representation of maximal surface, following [27], we have the following:

Proposition 2.2.5 *Let $h : \Omega \to \mathbb{C}$ be the complex coordinate of the isothermal representation of a generalized maximal surface $F = (h, w) : \Omega \to \mathbb{C} \times \mathbb{R} \simeq \mathbb{L}^3$. Then on $\Omega \subset \mathbb{C}$, we can write*

$$w(z) = 2Re \int_{z_0}^{z} \sqrt{h_z \overline{h_{\bar{z}}}} dz + w(z_0),$$

where the line integral is along any smooth curve starting from z_0 *and ending at* z*.*

Proof. The function $h_z\overline{h_{\bar{z}}}$ admits a continuous branch of square root in Ω. Let Γ be a closed curve in Ω. Consider

$$2Re\int_\Gamma \sqrt{h_z\overline{h_{\bar{z}}}}dz = \int_\Gamma \sqrt{h_z\overline{h_{\bar{z}}}}dz + \overline{\int_\Gamma \sqrt{h_z\overline{h_{\bar{z}}}}dz} = \int_\Gamma \omega_z dz + \int_\Gamma \overline{\omega_z}\overline{dz} = \int_\Gamma dw = 0.$$

Therefore we have for every closed curve $\Gamma \subset \Omega$,

$$Re\int_\Gamma \sqrt{h_z\overline{h_{\bar{z}}}}dz = 0.$$

This allows us to define $w(z) - w(z_0) = 2Re\int_{z_0}^{z} \sqrt{h_z\overline{h_{\bar{z}}}}dz$. This gives

$$w(z) = 2Re\int_{z_0}^{z} \sqrt{h_z\overline{h_{\bar{z}}}}dz + w(z_0).$$

2.2.1 Examples

The complex coordinate representation (as in Definition 2.2.2 and in Proposition 2.2.5) of the generalised maximal surface helps us to construct many examples of maximal surfaces. In particular if we take any complex harmonic map $h : \Omega \to \mathbb{C}$ such that $|h_z|$ is not identically same as $|h_{\bar{z}}|$, then the map $F : \Omega \to \mathbb{L}^3$, defined by $F(z) = \left(h(z), 2Re\int_{z_0}^{z} \sqrt{h_z\overline{h_{\bar{z}}}}dz\right)$ is a generalised maximal surface.

Example 2.2.3 If we take $h : \mathbb{C} \to \mathbb{C}$ defined by $h(z) = e^z + \bar{z}$. Then $h_z = e^z, h_{\bar{z}} = 1$ and hence $|h_z| = |h_{\bar{z}}| = 1$ on imaginary axis. Here $|h_z|$ is not identically equal to $|h_{\bar{z}}|$, by Proposition 2.2.5, we can determine the third real coordinate w to make (h, w) a maximal surface.

$$w(z) = 2Re\int \sqrt{h_z\overline{h_{\bar{z}}}}dz = 2(e^{\frac{z}{2}} + e^{\frac{\bar{z}}{2}}).$$

The map $F : \mathbb{C} \to \mathbb{L}^3$ given by $F(z) = (h(z), w(z))$ satisfies $h_z\overline{h_{\bar{z}}} - w_z^2 = 0$ (conformality relations) and $h_{z\bar{z}} = 0$, $w_{z\bar{z}} = 0$ (harmonicity) and hence defines a generalized maximal surface.

Example 2.2.4 If we take $h(z) = \frac{1}{2}\left(z - \frac{1}{\bar{z}}\right)$, by Proposition 2.2.5, we get $w(z) = \frac{1}{2}\log(z\bar{z})$. Then $F(z) = (h(z), w(z))$ defines what is known as a elliptic catenoid which is a generalised maximal surface with singular set the unit circle $\{|z| = 1\}$.

Example 2.2.5 (Lorentzian Helicoid) The Lorentzian helicoid can be expressed by $h(z) = \frac{i}{2}\left(z + \frac{1}{\bar{z}}\right)$ and $w(z) = -arg(z)$; z is a locally defined conformal parameter. We can obtain a global parametrization by replacing $z \in \mathbb{C} - \{0\}$ by $e^z, z \in \mathbb{C}$. The singularity set is $\{z \in \mathbb{C} | \Re(z) = 0\}$. Lorentzian Helicoid is *conjugate* to Elliptic Catenoid.

2.3 The singular Björling problem

Normal vector at a regular point of a generalised maximal surface can be given by a map $N : \Omega \to \mathbb{H}^2 := \{(x, y, t) \in \mathbb{L}^3 : x^2 + y^2 - t^2 = -1\}$ (see appendix **A.4**)

$$N(z) = \frac{F_x \times F_y}{|F_x \times F_y|} = \left(\frac{2\sqrt{h_z h_{\bar{z}}}}{|h_{\bar{z}}| - |h_z|}, \frac{|h_{\bar{z}}| + |h_z|}{|h_{\bar{z}}| - |h_z|}\right). \tag{2.6}$$

For a generalised maximal surface (F, Ω), Ω has two parts $\mathcal{A} := \{z : |h_{\bar{z}}| \neq |h_z|\}$ and $\mathcal{B} := \{z : |h_{\bar{z}}| = |h_z|\}$. Notice that $\mathcal{B}$ denotes the singular set of (F, Ω). The Gauss map at regular points (that is on $\mathcal{A}$) is obtained by stereographic projection of N as in (2.6) from the north pole of $\mathbb{H}^2$ to $\mathbb{C}$. It is given by (see appendix **A.5**)

$$\nu(z) = -\sqrt{\frac{h_{\bar{z}}}{h_z}} \quad \text{on} \quad \mathcal{A}. \tag{2.7}$$

Next we explain the singular Björling problem. Suppose we have given

$$\gamma(e^{i\theta}) = \left((\gamma_1 + i\gamma_2)(e^{i\theta}), \gamma_3(e^{i\theta})\right), \tag{2.8}$$

$$L(e^{i\theta}) = \left((L_1 + iL_2)(e^{i\theta}), L_3(e^{i\theta})\right),$$

and $\langle \gamma', L \rangle = 0$, where γ is a real analytic null closed curve, i.e., at each point of a null curve the induced metric degenerates (see appendix **A.1**), and L is a real analytic null vector field along γ and that atleast one of γ' and L is not identically zero, γ and L both are defined over S^1. The above data is known as singular Björling data. Kim and Yang in [32] studied the singular Björling problem in detail. In this section we discuss the same problem for a closed null curve from a different point of view. The singular Björling problem asks for the existence of a generalised maximal surface

$$F = (h, w) : A(r, R) \to \mathbb{L}^3$$

such that $F(e^{i\theta}) = \gamma(e^{i\theta})$ and $\left.\dfrac{\partial F}{\partial \rho}\right|_{e^{i\theta}} = (h_\rho(e^{i\theta}), w_\rho(e^{i\theta})) = L(e^{i\theta})$ with singular set atleast $\{|z| = 1\}$.

For the existence of maximal surface, having prescribed data as above, we will be looking for a complex harmonic function h and a real harmonic function w on some annulus $A(r, R)$, $r < 1 < R$ such that they satisfy

1. $h_z\overline{h_{\bar{z}}} - w_z^2 \equiv 0$,
2. $|h_z| = |h_{\bar{z}}|$ on $z = e^{i\theta}$,
3. $|h_z| - |h_{\bar{z}}|$ is not identically zero on $A(r, R)$.

We have the following relation between first order partial differentials in system $(z, \bar{z})$ to the first order partial differential in system (ρ, θ); where $z = \rho e^{i\theta}$:

$$h_z = \frac{1}{2}\left(h_\rho - \frac{i}{\rho}h_\theta\right)e^{-i\theta} \quad \text{and} \quad h_{\bar{z}} = \frac{1}{2}\left(h_\rho + \frac{i}{\rho}h_\theta\right)e^{i\theta}. \tag{2.9}$$

Here we have given $(h_\rho, w_\rho) = (L_1 + iL_2, L_3)$ and $(h_\theta, w_\theta) = ({\gamma_1}' + i{\gamma_2}', {\gamma_3}')$ on the unit circle. On $\{|z| = 1\}$, we define the map g as

$$g(e^{i\theta}) = \begin{cases} -\sqrt{\frac{L_1+iL_2}{L_1-iL_2}}, & \text{if } \gamma' \text{ vanishes identically} \\ -\sqrt{\frac{{\gamma_1}'+i{\gamma_2}'}{{\gamma_1}'-i{\gamma_2}'}}, & \text{otherwise.} \end{cases}$$

If there exists a generalised maximal surface (h, w) for the given Björling data, then analytic extension of g agrees with $\nu(z) = -\sqrt{\dfrac{\overline{h_{\bar{z}}}}{h_z}}$ on $\mathcal{A}$ (that is at those points of the domain where $|h_{\bar{z}}| \neq |h_z|$).

Now we state and prove the singular Björling theorem.

Theorem 2.3.1 *Given a real analytic null closed curve $\gamma : S^1 \to \mathbb{L}^3$ and a null vector field $L : S^1 \to \mathbb{L}^3$ such that $\langle \gamma', L \rangle = 0$; atleast one of γ' and L do not vanish identically. If $|g(z)|$ ($g(z)$ is analytic extension of $g(e^{i\theta})$) is not identically equal to 1, then there exists a unique generalised maximal surface $F := (h, w)$ defined on some annulus $A(r, R) := \{z : 0 < r < |z| < R\}; r < 1 < R$, such that*

1. $F(e^{i\theta}) = (h(e^{i\theta}), w(e^{i\theta})) = \gamma(e^{i\theta})$,

2. $\left.\dfrac{\partial F}{\partial \rho}\right|_{e^{i\theta}} = (h_\rho(e^{i\theta}), w_\rho(e^{i\theta})) = L(e^{i\theta})$,

with singular set atleast $\{|z| = 1\}$.

Proof. We will prove this theorem in two steps

1. We show the existence of generalised maximal surface $F = (h, w)$ containing the given singular Björling data.

2. Next we show that the determined generalized maximal surface will have singularity set atleast $\{|z| = 1\}$.

In the step 1, we find a complex harmonic function h and a real harmonic function w defined on some annulus $A(r, R)$, and show that $h_z\overline{h_{\bar z}} - w_z^2 \equiv 0$. To do this, we use an interesting fact about a harmonic function, that says any harmonic function (complex or real) which is defined over some annulus $A(r, R)$ can be expressed in the following form

$$\sum_{-\infty}^{\infty} a_n z^n + \frac{b_n}{\bar z^n} + c \ln |z|. \tag{2.10}$$

In particular, let us say for $h(z)$ we have

$$h(z) = \sum_{-\infty}^{\infty} a_n z^n + \frac{b_n}{\bar z^n} + c \ln |z|. \tag{2.11}$$

Also, $w(z)$ will have a similar expression, let us say

$$w(z) = \sum_{-\infty}^{\infty} c_n z^n + \frac{d_n}{\bar z^n} + d \ln |z|. \tag{2.12}$$

Now in terms of (ρ, θ) coordinates, on the unit circle, we get

$$h_\theta(e^{i\theta}) = i\sum_{-\infty}^{\infty} n(a_n + b_n)e^{in\theta}, \tag{2.13}$$

$$h_\rho(e^{i\theta}) = \sum_{-\infty}^{\infty} n(a_n - b_n)e^{in\theta} + c. \tag{2.14}$$

From the given data, $h(e^{i\theta}) = \gamma_1(\theta) + i\gamma_2(\theta)$, we know the left hand side of the equation (2.13) and as γ is analytic, $h_\theta(e^{i\theta})$ is analytic so the series (in equation (2.13)) in the right hand side converges.

Next we equate

$$h_\rho(e^{i\theta}) = L_1(\theta) + iL_2(\theta) \tag{2.15}$$

as above, as $L_1 + iL_2$ is analytic, h_ρ is analytic and hence the series in equation (2.14) converges. We have $n(a_n + b_n)$ as the fourier coefficients of h_θ in equation (2.13) for all n, and those for h_ρ are $n(a_n - b_n)$ in equation (2.14), for all n. Therefore, we can solve for a_n, b_n and c uniquely and hence we have determined $h(z)$ such that h is harmonic. In the same way, the harmonic function $w(z)$ can be determined, because we have given $w(e^{i\theta})$ and $w_\rho(e^{i\theta})$.

Next we show $h_z\overline{h_{\bar z}} - w_z^2 = 0$ on unit circle with given data. Indeed,

$$\overline{h_{\bar z}} = \frac{1}{2}\left(\overline{h_\rho} - \frac{i}{\rho}\overline{h_\theta}\right)e^{-i\theta}, \tag{2.16}$$

$$w_z = \frac{1}{2}\left(w_\rho - \frac{i}{\rho}w_\theta\right)e^{-i\theta}. \tag{2.17}$$

On unit circle we have

$$\begin{aligned} h_z\overline{h_{\bar z}} &= \frac{1}{4}(h_\rho - ih_\theta)(\overline{h_\rho} - i\overline{h_\theta})e^{-2i\theta} \\ &= \frac{1}{4}(L_1^2 + L_2^2 - \gamma'^2_1 - \gamma'^2_2 - i(L_1 + iL_2)(\gamma'_1 - i\gamma'_2) - i(\gamma'_1 + i\gamma'_2)(L_1 - iL_2))e^{-2i\theta}. \end{aligned}$$

As L and γ' are null vector fields we have $L_1^2 + L_2^2 = L_3^2$ and $\gamma'^2_1 + \gamma'^2_2 = \gamma'^2_3$, using these identities in above equation we get

$$h_z\overline{h_{\bar z}} = \frac{1}{4}(L_3^2 - \gamma'^2_3 - 2iL_3\gamma'_3)e^{-2i\theta}, \tag{2.18}$$

and

$$w_z^2 = \frac{1}{4}(w_\rho - iw_\theta)^2 e^{-2i\theta} = \frac{1}{4}(L_3^2 - \gamma'^2_3 - 2iL_3\gamma'_3)e^{-2i\theta}. \tag{2.19}$$

From equations (2.18) and (2.19) we see that $h_z\overline{h_{\bar{z}}} - w_z^2 = 0$ on the unit circle. As h and w are harmonic functions on an annulus $A(r, R)$, the function $h_z\overline{h_{\bar{z}}} - w_z^2$ is complex analytic on $A(r, R)$ which contains the unit circle and hence $h_z\overline{h_{\bar{z}}} - w_z^2 \equiv 0$ on annulus. We have also given that the analytic extension $g(z)$ of $g(e^{i\theta})$ is such that $|g(z)|$ is not identically 1, which is, equivalent to saying $|h_z|$ is not identically equal to $|h_{\bar{z}}|$. Hence, we have found the unique generalised maximal surface $F := (h, w)$.

