ARITHMETICAL PROPERTIES OF FOURIER COEFFICIENTS OF HILBERT MODULAR FORMS

By
RISHABH AGNIHOTRI
MATH08201704001

Harish-Chandra Research Institute, Prayagraj

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Prof. B. Ramakrishnan
Co-guide

Prof. Kalyan Chakraborty
Guide

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Rishabh Agnihotri

RISHABH AGNIHOTRI

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Rishabh Agnihotri

Rishabh Agnihotri

To my first teacher

my parents

Mamata Agnihotri

and

Ramkaran Agnihotri

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Contents

Notations

$\mathbb{N}$	The set of natural numbers.
$\mathbb{Z}$	The set of integers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
R	Commutative ring with identity.
$R^\times$	The multiplicative group of units of R.
$M(n, R)$	The set of all $n \times n$ matrices over the ring R.
$GL(n, R)$	The set of all invertible matrices in $M(n, R)$.
$SL(n, R)$	The set of all matrices in $GL(n, R)$ with determinant 1.
$SO(n)$	Special orthogonal group of degrre n.
$SU(n)$	Special unitary group of degree 2.
$I_{n\times n}$	The $n \times n$ identity matrix.
$\det A$	The determinant of the matrix A.
$\mathcal{O}_K$	The ring of integers of K.
$\mathfrak{m}$	An integral ideal of K.
$\mathfrak{D}_K$	The different ideal of K .
D	The discriminant of K.

ζ_K	The Dedekind zeta function of K.
h	The narrow class number of K.
R_K	The regulator of K.
$N(x)$	Norm of $x \in K$ over $\mathbb{Q}$.
$Tr(x)$	Trace of $x \in K$ over $\mathbb{Q}$.
$x \gg 0$	x is totally positive element this means $\sigma_i(x) > 0$ for all i.
X^+	Collection of totally positive elements of a set $X \subset K$.
$K_\wp$	$\wp$-adic completion of K.
$\mathcal{O}_\wp$	The ring of integers of $K_\wp$.
$\mathfrak{m}_\wp$	The maximal ideal of $\mathcal{O}_\wp$.

We also utilize the following notations frequently.

(1) Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that $g(x) > 0$ for all $x \in \mathbb{R}$. We write $f(x) = O(g(x))$ (read "$f(x)$ is big oh of $g(x)$") to mean that the quotient $\frac{f(x)}{g(x)}$ is bounded; that is there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all x. Moreover if there exsit constants $C_1, C_2 > 0$ such that

$$C_1 g(x) \leq |f(x)| \leq C_2 g(x),$$

then we write $f \asymp g$.

(2) By $f(x) \sim g(x)$ and $f(x) = o(g(x))$, we mean that

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1 \text{ and } \lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0,$$

respective.

Summary

This thesis contains four chapters that deal with some problems in the theory of Hilbert modular forms. The thesis begins with a brief introduction to the theory of Hilbert modular forms (Chapter 1).

Chapter 2 is concerned with the determination of modular forms, for example, "multiplicity one theorem", and a result of Dinakar Ramakrishnan. We determine a Hilbert modular form by the Fourier coefficients indexed by the square-free integral ideals. To prove our result, we use newform theory and a method of Balog and Ono.

Chapter 3 discusses the sign changes in the Fourier coefficients indexed by the square-free integral ideals and the integral ideals in an arithmetic progression. We use the adelic correspondence due to Shimura between classical Hilbert modular forms and adelic Hilbert modular forms to study the sign changes.

The last chapter is devoted to the study of the Lambert series associated with an adelic Hilbert cusp form. We establish an asymptotic relation between the Lambert series and the non-trivial zeros of the Dedekind zeta function. Such an asymptotic relation for Ramanujan τ-function was conjectured by Zagier and proved by Hafner and Stopple.

CHAPTER 1

Hilbert modular forms-a brief introdution

This chapter aims to collect some basic definitions and results that will be of importance to the rest of thesis but are also relevant in their own right. We do not give the proofs of statements as the proofs are easily available in the referred literature. In the first section, we develop a significant theory of classical Hilbert modular forms (HMF). In the second section, we talk about the association of classical HMF to adelic forms. The third section is devoted to defining various linear operators on the space of HMF. Then we briefly review the theory of newforms for HMF. At the end of this chapter, we focus on the Rankin-Selberg of HMF. Here we mainly follow [17], [18], [44] and [47].

1.1 Classical Hilbert Modular forms

Let K be a totally real number field of degree d over $\mathbb{Q}$. We denote the set of all real embeddings of K into $\mathbb{R}$ by $S_\infty = \{\sigma_1, \ldots, \sigma_d\}$. Naturally by these embeddings, K can be embedded inside $\mathbb{R}^d$ via the map $x \to (\sigma_1(x), \ldots, \sigma_d(x))$. Let us define some multindex notations that helps us to simplify longer statements to shorter ones, and these notations will be used throughout the thesis. For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d, \boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ and $x \in \mathbb{C}$, we define

$$\boldsymbol{\alpha}^{\boldsymbol{k}} := \prod_{n=1}^{d} \alpha_n^{k_n}, \qquad \{\boldsymbol{\alpha}\} := \sum_{n=1}^{d} \alpha_n, \qquad x^{\boldsymbol{\alpha}} := x^{\sum_{n=1}^{d} \alpha_n},$$

$$k_0 = \max\{k_1, k_2, \ldots, k_d\}, \quad e_K(\boldsymbol{\alpha}) := e(\{\boldsymbol{\alpha}\}) = \exp\left(2\pi i \sum_{n=1}^{d} \alpha_n\right).$$

1.1.1 Hilbert Modular Group

Let us begin with defining the group $GL^+(2, \mathbb{R})$,

$$GL^+(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R}, ad - bc > 0 \right\}.$$

The group $GL^+(2, \mathbb{R})$ acts on the Poincaré upper half-plane $\mathcal{H} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } y > 0\}$ via the fractional linear transformation. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{R})$ and z be in upper half-plane,

$$\gamma(z) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}.$$

Following the above action of $GL^+(2, \mathbb{R})$ on $\mathcal{H}$, we have an induced action of $GL^+(2,\mathbb{R})^d$ on $\mathcal{H}^d$, which is defined component-wise. More precisely for $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d) \in GL^+(2,\mathbb{R})^d$ and $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathcal{H}^d$

$$\boldsymbol{\gamma}(z_1, \ldots, z_d) := (\gamma_1(z_1), \ldots, \gamma_d(z_d)).$$

For a fixed choice of an embedding of K into $\mathbb{R}^d$, we obtain an embedding of $GL(2,K)$ into $GL(2,\mathbb{R})^d$ via the following map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \left\{ \begin{pmatrix} \sigma_j(a) & \sigma_j(b) \\ \sigma_j(c) & \sigma_j(d) \end{pmatrix} \right\}_{j=1}^{d}. \tag{1.1}$$

We now define the following two subgroups of $GL(2,K)$;

$$GL^+(2,K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,K) \;\middle|\; ad - bc \gg 0 \right\},$$

and

$$GL^+(2,\mathcal{O}_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathcal{O}_K) \;\middle|\; ad - bc \gg 0 \right\}.$$

Then the so defined embedding (1.1) gives an action of $GL^+(2,K)$ on $\mathcal{H}^d$. Let γ be an element of $GL^+(2,K)$ and $\boldsymbol{z} \in \mathcal{H}^d$,

$$\gamma(\boldsymbol{z}) := \left(\frac{\sigma_j(a)z_j + \sigma_j(b)}{\sigma_j(c)z_j + \sigma_j(d)} \right)_{j=1}^{d},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The group $GL^+(2,\mathcal{O}_K)$ is known as the ***full Hilbert modular group*** (attached to the field K).

1.1.2 Congruence Subgroup

Definition 1.1.1 Let N be a positive integer. We define

$$\Gamma_N = \{\gamma \in SL(2, \mathcal{O}_K) \mid \gamma - I_2 \in N \cdot M(2, \mathcal{O}_K)\}.$$

Γ_N is called the *"principal congruence subgroup"* of level N.

Definition 1.1.2 Let G and H be two groups. We say that the groups G and H are *commensurable* if $G \cap H$ has finite index in both the groups.

Definition 1.1.3 Let Γ be a subgroup of $GL^+(2, K)$. We say that Γ is a *congruence subgroup* if it contains Γ_N for some $N \in \mathbb{N}$, and $\Gamma/(\Gamma \cap K)$ is commensurable with $SL(2, \mathcal{O}_K)/ < \pm I_2 >$.

1.1.3 Class number and Cusps

The cusps of $GL^+(2, K)$ are the points $(\sigma_1(\alpha), \ldots, \sigma_d(\alpha)) \in \mathbb{R}^d \subset \partial\mathcal{H}^d$ for $\alpha \in K$, together with the point $\mathbf{i}\infty = (i\infty, \ldots, i\infty)$. Let Γ be a congruence subgroup of $GL^+(2, K)$, we say that two cusps s_1 and s_2 are Γ-*inequivalent* if their orbits under Γ are disjoint, that is Γs_1 and Γs_2 are disjoint. Now we state a theorem that relates the number of cusps and the class number of K.

Theorem 1.1.4 ([18]) *Let $SL(2, \mathcal{O}_K) \subset \Gamma \subset GL^+(2, \mathcal{O}_K)$. Let $h(K)$ be the absolute class number of K. Then there are exactly $h(K)$ number of Γ-inequivalent cusps.*

1.1.4 Classical Hilbert Modular Forms

Let f be a function defined from $\mathcal{H}^d$ to $\mathbb{C}$. For $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d) \in GL(2, \mathbb{R})^d$ and $\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, we define *stroke* operator on f as $f|_{\boldsymbol{k}}\boldsymbol{\gamma}$ on $\mathcal{H}^d$

$$(f|_{\boldsymbol{k}}\boldsymbol{\gamma})(z) := (\det \boldsymbol{\gamma})^{\frac{\boldsymbol{k}}{2}} \prod_{j=1}^{d} (c_j z_j + d_j)^{-k_j} f(\boldsymbol{\gamma} z),$$

where $\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ and $\det \boldsymbol{\gamma} = (\det \gamma_1, \ldots, \det \gamma_d)$.

Definition 1.1.5 Let Γ be a congruence subgroup of $GL^+(2, K)$. Let $\mathcal{M}_{\boldsymbol{k}}(\Gamma)$ be the space of all holomorphic functions f on $\mathcal{H}^d$ such that

(1) $(f|_{\boldsymbol{k}}\gamma)(\boldsymbol{z}) = f(\boldsymbol{z})$ for all $\gamma \in \Gamma$ and $\boldsymbol{z} \in \mathcal{H}^d$.

(2) f is holomorphic at cusps of Γ.

The elements of $\mathcal{M}_{\boldsymbol{k}}(\Gamma)$ are called ***Hilbert modular forms*** of weight $\boldsymbol{k}$ on Γ. We define $\mathcal{M}_{\boldsymbol{k}} = \mathcal{M}_{\boldsymbol{k}} \bigcup_{N=1}^{\infty} (\Gamma_N)$. Similar to the elliptic modular forms, the Hilbert modular forms also admit Fourier expansion as follows.

Proposition 1.1.6 ([18], [44]) *Let f be an element of $\mathcal{M}_{\boldsymbol{k}}(\Gamma)$. Then the Fourier expansion of f is given by,*

$$f(\boldsymbol{z}) = c_f(0) + \sum_{\xi \gg 0} c_f(\xi) e_K(\xi \boldsymbol{z}), \qquad \boldsymbol{z} \in \mathcal{H}^d,$$

where $c(\xi)$ are complex numbers, ξ runs over totally positive elements of a lattice, and

$$\xi \boldsymbol{z} = (\sigma_1(\xi) z_1, \ldots, \sigma_d(\xi) z_d).$$

The complex numbers $c_f(\xi)$ are known as *Fourier coefficients* of f.

Definition 1.1.7 An element of $\mathcal{M}_{\boldsymbol{k}}(\Gamma)$ is said to be a ***cusp form*** if the constant term of the Fourier expansion of $f|_{\boldsymbol{k}}\gamma$ is zero for all $\gamma \in GL^+(2, K)$. We denote the space of cusp forms by $\mathcal{S}_{\boldsymbol{k}}(\Gamma)$.

Proposition 1.1.8 ([17], [18]) *Let Γ be a congruence subgroup. Then the space $\mathcal{S}_{\boldsymbol{k}}(\Gamma)$ is a finite dimensional vector space over $\mathbb{C}$.*

Koecher's Principle

We know that there are non-trivial conditions imposed upon the Fourier coefficients to attain holomorphicity at cusps. According to Koecher's principle, if $d > 1$, the second condition in Definition 1.1.5 is redundant (stated explicitly in Proposition 1.1.9 below). Thus holomorphicity at cusps in the case of Hilbert modular forms is automatic ($d > 1$), while it is not trivial in the case of elliptic modular forms.