Now in this last step, we are going to show that the singular set of the generalised maximal surface $F := (h, w)$ contains atleast the set $\{|z| = 1\}$.
Since $L_3\gamma'_3 = L_1\gamma'_1 + L_2\gamma'_2$ and γ' and L are null vector field, we have $L_1\gamma'_2 = L_2\gamma'_1$. By using the equation (2.9) on the unit circle we have

$$|h_z| = \frac{1}{2}|h_\rho - ih_\theta| = |L_1 + \gamma'_2 + i(L_2 - \gamma'_1)|,$$

$$|h_z|^2 = \frac{L_1^2 + L_2^2 + \gamma'^2_1 + \gamma'^2_2}{4} + \frac{L_1\gamma'_2 - L_2\gamma'_1}{2}. \tag{2.20}$$

Similarly,

$$|h_{\bar{z}}|^2 = \frac{L_1^2 + L_2^2 + \gamma'^2_1 + \gamma'^2_2}{4} - \frac{L_1\gamma'_2 - L_2\gamma'_1}{2}. \tag{2.21}$$

Now subtracting equation (2.21) from (2.20), we get

$$|h_z|^2 - |h_{\bar{z}}|^2 = L_1\gamma'_2 - L_2\gamma'_1 = 0.$$

Thus $|h_z| = |h_{\bar{z}}|$ on unit circle, this proves that our unique generalised maximal surface $F := (h, w)$ will have singularity set atleast $\{|z| = 1\}$. This also completes the proof.

Next we give an example which illustrates the singular Björling problem.

Example 2.3.1 If $\gamma(\theta) = (c, c, c)$, then $\gamma'(\theta) = (0, 0, 0)$ and $\langle\gamma'(\theta), \gamma'(\theta)\rangle_L = 0$, i.e., the induced metric degenerates on the constant curve (c, c, c). Thus the constant curve is a null curve. Therefore, for any non vanishing null vector field $L(\theta)$ there exists a generalised maximal surface containing the constant curve as singularity. We give a particular example illustrating this and the proof of the above theorem. When $L(\theta) = (e^{i\theta}, 1) = (h_\rho, w_\rho)$ and $\gamma'(\theta) = (0 + i0, 0) = (h_\theta, w_\theta)$, we will get a generalised maximal surface known as elliptic catenoid. Recall the expressions (2.13) and (2.14), from these we have

$$0 = i \sum_{-\infty}^{\infty} n(a_n + b_n)e^{in\theta} \text{ and}$$

$$e^{i\theta} = \sum_{-\infty}^{\infty} n(a_n - b_n)e^{in\theta} + c$$

This gives $a_1 - b_1 = 1$ and $a_1 + b_1 = 0$ which imply $a_n = 0$, $b_n = 0$, $\forall\ n \neq 1$ and $c = 0$ and hence from the formula (2.11) we get $h(z) = \frac{1}{2}\left(z - \frac{1}{\bar{z}}\right)$. To obtain $w(z)$, we repeat the same step as in the case of obtaining $h(z)$, because here we know $w_\rho = 1$ and $w_\theta = 0$, from this we get $c = 1$ and $a_n = b_n = 0$, $\forall\ n$. This gives $w(z) = \frac{1}{2}log(z\bar{z})$. Expressions (h, w) together represents an elliptic catenoid.

2.4 Existence of maximal surface containing a prescribed curve and special singularity

2.4.1 Main Theorem

We start with an example to explain the problem and an approach for a solution to this problem. Let $\tilde{\gamma}(\theta) = (c_1 e^{i\theta}, c_2)$, c_1 and c_2 be some constants. We will see that if we take $c_1 = c_2 = 1$, i.e. $\tilde{\gamma}(\theta) = (e^{i\theta}, 1)$, then there does not exists any positive real $r_0 \neq 1$ and generalised maximal surface F as in the definition (2.2.2), defined on some annulus having $|z| = 1$ such that $F(r_0 e^{i\theta}) = (e^{i\theta}, 1)$ and F restricted to unit circle has a special singularity. While, in particular, if we take $c_1 = -\frac{3}{4}$ and $c_2 = \ln \frac{1}{2}$, then for $r_0 = \frac{1}{2}$, there is a generalised maximal surface $F : \mathbb{C} - \{0\} \to \mathbb{L}^3$ such that $F(r_0 e^{i\theta}) = \gamma(r_0 e^{i\theta}) := \tilde{\gamma}(\theta)$, maximal surface is the elliptic catenoid discussed in the example (2.2.1).

Now we justify our claim (made in the last paragraph). We are looking for a generalised maximal surface F such that

$$F(r_0e^{i\theta}) = (c_1e^{i\theta}, c_2); r_0 \neq 1, c_1, c_2 \quad \text{are constants} \quad \text{and} \quad F(e^{i\theta}) = (0,0,0). \tag{2.22}$$

Also on $|z| = 1$, F admits singularity. Suppose if we can find such a maximal surface $F(z) = (h(z), w(z))$ which satisfy the initial data given in (2.22), then h for that maximal surface over an annulus is of the form as in equation (2.11), and similarly for w. The initial condition $F(e^{i\theta}) = (0 + 0i, 0)$ will give us

$$a_n + b_n = 0; \quad \forall\, n \tag{2.23}$$

and the condition $F(r_0e^{i\theta}) = (c_1e^{i\theta}, c_2)$

$$a_nr_0^n + \frac{b_n}{r_0} = 0 \Rightarrow a_n = b_n = 0; \;\forall\, n \neq 1, 0. \tag{2.24}$$

$$a_0 + b_0 + c\log r_0 = 0 \Rightarrow c = 0 \quad \text{if} \quad r_0 \neq 1. \tag{2.25}$$

We use (2.23), (2.24) and (2.25) to get

$$a_1 = \frac{r_0}{r_0^2 - 1}c_1 \quad \text{and} \quad b_1 = -a_1, \tag{2.26}$$

hence

$$h(z) = \frac{r_0c_1}{r_0^2 - 1}\left(z - \frac{1}{\bar{z}}\right). \tag{2.27}$$

In a similar manner, for $w(z)$ using initial conditions (2.22) we get

$$c_n + d_n = 0; \quad \forall\, n \tag{2.28}$$

$$c_nr_0^n + \frac{d_n}{r_0^n} = 0 \Rightarrow c_n = d_n = 0; \quad \forall n \neq 1, 0. \tag{2.29}$$

$$c_0 + d_0 + d\log r_0 = c_2 \Rightarrow d = \frac{c_2}{\log r_0} \quad \text{if} \quad r_0 \neq 1. \tag{2.30}$$

$$w(z) = \left(\frac{c_2}{2\log r_0}\right)\log z\bar{z}. \tag{2.31}$$

Now in order to have $F(z) = (h(z), w(z))$ as the generalised maximal surface, h and w have to satisfy the conditions given in Definition 2.2.2. The relation $h_z\overline{h_{\bar{z}}} - w_z^2 \equiv 0$ gives us a relation between c_1, c_2 and r_0 as follows

$$\frac{c_1 r_0}{r_0^2 - 1} = \frac{c_2}{2\log r_0} \tag{2.32}$$

and we see for any set of constants c_1, c_2, r_0, satisfies above relation, $|h_z|$ is not identically same as $|h_{\bar{z}}|$. Therefore, if we have constants $(c_1, c_2, r_0 \neq 1)$ such that they satisfies (2.32), then there is a generalised maximal surface satisfying initial data (2.22) and having singularity on $|z| = 1$.

Moreover, we see that for the spacelike closed curve $\tilde{\gamma}(\theta) = (e^{i\theta}, 1)$, $c_1 = c_2 = 1$, mentioned in the beginning of this section, the equation (2.32) has no solution for any r_0. Therefore, there does not exists any generalised maximal surface F such that

$$F(r_0e^{i\theta}) = (e^{i\theta}, 1); r_0 \neq 1 \quad , \quad F(e^{i\theta}) = (0, 0, 0) \tag{2.33}$$

and F restricted to unit circle has a special singularity.

Based on above discussion, in general we can ask the following: Given a real analytic curve $\tilde{\gamma}(\theta)$, does there exists $F : A(r, R) \to \mathbb{L}^3$, a generalised maximal surface and $r_0 \neq 1$ such that $F(r_0e^{i\theta}) = \tilde{\gamma}(\theta)$ and F has a special singularity at $|z| = 1$.

For a curve $\tilde{\gamma}(\theta) = \gamma(r_0e^{i\theta}) = (f(r_0e^{i\theta}), g(r_0e^{i\theta}))$, $r_0 \neq 1$ (where $f(r_0e^{i\theta}) \in \mathbb{C}, g(r_0e^{i\theta}) \in \mathbb{R}$), we define the following modified Fourier coefficients (see appendix **A.6**) of f and g as

$$c = \frac{1}{2\pi\log r_0}\int_{-\pi}^{\pi} f(r_0e^{i\theta})d\theta; \quad d = \frac{1}{2\pi\log r_0}\int_{-\pi}^{\pi} g(r_0e^{i\theta})d\theta, \tag{2.34}$$

for $n \neq 0$;

$$c_n = \frac{r_0^n}{2\pi(r_0^{2n}-1)}\int_{-\pi}^{\pi} f(r_0e^{i\theta})e^{-in\theta}d\theta; \quad d_n = \frac{r_0^n}{2\pi(r_0^{2n}-1)}\int_{-\pi}^{\pi} g(r_0e^{i\theta})e^{-in\theta}d\theta. \tag{2.35}$$

We see that if $\tilde{\gamma}$ is real analytic (since c_n, d_n, c_{-n}, and d_{-n} all converges to 0), $\limsup |c_{-n}|^{\frac{1}{n}} = 0$, $\limsup |c_n|^{\frac{1}{n}} = 0$, $\limsup |d_{-n}|^{\frac{1}{n}} = 0$ and $\limsup |d_n|^{\frac{1}{n}} = 0$, therefore the following two series convreges for all $|z| \neq 0$,

$$h(z) = \sum_{-\infty}^{\infty} c_n \left(z^n - \frac{1}{\bar{z}^n}\right) + c\log|z|. \tag{2.36}$$

$$w(z) = \sum_{-\infty}^{\infty} d_n \left(z^n - \frac{1}{\bar{z}^n}\right) + d\log|z|, \tag{2.37}$$

Now we state the main theorem of this Chapter which is an application to the singular Björling Theorem 2.3.1.

Theorem 2.4.1 *Let $\tilde{\gamma}(\theta)$ be a nonconstant closed real analytic spacelike curve. Then there exists $s_0 \neq 1$ and a generalised maximal surface $F : \mathbb{C} - \{0\} \to \mathbb{L}^3$ such that $F(s_0e^{i\theta}) := \tilde{\gamma}(\theta)$ and having a special singularity at $(0,0,0) \in \mathbb{L}^3$ if and only if there exists $r_0 \neq 1$ and constants $c, c_n's, d, d_n's$ for the curve $\gamma(r_0e^{i\theta}) := \tilde{\gamma}(\theta) = (f(r_0e^{i\theta}), g(r_0e^{i\theta}))$, as defined in equations (2.34),(2.35) which satisfy the relations:*

$$\forall\, k \neq 0; \quad \sum_{-\infty}^{\infty} 4n(n-k)(c_n\bar{c}_{n-k} - d_nd_{n-k}) + 2k(c_k\bar{c} - c\bar{c}_{-k} - 2d_kd) = 0 \tag{2.38}$$

and

$$\sum 4n^2(c_n\bar{c}_n - d_n^2) + c\bar{c} - d^2 = 0. \tag{2.39}$$

2.4.2 Proof of the main theorem

(Also follow appendix **A.6**)

We start proving the "only if" part. Assume that the constants $c, c_n's, d, d_n's$ satisfies the conditions (2.38) and (2.39) for the curve $\gamma(r_0e^{i\theta})$. We claim that

h and w given by equation (2.36), (2.37) is the generalised maximal surface satisfying given data. We see that $h(|z|=1)=0$; $w(|z|=1)=0$ and $\gamma(r_0e^{i\theta})=(h(r_0e^{i\theta}),w(r_0e^{i\theta}))=(f(r_0e^{i\theta}),g(r_0e^{i\theta}))$. From equations (2.36) and (2.37), we have

$$h_\rho(e^{i\theta})=\sum_{-\infty}^{\infty}2nc_ne^{in\theta}+c;\ \text{and}$$

$$w_\rho(e^{i\theta})=\sum_{-\infty}^{\infty}2nd_ne^{in\theta}+d$$

$$\begin{aligned}h_\rho(e^{i\theta}).\bar{h}_\rho(e^{i\theta}) &= \left(\sum_{-\infty}^{\infty}2nc_ne^{in\theta}+c\right)\left(\sum_{-\infty}^{\infty}2n\bar{c}_ne^{-in\theta}+\bar{c}\right)\\ &= \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}4n(n-k)c_n\bar{c}_{n-k}+2k(c_k\bar{c}-c\bar{c}_{-k})\right)e^{ik\theta}+c\bar{c}.\end{aligned}$$

Similarly, we have

$$w_\rho^2(e^{i\theta})=\sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}4n(n-k)d_nd_{n-k}+4kd_kd\right)e^{ik\theta}+d^2.$$

All the series above converges absolutely as f and g are real analytic functions. The series conditions on the constants given in the theorem assures that $h_\rho\overline{h_\rho}-w_\rho^2=0$ for $z=e^{i\theta}$, that is to say that (h_ρ,w_ρ) is a null vector field along $|z|=1$. By the singular Björling Theorem 2.3.1 for closed curve $\alpha(e^{i\theta})=(0,0,0)$ and $L(e^{i\theta})=(h_\rho(e^\theta),w_\rho(e^\theta))$ we have a unique maximal surface (h',w') on some $A(r,R)$, $r<1<R$. On $A(r,R)$ we have (by uniqueness of (h,w) and (h',w')),

$$h_z\overline{h_{\bar{z}}}-w_z^2=h'_z\overline{h'_{\bar{z}}}-{w'_z}^2\equiv 0$$

But this being a complex analytic function on $\mathbb{C}-\{0\}$, $h_z\overline{h_{\bar{z}}}-w_z^2\equiv 0$. Also as $F(r_0e^{i\theta})$ is a spacelike curve, this gives that $|h_z|$ is not identically equal to $|h_{\bar{z}}|$, and it proves existence of the required generalised maximal surface. Here we can take $s_0=r_0$.

Now to show the "if" part, if the generalized maximal surface F is given such that $F(s_0e^{i\theta})=\gamma(\theta)$ and $F(|z|=1)=(0,0,0)$, then F has to be of the form (h,w) as given in the equations (2.36) and (2.37) with $c,c_n's,d,d_n's$ as in (2.34) and (2.35) with $r_0 = s_0$. In this form prescribing singularity set as $|z| = 1$ is same as asking for the vector $(h_\rho(e^{i\theta}), w_\rho(e^{i\theta}))$ is a null vector which gives series conditions as in (2.38) and (2.39). This completes the proof of the main theorem.

2.4.3 Examples

Example 2.4.1 We have seen that for $\tilde{\gamma}(\theta) = \left(-\frac{3}{4}e^{i\theta}, \ln\frac{1}{2}\right)$, if we take $r_0 = \frac{1}{2}$ then the constants as in equations (2.34), (2.35) are as follows $c_1 = \frac{1}{2}$, $d = 1$ and for all $n \neq 1$, $c_n's = 0$, $d_n = 0$ and $c = 0, d_1 = 0$ and these constants satisfies the series conditions as in equations (2.38) and (2.39). Therefore there exists a generalised maximal surface which is given by expression of h and w as in example (2.2.1) having special singularity.

Example 2.4.2 Consider the curve

$$\tilde{\gamma}(\theta) = (a_3e^{3i\theta} + a_1e^{-i\theta}, b_2e^{2i\theta} + b_2e^{-2i\theta}). \tag{2.40}$$

Below we analyze for which values of constants a_1, a_3, and b_2, does there exists $r_0 \neq 1$ and the generalised maximal surface of the type mentioned in the Theorem 2.4.1.

Recall the formulas (2.34) and (2.35), for $f(r_0e^{i\theta}) = a_3e^{3i\theta} + a_1e^{-i\theta}$and $g(r_0e^{i\theta}) = b_2e^{2i\theta} + b_2e^{-2i\theta}$, then we have $c_0 = 0, c = 0$, for $n \neq -1, 3$; $c_n = 0$ and

$$c_{-1} = \frac{r_0^{-1}}{r_0^{-2} - 1}a_1 \quad , \quad c_3 = \frac{r_0^3}{r_0^6 - 1}a_3. \tag{2.41}$$

Similarly $d = 0$, for $n \neq -2, 2$; $d_n = 0$ and

$$d_2 = \frac{r_0^2}{r_0^4 - 1}b_2 \quad , \quad d_{-2} = \frac{r_0^{-2}}{r_0^{-4} - 1}b_2. \tag{2.42}$$

Suppose the constants a_1, a_3 and b_2 are such that the curve $\tilde{\gamma}$ is spacelike, then there exists a generalised maximal surface F and r_0 as in Theorem 2.4.1 if and only if the conditions (2.38) and (2.39) are satisfied by the constants $c, c_n's, d, d_n's$. That is to say

$$\forall\, k \neq 0 \sum_{n=-2,-1,2,3} 4n(n-k)(c_nc_{n-k} - d_nd_{n-k}) = 0 \text{ and } \sum_{n=2,-2,-1,3} 4n^2(c_n^2 - d_n^2) = 0$$

which is equivalent to

$$4d_2d_{-2} - 3c_3c_{-1} = 0 \quad \text{and} \quad c_{-1}^2 + 9c_3^2 = 4(d_2^2 + d_{-2}^2). \tag{2.43}$$

Therefore, for $\gamma(r_0e^{i\theta}) := \tilde{\gamma}(\theta) = (a_3e^{3i\theta} + a_1e^{-i\theta}, b_2e^{2i\theta} + b_2e^{-2i\theta})$ if a_1, a_3, b_2 are such that $\tilde{\gamma}$ is spacelike then there exists a maximal surface $F : \mathbb{C} - \{0\} \to \mathbb{L}^3$ such that $F(r_0e^{i\theta}) = \tilde{\gamma}(\theta)$ and F has special singularity at $|z| = 1$ if and only if

$$4\frac{b_2}{\left(r_0^2 - \frac{1}{r_0^2}\right)}\frac{b_2}{\left(r_0^2 - \frac{1}{r_0^2}\right)} = \frac{3a_3}{\left(r_0^3 - \frac{1}{r_0^3}\right)}\frac{a_1}{\left(r_0 - \frac{1}{r_0}\right)} \quad \text{and} \tag{2.44}$$

$$\frac{a_1^2}{\left(r_0 - \frac{1}{r_0}\right)^2} + \frac{9a_3^2}{\left(r_0^3 - \frac{1}{r_0^3}\right)^2} = 2\frac{4b_2^2}{\left(r_0^2 - \frac{1}{r_0^2}\right)^2}. \tag{2.45}$$

In particular, for any given positive real $c \neq 1$, constants a_1, a_3 and b_2 as $a_1 = \frac{1}{2}\left(c - \frac{1}{c}\right)$, $a_3 = \frac{1}{6}\left(c^3 - \frac{1}{c^3}\right)$ and $b_2 = \frac{1}{4}\left(c^2 - \frac{1}{c^2}\right)$ satisfy the equations (2.44) and (2.45) with $r_0 = c$. Also for any positive $c \neq 1$, the curve

$$\tilde{\gamma}(\theta) = \left(\frac{1}{2}\left(c - \frac{1}{c}\right)e^{-i\theta} + \frac{1}{6}\left(c^3 - \frac{1}{c^3}\right)e^{3i\theta}, \frac{1}{2}\left(c^2 - \frac{1}{c^2}\right)\cos 2\theta\right)$$

is spacelike. Therefore, there is a generalised maximal surface as in Theorem 2.4.1 containing the curve $\tilde{\gamma}$ as above with special singularity at $|z| = 1$. The generalised maximal surface is given by

$$h(z) = \frac{1}{6}\left(z^3 - \frac{1}{\bar{z}^3}\right) + \frac{1}{2}\left(\bar{z} - \frac{1}{z}\right); \quad w(z) = \frac{1}{4}\left(z^2 - \frac{1}{\bar{z}^2} - \frac{1}{z^2} + \bar{z}^2\right).$$

2.4.4 Observation and Remark

Observation: For a given spacelike closed curve, $r_0(\neq 1)$ may not exist as in the above theorem unless it satisfies the series conditions and if such an r_0 exists, it need not be unique. For instance, we have seen that for the curve $\tilde{\gamma}(\theta) = (e^{i\theta}, 1)$, there does not exists a generalised maximal surface and $r_0 \neq 1$ such that $F(r_0e^{i\theta}) = \tilde{\gamma}(\theta)$; $F(e^{i\theta}) = (0, 0, 0)$ with singular set atleast $|z| = 1$. But if we do some small perturbations of this curve $\tilde{\gamma}$, i.e., for $\epsilon > 0$, let $F(r_0^{\epsilon}e^{i\theta}) = \tilde{\gamma}_{\epsilon}(\theta) = ((1 - \epsilon)e^{i\theta}, 1)$, compare this with (2.22), then $\frac{c_1}{c_2} = 1 - \epsilon$, and from the equation (2.32) we see that there are two choice of r_0^ϵ for fixed ϵ. Also, we can see that as $\epsilon \to 0, \tilde{\gamma}_\epsilon \to \tilde{\gamma}$ and $r_0^\epsilon \to 1$.

Remark 2.4.2 In the above theorem, fixing the special singularity at $(0,0,0)$ and asking for the existence of a generalised maximal surface is not necessary. We may ask for any point $(x_1, x_2, x_3) \in \mathbb{L}^3$ as the special singularity corresponding to $|z| = 1$. But then accordingly the expression of h and w as in (2.36) and (2.37) will change and the new series conditions (for e.g. (2.38) and (2.39)) can be found by posing condition that new (h_ρ, w_ρ) is a null vector along $|z| = 1$. We believe it is not the statement but the proof of the theorem that gives a handy way to check existence of the generalised maximal surface for a given spacelike closed curve.

CHAPTER 3

Various representations of maximal surfaces

3.1 Introduction

In this chapter, we obtain different representation formulas for maximal surfaces. First we rederive the Weierstrass-Enneper representation for maximal graphs (in neighbouhoods of points where the Gauss map for is one-one). For this we use the method of Barbishov and Chernikov [6] using hodographic coordinates which they used to find the solutions of Born-Infeld equation (to be discussed in the next chapter). We write Weierstrass-Enneper representation for maximal surfaces using hodographic coordinates. This method was used earlier by Dey [13] to rederive W-E representation for minimal surfaces away from umbilical points. Our method is analogous to this. Next we give an integral free representation formula for maximal surfaces, analogous to [41]. In the last section we briefly describe a method to construct a one parameter family of isometric maximal surfaces.

3.1.1 Classical Weierstrass-Enneper representation of Maximal surfaces

The complex representation formula which expresses a given maximal surface in $\mathbb{L}^3$ in terms of integrals involving a holomorphic function f and a meromorphic function g is known as Weierstrass-Enneper representation. Following is the

Weierstrass-Enneper representation [34]:

$$\Psi(\tau) = \Re \int (f(1+g^2), if(1-g^2), -2fg)d\tau, \tau \in D, D \subseteq \mathbb{C}$$

f is a holomorphic function on D, g is a meromorphic function on D, fg^2 is holomorphic on D and $|g(\tau)| \neq 1$ for $\tau \in D$.

3.1.2 Gauss map, Weierstrass-Enneper representation and Metric

For a spacelike surface in $\mathbb{L}^3$, the Gauss map G is defined as a map which assigns to a point of the surface S, the unit normal vector at that point. Therefore one can regard $G : S \longrightarrow \mathbb{H}^2$, $\mathbb{H}^2$ is a spacelike surface which has constant negative curvature -1 with respect to the induced metric. We can define a stereographic map σ for $\mathbb{H}^2$ as

$$\sigma : \mathbb{C} \setminus \{|\tau| = 1\} \longrightarrow \mathbb{H}^2 \text{ by}$$

$$\sigma(\tau) = \left(\frac{-2\Re\tau}{|\tau|^2 - 1}, \frac{-2\Im\tau}{|\tau|^2 - 1}, \frac{|\tau|^2 + 1}{|\tau|^2 - 1} \right) \quad \text{and} \quad \sigma(\infty) = (0, 0, 1).$$

Since any maximal surface can be given isothermal coordinates one can think of G as a map $G : D \subset \mathbb{C} \longrightarrow \mathbb{H}^2$, then the Gauss map G is given by $G(\tau) = \sigma(g(\tau)))$. Suppose that the Gauss map for our maximal surface is one-one, then from above expression for G we deduce that g is one-one and g^{-1} is holomorphic. Now by setting $\zeta = g$ as new variable, we define $fd\tau := M(\zeta)d\zeta$. Hence in this new variable ζ we can rewrite Weierstrass-Enepper representation for a maximal surface using just one meromorphic function $M(\zeta)$, [35].

$$\Psi(\zeta) = \Re \int (M(\zeta)(1+\zeta^2), iM(\zeta)(1-\zeta^2), -2M(\zeta)\zeta d\zeta. \tag{3.1}$$

The induced metric can be given in terms of the meromorphic function M as [34]

$$ds^2 = \left(\frac{|M(\zeta)|(1-|\zeta|^2)}{2} \right)^2 |d\zeta|^2. \tag{3.2}$$

3.2 Weierstrass-Enneper Representation for maximal graphs using hodograph transform

3.2.1 Maximal graph

A maximal surface in Lorentz-Minkowski space $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ is a spacelike surface whose mean curvature is zero everywhere. Any spacelike surface in $\mathbb{L}^3$ can be expressed locally as a graph $(x, y, \varphi(x, y))$ of some smooth function φ which satisfies $\varphi_x^2 + \varphi_y^2 < 1$. Then any graph in $\mathbb{L}^3$ is maximal if φ satisfies the following equation [35]

$$(1 - \varphi_x^2)\varphi_{yy} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_y^2)\varphi_{xx} = 0. \tag{3.3}$$

This equation is known as the maximal surface equation.

Remark 3.2.1 It has been known that the Born-Infeld equation

$$(1 + \varphi_x^2)\varphi_{yy} - 2\varphi_x\varphi_y\varphi_{xy} - (1 - \varphi_y^2)\varphi_{xx} = 0. \tag{3.4}$$

is related to the minimal surface equation (see [13])

$$(1 + \varphi_x^2)\varphi_{yy} - 2\varphi_x\varphi_y\varphi_{xy} + (1 + \varphi_y^2)\varphi_{xx} = 0. \tag{3.5}$$

via a Wick rotation in second variable 'y', i.e., if we replace y by iy in (3.4) we get (3.5).

Remark 3.2.2 We observe that if instead of second variable, if we make a wick rotation in first variable 'x', i.e., replacing x by ix in (3.4) we get the maximal surface equation (3.3) and vice-versa. We use this fact to obtain Weierstras-Enneper representation for maximal surfaces.

3.2.2 Rederiving Weierstrass-Enneper representation from Maximal surface equation

We begin with the complex coordinates

$$\tilde{\xi} = i(x - iy) = i\bar{z} \quad , \quad \tilde{\eta} = i(x + iy) = iz \quad , \quad \varphi_{i\bar{z}} = \tilde{u} = \frac{1}{i}\varphi_{\bar{z}} \quad , \quad \varphi_{iz} = \tilde{v} = \frac{1}{i}\varphi_z \tag{3.6}$$

$$\xi = x - iy = \bar{z} \quad , \quad \eta = x + iy = z \quad , \quad \varphi_{\bar{z}} = u \quad \text{and} \quad \varphi_z = v$$

The partial differentials in this new coordinates $(\tilde{\xi}, \tilde{\eta})$ will be related to partial differentials in the old coordinates (x, y) by the following relations

$\varphi_x = i(\tilde{u}+\tilde{v})$, $\varphi_y = \tilde{u}-\tilde{v}$, $\varphi_{xx} = -(\tilde{u}_{\tilde{\xi}}+2\tilde{v}_{\tilde{\xi}}+\tilde{v}_{\tilde{\eta}})$, $\varphi_{xy} = i(\tilde{u}_{\tilde{\xi}}-\tilde{v}_{\tilde{\eta}})$ and $\varphi_{yy} = (\tilde{u}_{\tilde{\xi}} - 2\tilde{v}_{\tilde{\xi}} + \tilde{v}_{\tilde{\eta}})$.