Proposition 1.1.9 ([17], [18]) *Assume that $[K : \mathbb{Q}] > 1$. Let f be a complex valued function which satisfies the first condition in Definition 1.1.5. Then f is holomorphic at cusps.*

Proposition 1.1.10 ([44]) *The space $\mathcal{M}_{\boldsymbol{k}}(\Gamma)$ is non-trivial only if $k_1 = \cdots = k_d \geq 0$ or all k_j are positive. Moreover the space $\mathcal{M}_{\boldsymbol{k}}(\Gamma) = \mathcal{S}_{\boldsymbol{k}}(\Gamma)$ unless $k_1 = \cdots = k_d$.*

1.1.5 The space $M_{\boldsymbol{k}}(\mathfrak{n}, \psi, \theta)$

Let $\mathfrak{n}$ be an integral and $\mathcal{I}$ be a fractional ideal in K. Consider the following congruence subgroup $\Gamma(\mathcal{I}, \mathfrak{n})$, defined by

$$\Gamma(\mathcal{I}, \mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, K) \;\middle|\; \substack{a,d \in \mathcal{O}_K, \;\; b \in \mathcal{I}^{-1}\mathfrak{D}_K^{-1} \\ c \in \mathcal{I}\mathfrak{n}\mathfrak{D}_K, \;\; ad-bc \in \mathcal{O}_K^\times} \right\}.$$

Let ψ be a finite character of $\left(\mathcal{O}_K/\mathfrak{n}\right)^\times$, known as *numerical character* modulo $\mathfrak{n}$. For a finite order character θ of $\mathcal{O}_K^{\times,+}$, the group of totally positive units of $\mathcal{O}_K$, define a character χ of $\Gamma(\mathcal{I},\mathfrak{n})$ as follows

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \psi(a)\theta(ad-bc).$$

Let

$$M_{\boldsymbol{k}}(\Gamma(\mathcal{I},\mathfrak{n}),\chi) := \left\{ f \in \mathcal{M}_{\boldsymbol{k}} \mid {}^{f|_{\boldsymbol{k}}\gamma = \chi(\gamma)f}_{\forall\, \gamma \in \Gamma(\mathcal{I},\mathfrak{n})} \right\}.$$

Here we remark that the space $M_{\boldsymbol{k}}(\Gamma(\mathcal{I},\mathfrak{n}),\chi)$ is trivial unless $\psi(\beta)\theta(\beta^2) = \mathrm{sgn}(\beta)^{\boldsymbol{k}}$ for all $\beta \in \mathcal{O}_K^\times$. Therefore, we always assume this parity condition. Note that there exists a $\boldsymbol{m} \in \mathbb{R}^d$ such that $\theta(\epsilon) = \epsilon^{i\boldsymbol{m}}$ for all $\epsilon \in \mathcal{O}_K^{\times,+}$. Though $\boldsymbol{m}$ is not unique, we fix this $\boldsymbol{m}$ throughout the thesis. For a set of representative of narrow class ideal of K say $\{\mathfrak{I}_1,\ldots,\mathfrak{I}_h\}$, $\Gamma_\lambda(\mathfrak{n}) = \Gamma(\mathfrak{I}_\lambda,\mathfrak{n})$, we define the space $M_{\boldsymbol{k}}(\mathfrak{n},\psi,\theta)$ as

$$M_{\boldsymbol{k}}(\mathfrak{n},\psi,\theta) := \prod_{\lambda=1}^{h} M_{\boldsymbol{k}}(\Gamma_\lambda(\mathfrak{n}),\psi,\theta).$$

1.2 Adelic Hilbert Modular Forms

In this section, we describe a lift of an h-tuple of classical Hilbert modular forms to an adelic form. Since the Hecke operators do not preserve the space of classical Hilbert modular forms, we need this lifting. We begin this section by defining the adeles and ideles of a number field.

Adeles and Ideles

Let F be a number field. We denote by S the set of the equivalence class of absolute values of F. We call an element v (or $\wp$) of S, **a place** of F. By

Ostrowski's theorem there are exactly three categories of inequivalent places. The first one corresponds to each prime ideal $\wp$ of $\mathcal{O}_F$, known as *finite place (non-archimedean)*, the second one corresponds to real embeddings of F into $\mathbb{C}$, and the third one corresponds to complex embeddings of F into $\mathbb{C}$. The latter type of places is known as *infinite places (archimedean place)*. Let v be a place of F, the completion of F at v is denoted by F_v and

$$F_v = \begin{cases} F_\wp & v = \wp \text{ is a prime ideal} \\ \mathbb{R} & v \text{ is real embedding} \\ \mathbb{C} & v \text{ is complex embedding} \end{cases},$$

where $F_\wp$ is the $\wp$-adic completion of F. For a finite place $\wp$, $\mathcal{O}_\wp$ and $\mathfrak{m}_\wp$ denote the ring of integers of $F_\wp$ and the unique maximal ideal of $\mathcal{O}_\wp$, respectively. The maximal ideal $\mathfrak{m}_\wp$ is generated by a single element. A generator $w_\wp$ (or $\pi_\wp$) of $\mathfrak{m}_\wp$ is called as *uniformizer* of $F_\wp$.

Definition 1.2.1 The *adele ring* $\mathbb{A}_F$ of a number field F is a restricted product of $F_\wp$ with respect to $\mathcal{O}_\wp$, where addition and multiplication are defined component-wise.

$$\mathbb{A}_F = \left\{ (a_\wp) \in \prod_{\wp \in S} F_\wp \;\middle|\; a_\wp \in \mathcal{O}_\wp \text{ for all but finitely many } \wp \right\}.$$

$\mathbb{A}_F$ assumes the structure of a topological ring. Note that F is embedded inside $\mathbb{A}_F$ diagonally. Thus $\mathbb{A}_F$ becomes a topological vector space over F.

Proposition 1.2.2 ([12], [15]) *For a number field F. The ring $\mathbb{A}_F$ is locally compact and Hausdorff. The field F sits inside $\mathbb{A}_F$ as a discrete subgroup, and the quotient topological group $\mathbb{A}_F/F$ is compact.*

Definition 1.2.3 The *idele group* $\mathbb{I}_F$ of F is the group $\mathbb{A}_F^\times$, the unit elements of the ring $\mathbb{A}_F$.

Theorem 1.2.4 ([12], [15]) *Let I_F be the set of all fractional ideals of F. Then there exists a surjective homomorphism between $\mathbb{I}_F$ and I_F, which is defined as*

$$\alpha = (\alpha_\wp) \mapsto \prod_{\wp<\infty} \wp^{v_\wp(\alpha)}.$$

Thus, the above result allows us to consider any idele as a fractional ideal of F. Also, we use the same notation for an idele and its associated fractional ideal.

Remark 1.2.5 Here $\mathbb{I}_K$ has a topology such that the inclusion map $i : \mathbb{I}_F \to \mathbb{A}_F^2$, given by $x \to (x, x^{-1})$, is continuous.

1.2.1 The Group $GL(2, \mathbb{A}_F)$

The group $GL(2, \mathbb{A}_F)$ makes sense as an abstract group. Since $\mathbb{A}_F^\times = \mathbb{I}_F$, therefore $GL(2, \mathbb{A}_F)$ is a set of 2×2 matrices over $\mathbb{A}_F$ with determinant in $\mathbb{I}_F$.

Set,

$$W_v = \begin{cases} GL(2, \mathcal{O}_\wp) & v = \wp \text{ is a prime ideal,} \\ SO(2) & v \text{ is a real embedding,} \\ SU(2) & v \text{ is a complex embedding.} \end{cases}$$

It is not difficult to see that

$$GL(2, \mathbb{A}_F) = \left\{ (\ldots, g_\wp, \ldots) \in \prod_{\wp \in S} GL(2, W_\wp) \;\middle|\; \text{for all but finitely many } g_\wp \in W_\wp \right\}.$$

Various subgroups

Let K be a totally real number field. For an integral ideal $\mathfrak{n}$ of K, we define two subsets $W_0(\mathfrak{n})$ and $Y(\mathfrak{n})$ of $GL(2,\mathbb{A}_K)$. For a finite place $\wp$, define subsets $Y_\wp(\mathfrak{n})$ and $W_\wp(\mathfrak{n})$ of $GL(2,K_\wp)$ as

$$Y_\wp(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \substack{a,d\in\mathcal{O}_\wp,\;\; c\in\mathfrak{n}\mathfrak{D}_K\mathcal{O}_\wp \\ b\in\mathfrak{D}_K^{-1}\mathcal{O}_\wp,\;\; (a\mathcal{O}_\wp,\mathfrak{n}\mathcal{O}_\wp)=1 \\ ad-bc\in K_\wp^\times} \right\},$$

$$W_\wp(\mathfrak{n}) := \left\{\gamma\in Y_\wp(\mathfrak{n}) \mid \det\gamma\in \mathcal{O}_\wp^\times\right\}.$$

Put

$$W_0(\mathfrak{n}) = GL^+(2,\mathbb{R})^d \times \prod_{\wp<\infty} W_\wp(\mathfrak{n}),$$
$$Y(\mathfrak{n}) = GL(2,\mathbb{A}_K)\cap\Big(GL^+(2,\mathbb{R})^d \times \prod_{\wp<\infty} Y_\wp(\mathfrak{n})\Big).$$

Remark 1.2.1 *It is worth to note that $W_\wp(\mathfrak{n})$ and $W_0(\mathfrak{n})$ are in fact subgroups of $GL(2,K_\wp)$ and $GL(2,\mathbb{A}_K)$, respectively, however $Y_\wp(\mathfrak{n})$ and $Y(\mathfrak{n})$ merely semi-groups.*

Decomposition of $GL(2,\mathbb{A}_K)$

Let $\{t_\lambda\}_{\lambda=1}^h$ be h elements of $\mathbb{I}_K$ such that at each infinite place t_λ has value 1 and the set $\{t_1\mathcal{O}_K,\dots,t_h\mathcal{O}_K\}$ forms a set of representatives for the narrow class group of K. Put

$$x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & t_\lambda \end{pmatrix}, \qquad x_\lambda^{-inv} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where *inv* denotes the involution on the set of 2×2 matrices. Now by strong approximation of $GL(2,\mathbb{A}_K)$, we see that $GL(2,\mathbb{A}_K)$ can be expressed as a disjoint union

$$GL(2,\mathbb{A}_K)=\bigcup_{\lambda=1}^{h} GL(2,K)x_\lambda W_0(\mathfrak{n})=\bigcup_{\lambda=1}^{h} GL(2,K)x_\lambda^{-inv}W_0(\mathfrak{n}).$$

Association of classical forms to adelic forms

To overcome the problem which arises in the case when the class number $h>1$, Shimura associates an h-tuple of classical Hilbert modular forms as a function on $GL(2,\mathbb{A}_K)$. For $x\in\mathbb{A}_K$, we define x_{fin}(respectively x_∞) to be an element of $\mathbb{A}_K$ by putting 1 at each infinite (respectively finite) place in x, thus $x=x_{\text{fin}}\cdot x_\infty$. In a similar manner for each integral ideal $\mathfrak{n}$ we define $x_{\mathfrak{n}}$ the $\mathfrak{n}$-part of x by putting 1 at each place v whenever $v\nmid\mathfrak{n}$. Given a numerical character ψ of $\big(\mathcal{O}/\mathfrak{n}\big)^\times$, following [44] we define a character of $Y(\mathfrak{n})$ by

$$\psi_Y\left(\begin{pmatrix} x & * \\ * & * \end{pmatrix}\right)=\psi(x_{\mathfrak{n}} \bmod (\mathfrak{n})).$$

Now we describe how to associate an h-tuple $(f_1,\ldots,f_h)$ of $M_{\boldsymbol{k}}(\mathfrak{n},\psi,\theta)$ to a function $\mathbf{f}$ on $GL(2,\mathbb{A}_K)$. By the decomposition of $GL(2,\mathbb{A}_K)$, the function $\mathbf{f}$ is given as follows

$$\mathbf{f}(\gamma x_\lambda^{-inv}w)=\psi_Y(w^{inv})\ \det\ (w_\infty)^{im}(f_\lambda|_k w_\infty)(\mathbf{i}),$$

where $w\in W_0(\mathfrak{n})$, $\mathbf{i}=(i,\ldots,i)$, and

$$f_\lambda \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\boldsymbol{z})=(ad-bc)^{\frac{\boldsymbol{k}}{2}}(cz+d)^{-\boldsymbol{k}}f_\lambda\Big(\frac{a\boldsymbol{z}+b}{c\boldsymbol{z}+d}\Big).$$

Following Shimura [44], [45], one can identify the space $M_{\boldsymbol{k}}(\mathfrak{n}, \psi, \theta)$ with a set of complex valued functions $\mathbf{f}$ on $GL(2, \mathbb{A}_K)$ which satisfy the following properties,

(1) $\mathbf{f}(\gamma x w) = \psi_Y(w^{inv}) f(x)$ for all $\gamma \in GL(2, K), x \in GL(2, \mathbb{A}_K)$, and $w \in W_0(\mathfrak{n})$ and $w_\infty = 1$.

(2) Corresponding to each λ there exists $f_\lambda \in M_{\boldsymbol{k}}$ such that

$$\mathbf{f}(x_\lambda^{-inv}\gamma) = \det(\gamma)^{i\boldsymbol{m}}(f_\lambda|\gamma)(\mathbf{i}),$$

for all $\gamma \in GL^+(2, \mathbb{R})^d$.