These identities will reduce maximal surface equation (3.3) to

$$\tilde{v}^2\tilde{u}_{\tilde{\xi}} - (1 + 2\tilde{u}\tilde{v})\tilde{u}_{\tilde{\eta}} + \tilde{u}^2\tilde{v}_{\tilde{\eta}} = 0 \quad \text{and} \quad \tilde{u}_{\tilde{\eta}} = \tilde{v}_{\tilde{\xi}}. \tag{3.7}$$

Now we can interchange the role of independent and dependent variables. i.e.

$$(\tilde{u}, \tilde{v}) \leftrightarrow (\tilde{\xi}, \tilde{\eta}).$$

We could do this since we only consider those maximal graphs whose Gauss map is one-one, and hence the Gaussian curvature $K \neq 0$. Also $1 - \varphi_x^2 - \varphi_y^2 \neq 0$ as $\varphi_x^2 + \varphi_y^2 < 1$. Therefore,

$$J = \tilde{u}_{\tilde{\xi}}\tilde{v}_{\tilde{\eta}} - \tilde{u}_{\tilde{\eta}}\tilde{v}_{\tilde{\xi}} = \frac{1}{4}(\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}) = \frac{K(1 - \varphi_x^2 - \varphi_y^2)^2}{4} \neq 0.$$

Then

$$\tilde{v}_{\tilde{\eta}} = J\tilde{\xi}_{\tilde{u}} \quad , \quad \tilde{v}_{\tilde{\xi}} = -J\tilde{\eta}_{\tilde{u}} \quad , \quad \tilde{u}_{\tilde{\eta}} = -J\tilde{\xi}_{\tilde{v}} \quad , \quad \tilde{u}_{\tilde{\xi}} = J\tilde{\eta}_{\tilde{v}}$$

will reduce equations (3.7)

$$\tilde{v}^2\tilde{\eta}_{\tilde{v}} + (1 + 2\tilde{u}\tilde{v})\tilde{\xi}_{\tilde{v}} + \tilde{u}^2\tilde{\xi}_{\tilde{u}} = 0 \quad \text{and} \quad \tilde{\xi}_{\tilde{v}} = \tilde{\eta}_{\tilde{u}}. \tag{3.8}$$

Now if we use the relations (3.6) in (3.8), we get

$$z_u - \bar{z}_v = 0 \quad \text{and} \quad v^2 z_v - (1 - 2uv)z_u + u^2 z_v = 0. \tag{3.9}$$

or,

$$\eta_u - \xi_v = 0 \quad \text{and} \quad v^2\eta_v - (1-2uv)\eta_u + u^2\eta_v = 0. \tag{3.10}$$

Differentiating equation (3.10) w.r.t u, we get a second order quasilinear pde

$$v^2\xi_{vv} - (1-2uv)\xi_{uv} + u^2\xi_{uu} = -2u\xi_u - 2v\xi_v.$$

Now assume that the solutions which we want to find are in hyperbolic regime, we will find the characteristics (see appendix **A.7**) for the above equation, they are integral curves of the following differential form

$$u^2dv^2 + (1-2uv)dudv + v^2du^2 = 0.$$

Characteristic curves are

$$\frac{1-\sqrt{1-4uv}}{2u} = c \quad , \quad \frac{1-\sqrt{1-4uv}}{2v} = c'.$$

Now if we introduce

$$\zeta = \frac{1-\sqrt{1-4uv}}{2v} \quad , \quad \bar{\zeta} = \frac{1-\sqrt{1-4uv}}{2u}$$

as new variables to replace u and v. From this we could see that we have

$$u = \frac{\zeta}{1+\zeta\bar{\zeta}} \quad , \quad v = \frac{\bar{\zeta}}{1+\zeta\bar{\zeta}}. \tag{3.11}$$

Thus we have the following lemma.

Lemma 3.2.3 *Equations (3.9) is equivalent to a single equation*

$$\zeta^2\bar{z}_\zeta - z_\zeta = 0.$$

Proof. Since $u = \dfrac{\zeta}{1+\zeta\bar{\zeta}}$, $v = \dfrac{\bar{\zeta}}{1+\zeta\bar{\zeta}}$ we get

$$z_\zeta = \frac{z_u - \bar{\zeta}^2 z_v}{(1+\zeta\bar{\zeta})^2} \quad , \quad \bar{z}_\zeta = \frac{\bar{z}_u - \bar{\zeta}^2 \bar{z}_v}{(1+\zeta\bar{\zeta})^2}$$

Using above values of z_ζ and $\bar{z}_\zeta$, we get

$$\zeta^2\bar{z}_\zeta - z_\zeta = v^2 z_v - (1-2uv)z_u + u^2 z_v,$$

this shows (3.9) is equivalent to

$$\zeta^2\bar{z}_\zeta - z_\zeta = 0.$$

Theorem 3.2.4 *Any maximal surface whose Gauss map is one-one will have a local Weierstrass-Enneper type representation of the following form*

$$x(\zeta) = x_0 + \Re(\int^{\zeta} M(\omega)(1+\omega^2)d\omega)$$

$$y(\zeta) = y_0 + \Re(\int^{\zeta} iM(\omega)(1-\omega^2)d\omega)$$

$$\varphi(\zeta) = \varphi_0 + \Re(\int^{\zeta} 2M(\omega)\omega d\omega)$$

where $M(\zeta)$ is, a meromorphic function, known as the Weierstrass data.

Remark 3.2.5 Observe that $\varphi \to -\varphi$ is a symmetry of the equation (3.3) if one keeps x and y invariant. Thus $\varphi(\zeta) = \tilde{\varphi}_0 + \Re(\int^{\zeta} -2M(\omega)\omega d\omega)$ is also an acceptable representation.

Proof. From above lemma, we have $\zeta^2 \bar{z}_{\zeta} - z_{\zeta} = 0$. This will imply

$$\bar{z}_{\zeta\bar{\zeta}} = 0 \Rightarrow \bar{z} = \bar{z}_0 + F(\zeta) + H(\bar{\zeta}).$$

Then

$$z = z_0 + \overline{F(\zeta)} + \overline{H(\bar{\zeta})}. \tag{3.12}$$

Now lemma also implies that

$$\overline{H(\bar{\zeta})} = \int^{\zeta} \omega^2 F'(\omega)d\omega.$$

Therefore,

$$\bar{z} = \bar{z}_0 + F(\zeta) + \int^{\zeta} \bar{\omega}^2 \overline{F'(\omega)} d\bar{\omega}.$$

Next, we have

$$\varphi_{\zeta} = \varphi_{i\bar{z}}(i\bar{z})_{\zeta} + \varphi_{iz}(iz)_{\zeta} = (u + v\zeta^2)F'(\zeta) = \zeta F'(\zeta).$$

Similarly,

$$\varphi_{\bar{\zeta}} = \bar{\zeta}\frac{d}{d\bar{\zeta}}(\overline{F(\zeta)}).$$

Hence

$$\varphi = \varphi_0 + \int^{\zeta} \omega F'(\omega)d\omega + \int^{\bar{\zeta}} \bar{\omega}\frac{d}{d\bar{\omega}}(\overline{F(\omega)})d\bar{\omega}. \tag{3.13}$$

Let $F'(\omega) = M(\omega)$.

By expanding z into its real and imaginary parts, also using $z + \bar{z} = 2\Re(z)$ we get

$$x(\zeta) = x_0 + \Re(\int^{\zeta} M(\omega)(1+\omega^2)d\omega),$$

$$y(\zeta) = y_0 + \Re(\int^{\zeta} iM(\omega)(1-\omega^2)d\omega),$$

$$\varphi(\zeta) = \varphi_0 + \Re(\int^{\zeta} 2M(\omega)\omega d\omega).$$

This completes a proof of the theorem.

3.2.3 Weierstrass-Enneper representation in hodographic coordinates

If $F'(\zeta) \neq 0$. Then let $H(\bar{\zeta}) = \overline{F(\zeta)} = \bar{\rho}$ and $F(\zeta) = \rho$, so that we can regard ρ and $\bar{\rho}$ as new variables, at least locally. Now in this new coordinate system ρ, the Weiersrtass-Enneper representation attains the following form

$$x(\rho) = \frac{\rho + \bar{\rho}}{2} + \frac{1}{2}\left(\int (F^{-1}(\rho))^2 d\rho + \int (H^{-1}(\bar{\rho}))^2 d\bar{\rho}\right), \tag{3.14}$$

$$y(\rho) = \frac{\bar{\rho} - \rho}{2i} + \frac{1}{2i}\left(\int (F^{-1}(\rho))^2 d\rho - \int (H^{-1}(\bar{\rho}))^2 d\bar{\rho}\right), \tag{3.15}$$

$$\varphi(\rho) = \int F^{-1}(\rho)d\rho + \int H^{-1}(\bar{\rho})d\bar{\rho}. \tag{3.16}$$

We also have

$$\varphi_\rho = F^{-1}(\rho) = \zeta \quad \text{and} \quad \varphi_{\bar{\rho}} = H^{-1}(\bar{\rho}) = \bar{\zeta}. \tag{3.17}$$

Now in terms of φ_ρ and $\varphi_{\bar{\rho}}$ equations (3.14),(3.15) and (3.16) reduces to

$$x(\rho) = \frac{\rho + \bar{\rho}}{2} + \frac{1}{2}\left(\int (\varphi_\rho)^2 d\rho + \int (\varphi_{\bar{\rho}})^2 d\bar{\rho}\right), \tag{3.18}$$

$$y(\rho) = \frac{\bar{\rho} - \rho}{2i} + \frac{1}{2i}\left(\int (\varphi_\rho)^2 d\rho - \int (\varphi_{\bar{\rho}})^2 d\bar{\rho}\right), \tag{3.19}$$

$$\varphi(\rho) = \varphi(\rho) + \varphi(\bar{\rho}). \tag{3.20}$$

Now if we write $\rho = \rho_1 + i\rho_2$. Then one can see that ρ_1 and ρ_2 are isothermal, i.e.,

$$|X_{\rho_1}|_L = |X_{\rho_2}|_L \text{ and } \langle X_{\rho_1}, X_{\rho_2}\rangle_L = 0$$

where $X = (x, y, \varphi)$ and $\langle , \rangle_L$ is the Lorentzian inner product.

Since this coordinate system ρ is related to the coordinate system ζ by a holomorphic map F. We conclude that the coordinate system $\zeta = \zeta_1 + i\zeta_2$ is also isothermal. Also the expression for unit normal to the maximal surface depends only on φ_ρ, as we have

$$N = \frac{X_{\rho_1} \times_L X_{\rho_2}}{|X_{\rho_1} \times_L X_{\rho_2}|_L} = \left(\frac{2\Re\varphi_\rho}{1-|\varphi_\rho|^2}, \frac{2\Im\varphi_\rho}{1-|\varphi_\rho|^2}, -\frac{1+|\varphi_\rho|^2}{1-|\varphi_\rho|^2}\right).$$

3.2.4 Integral free form of Weierstrass-Enneper representation

If we set $M(\omega) = \psi'''(\omega)$ where $\psi(\omega)$ is some function and $\omega = u + iv$. Then by applying integration by parts in the expressions for x, y, and z in Theorem 3.2.4, we obtain

$$x = \Re\{(1+\omega^2)\psi''(\omega) - 2\omega\psi'(\omega) + 2\psi(\omega)\},$$

$$y = \Re\{i(1-\omega^2)\psi''(\omega) + 2i\omega\psi'(\omega) - 2i\psi(\omega)\},$$

$$z = \Re\{-2\omega\psi''(\omega) + 2\psi'(\omega)\}.$$

If we define

$$g_1(\omega) := (1+\omega^2)\psi''(\omega) - 2\omega\psi'(\omega) + 2\psi(\omega),$$

$$g_2(\omega) := i(1-\omega^2)\psi''(\omega) + 2i\omega\psi'(\omega) - 2i\psi(\omega),$$

$$g_3(\omega) := -2\omega\psi''(\omega) + 2\psi'(\omega).$$

Then

$$\psi(\omega) = \frac{1}{4}(1+\omega^2)g_1(\omega) - \frac{i}{4}(\omega^2-1)g_2(\omega) + \frac{1}{2}\omega g_3(\omega).$$

Example 3.2.1 (Enneper's surface of first kind) If we take $\psi(\omega) = \frac{\omega^3}{6}$, then using the integral free form of Weierstrass-Enneper representation we get

$$x(\omega) = \Re\left(\omega + \frac{\omega^3}{3}\right) = u + \frac{u^3}{3} - uv^2,$$

$$y(\omega) = -\Im\left(\omega - \frac{\omega^3}{3}\right) = -v - \frac{v^3}{3} + u^2 v,$$

$$z(\omega) = \Re(-\omega^2) = v^2 - u^2.$$

3.3 Examples

Lorentzian Catenoid [34]: Consider

$$\varphi(x, y) = \sinh^{-1}(\sqrt{x^2 + y^2}) = \sinh^{-1}(\sqrt{z\bar{z}}) \tag{3.21}$$

which is a maximal graph in the Lorentz-Minkowski space whose Gauss map is one-one. Then

$$\varphi_z = \frac{\bar{z}}{2|z|\sqrt{|z|^2+1}} \quad , \quad \varphi_{\bar{z}} = \frac{z}{2|z|\sqrt{|z|^2+1}}.$$

Recall (3.6) and (3.11), we get

$$\frac{u}{v} = \frac{z}{\bar{z}} = \frac{\zeta}{\bar{\zeta}}. \tag{3.22}$$

Next we have

$$\frac{\zeta}{1+\zeta\bar{\zeta}} = \frac{z}{2|z|\sqrt{|z|^2+1}}, \tag{3.23}$$

now use this equation to obtain z in terms of ζ and then from this we get to know the single holomorphic function $F(\zeta)$. Squaring both the sides of equation (3.23) and using the relations (3.22) in between, we get

$$z^2 = \left(\frac{1}{2}\left(\frac{1}{\bar{\zeta}} - \zeta\right)\right)^2$$

taking positive square root

$$z = \frac{1}{2}\left(\frac{1}{\bar{\zeta}} - \zeta\right).$$

Comparing this with (3.12), we obtain $\overline{F(\zeta)} = \frac{1}{2\bar{\zeta}}$, so we have $F(\zeta) = \frac{1}{2\zeta}$. Therefore we can compute the Weierstrass data as $M(\zeta) = F'(\zeta) = \frac{-1}{2\zeta^2}$. Now $\varphi(\zeta, \bar{\zeta})$ can be computed by the formula (3.13). Infact

$$\varphi(\zeta, \bar{\zeta}) = -\frac{1}{2}\log(\zeta\bar{\zeta}) \tag{3.24}$$

$$x = -\frac{1}{2}Re\left(\zeta - \frac{1}{\zeta}\right); \quad y = -\frac{1}{2}Im\left(\zeta + \frac{1}{\zeta}\right).$$

This is Weierstrass-Enneper representation in terms of the coordinates $(\zeta, \overline{\zeta})$. Next we write (x, y, φ) in terms of hodographic coordinates $(\rho, \overline{\rho})$ (see (3.17)).

$$\varphi(\rho, \bar{\rho}) = \frac{1}{2}(\log(2\rho) + \log(2\bar{\rho})) \tag{3.25}$$

$$x = -\frac{1}{2}Re\left(\frac{1}{2\rho} - 2\rho\right); \quad y = -\frac{1}{2}Im\left(\frac{1}{2\rho} + 2\rho\right).$$

Lorentzian Helicoid [34][1]: Consider

$$\varphi(x, y) = \frac{\pi}{2} + \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \tan^{-1}\left(\frac{1}{i}\left(\frac{z - \bar{z}}{z + \bar{z}}\right)\right). \tag{3.26}$$

Then

$$u = \varphi_{\bar{z}} = \frac{i}{2\bar{z}} \quad \text{and} \quad v = \varphi_z = \frac{-i}{2z}. \tag{3.27}$$

Again we have

$$\frac{u}{v} = \frac{-z}{\bar{z}} = \frac{\zeta}{\bar{\zeta}}. \tag{3.28}$$

[1]Plane and Helicoid are the only maximal surfaces in Lorentz-Minkowski space which are also minimal surfaces in Euclidean space.

Using relations (3.11), (3.12) and (3.27), we found

$$\overline{F(\zeta)} = \frac{-i}{2\bar{\zeta}} \quad \text{and} \quad F(\zeta) = \frac{i}{2\zeta} \tag{3.29}$$

and hence Weierstrass data $M(\zeta) = F'(\zeta) = \frac{-i}{2\zeta^2}$. Therefore

$$\varphi(\zeta, \bar{\zeta}) = -\frac{i}{2} \log\left(\frac{\zeta}{\bar{\zeta}}\right) \tag{3.30}$$

$$x = \frac{1}{2} Im\left(\zeta - \frac{1}{\zeta}\right); \quad y = -\frac{1}{2} Re\left(\zeta + \frac{1}{\zeta}\right).$$

Now we write (x, y, ϕ) in terms of hodographic coordinates ρ and $\bar{\rho}$.

$$x = \frac{1}{2} Im\left(\frac{i}{2\rho} - \frac{2\rho}{i}\right); \; y = -\frac{1}{2} Re\left(\frac{i}{2\rho} + \frac{2\rho}{i}\right)$$

$$\phi = \frac{-i}{2} \log\left(\frac{-\bar{\rho}}{\rho}\right).$$

3.4 One parameter family of isometric maximal surfaces

In the previous section of examples we have also computed the Weierstrass data $M_c(\zeta) = \dfrac{-1}{2\zeta^2}$ for Lorentzian catenoid and $M_h(\zeta) = \dfrac{-i}{2\zeta^2}$ for Lorentzian helicoid. Now if we define $\forall\ \theta$, such that $0 \leq \theta \leq \frac{\pi}{2}$

$$M_\theta(\zeta) = e^{i\theta} M(\zeta) \quad \text{,where} \quad M(\zeta) = \frac{-1}{2\zeta^2} \tag{3.31}$$

then $M_\theta(\zeta)$ becomes the Weierstrass data for a maximal surface which can be obtained using Theorem 3.2.4. In particular, when $\theta = 0$, $M_0(\zeta) = M_c(\zeta)$, we get back Lorentzian catenoid and when $\theta = \frac{\pi}{2}$, $M_{\frac{\pi}{2}}(\zeta) = M_h(\zeta)$, we get the Lorentzian helicoid. Now we recall the expression for the metric (3.2), here we see that the metric depends only on the modulus of Weierstrass data $M(\zeta)$, so if we replace $M(\zeta)$ by $e^{i\theta}M(\zeta)$ in the expression of the metric, the form of the metric remains unchanged because $|M(\zeta)| = |e^{i\theta}M(\zeta)|$. This tells us that by varying θ, we get a one parameter family of isometric maximal surfaces in

Lorentz-Minkowski space.

In general, if one starts with a Weierstrass data for a given maximal surface, one can construct a one parameter family of isometric maximal surfaces, by following the procedure described in previous paragraph, starting from the given surface.

CHAPTER 4

A family of solitons

In the year 1954, the physicist G.C. Wick in his paper [51], introduced a transformation which involves replacing real time variable t by the imaginary time variable it. This process of changing a real parameter to an imaginary parameter is what is known as *Wick rotation*. In past severals authors have used this technique of Wick rotation to prove certain kinds of dualities among minimal surfaces in $\mathbb{E}^3$, maximal surfaces and timelike minimal surfaces in $\mathbb{L}^3$ [26, 29, 33, 3, 16]. For instance, in [33], Kim and his collaborators had shown that spacelike maximal surfaces and timelike minimal surfaces in $\mathbb{L}^3$ can be transformed to each other, when one considers their parametric representaions, by taking a Wick rotation in one of the parametrising coordinates. In this Chapter, we first prove a proposition which states that a solution to Born-Infeld equation can also be thought of as a spacelike minimal graph or timelike minimal graph over a domain of timelike plane or a combination of both away from singular points in Lorentz-Minkowski space $\mathbb{L}^3$. In the last Chapter we obtained Weierstrass-Enneper representation for maximal surfaces (in neighbourhood of points assuming that the Gauss map is one-one) in a complex isothermal coordinate system $(\zeta, \bar{\zeta})$ using maximal surface equation. Here we make an observation that the maximal surface equation is related to Born-Infeld equation via a wick rotation in first variable x. We use this observation to obtain some solutions of the Born-Infeld equation from already known solutions to the maximal surface equation. We give a method to construct a one parameter family of complex solitons from a given one parameter family of maximal surfaces, for this we use the Weierstrass-Enneper representation of given *conjugate* maximal surfaces. This construction is analogous to [16].

4.1 Born-Infeld Solitons

4.1.1 Introduction

Any smooth function $\varphi(x,t)$ which is a solution to Born-Infeld equation (see [50])

$$(1+\varphi_x^2)\varphi_{tt} - 2\varphi_x\varphi_t\varphi_{xt} + (\varphi_t^2 - 1)\varphi_{xx} = 0. \tag{4.1}$$

is known as a Born-Infeld soliton.
The general solution may be taken as

$$x - t = F(r) - \int s^2 G'(s)ds \quad , \quad x + t = G(s) - \int r^2 F'(r)dr$$

where $F(r)$, $G(s)$ are arbitrary functions. The corresponding expression for φ is

$$\varphi = \int rF'(r)dr + \int sG'(s)ds.$$

4.1.2 Born-Infeld soliton as a geometric object

Consider the Lorentz-Minkowski space $\mathbb{L}^3$, assuming that the cartesian coordinates are (x,y,z), then the Lorentzian metric is denoted by $dx^2 + dy^2 - dz^2$ or $\langle , \rangle_L$. Then a graph in $\mathbb{L}^3$ over a domain of the timelike plane $\{x = 0\}$ has the form

$$X(y,z) = (\varphi(y,z), y, z), \tag{4.2}$$

where $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a smooth function [37]. For X we can compute the coefficients of first fundamental form as

$$E = \langle X_y, X_y\rangle_L \quad , \quad F = \langle X_y, X_z\rangle_L \quad \text{and} \quad G = \langle X_z, X_z\rangle_L. \tag{4.3}$$

Similarly, we can compute the coefficients of second fundamental form as

$$e = \langle X_{yy}, N\rangle_L \quad , \quad f = \langle X_{yz}, N\rangle_L \quad \text{and} \quad g = \langle X_{zz}, N\rangle_L.$$

The mean curvature H for a graph in $\mathbb{L}^3$ is given by

$$H = \frac{\epsilon}{2}\left(\frac{eG - 2fF + gE}{EG - F^2}\right),$$

where $\epsilon = 1$ if the graph is timelike, $\epsilon = -1$ if the graph is spacelike.

A graph in $\mathbb{L}^3$ is said to be *minimal* if its mean curvature vanishes everywhere (i.e. $H \equiv 0$).

Proposition 4.1.1 *The solutions of (4.1), i.e., Born-Infeld solitons can be represented as a spacelike minimal graph or timelike minimal graph over a domain in timelike plane or a combination of both away from singular points (points where tangent plane degenerates), i.e., points where the determinant of the coefficients of first fundamental form vanishes.*

Proof. Coefficients of first fundamental form for (4.2) are

$$E = \varphi_y^2 + 1 \quad , \quad G = \varphi_z^2 - 1 \quad , \quad F = \varphi_y \varphi_z$$

and determinant of the coefficients of the first fundamental form is $EG - F^2 = -\varphi_y^2 + \varphi_z^2 - 1$.

In general, we can have $-\varphi_y^2 + \varphi_z^2 - 1 = 0$ (tangent plane degenerates). But when $-\varphi_y^2 + \varphi_z^2 - 1 \neq 0$, one can define the normal vector N and it is given by

$$N = \left(\frac{1}{\sqrt{|1 + \varphi_y^2 - \varphi_z^2|}}, \frac{-\varphi_y}{\sqrt{|1 + \varphi_y^2 - \varphi_z^2|}}, \frac{\varphi_z}{\sqrt{|1 + \varphi_y^2 - \varphi_z^2|}}\right).$$

Therefore

$$\langle N, N\rangle_L = \frac{1 + \varphi_y^2 - \varphi_z^2}{|1 + \varphi_y^2 - \varphi_z^2|}.$$

If $1 + \varphi_y^2 - \varphi_z^2 > 0$, we have $\langle N, N\rangle_L = 1$, then the graph is timelike. On the other hand if $\langle N, N\rangle_L = -1$, i.e. $1 + \varphi_y^2 - \varphi_z^2 < 0$, then the graph is spacelike.

Now we can easily compute coefficients of second fundamental form, they are given by

$$e = \frac{\varphi_{yy}}{\sqrt{|1 + \varphi_y^2 - \varphi_z^2|}} \quad , \quad g = \frac{\varphi_{zz}}{\sqrt{|1 + \varphi_y^2 - \varphi_z^2|}} \quad , \quad f = \frac{\varphi_{yz}}{\sqrt{|1 + \varphi_y^2 - \varphi_z^2|}},$$

here we see $EG - F^2 = -1 - \varphi_y^2 + \varphi_z^2$, and if $EG - F^2 > 0$, i.e. $1 + \varphi_y^2 - \varphi_z^2 < 0$ the graph is spacelike and if $EG - F^2 < 0$, i.e., $1 + \varphi_y^2 - \varphi_z^2 > 0$, then the graph is timelike. In any case we know that the mean curvature for a surface in $\mathbb{L}^3$ is given by (see page no. 40 of [36]).

$$H = \frac{\epsilon}{2}\left(\frac{eG - 2fF + gE}{EG - F^2}\right),$$

where $\epsilon = 1$ if the surface is timelike, $\epsilon = -1$ if the surface is spacelike. So for the spacelike graph over a timelike plane, we have

$$H = -\frac{1}{2}\frac{(1+\varphi_y^2)\varphi_{zz} - 2\varphi_y\varphi_z\varphi_{yz} + (\varphi_z^2 - 1)\varphi_{yy}}{(-1 - \varphi_y^2 + \varphi_z^2)^{\frac{3}{2}}},$$

and for the timelike graph over timelike plane, we have

$$H = -\frac{1}{2}\frac{(1+\varphi_y^2)\varphi_{zz} - 2\varphi_y\varphi_z\varphi_{yz} + (\varphi_z^2 - 1)\varphi_{yy}}{(1 + \varphi_y^2 - \varphi_z^2)^{\frac{3}{2}}}.$$

So if the mean curvature H for the spacelike graph or timelike graph over a timelike plane is zero, we get

$$(1+\varphi_y^2)\varphi_{zz} - 2\varphi_y\varphi_z\varphi_{yz} + (\varphi_z^2 - 1)\varphi_{yy} = 0.$$

By renaming the variables y, z as x, t, we get

$$(1+\varphi_x^2)\varphi_{tt} - 2\varphi_x\varphi_t\varphi_{xt} + (\varphi_t^2 - 1)\varphi_{xx} = 0.$$

Hence we obtain the Born-Infeld equation.

Now we illustrate with an example that a Born-Infeld soliton in general can have some points where the determinant of the coefficients of first fundamental form vanishes, i.e., it has the points where the tangent plane degenerates.

Example 4.1.1 Consider the graph $X(y,z) = (x = \sinh^{-1}(\sqrt{z^2 - y^2}), y, z)$. Then we can easily check that it satisfies the Born-Infeld equation. Also, its tangent planes degenerates precisely at the points $(x, y, z) \in \mathbb{L}^3$ where $x = 0$ and $y = \pm z$. This Born-Infeld soliton can also be obtained from the elliptic catenoid (a maximal surface, see [40]), by wick rotation (a concept which we describe in the next section) and renaming the variables.

4.2 Maximal surface equation and Wick rotation

A graph $(x, t, f(x,t))$ in Lorentz-Minkowski space $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dt^2 - dz^2)$ is maximal if it satisfies

$$(1 - f_x^2)f_{tt} + 2f_x f_t f_{xt} + (1 - f_t^2)f_{xx} = 0, \tag{4.4}$$

for some smooth function $f(x,t)$ satisfying $f_x^2 + f_t^2 < 1$, see [35] ; the last condition on f is just to ensure the spacelike nature of maximal graphs, i.e., the induced metric on graph is Riemannian. This equation is known as maximal surface equation.

Now we would like to obtain some solutions to the Born-Infeld equation (4.1) from some of the already known solutions to the maximal surface equation (4.4). For that first we observe that if one replaces x (which is the first parametrizing variable of the graph $(x, t, f(x,t))$) by ix then maximal surface equation changes to Born-Infeld equation and vice-versa. Suppose if $f(x,t)$, is a solution to the maximal surface equation (4.4), then we obtain a solution to Born-Infeld equation (4.1) by defining $\varphi(x,t) := f(ix,t)$. This process of replacing x by ix is what is known as *Wick rotation*. Now we give some solutions of maximal surface equation (maximal graph) and then we wick rotate it to obtain some solutions of Born-Infeld equation (Born-Infeld soliton). In general, a solution to the Born-Infeld equation obtain this way may be complex.

Wick rotated helicoid of the first kind: Consider helicoid of the first kind (see [40])

$$f(x,t) = \frac{1}{k}\tan^{-1}\left(\frac{t}{x}\right), k \neq 0 \quad \text{and} \quad k \in \mathbb{R}.$$

which is a solution to maximal surface equation. Then the wick rotated helicoid of first kind

$$\varphi(x,t) := f(ix,t) = -\frac{i}{k}\tanh^{-1}\left(\frac{t}{x}\right)$$

is a complex-valued solution to the Born-Infeld equation.