Hereafter, the space of functions on $GL(2, \mathbb{A}_K)$ satisfying property (1) and (2) will be denoted by $M_{\boldsymbol{k}}(\mathfrak{n}, \psi, \boldsymbol{m})$ with fixed $\boldsymbol{m} \in \mathbb{R}^d$ such that $\theta(\epsilon) = \epsilon^{i\boldsymbol{m}}$ for all totally positive units in $\mathcal{O}_K$. We denote the subspace of cusp forms of $M_{\boldsymbol{k}}(\mathfrak{n}, \psi, \boldsymbol{m})$ by $S_{\boldsymbol{k}}(\mathfrak{n}, \psi, \boldsymbol{m})$.

1.2.2 The space $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$

In this thesis, we mainly deal with the problems related to the space $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$, which will be defined in this section. Now we have collected almost all ingredients to define the space $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$. We begin with the definition of Hecke character.

Definition 1.2.6 A *Hecke character* Ψ is a continuous homomorphism from $\mathbb{I}_K$ to $\mathbb{C}^\times$ such that Ψ is trivial on $K^\times$.

Action of $\mathbb{I}_K$ on $M_{\boldsymbol{k}}(\mathfrak{n}, \psi, \boldsymbol{m})$

For any $s \in \mathbb{I}_K$, one can think of s as an endomorphism of $M_k(\mathfrak{n}, \psi, \boldsymbol{m})$ given by $\mathbf{f}(x) \to \mathbf{f}(sx)$. This action induces an unitary representation of $\mathbb{I}_K$ in

$M_{\boldsymbol{k}}(\mathfrak{n},\psi,\boldsymbol{m})$. Since $\mathbb{I}_K$ is an abelian group, by Schur's lemma this representation decomposes into a direct sum of irreducible representations of dimension 1. Let Ψ be a character (not necessarily Hecke character) of $\mathbb{I}_K$. Now we define the space $M_{\boldsymbol{k}}(\mathfrak{n},\Psi)$ as

$$M_{\boldsymbol{k}}(\mathfrak{n},\Psi) := \left\{\mathbf{f} \in M_{\boldsymbol{k}}(\mathfrak{n},\psi,\boldsymbol{m}) \;\middle|\; \mathbf{f}(sx) = \Psi(s)\mathbf{f}(x)\right\},$$

for all $s \in \mathbb{I}_K$ and $x \in GL(2,\mathbb{A}_K)$. Note that by definition, $\mathbf{f}(sx) = \mathbf{f}(x)$ for all $s \in K^\times$. This forces $M_{\boldsymbol{k}}(\mathfrak{n},\Psi)$ to be non trivial only when Ψ is trivial on $K^\times$, consequently Ψ is a Hecke character. Let $S_{\boldsymbol{k}}(\mathfrak{n},\Psi)$ denotes the space of cusp forms of $M_{\boldsymbol{k}}(\mathfrak{n},\Psi)$. Let $\boldsymbol{m}$ be as above, we define a homomorphism ψ_∞ from $\mathbb{I}_K$ to $\mathbb{C}^\times$ by $\psi_\infty(a) = \operatorname{sgn}(a_\infty)^{\boldsymbol{k}}|a_\infty|^{2i\boldsymbol{m}}$.

Remark 1.2.7 In the view of the above definitions of the spaces $M_{\boldsymbol{k}}(\mathfrak{n},\psi,\boldsymbol{m})$ and $M_{\boldsymbol{k}}(\mathfrak{n},\Psi)$, we have $\Psi(a) = \psi(a_{\mathfrak{n}})\psi_\infty(a)$ for all $a \in (\mathbb{R}^\times)^n \times \prod_{\wp<\infty}\mathcal{O}_\wp^\times$ see [equation 9.22 of [45]].

We say that a Hecke character Ψ is an extension of $\psi\psi_\infty$ if $\Psi(a) = \psi(a_{\mathfrak{n}})\psi_\infty(a)$ for all $a \in (\mathbb{R}^\times)^n \times \prod_{\wp<\infty}\mathcal{O}_\wp^\times$. Then one gets that there are only finitely many such Hecke characters.

Ideal character

Let $\mathfrak{n}_\infty$ be the product of all infinite places. For a character ψ of $(\mathcal{O}/\mathfrak{n})^\times$, the conductor of a Hecke character extending $\psi\psi_\infty$ is a divisor of $\mathfrak{n}\mathfrak{n}_\infty$. Following Shimura, we define an ideal character Ψ^* modulo $\mathfrak{n}\mathfrak{n}_\infty$ as follows. For a prime ideal $\wp$ of K

$$\Psi^*(\wp) := \begin{cases} \Psi(\pi_\wp) & \wp \nmid \mathfrak{n} \text{ and } \pi\mathcal{O} = \wp \\ 0 & \wp \mid \mathfrak{n} \end{cases}.$$

Note that for an integral ideal $\mathfrak{a}$ such that $(\mathfrak{a}, \mathfrak{n}) \neq 1$, we have $\Psi^*(\mathfrak{a}) = 0$ independent of whether Ψ is trivial character or not.

1.2.3 Fourier Coefficients of adelic Hilbert modular forms

Let $\mathbf{f}$ be an element of $M_{\boldsymbol{k}}(\mathfrak{n}, \psi, \boldsymbol{m})$, by the correspondence described earlier there is an h-tuple $(f_1, \ldots, f_h)$ of classical Hilbert modular forms where each f_i has the following Fourier series expansion,

$$f_\lambda(z) = a_\lambda(0) + \sum_{0 \ll \xi \in \mathfrak{I}_\lambda} a_\lambda(\xi) \exp(2\pi i Tr(\xi z)).$$

Now following [44], the Fourier coefficients of $\mathbf{f}$ which are indexed by integral ideals of K is given by

$$C(\mathbf{f}, \mathfrak{m}) = \begin{cases} N(\mathfrak{m})^{\frac{1}{2}} a_\lambda(\xi) \xi^{\frac{-\boldsymbol{k}}{2}} & \text{if } \mathfrak{m} = \xi \mathfrak{I}_\lambda^{-1} \subset \mathcal{O}_K \\ 0 & \text{otherwise} \end{cases}, \tag{1.2}$$

where $\mathfrak{m}$ is an integral ideal of K and k_0 is the maximum of $\{k_1, \ldots, k_d\}$.

Petersson Inner Product

Let $\mathbf{f} = (f_1, \ldots, f_h)$ and $\mathbf{g} = (g_1, \ldots, g_h)$ be two elements of $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$. The Petersson inner product of $\mathbf{f}$ and $\mathbf{g}$ is given by

$$\langle \mathbf{f}, \mathbf{g} \rangle := \sum_{\mathfrak{v}=1}^{h} \langle f_\mathfrak{v}, g_\mathfrak{v} \rangle_\mathfrak{n} = \sum_{\mathfrak{v}=1}^{h} \frac{1}{\mu(\Gamma_\mathfrak{v}(\mathfrak{n}) \backslash \mathcal{H}^d)} \int_{\Gamma_\mathfrak{v}(\mathfrak{n}) \backslash \mathcal{H}^d} f_\mathfrak{v}(z) \overline{g_\mathfrak{v}(z)} y^{\boldsymbol{k}} d\mu(z),$$

where $d\mu(z) = \prod_{j=1}^d y_j^{-2} dx_j dy_j$. The inner product is well-defined if $f_\mathfrak{v} g_\mathfrak{v}$ is cusp form for all $1 \leq \mathfrak{v} \leq h$.

1.3 Hecke operators

In this section we introduce some linear operators on the space $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ called as *Hecke operators*. These operators are indexed by integral ideals. The family $\{T_{\mathfrak{l}}\}_{\mathfrak{l}}$ forms a commutative ring of normal operators on the space $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$. Some of the properties of these operators are stated in the following theorem.

Theorem 1.3.1 ([44], [47]) *Let* $\mathbf{f}$ *be an element of* $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$. *Then the following statements hold.*

(1) $T_{\mathfrak{l}}$ *maps* $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ *to* $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$, *cusp forms to cusp forms, and is independent of whether* $(\mathfrak{n}, \mathfrak{l}) = 1$ *or not.*

(2) The effect of $T_{\mathfrak{l}}$ *on the Fourier coefficients of* $\mathbf{f}$ *is given by*

$$C(T_{\mathfrak{l}}\mathbf{f}, \mathfrak{m}) = \sum_{\mathfrak{m}+\mathfrak{l} \subset \mathfrak{a}} \Psi^*(\mathfrak{a}) N(\mathfrak{a})^{k_0-1} C(\mathbf{f}, \mathfrak{a}^{-2}\mathfrak{l}\mathfrak{m}).$$

(3) If $\mathbf{f}$ *is an eigenfunction for all Hecke operators* $T_{\mathfrak{l}}$ *with* $C(\mathbf{f}, \mathcal{O}_K) = 1$, *then*

$$\begin{aligned} C(\mathbf{f}, \mathfrak{m}\mathfrak{m}') &= C(\mathbf{f}, \mathfrak{m})C(\mathbf{f}, \mathfrak{m}') \qquad \text{if } (\mathfrak{m}, \mathfrak{m}') = 1, \\ C(\mathbf{f}, \wp^r) &= C(\mathbf{f}, \wp)C(\mathbf{f}, \wp^{r-1}) - \Psi(\wp)N(\wp)^{k_0-1}C(\mathbf{f}, \wp^{r-2}), \end{aligned} \tag{1.3}$$

where $\wp$ *is a prime ideal and* $r \geq 2$.

(4) The Hecke operators are Ψ *hermitian in the sense that*

$$\Psi^*(\mathfrak{l})\langle T_{\mathfrak{l}}\mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{f}, T_{\mathfrak{l}}\mathbf{g}\rangle,$$

for all integral ideals $\mathfrak{l}$ *with* $(\mathfrak{n}, \mathfrak{l}) = 1$.

(5) The shift operator $B_{\mathfrak{l}}$ is defined by

$$C(\mathbf{f}|B_{\mathfrak{l}}, \mathfrak{m}) = C\left(\mathbf{f}, \frac{\mathfrak{m}}{\mathfrak{l}}\right).$$

(6) The shift operator $B_{\mathfrak{l}}$ maps $M_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ to $M_{\boldsymbol{k}}(\mathfrak{l}\mathfrak{n}, \Psi)$ and cusp forms to cusp forms.

1.3.1 Newforms in $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$

The aim of this section is to briefly discuss the newform theory for the space $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ developed by Shemanske and Walling in the context of Hilbert modular forms [47]. First, we define $S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi)$, a subspace of $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$, generated by the forms $\mathbf{g}|B_{\mathfrak{l}}$, where $\mathbf{g} \in S_{\boldsymbol{k}}(\mathfrak{n}', \Psi)$ with $\mathfrak{n} \neq \mathfrak{n}', \mathfrak{n}' \mid \mathfrak{n}$ and $\mathfrak{l} \mid \left(\frac{\mathfrak{n}}{\mathfrak{n}'}\right)$. The significant property of the space $S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi)$ is that it remains invariant under the action of all Hecke operators $T_{\mathfrak{l}}$ for $(\mathfrak{n}, \mathfrak{l}) = 1$. Let $S_{\boldsymbol{k}}^{+}(\mathfrak{n}, \Psi)$ be the orthogonal complement of $S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi)$ in $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ with respect to the Petersson inner product. Note that for each integral ideal $\mathfrak{l}$ such that $(\mathfrak{n}, \mathfrak{l}) = 1$, the operator $T_{\mathfrak{l}}$ maps $S_{\boldsymbol{k}}^{+}(\mathfrak{n}, \Psi)$ to itself, which is a consequence of the fact that $T_{\mathfrak{l}}$ is a hermitian operator and maps $S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi)$ to $S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi)$.

Definition 1.3.2 A form $\mathbf{f} \in S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ is said to be a *new form* if $\mathbf{f} \in S_{\boldsymbol{k}}^{+}(\mathfrak{n}, \Psi)$ and $\mathbf{f}$ is an eigenfunction for all Hecke operators $T_{\wp}$, where $\wp$ is a prime ideal not dividing $\mathfrak{n}$. We say that $\mathbf{f}$ is a *normalized newform* (or a *primitive form*) if $C(\mathcal{O}_K, \mathbf{f}) = 1$.

Proposition 1.3.3 ([47]) *For the space $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ the following statements hold.*

(1) $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi) = S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi) \oplus S_{\boldsymbol{k}}^{+}(\mathfrak{n}, \Psi)$.

(2) The spaces $S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi)$ and $S_{\boldsymbol{k}}^{+}(\mathfrak{n}, \Psi)$ are stable under the action of $T_{\mathfrak{l}}$ with $(\mathfrak{l}, \mathfrak{n}) = 1$.

(3) The spaces $S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi)$ and $S_{\boldsymbol{k}}^{+}(\mathfrak{n}, \Psi)$ have an orthogonal basis which is formed by the eigenforms of all the Hecke operators $T_{\mathfrak{l}}$ with $(\mathfrak{n}, \mathfrak{l}) = 1$.