Wick rotated helicoid of the second kind: Next consider helicoid of the second kind (see [40])

$$f(x,t) = x\tanh kt, k \neq 0 \quad \text{and} \quad k \in \mathbb{R}$$

$$\varphi(x,t) := f(ix,t) = ix \tanh kt$$

.

which is again a complex valued solution to the Born-Infeld equation.

Wick rotated Scherk's surface of the first kind: Consider (see [34])

$$f(x,t) = \ln\left(\frac{\cosh t}{\cosh x}\right)$$

$$\varphi(x,t) := f(ix,t) = \ln\left(\frac{\cosh t}{\cos x}\right).$$

Since $\cosh t$ is always positive, this solution is conditionally real-valued, depending on the sign of $\cos x$.

4.3 One parameter family of complex solitons

In this section we define the notion of conjugate maximal graphs, which helps to construct a one parameter family of maximal graphs. Further using this one parameter family of maximal graphs we show the construction of one parameter family of Born-Infeld solitons. In general, this way of constructing one parameter family of solitons give complex solitons.

4.3.1 Conjugate maximal graphs

Definition 4.3.1 We say that two maximal graphs

$$X_1(\tau,\bar{\tau}) = (x_1(\tau,\bar{\tau}), t_1(\tau,\bar{\tau}), f_1(\tau,\bar{\tau})) \ \text{ and } \ X_2(\tau,\bar{\tau}) = (x_2(\tau,\bar{\tau}), t_2(\tau,\bar{\tau}), f_2(\tau,\bar{\tau}))$$

given in isothermal parametrization are conjugate if

$$X := X_1 + iX_2 : \Omega \subseteq \mathbb{C} \to \mathbb{C}^3$$

defines a holomorphic mapping, where $X_j(\tau,\bar{\tau}) : \Omega \subseteq \mathbb{C} \to \mathbb{L}^3, \tau = \tilde{u} + i\tilde{v} \in \Omega$; $j = 1, 2$ and $\tilde{u}, \tilde{v}$ are isothermal parameters.

Note that if the Gauss map of a given maximal graph in $\mathbb{L}^3$ is one-one, then its conjugate maximal graph always exists.

If $X_1(\tau,\bar\tau) = (x_1(\tau,\bar\tau), t_1(\tau,\bar\tau), f_1(\tau,\bar\tau))$ is a maximal surface and $X_2(\tau,\bar\tau) = (x_2(\tau,\bar\tau), t_2(\tau,\bar\tau), f_2(\tau,\bar\tau))$ its conjugate maximal surface, where $\tau = \tilde u + i\tilde v$ is an isothermal coordinate system. Then it can be easliy shown that

$$X_\theta(\tau,\bar\tau) := X_1(\tau,\bar\tau)\cos\theta + X_2(\tau,\bar\tau)\sin\theta$$

also defines a maximal surface for each θ.

4.3.2 Complex solitons

As we have seen in the last section, if $(x, t, f(x,t))$ is a solution to maximal surface equation (4.4), then $(ix, t, \varphi(x,t) := f(ix,t))$ is a solution for Born-Infeld equation (4.1).

Next, if $X_1 = (x_1, t_1, f_1)$ and $X_2 = (x_2, t_2, f_2)$ are conjugate maximal graphs, then we define $X_1^s = (ix_1, t_1, \varphi_1)$, $X_2^s = (ix_2, t_2, \varphi_2)$ as conjugate Born-Infeld Solitons.

Now we digress a little. From the last chapter we know that if

$$X_j(\tau,\bar\tau) = (x_j(\tau,\bar\tau), t_j(\tau,\bar\tau), f_j(\tau,\bar\tau))$$

for $j = 1, 2$ are two maximal graphs in isothermal coordinates, then

$$x_j - it_j = F_j(\tau) + \int \bar\tau^2 \overline{F_j'(\tau)} d\bar\tau, \qquad x_j + it_j = \overline{F_j(\tau)} + \int \tau^2 F_j'(\tau) d\tau,$$

$$f_j = \int \tau F_j'(\tau) d\tau + \int \bar\tau (\overline{F_j(\tau)})' d\bar\tau.$$

where F_j are functions which can be derived from the Weierstrass-Enneper data. Then

$$ix_j + t_j = iF_j(\tau) + i\int \bar\tau^2 \overline{F_j'(\tau)} d\bar\tau, \qquad ix_j - t_j = i\overline{F_j(\tau)} + i\int \tau^2 F_j'(\tau) d\tau,$$

$$f_j = \int \tau F_j'(\tau) d\tau + \int \bar\tau (\overline{F_j(\tau)})' d\bar\tau.$$

Now we make an isothermal change of coordinates i.e. replacing τ by $i\zeta$ and $\bar\tau$ by $-i\bar\zeta$. Then

$$ix_j + t_j = iF_j(i\zeta) - \int \bar{\zeta}^2 d(i\overline{F_j(i\zeta)}) = H_j(\zeta) - \int \bar{\zeta}^2 G_j'(\bar{\zeta})d\bar{\zeta}, \quad (4.5)$$

$$ix_j - t_j = i\overline{F_j(i\zeta)} - \int \zeta^2 d(iF_j(i\zeta)) = G_j(\bar{\zeta}) - \int \zeta^2 H_j'(\zeta)d\zeta, \quad (4.6)$$

$$f_j = \int \zeta d(iF_j(i\zeta)) + \int \bar{\zeta} d(-i\overline{F_j(i\zeta)}) = \int \zeta H_j'(\zeta)d\zeta + \int \bar{\zeta}(-G_j'(\bar{\zeta}))d\bar{\zeta}, \quad (4.7)$$

where $H_j(\zeta) = iF_j(i\zeta)$ and $G_j(\bar{\zeta}) = i\overline{F_j(i\zeta)}$ and they satisfy $\overline{G_j(\bar{\zeta})} = -H_j(\zeta)$.

To come back to solitons, define

$$X_\theta^s(\zeta,\bar{\zeta}) = X_1^s(\zeta,\bar{\zeta})\cos\theta + X_2^s(\zeta,\bar{\zeta})\sin\theta,$$

then

$$X_\theta^s = (ix_1, t_1, \varphi_1)\cos\theta + (ix_2, t_2, \varphi_2)\sin\theta,$$

we let

$$X_\theta^s = (i(x_1\cos\theta + x_2\sin\theta), (t_1\cos\theta + t_2\sin\theta), (\varphi_1\cos\theta + \varphi_2\sin\theta)) = (x_\theta^s, t_\theta^s, \varphi_\theta^s).$$

Now we prove the main theorem of this Chapter:

Theorem 4.3.2 *Let $X_1 = (x_1, t_1, f_1)$ and $X_2 = (x_2, t_2, f_2)$ be two conjugate maximal graphs and let $X_\theta = (x_1\cos\theta + x_2\sin\theta, t_1\cos\theta + t_2\sin\theta, f_1\cos\theta + f_2\sin\theta)) = (x_\theta, t_\theta, f_\theta)$ denotes the one parameter family of maximal graphs corresponding to X_1 and X_2. Then $X_\theta^s = (i(x_1\cos\theta + x_2\sin\theta), (t_1\cos\theta + t_2\sin\theta), (\varphi_1\cos\theta+\varphi_2\sin\theta)) = (x_\theta^s, t_\theta^s, \varphi_\theta^s)$, where $\varphi_j(x_j, t_j) := f_j(ix_j, t_j)$, $j = 1, 2$ give us a one parameter family of complex solitons i.e. for each θ we have a complex solution to the Born-Infeld equation (4.1).*

Proof. To show this, we show that $X_\theta^s = (x_\theta^s, t_\theta^s, \varphi_\theta^s)$ will give us the general solution of the Born-Infeld equation as discussed in the starting of this Chapter.

One can find a discussion on this in [50]: Consider

$$\begin{aligned} x^s_\theta - t^s_\theta =& (ix_1 - t_1)\cos\theta + (ix_2 - t_2)\sin\theta \\ =& (G_1(\bar\zeta)\cos\theta + G_2(\bar\zeta)\sin\theta) - \int (\zeta^2 H_1'(\zeta)\cos\theta + \zeta^2 H_2'(\zeta)\sin\theta) d\zeta, \end{aligned}$$

where last line is obtained using (4.6).
If we define $G^s_\theta(\bar\zeta) := G_1(\bar\zeta)\cos\theta + G_2(\bar\zeta)\sin\theta$ and $H^s_\theta(\zeta) := H_1(\zeta)\cos\theta + H_2(\zeta)\sin\theta$, then $\overline{G^s_\theta(\bar\zeta)} = -H^s_\theta(\zeta)$. Therefore

$$x^s_\theta - t^s_\theta = G^s_\theta(\bar\zeta) - \int \zeta^2 {H^s_\theta}'(\zeta) d\zeta, \tag{4.8}$$

in a similar manner, we can show

$$x^s_\theta + t^s_\theta = H^s_\theta(\zeta) - \int \bar\zeta^2 {G^s_\theta}'(\bar\zeta) d\bar\zeta \tag{4.9}$$

and

$$\varphi^s_\theta = \int \zeta {H^s_\theta}'(\zeta) d\zeta + \int \bar\zeta(-{G^s_\theta}'(\bar\zeta)) d\bar\zeta \tag{4.10}$$

Now the expressions (4.8), (4.9) and (4.10) describes the general solution for Born-Infeld equation, see [50], where $G^s_\theta(\bar\zeta)$ and $H^s_\theta(\zeta)$ are such that they satisfy $\overline{G^s_\theta(\bar\zeta)} = -H^s_\theta(\zeta)$.

4.4 Example

In this section we start with a pair of conjugate maximal graphs and show the construction of corresponding one parameter family of Born-Infeld solitons. This also explain the method of the proof of the Theorem 4.3.2.

Consider the Lorentzian helicoid

$$f_1(x_1, t_1) = \frac{\pi}{2} + \tan^{-1}\left(\frac{t_1}{x_1}\right).$$

which is a maximal graph in the Lorentz-Minkowski space whose Gauss map is one-one. Then the W-E representation in terms of the coordinates $(\tau, \overline{\tau})$ is given by $X_1(\tau,\bar\tau) = (x_1(\tau,\bar\tau), t_1(\tau,\bar\tau), f_1(\tau,\bar\tau))$ where (for details see the last chapter)

$$x_1(\tau,\bar\tau) = \frac{1}{2} Im\left(\tau - \frac{1}{\tau}\right), \qquad t_1(\tau,\bar\tau) = -\frac{1}{2} Re\left(\tau + \frac{1}{\tau}\right),$$

$$f_1(\tau,\bar\tau) = -\frac{i}{2}\ln\left(\frac{\tau}{\bar\tau}\right).$$

and similarly for Lorentzian catenoid

$$f_2(x_2,t_2) = \sinh^{-1}(\sqrt{x_2^2 + t_2^2}),$$

which is a conjugate to Lorentzian helicoid. we have (for details see the last chapter) , $X_2(\tau,\bar\tau) = (x_2(\tau,\bar\tau), t_2(\tau,\bar\tau), f_2(\tau,\bar\tau))$, where

$$x_2(\tau,\bar\tau) = -\frac{1}{2} Re\left(\tau - \frac{1}{\tau}\right), \qquad t_2(\tau,\bar\tau) = -\frac{1}{2} Im\left(\tau + \frac{1}{\tau}\right),$$

$$f_2(\tau,\bar\tau) = -\frac{1}{2}\ln(\tau\bar\tau).$$

Then

$$x_1 + ix_2 = \frac{-i}{2}\left(\tau - \frac{1}{\tau}\right), \qquad t_1 + it_2 = \frac{-1}{2}\left(\tau + \frac{1}{\tau}\right), \qquad f_1 + if_2 = -i\ln\tau.$$

Thus we see that $X_1 + iX_2 := \left(\frac{-i}{2}\left(\tau - \frac{1}{\tau}\right), \frac{-1}{2}\left(\tau + \frac{1}{\tau}\right), -i\ln\tau\right)$ is a holomorphic mapping on a common domain of $\mathbb{C} - \{0\}$. Therefore, the Lorentzian helicoid and Lorentzian catenoid are conjugate maximal graphs. Now

$$X_\theta(\tau,\bar\tau) := X_1(\tau,\bar\tau)\cos\theta + X_2(\tau,\bar\tau)\sin\theta$$

gives a one parameter family of maximal surfaces. We have

$$ix_1 - t_1 = \frac{1}{2}\left(\frac{1}{\bar\tau} + \tau\right), \quad \text{and} \quad ix_1 + t_1 = -\frac{1}{2}\left(\frac{1}{\tau} + \bar\tau\right).$$

and

$$ix_2 - t_2 = \frac{i}{2}\left(\frac{1}{\bar\tau} - \tau\right), \quad \text{and} \quad ix_2 + t_2 = \frac{i}{2}\left(\frac{1}{\tau} - \bar\tau\right).$$

If we replace τ by $i\zeta$ and $\bar{\tau}$ by $-i\bar{\zeta}$ we get

$$ix_1 - t_1 = \frac{i}{2}\left(\frac{1}{\bar{\zeta}} + \zeta\right) \quad ; \quad ix_1 + t_1 = \frac{i}{2}\left(\frac{1}{\zeta} + \bar{\zeta}\right) \tag{4.11}$$

and

$$f_1(\zeta, \bar{\zeta}) = -\frac{i}{2}\ln\left(\frac{\zeta}{\bar{\zeta}}\right). \tag{4.12}$$

$$ix_2 - t_2 = -\frac{1}{2}\left(\frac{1}{\bar{\zeta}} - \zeta\right) \quad ; \quad ix_2 + t_2 = \frac{1}{2}\left(\frac{1}{\zeta} - \bar{\zeta}\right) \tag{4.13}$$

$$f_2(\zeta, \bar{\zeta}) = -\frac{1}{2}\ln(\zeta\bar{\zeta}). \tag{4.14}$$

Now we are going to compute the functions $G^s_\theta(\bar{\zeta})$ and $H^s_\theta(\zeta)$ which will give our required one parameter family of complex solitons corresponding to the one parameter family of maximal surfaces mentioned above. We first compute

$$\begin{aligned} x^s_\theta - t^s_\theta =& (ix_1 - t_1)\cos\theta + (ix_2 - t_2)\sin\theta \\ =& \frac{i}{2\bar{\zeta}}e^{i\theta} + \frac{i\zeta}{2}e^{-i\theta}, \end{aligned} \tag{4.15}$$

next we compute

$$\begin{aligned} x^s_\theta + t^s_\theta =& (ix_1 + t_1)\cos\theta + (ix_2 + t_2)\sin\theta \\ =& \frac{i}{2\zeta}e^{-i\theta} + \frac{i\bar{\zeta}}{2}e^{i\theta}. \end{aligned} \tag{4.16}$$

Here we get $G^s_\theta(\bar{\zeta}) = \frac{i}{2\bar{\zeta}}e^{i\theta}$ and $H^s_\theta(\zeta) = \frac{i}{2\zeta}e^{-i\theta}$ they also satisfy $\overline{G^s_\theta(\bar{\zeta})} = -H^s_\theta(\zeta)$. Hence

$$\varphi^s_\theta = -\frac{i}{2}\ln(\zeta)e^{-i\theta} + \frac{i}{2}\ln(\bar{\zeta})e^{i\theta}. \tag{4.17}$$

Equations (4.15), (4.16) and (4.17) describes the general solution of Born-Infeld equation (4.1). Therefore, $X^s_\theta := (x^s_\theta, t^s_\theta, \varphi^s_\theta)$ gives a one parameter family of Born-Infeld solitons.

CHAPTER 5

Ramanujan's type identities

5.1 Introduction

In this chapter, we use implicit representation (relation among x, y, z) of some of the well known maximal surfaces and then apply Weierstrass-Enneper representation of such maximal surfaces together with certain Ramanujan's identities to obtain further nontrivial identities. Earlier in [17], Dey had also obtained new identities with the help of Weierstrass-Enneper representation of minimal surfaces and the same Ramanujan's identities.

Here we state some of the identities which were obtained by Srinivasa Ramanujan [45]. Suppose we have X and A as complex numbers, where A is not an odd multiple of $\frac{\pi}{2}$, then

$$\frac{\cos(X+A)}{\cos(A)} = \prod_{k=1}^{\infty}\left\{\left(1-\frac{X}{(k-\frac{1}{2})\pi - A}\right)\left(1+\frac{X}{(k-\frac{1}{2})\pi + A}\right)\right\}. \tag{5.1}$$

Now if X and A are real, then

$$\tan^{-1}(\tanh X \cot A) = \tan^{-1}\left(\frac{X}{A}\right)+\sum_{k=1}^{\infty}\left(\tan^{-1}\left(\frac{X}{k\pi + A}\right) - \tan^{-1}\left(\frac{X}{k\pi - A}\right)\right). \tag{5.2}$$

Recall that (see 3rd Chapter), for a maximal surface (x, y, z), in Lorentz-Minkowski space $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$, whose Gauss map is one-one is given by [35],

$$x(\zeta) = Re(\int^{\zeta} M(\omega)(1+\omega^2)d\omega) \; ; \; y(\zeta) = Re(\int^{\zeta} iM(\omega)(1-\omega^2)d\omega)$$

$$z(\zeta) = Re(\int^{\zeta} -2M(\omega)\omega d\omega), \text{ where } \zeta = u + iv.$$

In the next Section we derive non trivial identities which corresponds to different maximal surfaces in $\mathbb{L}^3$. In particular, we obtain three different identities each corresponds to Scherk's surface of first kind, helicoid of second kind and Lorentzian helicoid respectively.

5.2 Identities

5.2.1 Scherk's surface of first kind

Proposition 5.2.1 *For $\zeta \in \Omega \subset \mathbb{C} - \{\pm 1, \pm i\}$, we have the following identity*

$$\ln|\frac{\zeta^2-1}{\zeta^2+1}| = \sum_{k=1}^{\infty} \ln\left(\frac{(k-\frac{1}{2})\pi - i\ln|\frac{\zeta-i}{\zeta+i}|}{(k-\frac{1}{2})\pi - i\ln|\frac{\zeta+1}{\zeta-1}|}\right)\left(\frac{(k-\frac{1}{2})\pi + i\ln|\frac{\zeta-i}{\zeta+i}|}{(k-\frac{1}{2})\pi + i\ln|\frac{\zeta+1}{\zeta-1}|}\right). \tag{5.3}$$

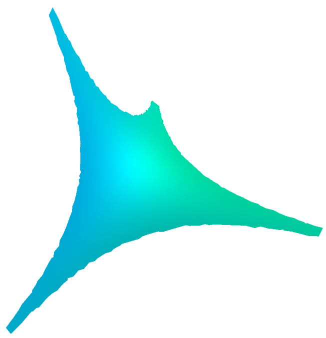

Figure 5.1: Scherk's surface of first kind

Before we proceed to give a proof, it is easy to see for instance for $\zeta = 0$ this identity holds.

Proof. For Scherk's surface of first kind [34], which in non-parametric form is, defined by,

$$z = \ln(\cosh y) - \ln(\cosh x) \quad , \quad (\cosh^{-2} x + \cosh^{-2} y > 1) \tag{5.4}$$

If we take the Weierstrass data, $M(\omega) = \dfrac{2}{1-\omega^4}$. Then using the Weierstrass-Enneper representation, we can write Scherk's surface in parametric form as

$$x(\zeta) = \ln|\frac{\zeta+1}{\zeta-1}|, \tag{5.5}$$

$$y(\zeta) = \ln|\frac{\zeta-i}{\zeta+i}|, \tag{5.6}$$

$$z(\zeta) = \ln|\frac{\zeta^2-1}{\zeta^2+1}|. \tag{5.7}$$

This parametrization is well defined on $\Omega \subset \mathbb{C} - \{\pm i, \pm i\}$. We easily compute that

$$x(\zeta) = \frac{1}{2}\ln\left(\frac{(u+1)^2+v^2}{(u-1)^2+v^2}\right),$$

$$y(\zeta) = \frac{1}{2}\ln\left(\frac{u^2+(v-1)^2}{u^2+(v+1)^2}\right),$$

$$z(\zeta) = \frac{1}{2}\ln\left(\frac{(u^2-v^2-1)^2+4u^2v^2}{(u^2-v^2+1)^2+4u^2v^2}\right).$$

One can easily verify from the expressions for x, y, and z that

$$z = \ln(\cosh y) - \ln(\cosh x).$$

Now if we take the logarithm on both sides of the identity (5.1), we get

$$\ln\left(\frac{\cos(X+A)}{\cos(A)}\right) = \sum_{k=1}^{\infty} \ln\left(\frac{(k-\frac{1}{2})\pi-(X+A)}{(k-\frac{1}{2})\pi - A}\right)\left(\frac{(k-\frac{1}{2})\pi+(X+A)}{(k-\frac{1}{2})\pi + A}\right). \tag{5.8}$$

If we put $X + A = iy$ and $A = ix$ in (5.8), where ix is not an odd multiple of

$-\frac{i\pi}{2}$, we obtain

$$z = \ln\left(\frac{\cosh y}{\cosh x}\right) = \ln\left(\frac{\cos iy}{\cos ix}\right) = \sum_{k=1}^{\infty} \ln\left(\frac{(k-\frac{1}{2})\pi - iy}{(k-\frac{1}{2})\pi - ix}\right)\left(\frac{(k-\frac{1}{2})\pi + iy}{(k-\frac{1}{2})\pi + ix}\right). \tag{5.9}$$

Now we use (5.5), (5.6), and (5.7) in (5.9), we will get our first identity (5.3).

5.2.2 Helicoid of second kind.

Proposition 5.2.2 *For $\zeta \in \Omega \subset \mathbb{C} - \{0\}$, we have the following identity*

$$\frac{Im\left(\zeta + \frac{1}{\zeta}\right)}{Im\left(\zeta - \frac{1}{\zeta}\right)} = \frac{1}{i}\prod_{k=1}^{\infty}\left\{\left(\frac{(k-1)\pi + i\ln|\zeta|}{(k-\frac{1}{2})\pi + i\ln|\zeta|}\right)\left(\frac{k\pi - i\ln|\zeta|}{(k-\frac{1}{2})\pi - i\ln|\zeta|}\right)\right\}. \tag{5.10}$$

Figure 5.2: Helicoid of second kind

Before we proceed to the proof it is easy to check that for instance $\zeta = i$ satisfies this identity.

Proof. The helicoid of second kind is a ruled surface, which in non-parametric form, is given by [34]

$$z = -x \tanh y, \qquad (x^2 \leq \cosh^2 y). \tag{5.11}$$

Here, we use a variant of Weierstrass-Enneper representation given by [34]

$$x(\zeta) = Re(\int^{\zeta} M(\omega)(1+\omega^2)d\omega) \quad ; \quad y(\zeta) = Re(\int^{\zeta} 2iM(\omega)\omega d\omega)$$

$$z(\zeta) = Re(\int^{\zeta} M(\omega)(\omega^2 - 1)d\omega).$$

and we take the Weierstrass data as, $M(\omega) = \dfrac{i}{2\omega^2}$. Then we get a parametric representation of (5.11), valid in a domain $\Omega \subset \mathbb{C} - \{0\}$, as follows,

$$x(\zeta) = -\frac{1}{2} Im\left(\zeta - \frac{1}{\zeta}\right), \qquad y(\zeta) = -\ln|\zeta|, \quad z(\zeta) = -\frac{1}{2} Im\left(\zeta + \frac{1}{\zeta}\right). \tag{5.12}$$

Now we write equation (5.11) as $-\dfrac{z}{ix} = \dfrac{\cos(iy+\frac{\pi}{2})}{\cos(iy)}$, next replace X by $\frac{\pi}{2}$ and A by iy, in Ramanujan identity (5.1), and then use equations (5.12) to get the desired identity (5.10).

5.2.3 Lorentzian helicoid

Proposition 5.2.3 *For $\zeta = u + iv$, such that $\zeta \in \Omega \subset \mathbb{C} - \{0\}$, we have the following identity*

$$Im(\ln(\zeta)) - \tan^{-1}\left(\tanh\left(-\frac{1}{2}Re\left(\zeta + \frac{1}{\zeta}\right)\right)\cot\left(\frac{1}{2}Im\left(\zeta - \frac{1}{\zeta}\right)\right)\right) \tag{5.13}$$

$$= \pm\frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\tan^{-1}\left(\frac{Re(\zeta+\frac{1}{\zeta})}{Im(\zeta-\frac{1}{\zeta}) - 2k\pi}\right) + \tan^{-1}\left(\frac{Re(\zeta+\frac{1}{\zeta})}{Im(\zeta-\frac{1}{\zeta}) + 2k\pi}\right)\right),$$

where the constant term is $\frac{\pi}{2}$, when either $u > 0$ and $v > 0$ or $u < 0$ and $v < 0$ or $u = 0$ or $v = 0$ and the constant term is $-\frac{\pi}{2}$ otherwise.

Before proceeding to the proof we can easily see for instance that $\zeta = 1$ satisfies this identity.

Proof. Consider a Lorentzian helicoid, $z = \pm\frac{\pi}{2} + \tan^{-1}(\frac{y}{x})$, we have Weierstrass-

Figure 5.3: Lorentzian helicoid

Enneper representation for this valid in a domain $\Omega \subset \mathbb{C} - \{0\}$, given by [48]

$$x(\zeta) = \frac{1}{2} Im(\zeta - \frac{1}{\zeta}), \qquad y(\zeta) = -\frac{1}{2} Re(\zeta + \frac{1}{\zeta}), \quad z(\zeta) = Im(\ln(\zeta)) \tag{5.14}$$

In the parameter $\zeta = u + iv$, we have $z = \tan^{-1}(\frac{v}{u})$ and $\tan^{-1}(\frac{y}{x}) = -\tan^{-1}(\frac{u}{v})$. Now we see that

$$z = \frac{\pi}{2} + \tan^{-1}(\frac{y}{x}),$$

only when either $u > 0$ and $v > 0$ *or* $u < 0$ and $v < 0$ *or* $u = 0$ *or* $v = 0$. For other values of u, v i.e. when either $u < 0$ and $v > 0$ *or* $u > 0$ and $v < 0$, we get

$$z = -\frac{\pi}{2} + \tan^{-1}(\frac{y}{x}).$$

Next we use equations (5.14) in Ramanujan identity (5.2) to obtain the identity (5.13).

5.3 Conclusion

It is quite remarkable to see that how Weierstrass-Enneper representation of maximal surfaces together with certain Ramanujan's identities give us further non- trivial identities. This shows that there is a beautiful connection between the geometry of maximal surfaces and analytic number theory through these Ramanujan's identities. However, this is just one example. It would be an interesting problem to see whether there exist further such connections between the maximal surfaces and analytic number theory.

APPENDIX A

Appendix

A.1 Causal characters

Definition A.0.1 A vector $v \in \mathbb{L}^3 := (\mathbb{R}^3, \langle , \rangle_L := dx^2 + dy^2 - dz^2)$ is said to be a *spacelike* (respectively *timelike*, *lightlike*) if $\langle v, v \rangle_L > 0$ or $v = 0$ (respectively $\langle v, v \rangle_L < 0$, $\langle v, v \rangle_L = 0$ and $v \neq 0$).

Proposition A.0.2 *Two lightlike (null) vectors $u, v \in \mathbb{L}^3$ are linearly dependent if and only if $\langle u, v \rangle_L = 0$.*

Let $\gamma : I \subset \mathbb{R} \to \mathbb{L}^3$ be a curve. Then in order to classify the manifold we need to know the sign of $\langle \gamma'(t), \gamma'(t) \rangle_L$. Thus

- If $\langle \gamma'(t), \gamma'(t) \rangle_L > 0$, $(I, \gamma^* \langle , \rangle_L)$ is a Riemannian manifold, i.e., the induced metric is positive definite.
- If $\langle \gamma'(t), \gamma'(t) \rangle_L < 0$, $(I, \gamma^* \langle , \rangle_L)$ is a Lorentzian manifold, i.e., the induced metric is negative definite.
- If $\langle \gamma'(t), \gamma'(t) \rangle_L = 0$, $(I, \gamma^* \langle , \rangle_L)$ is a degenerate manifold, i.e., the induced metric degenerates.

This classification justifies the following definition

Definition A.0.3 A curve γ in $\mathbb{L}^3 := (\mathbb{R}^3, \langle , \rangle_L := dx^2 + dy^2 - dz^2)$ is said to be *spacelike* (respectively *timelike*, *lightlike*) at t if $\gamma'(t)$ is a spacelike (respectively timelike, lightlike) vector. The curve $\gamma : I \to \mathbb{L}^3$ is *spacelike* (respectively *timelike*, *lightlike*) if it is spacelike (respectively timelike, lightlike) for all $t \in I$.

Definition A.0.4 Given a two dimensional vector subspace T of $\mathbb{R}^3$, we consider the metric, denoted by $\langle,\rangle_L^T$, on T which is induced from Lorentzian metric $\langle,\rangle_L$. Then we say the vector subspace T is *spacelike* (respectively *timelike*, *lightlike*) if the induced metric is positive definite (respectively *index one*, *degenerate*).