(4) We have

$$S_{\boldsymbol{k}}(\mathfrak{n}, \Psi) = \oplus_{\mathfrak{n}_0 | \frac{\mathfrak{n}}{\mathfrak{n}'}} \oplus_{\mathfrak{q} | \frac{\mathfrak{n}}{\mathfrak{n}'}} S_{\boldsymbol{k}}^{+}(\mathfrak{n}', \Psi) | B(\mathfrak{q}), \tag{1.4}$$

where $\mathfrak{n}_0$ is conductor of Ψ and $B(\mathfrak{q})$ is the shift operator.

Remark 1.3.4 The spaces $S_{\boldsymbol{k}}^{-}(\mathfrak{n}, \Psi)$ and $S_{\boldsymbol{k}}^{+}(\mathfrak{n}, \Psi)$ are known as the space of oldforms and the space of newforms, respectively. Note that not every form in $S_{\boldsymbol{k}}^{+}(\mathfrak{n}, \Psi)$ is a newform but only those that are eigenfunction of the $T_{\mathfrak{l}}$ with $(\mathfrak{l}, \mathfrak{n}) = 1$.

L-function

In 2006, Blasius proved the Ramanujan-Petersson conjecture for Hilbert modular forms [11]. For a primitive form $\mathbf{f}$ in $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ and for any $\epsilon > 0$, we have

$$C(\mathbf{f}, \mathfrak{m}) \ll_{\epsilon} N(\mathfrak{m})^{\epsilon}. \tag{1.5}$$

The L-series associated with $\mathbf{f}$ is given by

$$L(s, \mathbf{f}) = \sum_{\mathfrak{m}} \frac{C(\mathbf{f}, \mathfrak{m})}{N(\mathfrak{m})^s}.$$

By the Ramanujan bound $L(s, \mathbf{f})$ is absolutely convergent for $\mathrm{Re}(s) > 1$.

Theorem 1.3.5 ([44]) *Let $\mathbf{f}$ be a primitive form in $S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$. Then*

(1) $L(s,\mathbf{f})$ *has the following Euler product which is*

$$L(s,\mathbf{f})=\prod_{\wp\mid\mathfrak{n}}(1-C(\mathbf{f},\wp)N(\wp)^{-s})^{-1}\prod_{\wp\nmid\mathfrak{n}}(1-C(\mathbf{f},\wp)N(\wp)^{-s}+\Psi^*(\wp)N(\wp)^{-2s})^{-1}.$$

(2) For a trivial character Ψ*, we have* $L(s,\mathbf{f})$ *has an analytic continuation to the entire complex plane. Set*

$$\Lambda(s,\mathbf{f})=N(\mathfrak{n}\mathfrak{D}_K^2)(2\pi)^{-ds}\prod_{j=1}^{d}\Gamma\left(s+\frac{k_j-1}{2}\right)L(s,\mathbf{f}).$$

Then

$$\Lambda(s,\mathbf{f})=w_{\mathbf{f}}\Lambda(1-s,\mathbf{f}),$$

where $w_{\mathbf{f}}$ *is a root of unity.*

Twist of a newform

For a Hecke character Φ with conductor $\mathfrak{l}_0$ and $\mathbf{f}\in S_{\boldsymbol{k}}(\mathfrak{n},\Psi)$, the twist $\mathbf{f}|_\Phi$ of $\mathbf{f}$ with respect to Φ is given by

$$C(\mathbf{f}|_\Phi,\mathfrak{m})=\Phi^*(\mathfrak{m})C(\mathbf{f},\mathfrak{m}).$$

It is well known that $\mathbf{f}|_\Phi\in S_{\boldsymbol{k}}(\mathfrak{l}',\Psi\Phi^2)$, where $\mathfrak{l}'=\operatorname{lcm}(\mathfrak{n},\mathfrak{l}_0\mathfrak{n}_0,\mathfrak{l}_0^2)$ and $\mathfrak{n}_0$ the conductor of Ψ. The following theorem tells us that when a twist of a newform is a newform.

Theorem 1.3.6 ([47]) *Let* Φ *be a Hecke character with conductor* $\mathfrak{l}$*. For a normalized form* $\mathbf{f}\in S_{\boldsymbol{k}}(\mathfrak{n},\Psi)$*, the twist* $\mathbf{f}|_\Phi$ *is a normalized newform in* $S_{\boldsymbol{k}}(\mathfrak{n}\mathfrak{l}^2,\Psi\Phi^2)$ *whenever* $(\mathfrak{l},\mathfrak{n})=1$.

1.3.2 The Rankin-Selberg Method

Given two primitive forms $\mathbf{f}$ and $\mathbf{g}$ in $S_{\boldsymbol{k}}(\mathfrak{n})$, let $\alpha_1(\wp), \alpha_2(\wp)$ and $\beta_1(\wp), \beta_2(\wp)$ be the roots of the quadratic polynomials $x^2 - C(\mathbf{f}, \wp)x + \Psi(\wp)$ and $x^2 - C(\mathbf{g}, \wp)x + \Psi(\wp)$, where $\Psi(\wp)$ is either 0 or 1 accordingly as prime ideal $\wp$ divides $\mathfrak{n}$ or not. The Rankin-Selberg convolution $L(s, \mathbf{f} \otimes \mathbf{g})$ of $\mathbf{f}$ and $\mathbf{g}$ is defined as

$$L(s, \mathbf{f} \otimes \mathbf{g}) := \prod_{\wp} \prod_{i,j=1}^{2} \left(1 - \frac{\alpha_i(\wp)\overline{\beta_j(\wp)}}{N(\wp)^s}\right)^{-1}. \tag{1.6}$$

Due to the multiplicative nature of Fourier coefficients of primitive forms, the following equality holds

$$L(s, \mathbf{f} \otimes \mathbf{g}) = \zeta_K^{\mathfrak{n}}(2s) \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ \mathfrak{m} \neq \{0\}}} \frac{C(\mathbf{f}, \mathfrak{m})C(\mathbf{g}, \mathfrak{m})}{N(\mathfrak{m})^s} = \zeta_K^{\mathfrak{n}}(2s) L_1(s, \mathbf{f} \otimes \mathbf{g}), \tag{1.7}$$

where

$$\zeta_K^{\mathfrak{n}}(2s) = \zeta_K(2s) \prod_{\mathfrak{l} | \mathfrak{n}} (1 - N(\mathfrak{l})^{-2s}), \qquad L_1(s, \mathbf{f} \otimes \mathbf{g}) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ \mathfrak{m} \neq \{0\}}} \frac{C(\mathbf{f}, \mathfrak{m})C(\mathbf{g}, \mathfrak{m})}{N(\mathfrak{m})^s}.$$

Let

$$L_\infty(s, \mathbf{f} \otimes \mathbf{g}) = \prod_{j=1}^{d} (2\pi)^{-2s-k_j} \Gamma(s)\Gamma(s - 1 + k_j),$$

and let

$$\Lambda(s, \mathbf{f} \otimes \mathbf{g}) = N(\mathfrak{D}_K^2 \mathfrak{n})^s L_\infty(s, \mathbf{f} \otimes \mathbf{g}) L(s, \mathbf{f} \otimes \mathbf{g}). \tag{1.8}$$

Then $\Lambda(s, \mathbf{f} \otimes \mathbf{g})$ has a meromorphic continuation to the whole of $\mathbb{C}$, and satisfies the following functional equation

$$\Lambda(s, \mathbf{f} \otimes \mathbf{g}) = \Lambda(1 - s, \mathbf{f} \otimes \mathbf{g}). \tag{1.9}$$

We end this chapter with the following theorem due to Shimura [44].

Theorem 1.3.7 ([44]) *The function* $\Lambda(s, \mathbf{f} \otimes \mathbf{g})$ *has an analytic continuation to the whole of* $\mathbb{C}$ *if* $\mathbf{f} \neq \mathbf{g}$*, otherwise it has a meromorphic continuation to the whole plane, with possible simple poles at* $s = 1$ *and* $s = 0$*. The residue of* $L(s, \mathbf{f} \otimes \mathbf{f})$ *at* $s = 1$ *is*

$$2^{d-1}(4\pi)^{\boldsymbol{k}}\zeta_K^{\mathfrak{n}}(2)\Gamma(\boldsymbol{k})^{-1}R_K[\mathcal{O}_K^{\times +} : \mathcal{O}_K^{\times 2}]^{-1}\langle \mathbf{f}, \mathbf{f}\rangle_{\mathfrak{n}}.$$

CHAPTER 2

Sturm-like bound for square-free Fourier coefficients

This chapter investigates the problem of determining a Hilbert modular form by its Fourier coefficients indexed by square-free integral ideals. We obtain an upper bound (say B) that depends only on the level and weight such that if the Fourier coefficients of a Hilbert modular form indexed by square-free integral ideals having norm less or equal to B are zero, then the form is identically zero. The main theorem of this chapter is a generalization of the result obtained by S. Das and P. Anamby [4] and [5] in the context of elliptic modular forms. The results of this chapter have been published in [2].

2.1 Introduction

Let k and N be two positive integers, let $M_k(\Gamma_0(N))$ be the space of elliptic modular forms of weight k on $\Gamma_0(N)$. It is well known that any element f of

$M_k(\Gamma_0(N))$ has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n, \tag{2.1}$$

where $z \in \mathcal{H}$ and $q = \exp(2\pi i z)$. The Fourier coefficients of a modular form completely determine the modular form in the sense that two modular forms are equal if and only if they have the same Fourier coefficients. One can ask the following natural questions.

(1) *Can one determine modular forms by a proper subset of all Fourier coefficients?*

(2) *If yes, then what is its size, is it finite or infinite?*

(3) *If it is finite then what is its cardinality?*

It is not always possible to obtain the exact cardinality of a set using analytical techniques, and hence one naturally looks for an upper or lower bound of cardinality in terms of some known parameters of the set. The well known bound for $M_k(\Gamma_0(N))$ is the *"Sturm's bound"* that was obtained by E. Hecke. Sturm's bound tells us that if all the Fourier coefficients of f up to $\frac{k}{12}[SL(2,\mathbb{Z}) : \Gamma_0(N)]$ are zero then the form f is identically zero [23]. There are many such results in the literature, for example, "the multiplicity one theorem" which asserts that if two normalized Hecke eigenforms have the same eigenvalues at Hecke operators T_p for all but finitely many primes p then both forms are equal. D. Ramakrishnan [42] strengthened the result by replacing the need for all most all primes with only those set of primes whose Dirichlet density is greater than $\frac{7}{8}$, and a similar result for Siegel modular forms was proved by Siegel himself. Regardless our main focus is to determine modular forms by their Fourier coefficients indexed by square-free positive integers.

2.2 Overview of previous work

The question of determining a modular form by its Fourier coefficients indexed by square-free positive integers was first raised by S. Das and P. Anmby [4], [5]. More concretely they asked the following question. Let k and N be two positive integers and χ be a Dirichlet character modulo N. Let $S_k(N, \chi)$ be the space of cusp forms of weight k on $\Gamma_0(N)$ with nebentypus χ of conductor N_0.

Question *Let $f \in S_k(N, \chi)$ have a Fourier expansion as in (2.1). If $\frac{N}{N_0}$ is square-free, does there exists a positive real number A depending only on k and N such that if $a_f(m) = 0$ for all square-free positive integers $m \leq A$, then $f = 0$?*

Anamby et al. answered the above question affirmatively using the prime number theorem for L-function associated to a newform and applying an argument of Balog and Ono [8]. Before, we state their result, we define the following.

Let $A_f(k, N)$ to be the smallest positive integer such that if $f \in M_k(\Gamma_0(N))$, $\frac{N}{N_0}$ is square-free, and $a_f(m) = 0$ for all square-free positive integers $m \leq A_f(k, N)$, then $f = 0$. More precisely, they proved the following theorem.

Theorem 2.2.1 ([4]) *Let N be a positive square-free integer and $k \geq 2$. Then*

$$A_f(k, N) \leq A_0 2^{\frac{r(r-1)}{2}} N \exp(4r \log^2(7k^2 N)),$$

where $r = \frac{k-1}{2} N$ and $A_0 \in \mathbb{R}^+$ is an absolute constant.

Note that the above bound is exponential in k and N. Later Anamby et al. [5] improved the above result by replacing the prime number theorem with the Rankin-Selberg method and then invoking newform theory. We now state the improved result of Anamby et al.

Theorem 2.2.2 ([5]) *Let k and N be two positive integers and χ be a Dirichlet character modulo N with conductor N_0 such that $\frac{N}{N_0}$ is square-free. Assume that f is a non-zero element of $S_k(\Gamma_0(n), \chi)$. Then for any $\epsilon > 0$, there exists a square-free positive integer $m \ll k^{3+\epsilon} N^{\frac{7}{2}+\epsilon}$ such that $a_f(m) \neq 0$, and the implied constant depends only on ϵ.*

Here, we would like to remark that the upper bound obtained in Theorem 2.2.2 is a significant improvement of the bound in Theorem 2.2.1.

We use the notation as in Chapter 1. Throughout this chapter, K is a totally real field of degree d over $\mathbb{Q}$. We now state the main result of this chapter.