Definition A.0.5 Let S be a surface. An immersion $F : S \to \mathbb{L}^3$ is said to be *spacelike* (respectively *timelike*, *lightlike*) if all the tangent planes $(T_pS, F^*\langle,\rangle_L^p)$ (pullback metric), where $p \in S$, are *spacelike* (respectively *timelike*, *lightlike*).

Definition A.0.6 Let S be a Riemann surface and $X = (X_1, X_2, X_3) : S \to \mathbb{L}^3$ a non-constant harmonic mapping and if for every point $p \in S$ there exists a (complex) coordinate neighbourhood (U, z) such that the complexified derivatives $\phi_i = \frac{\partial X_i}{\partial z}$; $i = 1, 2, 3$ satisfy

$$\phi_1^2 + \phi_2^2 - \phi_3^2 = 0 \text{ (Conformality)},$$

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 \not\equiv 0 \text{ (Metric)}.$$

Then (X, S) is said to be a generalized maximal surface. Note that the functions X_k are harmonic functions on S, hence S cannot be compact.

A.2 Singular Bjölring problem for generalized maximal surfaces

Given a real analytic null curve $\gamma : I \to \mathbb{L}^3$ and a real analytic null vector field $L : I \to \mathbb{L}^3$ such that $\gamma'(u)$ and $L(u)$ are proportional for all $u \in I$ and that at least one of $\gamma'(u)$ and $L(u)$ is not identically 0, the problem is to find a generalized maximal surface $X(u, v)$ with (u, v) as a conformal parameter whose u-parameter curve $X(u, 0)$ is $\gamma(u)$ and whose coordinate vector field $X_v(u, 0)$ along γ is $L(u)$ for all $u \in I$. Then solution to the singular Björling problem is stated as follows:

Theorem A.0.7 *Given* $u, I, \gamma = (\gamma_1, \gamma_2, \gamma_0)$ *and* $L = (L_1, L_2, L_0)$ *as above, define*

$$g(u) := \begin{cases} \frac{\gamma_1'(u) + i\gamma_2'(u)}{\gamma_0'(u)}, & \text{if } \gamma' \not\equiv 0 \\ \frac{L_1(u) + iL_2(u)}{L_0(u)}, & \text{if } \gamma' \equiv 0. \end{cases}$$

If the analytic extension $g(z)$ of $g(u)$, where $z = u + iv \in U \subset \mathbb{C}$, $I \subset U$, satisfies

$$|g(z)| \not\equiv 1,$$

then there is exactly one generalized maximal surface $X : U \to \mathbb{L}^3$ with $u + iv$ as a conformal parameter and $X(u, 0) = \gamma(u)$, $X_v(u, 0) = L(u)$. It is given by

$$X(u + iv) = \gamma(u_0) + Re \int_{u_0}^{z} (\gamma'(w) - iL(w))dw$$

where $u_0 \in I$ is fixed.

A.3 Conformal relations

For the complex variable $z = x + iy$, the two Wirtinger differential operators are

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

For the coordinate functions $F = (h := u + iv, w)$, where h is a complex coordinate and w is a real coordinate

$$h_z = u_z + iv_z,\ h_{\bar{z}} = u_{\bar{z}} + iv_{\bar{z}},\ \overline{u_z} = u_{\bar{z}},\ \overline{v_z} = v_{\bar{z}}, \overline{h_z} = u_{\bar{z}} - iv_{\bar{z}}$$

$$\overline{h_{\bar{z}}} = \overline{u_{\bar{z}}} - i\overline{v_{\bar{z}}} = u_z - iv_z.$$

Here $A_z := \frac{\partial A}{\partial z}$ and $A_{\bar{z}} := \frac{\partial A}{\partial \bar{z}}$ where $A = u, v, w, h$.

The conformality relations in the definition of generalised maximal surface is equivalent to a single complex equation

$$u_x^2 + v_x^2 - w_x^2 = u_y^2 + v_y^2 - w_y^2 \ \text{and} \ u_x u_y + v_x v_y - w_x w_y = 0 \Leftrightarrow u_z^2 + v_z^2 - w_z^2 = 0$$

$$\Leftrightarrow (u_z + iv_z)(u_z - iv_z) - w_z^2 = 0 \Leftrightarrow h_z\overline{h_{\bar{z}}} - w_z^2 = 0.$$

Also, the expression $|u_z|^2 + |v_z|^2 - |w_z|^2 = 2(|h_z| - |h_{\bar{z}}|)^2$ (use complex conformality relation). $|u_z|^2 + |v_z|^2 - |w_z|^2$ also represents the coefficients of induced metric on a generalised maximal surface.

A.4 Normal vector to a maximal surface in $\mathbb{L}^3$

Let $F = (u, v, w) : \Omega \subset \mathbb{C} \to \mathbb{L}^3$ be a generalised maximal surface. Then the normal vector field along (F, Ω) is a map $N : \Omega \subset \mathbb{C} \to \mathbb{H}^2$ defined by

$$N(z) = \frac{F_x \times F_y}{|F_x \times F_y|_L} \quad \text{where} \quad |F_x \times F_y|_L = \sqrt{|\langle F_x \times F_y, F_x \times F_y \rangle_L|}$$

and $\langle . \rangle_L = dx_1^2 + dx_2^2 - dx_3^2$ stands for the Lorentzian metric in $\mathbb{R}^3$.

For $F_x, F_y \in \mathbb{L}^3$, the cross-product $F_x \times F_y \in \mathbb{L}^3$ is given by

$$F_x \times F_y = (v_x w_y - v_y w_x, u_y w_x - u_x w_y, u_y v_x - u_x v_y)$$

Let $U = v_x w_y - v_y w_x$, $V = u_y w_x - u_x w_y$ and $W = u_y v_x - u_x v_y$. We also have

$$h = u + iv,\ h_x = u_x + iv_x,\ h_y = u_y + iv_y,\ u_x = u_z + u_{\bar{z}},\ u_y = i(u_z - u_{\bar{z}})$$

$$v_x = v_z + v_{\bar{z}},\ v_y = i(v_z - v_{\bar{z}}),\ w_x = w_z + w_{\bar{z}},\ w_y = i(w_z - w_{\bar{z}}).$$

Now let us consider $H = U + iV$, then

$$U + iV = -iw_y h_x + iw_x h_y = 2\sqrt{h_z h_{\bar{z}}}(|h_{\bar{z}}| - |h_z|)$$

and $W = u_y v_x - u_x v_y = |h_{\bar{z}}|^2 - |h_z|^2$. Thus

$$U^2 + V^2 - W^2 = (U + iV)(U - iV) = -(|h_{\bar{z}}| - |h_z|)^4.$$

Hence

$$|F_x \times F_y|_L = \sqrt{|U^2 + V^2 - W^2|} = (|h_{\bar{z}}| - |h_z|)^2.$$

Therefore, normal vector is given by

$$N(z) = \left(\frac{U + iV}{\sqrt{|U^2 + V^2 - W^2|}}, \frac{W}{\sqrt{|U^2 + V^2 - W^2|}} \right) = \left(\frac{2\sqrt{h_z h_{\bar{z}}}}{|h_{\bar{z}}| - |h_z|}, \frac{|h_{\bar{z}}| + |h_z|}{|h_{\bar{z}}| - |h_z|} \right).$$

A.5 Gauss map

Suppose $(x,y,t) \in \mathbb{H}^2 := \{(x,y,t) \in \mathbb{L}^3 : x^2+y^2-t^2 = -1\}$. Note that $\mathbb{H}^2$ has two connected components $\mathbb{H}^2_+ := \mathbb{H}^2 \cap \{t \geq 1\}$ and $\mathbb{H}^2_- := \mathbb{H}^2 \cap \{t \leq -1\}$. Let σ_N be the stereographic projection from the north pole $(0,0,1)$ of $\mathbb{H}^2$, then $\sigma_N : \mathbb{H}^2 \to \mathbb{R}^2 \setminus \{x^2+y^2=1\} \cong \mathbb{C} \setminus \{|z|=1\}$ is defined by

$$\sigma_N(x,y,t) = \left(\frac{x}{1-t}, \frac{y}{1-t}\right) \quad \text{or} \quad \sigma_N(x+iy,t) = \left(\frac{x+iy}{1-t}\right),$$

$$\sigma_N(0,0,1) = \infty \quad , \quad \sigma_N(\mathbb{H}^2_+) = \{|z|>1\} \quad \text{and} \quad \sigma_N(\mathbb{H}^2_-) = \{|z|<1\}.$$

In a similar way one can also define stereographic projection from the south pole $(0,0,-1)$ of $\mathbb{H}^2$ which is given by

$$\sigma_S(x+iy,t) = \left(\frac{x+iy}{1+t}\right) \quad \text{and} \quad \sigma_S(0,0,-1) = \infty.$$

Now one can define Gauss map of a spacelike surface either by composing σ_N with N or σ_S with N. Only the expression of the Gauss map changes in these cases. This does not effect our study. We need this while proving singular Björling problem in order to define a map on the unit circle using initial data $\gamma'(e^{i\theta})$ and $L(e^{i\theta})$ so that we can talk about its analytic extension on some annulus $A(r,R)$ which contains the unit circle.

In our case we can take $\nu := \sigma_S \circ N : \Omega \subset \mathbb{C} \to \mathbb{C} \setminus \{|z|=1\} \cup \{\infty\}$ as Gauss map, then we get

$$\nu(z) = -\sqrt{\frac{h_{\bar{z}}}{\overline{h_z}}} \quad \text{where} \quad N(z) = \left(\frac{2\sqrt{h_z h_{\bar{z}}}}{|h_{\bar{z}}|-|h_z|}, \frac{|h_{\bar{z}}|+|h_z|}{|h_{\bar{z}}|-|h_z|}\right) \quad \text{and} \quad |h_z| \neq |h_{\bar{z}}|.$$

A.6 Modified Fourier coefficients

We claim that the required harmonic map $F(z) = (h(z), w(z))$ in our Main Theorem of the Chapter 2 must be of the form

$$h(z) = \sum_{-\infty}^{\infty} c_n \left(z^n - \frac{1}{\bar{z}^n}\right) + c \log|z|, \tag{A.1}$$

$$w(z) = \sum_{-\infty}^{\infty} d_n \left(z^n - \frac{1}{\bar{z}^n} \right) + d \log |z|. \tag{A.2}$$

and the coefficients c_n, c, d_n, d (which we call modified Fourier coefficients) are given by

$$c = \frac{1}{2\pi \log r_0} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) d\theta; \quad d = \frac{1}{2\pi \log r_0} \int_{-\pi}^{\pi} g(r_0 e^{i\theta}) d\theta, \tag{A.3}$$

for $n \neq 0$;

$$c_n = \frac{r_0^n}{2\pi (r_0^{2n} - 1)} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) e^{-in\theta} d\theta; \quad d_n = \frac{r_0^n}{2\pi (r_0^{2n} - 1)} \int_{-\pi}^{\pi} g(r_0 e^{i\theta}) e^{-in\theta} d\theta. \tag{A.4}$$

In the next few paragraphs we will try to explain this point. We have given $F(r_0 e^{i\theta}) = (h(r_0 e^{i\theta}), w(r_0 e^{i\theta})) = (f(r_0 e^{i\theta}), g(r_0 e^{i\theta}))$. From this we get

$$\begin{aligned} h(r_0 e^{i\theta}) &= \sum_{-\infty}^{\infty} c_n \left(r_0^n - \frac{1}{r_0^n} \right) e^{in\theta} + c \log r_0 \\ &= \sum_{-\infty}^{\infty} \hat{f}_n e^{in\theta} = f(r_0 e^{i\theta}) \end{aligned}$$

here $f(r_0 e^{i\theta})$ is real analytic and $\hat{f}_n$ are the Fourier coefficients for f. From this we would like to find c_n and c which we will call modified Fourier coefficients. A similar calculation for w will give us $w(r_0 e^{i\theta}) = g(r_0 e^{\theta})$ and hence we can compute modified Fourier coefficients d_n and d.

Now we explain how to compute c_n, c and d_n, d from the given data for real analytic spacelike closed curve $(f(r_0 e^{i\theta}), g(r_0 e^i \theta))$. We know if we have $F = (h, w)$, h (complex harmonic) and w are two harmonic functions.

Then h is of the form

$$h(z) = \sum_{-\infty}^{\infty} a_n z^n + \frac{b_n}{\bar{z}^n} + c \ln |z|. \tag{A.5}$$

defined on some annulus. The real harmonic function w will have a similar series representation. But according to the assumption of our problem we must have

$F(e^{i\theta}) = (0,0,0),\ F(r_0 e^{i\theta}) = (f(r_0 e^{i\theta}), g(r_0 e^{i\theta}))$. This gives

$$\forall n, \quad n(a_n + b_n) = 0.$$

Thus $c_n = a_n = -b_n$ and

$$\forall n \neq 0, a_n r_0^n + \frac{b_n}{r_0^n} = \hat{f}_n.$$

This implies $a_n(r_0^{2n} - 1) = r_0^n \hat{f}_n$ and we get $c_n = \dfrac{r_0^n}{r_0^{2n} - 1} \hat{f}_n$ where $\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) e^{-in\theta} d\theta$.

Also for

$$n = 0, \quad \text{we have} \quad a_0 + b_0 + c \log r_0 = \hat{f}_0.$$

This implies $c = \frac{\hat{f}_0}{\log r_0}$, where $\hat{f}_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) d\theta$. This gives h of the form as in equation (A.1) with c_n and c for $h(z)$ as obtained.

Now one have to do a similar calculations in order to obtain modified Fourier coefficients d_n and d for $w(z)$ as in (A.2).

A.7 Characteristic curves

Let us consider a second order PDE

$$\begin{aligned} A(x, y, u, u_x, u_y) u_{xx} &+ 2B(x, y, u, u_x, u_y) u_{xy} + C(x, y, u, u_x, u_y) u_{yy} \\ &= f(x, y, u, u_x, u_y) \end{aligned} \tag{A.6}$$

and its associated differential form is given by

$$A(dy)^2 - 2B dx dy + C(dx)^2 = 0.$$

Then we have the following classification. The equation

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$

has two roots

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

Now we have two ODEs and we can solve them to get two sets of curves $\xi(x, y) = c$ (constant) and $\eta(x, y) = c'$ (constant) known as characteristic curves. We can decide the type of a PDE based on the sign of $B^2 - AC$:

(a) when $B^2 > AC$, the PDE is hyperbolic and hence we get two different real characteristics curves.

(b) when $B^2 < AC$, the PDE is elliptic.

(c) when $B^2 = AC$, the PDE is parabolic.

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