2.3 Main Result

Theorem 2.3.1 ([2]) *Let $\mathfrak{n}$ be a square-free integral ideal of K. Let $\mathbf{f} \in S_{\boldsymbol{k}}(\mathfrak{n})$ be a non-zero adelic Hilbert modular form with Fourier coefficients $C(\mathbf{f}, \mathfrak{m})$. Then there exists a square-free integral ideal $\mathfrak{m}$ with $N(\mathfrak{m}) \ll k_0^{3d+\epsilon} N(\mathfrak{n})^{\frac{6d^2+1}{2}+\epsilon}$ such that $C(\mathbf{f}, \mathfrak{m}) \neq 0$. The implied constant depends only on ϵ and K.*

Note that the above Theorem 2.3.1 is a generalization of the Theorem 2.2.2 in the context of Hilbert modular forms.

2.4 Preliminaries

In this section, we collect all the ingredients that we need to prove our main Theorem 2.3.1.

2.4.1 Some Dirichlet series

We recall the theory of L-function attached to a normalized newform. Let $\mathbf{f} \in S_{\boldsymbol{k}}(\mathfrak{n})$ be a normalized newform having Fourier coefficients $C(\mathbf{f}, \mathfrak{m})$ and $\wp$ be a prime ideal of $\mathcal{O}_K$. The L-function associated to $\mathbf{f}$ is given by

$$L(s, \mathbf{f}) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ \mathfrak{m} \neq \{0\}}} \frac{C(\mathbf{f}, \mathfrak{m})}{N(\mathfrak{m})^s}.$$

We also assume that Fourier coefficients $C(\mathbf{f}, \mathfrak{m})$ are normalized in the sense that $L(s, \mathbf{f})$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Let $\mathbf{f}$ and $\mathbf{g}$ be two normalized newforms in $S_{\boldsymbol{k}}(\mathfrak{n})$. Let $L(s, \mathbf{f} \otimes \mathbf{g})$ be the Rankin-Selberg of $\mathbf{f}$ and $\mathbf{g}$,

$$L(s, \mathbf{f} \otimes \mathbf{g}) := \prod_{\wp} \prod_{i,j=1}^{2} (1 - \alpha_i(\wp)\overline{\beta_j(\wp)} N(\wp)^{-s})^{-1}, \tag{2.2}$$

where $\alpha_1(\wp), \alpha_2(\wp)$ and $\beta_1(\wp), \beta_2(\wp)$ are the roots of the following quadratic polynomials

$$x^2 - C(\mathbf{f}, \wp)x + \Psi^*(\wp) \qquad \text{and} \qquad x^2 - C(\mathbf{g}, \wp)x + \Psi^*(\wp),$$

respectively, with

$$\Psi^*(\wp) = \begin{cases} 1 & \wp \nmid \mathfrak{n}, \\ 0 & \wp \mid \mathfrak{n}. \end{cases}$$

Using the multiplicative property of Fourier coefficients, we can write $L(s, \mathbf{f} \otimes \mathbf{g})$ as follows

$$L(s, \mathbf{f} \otimes \mathbf{g}) = \zeta_K^{\mathfrak{n}}(2s) \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ \mathfrak{m} \neq \{0\}}} \frac{C(\mathbf{f}, \mathfrak{m}) C(\mathbf{g}, \mathfrak{m})}{N(\mathfrak{m})^s} = \zeta_K^{\mathfrak{n}}(2s) L_1(s, \mathbf{f} \otimes \mathbf{g}), \tag{2.3}$$

where

$$\zeta_K^{\mathfrak{n}}(2s) = \zeta_K(2s) \prod_{\wp | \mathfrak{n}} (1 - N(\wp)^{-2s}), \quad L_1(s, \mathbf{f} \otimes \mathbf{g}) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ \mathfrak{m} \neq \{0\}}} \frac{C(\mathbf{f}, \mathfrak{m}) C(\mathbf{g}, \mathfrak{m})}{N(\mathfrak{m})^s}.$$

Define

$$L_{\mathfrak{n}}(s, \mathbf{f} \otimes \mathbf{g}) := \prod_{\wp \nmid \mathfrak{n}} \left(1 - N(\wp)^{-2s}\right) \prod_{i,j=1}^{2} \left(1 - \alpha_i(\wp)\overline{\beta_j(\wp)} N(\wp)^{-s}\right)^{-1}, \tag{2.4}$$

and

$$L^{\#}(s, \mathbf{f} \otimes \mathbf{g}) = \prod_{\wp \nmid \mathfrak{n}} \left(1 + \frac{C(\mathbf{f}, \wp) C(\mathbf{g}, \wp)}{N(\wp)^s}\right) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ (\mathfrak{m}, \mathfrak{n}) = \mathcal{O}_K}}^{\#} \frac{C(\mathbf{f}, \mathfrak{m}) C(\mathbf{g}, \mathfrak{m})}{N(\mathfrak{m})^s}, \tag{2.5}$$

where # indicates that the sum is over the square-free integral ideals of K.

We now relate $L(s, \mathbf{f} \otimes \mathbf{g})$ and $L_{\mathfrak{n}}(s, \mathbf{f} \otimes \mathbf{g})$ by multiplying and dividing $L_{\mathfrak{n}}(s, \mathbf{f} \otimes \mathbf{g})$ by $\prod_{\wp | \mathfrak{n}} \prod_{i,j=1}^{2} (1 - \alpha_i(\wp)\overline{\beta_j(\wp)} N(\wp)^{-s})^{-1}$

$$\begin{aligned} L_{\mathfrak{n}}(s, \mathbf{f} \otimes \mathbf{g}) &= \left(\prod_{\wp} \prod_{i,j=1}^{2} (1 - \alpha_i(\wp)\overline{\beta_j(\wp)} N(\wp)^{-s})^{-1}\right)\left(\prod_{\wp \nmid \mathfrak{n}} (1 - N(\wp)^{-2s}\right) \\ &\quad \left(\prod_{\wp | \mathfrak{n}} \prod_{i,j=1}^{2} (1 - \alpha_i(\wp)\overline{\beta_j(\wp)} N(\wp)^{-s})\right) \\ &= L(s, \mathbf{f} \otimes \mathbf{g}) F(s), \end{aligned} \tag{2.6}$$

where $F(s)$ is the following absolutely convergent Dirichlet series for $\mathrm{Re}(s) > \frac{1}{2}$

$$F(s) = \left(\prod_{\wp \nmid \mathfrak{n}} (1 - N(\wp)^{-2s})\right)\left(\prod_{\wp | \mathfrak{n}} \prod_{i,j=1}^{2} (1 - \alpha_i(\wp)\overline{\beta_j(\wp)} N(\wp)^{-s})\right).$$

Analytic conductor

Let us define the analytic conductor of $L(s, \mathbf{f} \otimes \mathbf{g})$. Following [26], the analytic conductor $\mathfrak{q}(s, \mathbf{f} \otimes \mathbf{g})$ of $L(s, \mathbf{f} \otimes \mathbf{g})$ is given by

$$\mathfrak{q}(s, \mathbf{f} \otimes \mathbf{g}) = q(\mathbf{f} \otimes \mathbf{g})\mathfrak{q}_{\infty}(s),$$

where $q(\mathbf{f} \otimes \mathbf{g}) = N(\mathfrak{n}\mathcal{O}_K^2)^2$ and

$$\mathfrak{q}_{\infty}(s) = \prod_{j=1}^{d}(|s|+3)(|s+1|+3)(|s+k_j-1|+3)(|s+k_j|+3).$$

We also define

$$\mathfrak{q}(\mathbf{f} \otimes \mathbf{g}) = \mathfrak{q}(0, \mathbf{f} \otimes \mathbf{g}) = q(\mathbf{f} \otimes \mathbf{g}) \prod_{j=1}^{d}(3)(|1|+3)(|k_j-1|+3)(|k_j|+3),$$

$$\mathfrak{q}(\mathbf{f} \otimes \mathbf{g}) \asymp 4^{2d} N(\mathfrak{n}\mathcal{O}_K^2)^2 (\prod_{j=1}^{d}(k_j))^2.$$

Now by [26] we have,

$$\begin{aligned}\mathfrak{q}(s, \mathbf{f} \otimes \mathbf{g}) &\leq N(\mathfrak{n}\mathcal{O}_K^2)^2 4^{2d} (\prod_{j=1}^{d}(k_j))^2 (|s|+3)^{4d} \\ &\leq N(\mathfrak{n}\mathcal{O}_K^2)^2 4^{2d} k_0^{2d} (|s|+3)^{4d}.\end{aligned} \tag{2.7}$$

Smooth cut off function

Let g be a positive and smooth function supported on $[\frac{1}{2}, 1]$. The Mellin transform $M(g)(s)$ of g is given by,

$$M(g)(s) = \int_0^{\infty} g(y) y^{s-1} dy.$$

Note that $M(g)$ converges for any $s \in \mathbb{C}$, thus $M(g)(s)$ defines an entire function and satisfies

$$M(g)(s) \ll |s|^{-A-1},$$

for any $A > 0$ (for instance, see [5]). As g is a smooth and compactly supported function, the Mellin inversion formula holds. Moreover one can recover g from $M(g)(s)$ via the inversion formula given by,

$$g(y) = \frac{1}{2\pi i} \int_{(\sigma)} M(g)(s) y^{-s} ds, \tag{2.8}$$

where (σ) denotes that the integration is taken over the vertical line $\mathrm{Re}(s) = \sigma$. For more details we refer [26].

2.4.2 Few lemmas

Let us recall a result of Harcos [22] (see also Qu [41]). It is well known that $L(s, \mathbf{f} \otimes \mathbf{g})$ is automorphic [43]. We now state a lemma that talks about the growth of $L(s, \mathbf{f} \otimes \mathbf{g})$ in a vertical strip.

Lemma 2.4.1 ([22]) *If $s = \sigma + it$ and $0 < \sigma < 1$, then for any $\epsilon > 0$ we have,*

$$L(s, \mathbf{f} \otimes \mathbf{g}) \ll_{\epsilon, K} (\mathfrak{q}(s, \mathbf{f} \otimes \mathbf{g}))^{\frac{1-\sigma}{2} + \epsilon}.$$

Lemma 2.4.2 *Let $L_{\mathfrak{n}}(s, \mathbf{f} \otimes \mathbf{g})$ and $L^{\#}(s, \mathbf{f} \otimes \mathbf{g})$ be as defined in (2.4) and (2.5), respectively. Then*

$$L^{\#}(s, \mathbf{f} \otimes \mathbf{g}) = L_{\mathfrak{n}}(s, \mathbf{f} \otimes \mathbf{g}) H(s),$$

where $H(s)$ is an absolutely convergent Dirichlet series for $Re(s) > \frac{1}{2}$.

Proof. It suffices to prove the above result for prime factors $\wp$. The $\wp$-factor of

$L^{\#}(s, \mathbf{f} \otimes \mathbf{g})\big(L_{\mathfrak{n}}(s, \mathbf{f} \otimes \mathbf{g})\big)^{-1}$ is given by,

$$\begin{aligned}
\frac{L^{\#}_{\wp}(s, \mathbf{f} \otimes \mathbf{g})}{L_{\mathfrak{n},\wp}(s, \mathbf{f} \otimes \mathbf{g})} &= \big(1 + C(\mathbf{f}, \wp)C(\mathbf{g}, \wp)N(\wp)^{-s}\big)\big(1 - N(\wp)^{-2s}\big)^{-1} \\
&\quad \times \prod_{i,j=1}^{2} \big(1 - \alpha_i(\wp)\overline{\beta_j(\wp)}N(\wp)^{-s}\big) \\
&= \big(1 - N(\wp)^{-2s}\big)^{-1}\big(1 + C(\mathbf{f}, \wp)C(\mathbf{g}, \wp)N(\wp)^{-s}\big) \\
&\quad \times \Big(\big(1 - C(\mathbf{f}, \wp)C(\mathbf{g}, \wp)N(\wp)^{-s}\big) + O\big(N(\wp)^{-2s}\big)\Big) \\
&= \big(1 - N(\wp)^{-2s}\big)^{-1}\big(1 + O\big(N(\wp)^{-2s}\big)\big) \\
&= H_{\wp}(s).
\end{aligned}$$

It is easy to see that $\prod_{\wp \nmid \mathfrak{n}} H_{\wp}(s) = H(s)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$.

□

Lemma 2.4.3 *Let ϵ be any positive real number. Then*

$$Res_{s=1} L(s, \mathbf{f} \otimes \mathbf{g}) \gg_{\epsilon,K} (k_0 N(\mathfrak{n}))^{-\epsilon}.$$

Proof. Note that $L(s, \mathbf{f} \otimes \mathbf{g})$ does not have any real zero in in the interval $(0, 1)$, the proof now can easily be completed from [24]. □

Lemma 2.4.4 *Let $\mathfrak{m}$ be an integral ideal. Let $d(\mathfrak{m})$ be the number of integral ideal dividing $\mathfrak{m}$. Then for any $\epsilon > 0$,*

$$d(\mathfrak{m}) \ll_{\epsilon} (N(\mathfrak{m}))^{\epsilon}.$$

Proof. To prove the assertion, we follow [38]. Let $\mathfrak{m}$ be an integral ideal,

$$\mathfrak{m} = \wp_1^{e_1} \wp_2^{e_2} \cdots \wp_r^{e_r},$$

where $\wp_j$'s are distinct prime ideals. Since the divisor function and the norm map are completely multiplicative function, we have

$$\frac{d(\mathfrak{m})}{N(\mathfrak{m})^{\epsilon}} = \prod_{j=1}^{r} \frac{e_r + 1}{N(\wp_j)^{e_r \epsilon}}.$$

Next, we decompose the above product into two parts; first $N(\wp_j) < 2^{\frac{1}{\epsilon}}$ and then $N(\wp_j) \geq 2^{\frac{1}{\epsilon}}$. For the primes with $N(\wp_j) \geq 2^{\frac{1}{\epsilon}}$, we have $N(\wp_j)^{e_r \epsilon} \geq 2^{e_r}$, and hence

$$\frac{e_r + 1}{N(\wp_j)^{e_r \epsilon}} \leq \frac{e_r + 1}{2^{e_r}} \leq 1,$$

in the above equality. We now consider the primes satisfying $N(\wp_j) < 2^{\frac{1}{\epsilon}}$. Since $N(\wp_j) \geq 2$, we have $e_r \epsilon \log 2 \leq 2^{e_r \epsilon} \leq N(\wp_j)^{e_r \epsilon}$, and hence

$$\frac{e_r + 1}{N(\wp_j)^{e_r \epsilon}} \leq \; 1 + \frac{e_r}{N(\wp_j)^{e_r \epsilon}} \leq \left(1 + \frac{1}{\epsilon \log 2}\right).$$

From the above arguments, we get $d(\mathfrak{m}) \ll_{\epsilon} (N(\mathfrak{m}))^{\epsilon}$. □

Remark 2.4.5 Let $\mathfrak{v}(\mathfrak{m})$ be the number of distinct prime ideals dividing the integral ideal $\mathfrak{m}$. Since $\mathfrak{v}(\mathfrak{m}) \leq d(\mathfrak{m})$, we have $\mathfrak{v}(\mathfrak{m}) \ll_{\epsilon} (N(\mathfrak{m}))^{\epsilon}$, .

2.4.3 The fundamental result

The following proposition reveals something specific about the normalized newforms of the space $S_k(\mathfrak{n})$, which we shall later exploit to say about any form in $S_k(\mathfrak{n})$.

Proposition 2.4.6 (Fundamental result) *Let* $\mathbf{f}, \mathbf{g} \in S_{\boldsymbol{k}}(\mathfrak{n})$ *be two normalized newforms. Then for every* $\frac{1}{2} < a < 1$ *and for any* $\epsilon > 0$*, we have* *(i) If* $\mathbf{f} = \mathbf{g}$*, then there exists a constant* $A(\mathbf{f}, g) > 0$ *such that,*

$$\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} |C(\mathbf{f},\mathfrak{m})|^2 g\Big(\frac{N(\mathfrak{m})}{x}\Big) = A(\mathbf{f}, g)x + O\big(x^a k_0^{d(1-a)+\epsilon} N(\mathfrak{n})^{2(\frac{1-a}{2}+\epsilon)}\big).$$

(ii) If $\mathbf{f} \neq \mathbf{g}$*, then*

$$\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m}) g\Big(\frac{N(\mathfrak{m})}{x}\Big) = O\big(x^a k_0^{d(1-a)+\epsilon} N(\mathfrak{n})^{2(\frac{1-a}{2}+\epsilon)}\big).$$

The implied constant depends only on ϵ *and* K*. Also,* $A(\mathbf{f}, g) \gg_{\epsilon,K} (k_0 N(\mathfrak{n}))^{-\epsilon}$*.*

Proof. Let I be the integral $I := \frac{1}{2\pi i}\int_{(2)} L^{\#}(s, \mathbf{f}\otimes\mathbf{g})x^s M(g)(s)ds$, where (2) indicates that the integration is taken over the vertical line $\mathrm{Re}(s) = 2$. On substituting $L^{\#}(s, \mathbf{f}\otimes\mathbf{g})$ (from (2.5)) in I, we get

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{(2)} \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m})N(\mathfrak{m})^{-s}x^s M(g)(s)ds \\ &= \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m})\frac{1}{2\pi i}\int_{(2)} \Big(\frac{x}{N(\mathfrak{m})}\Big)^s M(g)(s)ds \\ &= \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m}) g\Big(\frac{N(\mathfrak{m})}{x}\Big). \end{aligned}$$

In above equality we used (2.8), the fact that the Dirichlet series $L^{\#}(s, \mathbf{f}\otimes\mathbf{g})$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Now by Lemma 2.4.2 and (2.6), I becomes

$$I = \frac{1}{2\pi i}\int_{(2)} L(s, \mathbf{f}\otimes\mathbf{g})F(s)H(s)x^s M(g)(s)ds,$$

where $H(s)$ and $F(s)$ are absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$. Our main goal is to compute

$$I = \frac{1}{2\pi i}\int_{(2)} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^s M(g)(s)ds.$$

In order to compute I, we consider a contour C oriented in counter clock-wise, having line segments $[2-iT, 2+iT], [2+iT, a+iT], [a+iT, a-iT]$ and $[a-iT, 2-iT]$, where a is a real number with $\frac{1}{2} < a < 1$ and T a large enough positive real number. We now consider the following contour integral

$$\frac{1}{2\pi i}\int_{C} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^s M(g)(s)ds. \tag{2.9}$$

To compute (2.9), we divide the above integral into the following four parts:

$$\int_C ds = \int_{2-iT}^{2+iT} ds + \int_{2+iT}^{a+iT} ds + \int_{a+iT}^{a-iT} ds + \int_{a-iT}^{2-iT} ds. \tag{2.10}$$

Put

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\int_{2+iT}^{a+iT} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^s M(g)(s)ds,\\ I_2 &= \frac{1}{2\pi i}\int_{a-iT}^{2-iT} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^s M(g)(s)ds,\\ I_3 &= \frac{1}{2\pi i}\int_{a+iT}^{a-iT} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^s M(g)(s)ds. \end{aligned}$$

Taking $T\to\infty$ in (2.10), we get

$$\begin{aligned} &\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^s M(g)(s)ds\\ &\qquad = -\lim_{T\to\infty}\left(I_1+I_2+I_3-\frac{1}{2\pi i}\int_C L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^s M(g)(s)ds\right). \end{aligned} \tag{2.11}$$

Now we make use of Theorem 1.3.7 and the well known Cauchy's integral formula in (2.9), to get

$$\frac{1}{2\pi i}\int_C L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^sM(g)(s)ds = \delta(\mathbf{f},\mathbf{g})\mathrm{Res}_{s=1}\Big(L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s) \times x^sM(g)(s)\Big).$$

Since the function $L(s,\mathbf{f}\otimes\mathbf{g})$ is polynomially bounded on vertical strips and $M(g)(s)$ has rapid decay (see also in [26]). Thus as $T\to\infty$, the horizontal integrals I_1, I_2 will vanish. Therefore (2.11) reduces to

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^sM(g)(s)ds\\
&\quad= \delta(\mathbf{f},\mathbf{g})\mathrm{Res}_{s=1}\Big(L(s,\mathbf{f}\otimes\mathbf{g})F(s)\times H(s)x^sM(g)(s)\Big)\\
&\quad+\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^sM(g)(s)ds\\
&\quad= \delta(\mathbf{f},\mathbf{g})x\mathrm{Res}_{s=1}\Big(L(s,\mathbf{f}\otimes\mathbf{g})F(s)\times H(s)M(g)(s)\Big)+G(s),
\end{aligned}$$

where $G(s)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} L(s,\mathbf{f}\otimes\mathbf{g})F(s)H(s)x^sM(g)(s)ds$. Let us now compute $G(s)$. By the combined application of Lemma 2.4.1, (2.7) and the fact that $F(s), H(s)$ are absolutely convergent for $\mathrm{Re}(s)>\frac{1}{2}$. We get,

$$\begin{aligned}
\Big|G(s)\Big| &\ll_{\epsilon,K} \Big|\frac{x^a}{2\pi i}\Big|\int_{-\infty}^{\infty}\Big|\mathfrak{q}(a+it,\mathbf{f}\otimes\mathbf{g})^{\frac{1-(a+it)}{2}+\epsilon}\Big||a+it|^{-A-1}|F(a+it)||H(a+it)|dt,\\
&\ll_{\epsilon,K} x^a\int_{-\infty}^{\infty}\Big(N(\mathfrak{n}\mathfrak{D}_K^2)^2 4^{2d}k_0^{2d}(1+|t|)^{4d}\Big)^{\frac{1-a}{2}+\epsilon}(1+|t|)^{-A-1}dt,\\
&= O\Big(x^a k_0^{d(1-a)+\epsilon}N(\mathfrak{n})^{1-a+\epsilon}\int_0^{\infty}(1+|t|)^{2d(1-a)-A-1+\epsilon}dt\Big).
\end{aligned}$$

The choice $A=2(d(1-a)+\epsilon)>0$ makes the integral $\int_0^{\infty}(1+|t|)^{2d(1-a)-A-1+\epsilon}dt$

absolutely convergent. Consequently with this choice of A, we get

$$G(s) = O\big(x^a k_0^{d(1-a)+\epsilon} N(\mathfrak{n})^{1-a+\epsilon}\big),$$

where the implied constant depend on ϵ and K only.

Next we compute $A(\mathbf{f}, g) = \mathrm{Res}_{s=1}\Big(L(s, \mathbf{f}\otimes\mathbf{g})F(s)H(s)M(g)(s)\Big)$. Note that $1 \ll H(1)$ (from Lemma 2.4.2) and depending constant is absolute. Further from (2.6) we see that,

$$F(1) = \prod_{\wp\nmid\mathfrak{n}}(1 - N(\wp)^{-2})\Big(\prod_{\wp|\mathfrak{n}}\prod_{i,j=1}^{2}(1 - \alpha_i(\wp)\overline{\beta_j(\wp)}N(\wp)^{-1})\Big)$$
$$F(1) \gg \prod_{\wp|\mathfrak{n}}\left(1 - \frac{1}{N(\wp)}\right)^4 \gg 2^{-4\mathfrak{v}(\mathfrak{n})}.$$

By Lemma 2.4.4, $\mathfrak{v}(\mathfrak{n}) \ll_\epsilon (N(\mathfrak{n}))^\epsilon$, and on applying Lemma 2.4.3 we get,

$$\begin{aligned}\mathrm{Res}_{s=1}\Big(L(s, \mathbf{f}\otimes\mathbf{g})F(s)H(s)M(g)(s)\Big) &= F(1)H(1)M(f)(1)\mathrm{Res}_{s=1}(L(s, \mathbf{f}\otimes\mathbf{g}) \\ &\gg_{\epsilon,K} (k_0 N(\mathfrak{n}))^{-\epsilon}.\end{aligned}$$

2.5 Proof of the Theorem 2.3.1

In order to prove the main theorem, we invoke newform theory for Hilbert Modular forms. By Proposition 1.3.3 there exists a basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_l\}$ of $S_{\boldsymbol{k}}(\mathfrak{n})$ consisting of newforms of weight $\boldsymbol{k}$ and level dividing $\mathfrak{n}$. Let $\mathbf{f}$ be a non-zero form in $S_{\boldsymbol{k}}(\mathfrak{n})$. Then by newform theory

$$\mathbf{f} = \sum_{i=1}^{l}\sum_{\mathfrak{q}|\mathfrak{n}} a_{i,\mathfrak{q}}\mathbf{f}_i|B_{\mathfrak{q}}, \tag{2.12}$$

where $a_{i,\mathfrak{q}}$ is zero whenever $\mathbf{f}_i$ is not a newform of level $\frac{\mathfrak{n}}{\mathfrak{q}}$. Let $\mathfrak{q}_0$ be an integral ideal such that $\mathfrak{q}_0 \mid \mathfrak{n}$, $N(\mathfrak{q}_\circ) \leq N(\mathfrak{q}')$ for any ideal $\mathfrak{q}'(\neq \mathfrak{q}_\circ)$ dividing the level $\mathfrak{n}$, and $a_{i,\mathfrak{q}_0} \neq 0$ for some i. Now using Theorem 1.3.1, and comparing the Fourier coefficients indexed by $\mathfrak{q}_0\mathfrak{m}$ in (2.12), we get

$$C(\mathbf{f}, \mathfrak{q}_0\mathfrak{m}) = \sum_{i=1}^{l} \sum_{\mathfrak{q}|\mathfrak{n}} a_{i,\mathfrak{q}} C\Big(\mathbf{f}_i, \frac{\mathfrak{q}_0\mathfrak{m}}{\mathfrak{q}}\Big), \tag{2.13}$$

where $\mathfrak{m}$ is a square-free integral ideal and co-prime to the level $\mathfrak{n}$. Note that if $N(\mathfrak{q}) < N(\mathfrak{q}_0)$, then $a_{i,\mathfrak{q}} = 0$ (by our choice of $\mathfrak{q}_0$) and also if $\mathfrak{q} \neq \mathfrak{q}_0$, then $C_{\mathbf{f}_i}\Big(\frac{\mathfrak{m}\mathfrak{q}_0}{\mathfrak{q}}\Big) = 0$ (see equation (1.2)). Thus for some positive integer $m \leq l$, (2.13) becomes

$$C(\mathbf{f}, \mathfrak{q}_0\mathfrak{m}) = \sum_{i=1}^{m} a_{i,\mathfrak{q}_0} C(\mathbf{f}_i, \mathfrak{m}).$$

We now consider the following sum,

$$\begin{aligned}
\sideset{}{^\#}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}} |C(\mathbf{f}, \mathfrak{q}_0\mathfrak{m})|^2 g\Big(\frac{N(\mathfrak{m})}{x}\Big) &= \sideset{}{^\#}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}} \Big(\sum_{i=1}^{m} a_{i,\mathfrak{q}_0} C(\mathbf{f}_i, \mathfrak{m}) \cdot \overline{\sum_{i=1}^{m} a_{i,\mathfrak{q}_0} C(\mathbf{f}_i, \mathfrak{m})}\Big) \\
&\quad \times g\Big(\frac{N(\mathfrak{m})}{x}\Big) \qquad (2.14)\\
&= \sideset{}{^\#}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}} \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 |C(\mathbf{f}_i, \mathfrak{m})|^2 g\Big(\frac{N(\mathfrak{m})}{x}\Big) \\
&\quad + \sideset{}{^\#}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}} \sum_{\substack{i,j=1\\i\neq j}}^{m} a_{i,\mathfrak{q}_0} \cdot \overline{a_{j,\mathfrak{q}_0}} C(\mathbf{f}_i, \mathfrak{m}) \overline{C(\mathbf{f}_j, \mathfrak{m})} \times g\Big(\frac{N(\mathfrak{m})}{x}\Big).
\end{aligned}$$

Let us Interchange the summation in (2.14) first and then apply Proposition 2.4.6, we get

$$\sideset{}{^\#}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}} |C(\mathbf{f}, \mathfrak{q}_0\mathfrak{m})|^2 g\Big(\frac{N(\mathfrak{m})}{x}\Big)$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 \Big(A(\mathbf{f}_i, g)x + O\Big(x^a k_0^{d(1-a)+\epsilon} N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big) \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{m} a_{i,\mathfrak{q}_0} \cdot \overline{a_{j,\mathfrak{q}_0}} \Big(O\Big(x^a k_0^{d(1-a)+\epsilon} N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big) \\
&\geq \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 \Big(A(\mathbf{f}_i, g)x + O\Big(x^a k_0^{d(1-a)+\epsilon} N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big) \\
&- \sum_{\substack{i,j=1\\ i\neq j}}^{m} |a_{i,\mathfrak{q}_0} \cdot \overline{a_{j,\mathfrak{q}_0}}| \Big(O\Big(x^a k_0^{d(1-a)+\epsilon} N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big) \\
&\geq \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 A(\mathbf{f}_i, g)x + \Big(\sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{m} |a_{i,\mathfrak{q}_0} \cdot \overline{a_{j,\mathfrak{q}_0}}| \Big). \\
&\Big(O\Big(x^a k_0^{d(1-a)+\epsilon} N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big) \\
&\geq \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 A(\mathbf{f}_i, g)x - \Big(\sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{m} |a_{i,\mathfrak{q}_0} \cdot \overline{a_{j,\mathfrak{q}_0}}| \Big). \\
&\Big(O\Big(x^a k_0^{d(1-a)+\epsilon} N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big) \\
&\geq \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 A(\mathbf{f}_i, g)x - |\sum_{i=1}^{m} a_{i,\mathfrak{q}_0}|^2 \Big(O\Big(x^a k_0^{d(1-a)+\epsilon} \\
&\times N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big).
\end{aligned}
$$

Using the bound $A(\mathbf{f}_i, g) \gg_{\epsilon,K} \Big(k_0 N\big(\frac{\mathfrak{n}}{\mathfrak{q}_o}\big)\Big)^{-\epsilon}$ (by Lemma 2.4.3), and by applying the Cauchy-Schwarz inequality in the second term $|\sum_{i=1}^{m} a_{i,\mathfrak{q}_0}|^2$, we get

$$
\begin{aligned}
\sideset{}{^\#}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ (\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}} |C(\mathbf{f}, \mathfrak{q}_0\mathfrak{m})|^2 g\Big(\frac{N(\mathfrak{m})}{x}\Big) \geq \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 \Bigg(\Big(k_0 N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_o}\Big)\Big)^{-\epsilon} x - \\
m\Big(O\Big(x^a k_0^{d(1-a)+\epsilon} N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big)\Bigg). \qquad (2.15)
\end{aligned}
$$

Now we make use of an upper bound for $r \ll k_0^d N\big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\big)^{3d^2}$ [Corollary 15, [48]]. Therefore (2.15) becomes

$$\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} |C(\mathbf{f},\mathfrak{q}_0\mathfrak{m})|^2 g\Big(\frac{N(\mathfrak{m})}{x}\Big) \gg \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 \Bigg(\Big(k_0 N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_o}\Big)\Big)^{-\epsilon} x - \Big(k_0^d N\big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\big)^{3d^2}\big(x^a k_0^{d(1-a)+\epsilon} N\big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\big)^{2(\frac{1-a}{2}+\epsilon)}\big)\Big)\Bigg). \tag{2.16}$$

Note that (2.16) is independent of the choice of the weight function g, and hence one can choose $0 \leq g \leq 1$ such that the following holds

$$\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} |C(\mathbf{f},\mathfrak{q}_0\mathfrak{m})|^2 g\Big(\frac{N(\mathfrak{m})}{x}\Big) \leq \sum_{\substack{\frac{x}{2}<N(\mathfrak{m})<x\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} |C(\mathbf{f},\mathfrak{q}_0\mathfrak{m})|^2. \tag{2.17}$$

Now (2.16) and (2.17) together imply

$$\sum_{\substack{\frac{x}{2}<N(\mathfrak{m})<x\\(\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} |C(\mathbf{f},\mathfrak{q}_0\mathfrak{m})|^2 \gg \sum_{i=1}^{m} |a_{i,\mathfrak{q}_0}|^2 \Bigg(\Big(k_0 N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_o}\Big)\Big)^{-\epsilon} x - \Big(k_0^d N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{3d^2}\Big(x^a k_0^{d(1-a)+\epsilon} N\Big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\Big)^{2(\frac{1-a}{2}+\epsilon)}\Big)\Big)\Bigg).$$

We see that in the above inequality the RHS is positive whenever $x \geq k_0^{d(3-2\epsilon)+4\epsilon} N\big(\frac{\mathfrak{n}}{\mathfrak{q}_0}\big)^{\frac{6d^2+1+4\epsilon}{2}}$ with $a = \frac{1}{2}+\epsilon$. This completes the proof. □

CHAPTER 3

Sign changes in restricted coefficients of Hilbert modular forms

In this chapter, we study the sign changes in the Fourier coefficients of $\mathbf{f}$ *when restricted to square-free integral ideals and integral ideals in an "arithmetic progression". In both cases, we obtain qualitative results, and in the former case, we obtain a quantitative result as well. These results are general in the sense that we do not impose any restriction on the number field* K*, the weight* $\boldsymbol{k}$ *or the level* $\mathfrak{n}$*. The results of this chapter are in [3].*

3.1 Introduction and overview of previous work

In the last chapter, we have seen that the Fourier coefficients completely determine the modular form. The Fourier coefficients have great importance as they encoded many arithmetic and algebraic properties. Indian mathematician Srinivasan Ramanujan was the first who observed some of the important facts about the coefficients of the well known Ramanujan delta function Δ, latter proved by Mordell and Deligne [16]. In recent years, the study of sign change of Fourier coefficients has been considered by many mathematicians in various aspects. In the literature, the problem was first initiated by Ram Murty [37] for integral weight modular forms, where he proved the sign change among the Fourier coefficients indexed by prime numbers. In particular, he proved the following theorem. Let k, N be two positive integers, $S_k(N)$ denotes the space of cusp forms of weight k on $\Gamma_0(N)$.

Theorem 3.1.1 ([37]) *Let f be a non-zero element of $S_k(N)$ having Fourier expansion,*

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n.$$

Then either $\{Re(a_f(p))\}_p$ or $\{Im(a_f(p))\}_p$ changes sign infinitely many often, where p is prime number. Moreover there exists a small positive real number δ such that the number of sign change for $p \leq X$ is at least αX^δ for some positive real number α.

It is also interesting to find out when the first sign change will occur. In 2006, Sengupta et al. [31] addressed this problem and proved the following result.

Theorem 3.1.2 ([31]) *Let f be a normalized newform with real Fourier coefficients $a_f(n)$. Assume that N is square-free positive integer. Then there exists $n \in \mathbb{N}$ with*

$$n \leq \alpha_1 K N \log^A N \exp\left(\alpha_2 \sqrt{\frac{\log(N+1)}{\log\log(N+2)}}\right), \qquad (n, N) = 1$$

such that $a_f(n) < 0$*, for some real number* $A > 26, \alpha_1$ *is a constant depending only on* A*, and* α_2 *is an absolute constant.*

Similar result for half integral weight modular form can be found in [9], [19], [25], [34] and [35].

In the context of Hilbert modular forms, Meher et al. [36] proved the following result.

Theorem 3.1.3 ([36]) *Let* $\mathbf{f}$ *be a non-zero adelic Hilbert cusp form of weight* $\boldsymbol{k} = (k_1, k_2, \ldots, k_d)$ *and level* $\mathfrak{n}$*. Let* $C(\mathbf{f}, \mathfrak{m})$ *be the Fourier coefficients at each integral ideal* $\mathfrak{m}$*. If* $\{C(\mathbf{f}, \mathfrak{m})\}_{\mathfrak{m}}$ *are all real numbers, then there are infinitely many sign change in the sequence* $\{C(\mathbf{f}, \mathfrak{m})\}_{\mathfrak{m}}$*.*

In the same paper [36], Meher and Tanabe also considered the problem of first sign change and proved the following theorem.

Theorem 3.1.4 ([36]) *Let* $\mathbf{f}$ *be a primitive cusp form in* $S_{\mathbf{k}}(\mathfrak{n})$*. Let* $Q_{\mathbf{f}}$ *be the analytic conductor of* $\mathbf{f}$*. Then there exists an integral ideal* $\mathfrak{m}$ *with*

$$N(\mathfrak{m}) \ll_{d,\epsilon} Q_{\mathbf{f}}^{1+\epsilon},$$

such that $C(\mathbf{f}, \mathfrak{m}) < 0$*.*

In [40], Ritwik Pal improved the upper bound obtained in the above theorem. More precisely he proved the following theorem.

Theorem 3.1.5 ([40]) *Let* $\mathbf{f}$ *be a primitive cusp form of weight* $\boldsymbol{k} = (k_1, k_2, \ldots, k_d)$ *and full level. Then for any arbitrary* $\epsilon > 0$*,*

(1) when $k_1, k_2, \ldots, k_d$ *are all even, we have* $C(\mathbf{f}, \mathfrak{m}) < 0$ *for some ideal* $\mathfrak{m}$ *with* $N(\mathfrak{m}) \ll_{d,\epsilon} Q_{\mathbf{f}}^{\frac{9}{20}+\epsilon}$*;*

(2) otherwise we have $C(\mathbf{f}, \mathfrak{m}) < 0$ *for some ideal* $\mathfrak{m}$ *with* $N(\mathfrak{m}) \ll_{d,\epsilon} Q_{\mathbf{f}}^{\frac{1}{2}+\epsilon}$.

More results on sign change in the context of Hilbert modular forms can be found in [29].

3.2 Main results

The first result of this chapter is the study of sign changes in the Fourier coefficients of a primitive Hilbert cusp form indexed by square-free integral ideals. More precisely, we prove the following theorem.

Theorem 3.2.1 ([3]) *Let* $\mathbf{f}$ *be a primitive adelic Hilbert cusp form of weight* $\boldsymbol{k} = (k_1, k_2, \ldots, k_d)$ *and level* $\mathfrak{n}$ *with trivial Hecke character* Ψ *modulo* $\mathfrak{n}$*. Then the sequence* $\{C(\mathbf{f}, \mathfrak{m})\}_{\mathfrak{m}}$ *has infinitely many sign changes where* $\mathfrak{m}$ *runs through the square-free integral ideals of* K*. Furthermore the number of sign changes in* $C(\mathbf{f}, \mathfrak{m})$ *with* $N(\mathfrak{m}) \leq X$ *is* $\gg X^{1/2}$ *for all large enough* X*.*

The next result talks about the sign changes in arithmetic progressions. Let $\mathfrak{m}$ be an integral ideal co-prime to $\mathfrak{n}$. Let $\mathfrak{R}^{+}_{K,\mathfrak{m}}$ be the strict ray class group modulo $\mathfrak{m}\mathfrak{m}_\infty$ where $\mathfrak{m}_\infty$ is a formal product of all real embeddings of K into $\mathbb{C}$. We now state the second result of this chapter.

Theorem 3.2.2 ([3]) *Let* $\mathbf{f}$ *be a primitive adelic Hilbert cusp form of weight* $\boldsymbol{k} = (k_1, k_2, \ldots, k_d)$ *and level* $\mathfrak{n}$ *with trivial Hecke character* Ψ *modulo* $\mathfrak{n}$*. Then for any given* $\mathfrak{m}$ *(as above) co-prime to the level* $\mathfrak{n}$ *and for any ideal class* $[\mathfrak{a}]$ *in* $\mathfrak{R}^{+}_{K,\mathfrak{m}}$*, the sequence* $\{C(\mathbf{f}, \mathfrak{l})\}_{\mathfrak{l}}$ *has infinitely many sign changes where* $\mathfrak{l}$ *runs through the integral ideals lying in the class* $[\mathfrak{a}]$*.*

3.3 Preliminaries

This section is devoted to collecting all the preliminary results which will be needed to prove the aforementioned theorems. Let us begin this section by defining generalized Möbius function μ (we use same notation) as a function on all the integral ideals of $\mathcal{O}_K$ as follows

$$\mu(\mathfrak{m}) = \begin{cases} (-1)^n & \text{if } \mathfrak{m} = \prod_{i=1}^{n} \wp_i \text{ and } \wp_i \neq \wp_j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have the following identity for the generalized Möbius function

$$\sum_{\mathfrak{r}^2 | \mathfrak{m}} \mu(\mathfrak{r}) = \begin{cases} 1 & \mathfrak{m} \text{ is square-free,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

The proof of the above identity follows in the same line as the classical Möbius function defined on natural numbers (see for a proof [38]). Let $\mathbf{f} \in S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ be a normalized newform with Fourier coefficients $\{C(\mathbf{f}, \mathfrak{m})\}_{\mathfrak{m}}$. The L-function associated to $\mathbf{f}$ is given by

$$L(s, \mathbf{f}) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ \mathfrak{m} \neq \{0\}}} \frac{C(\mathbf{f}, \mathfrak{m})}{N(\mathfrak{m})^s}. \tag{3.2}$$

From Theorem 1.3.5, $L(s, \mathbf{f})$ has the following Euler product expansion

$$\begin{aligned} L(s, \mathbf{f}) &= \prod_{\wp | \mathfrak{n}} (1 - C(\mathbf{f}, \wp) N(\wp)^{-s})^{-1} \prod_{\wp \nmid \mathfrak{n}} (1 - C(\mathbf{f}, \wp) N(\wp)^{-s} + \Psi^*(\wp) N(\wp)^{-2s})^{-1} \\ &= \prod_{\wp} \left(1 - C(\mathbf{f}, \wp) N(\wp)^{-s} + \Psi^*(\wp) N(\wp)^{-2s}\right)^{-1} \end{aligned}$$

$$= \prod_{\wp} \left(1 + \frac{C(\mathbf{f},\wp)}{N(\wp)^s} + \frac{C(\mathbf{f},\wp^2)}{N(\wp^2)^s} + \dots\right),$$

where $\Psi^*(\wp) = 0$ whenever $\wp \mid \mathfrak{n}$. Define

$$\mathbf{Sq}(s) = \sideset{}{^\#}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}} \frac{C(\mathbf{f},\mathfrak{m})}{N(\mathfrak{m})^s} = \prod_{\wp \text{ prime}} \left(1 + \frac{C(\mathbf{f},\wp)}{N(\wp)^s}\right),$$

where # indicates that the sum runs over the square-free integral ideals. Using the Ramanujan bound, the Dirichlet series $\mathbf{Sq}(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Due to the requirement of some reasonable computations, we present a slightly different but a longer proof of this fact.

3.3.1 Analytic continuation and some estimations

Lemma 3.3.1 *The above Dirichlet series* $\boldsymbol{Sq}(s)$ *is absolutely convergent for* $Re(s) > 1$.

Proof. From (3.1) it is clear that,

$$\mathbf{Sq}(s) = \sideset{}{^\#}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}} \frac{C(\mathbf{f},\mathfrak{m})}{N(\mathfrak{m})^s} = \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}} \frac{C(\mathbf{f},\mathfrak{m})}{N(\mathfrak{m})^s} \left(\sum_{\mathfrak{r}^2\mid\mathfrak{m}} \mu(\mathfrak{r})\right) = \sum_{\mathfrak{r}} \mu(\mathfrak{r}) D_{\mathfrak{r}^2}(s),$$

where we have set

$$D_{\mathfrak{r}}(s) := \sum_{\mathfrak{r}\mid\mathfrak{m}} \frac{C(\mathbf{f},\mathfrak{m})}{N(\mathfrak{m})^s} = \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}} \frac{C(\mathbf{f},\mathfrak{r}\mathfrak{m})}{N(\mathfrak{r}\mathfrak{m})^s} \tag{3.3}$$

for an integral ideal $\mathfrak{r}$. By the Ramanujan bound,

$$C(\mathbf{f},\mathfrak{m}) \ll_\epsilon N(\mathfrak{m})^\epsilon,$$

we deduce that,

$$|D_{\mathfrak{r}}(1+\epsilon+it)| \ll \sum_{\mathfrak{m}} \left| \frac{C(\mathbf{f},\mathfrak{rm})}{N(\mathfrak{rm})^{1+\epsilon+it}} \right| \ll \frac{1}{N(\mathfrak{r})^{1+\epsilon}} \sum_{\mathfrak{m}} \left| \frac{C(\mathbf{f},\mathfrak{rm})}{N(\mathfrak{m})^{1+\epsilon+it}} \right|.$$

The right most summation in the previous inequality is $\ll_\epsilon N(\mathfrak{r})^\epsilon$. Therefore, for every $\epsilon > 0$, we get

$$|D_{\mathfrak{r}}(1+\epsilon+it)| \ll_\epsilon \frac{1}{N(\mathfrak{r})^{1+\epsilon}}. \tag{3.4}$$

In particular, we have

$$|D_{\mathfrak{r}^2}(1+\epsilon+it)| \ll_\epsilon \frac{1}{N(\mathfrak{r})^{2+\epsilon}}. \tag{3.5}$$

Thus from (3.5) we can see that $\mathbf{Sq}(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$.

□

Next we show that $\mathbf{Sq(s)}$ has an analytic continuation to $\mathrm{Re}(s) > \frac{1}{2}$.

Lemma 3.3.2 *Let $\wp$ be a prime ideal of $\mathcal{O}_K$ and $s \in \mathbb{C}$, we define*

$$S_\wp(s) := \left(-\frac{C(\mathbf{f},\wp)^2}{N(\wp)^{2s}} + \Psi^*(\wp) N(\wp)^{-2s} \left(\frac{C(\mathbf{f},\wp)}{N(\wp)^s} + 1 \right) \right).$$

Then we have

$$\boldsymbol{Sq}(s) = L(s,\mathbf{f}) \prod_{\wp} (1 + S_\wp(s)),$$

where product runs over all prime ideals $\wp$ of $\mathcal{O}_K$. In particular, $\boldsymbol{Sq}(s)$ can be analytically continued to the half plane $Re(s) > \frac{1}{2}$.

Proof. Let $\mathbf{f}$ be a normalized newform in $S_{\boldsymbol{k}}(\mathfrak{n},\Psi)$ and let $L_\wp(s)$ denote the

$\wp$-factor of $L(s, \mathbf{f})$. Then

$$\begin{aligned} L_\wp(s) &= \left(1 + \frac{C(\mathbf{f},\wp)}{N(\wp)^s} + \frac{C(\mathbf{f},\wp^2)}{N(\wp^2)^s} + \dots\right) \\ &= \frac{1}{(1 - C(\mathbf{f},\wp)N(\wp)^{-s} + \Psi^*(\wp)N(\wp)^{-2s})}. \end{aligned} \tag{3.6}$$

Let r be a non-negative integer, we define $L_\wp^{(r)}(s)$ to be the r tail of $L_\mathfrak{p}$,

$$L_\wp^{(r)}(s) = \sum_{n=r}^{\infty} \frac{C(\mathbf{f},\wp^n)}{N(\wp^n)^s}. \tag{3.7}$$

Note that $L_\wp^{(0)}(s) = L_\wp(s)$ by definition.

Let $\mathfrak{r} = \prod_\wp \wp^{e_\wp}$, where $e_\wp = 0$ for all but finitely many $\wp$. Using the fact that $C(\mathbf{f}, \mathfrak{m})$ are multiplicative and any ideal has unique factorization into prime ideals. We have

$$\begin{aligned} D_\mathfrak{r}(s) &= \prod_\wp \sum_{r=e_\wp}^{\infty} \frac{C(\mathbf{f},\wp^r)}{N(\wp^r)^s} \\ &= \prod_\wp L_\wp^{(e_\wp)}(s) \\ &= L(s,\mathbf{f})/L(s,\mathbf{f}) \prod_\wp L_\wp^{(e_\wp)}(s) \\ &= L(s,\mathbf{f}) \prod_\wp \frac{L_\wp^{(e_\wp)}(s)}{L_\wp(s)} \\ &= L(s,\mathbf{f}) \prod_{\wp|\mathfrak{r}} \frac{L_\wp^{(e_\wp)}(s)}{L_\wp(s)} \prod_{\wp\nmid\mathfrak{r}} \frac{L_\wp^{(e_\wp)}(s)}{L_\wp(s)}. \end{aligned}$$

If $\mathfrak{r}$ is a square-free ideal, then

$$D_{\mathfrak{r}^2}(s) = L(s,\mathbf{f}) \prod_{\wp|\mathfrak{r}} \frac{L_\wp^{(2)}(s)}{L_\wp(s)}$$

$$
\begin{aligned}
&= L(s,\mathbf{f})\prod_{\wp|\mathfrak{r}}\left(1-\frac{1}{L_\wp(s)}-\frac{C(\mathbf{f},\wp)}{N(\wp)^sL_\wp(s)}\right)\\
&= L(s,\mathbf{f})\prod_{\wp|\mathfrak{r}}\Bigg(1-\left(1-C(\mathbf{f},\wp)N(\wp)^{-s}+\Psi^*(\wp)N(\wp)^{-2s}\right)-\\
&\qquad \frac{C(\mathbf{f},\wp)}{N(\wp)^s}\left(1-C(\mathbf{f},\wp)N(\wp)^{-s}+\Psi^*(\wp)N(\wp)^{-2s}\right)\Bigg)\\
&= L(s,\mathbf{f})\prod_{\wp|\mathfrak{r}}\left(\frac{C(\mathbf{f},\wp)^2}{N(\wp)^{2s}}-\Psi^*(\wp)N(\wp)^{-2s}\left(\frac{C(\mathbf{f},\wp)}{N(\wp)^s}+1\right)\right).
\end{aligned}
$$

In above equality, we make use of the fact that $e_\wp = 2$ and $e_\wp = 0$ whenever $\wp \mid \mathfrak{r}$ and $\wp \nmid \mathfrak{r}$, respectively. We also utilize the fact that $L^{(0)}_\wp(s) = L_\wp(s)$ and (3.6), (3.7). Since $\mu(\mathfrak{r}) = \prod_{\wp|\mathfrak{r}}\mu(\wp)$, we have

$$
\begin{aligned}
\mu(\mathfrak{r})D_{\mathfrak{r}^2}(s) &= L(s,\mathbf{f})\prod_{\wp|\mathfrak{r}}\mu(\wp)\left(\frac{C(\mathbf{f},\wp)^2}{N(\wp)^{2s}}-\Psi^*(\wp)N(\wp)^{-2s}\left(\frac{C(\mathbf{f},\wp)}{N(\wp)^s}+1\right)\right)\\
&= L(s,\mathbf{f})\prod_{\wp|\mathfrak{r}}\left(-\frac{C(\mathbf{f},\wp)^2}{N(\wp)^{2s}}+\Psi^*(\wp)N(\wp)^{-2s}\left(\frac{C(\mathbf{f},\wp)}{N(\wp)^s}+1\right)\right).
\end{aligned}
$$

Now recall that

$$
\begin{aligned}
\mathbf{Sq}(s) &= \sum_{\mathfrak{r}}\mu(\mathfrak{r})D_{\mathfrak{r}^2}(s)\\
&= \sum_{\mathfrak{r}}{}^{\#}\, L(s,\mathbf{f})\prod_{\wp|\mathfrak{r}}\left(-\frac{C(\mathbf{f},\wp)^2}{N(\wp)^{2s}}+\Psi^*(\wp)N(\wp)^{-2s}\left(\frac{C(\mathbf{f},\wp)}{N(\wp)^s}+1\right)\right)\\
&= \sum_{\mathfrak{r}}{}^{\#}\, L(s,\mathbf{f})\prod_{\wp|\mathfrak{r}}S_\wp(s) = L(s,\mathbf{f})\prod_{\wp}\left(1+S_\wp(s)\right). \qquad (3.8)
\end{aligned}
$$

In the last line, we have put $\prod_{\wp|\mathfrak{r}}S_\mathfrak{p}(s) = 1$ for $\mathfrak{r} = \mathcal{O}_K$. The product on the right hand side of (3.8) is absolutely convergent for $\mathrm{Re}(s) > 1/2$. Therefore we have an analytic continuation for $\mathbf{Sq}(s)$ to the plane $\mathrm{Re}(s) > 1/2$. □

Lemma 3.3.3 *Let* $\mathbf{f}$ *be a primitive form. Then,*

$$\sum_{\substack{\mathfrak{m} \\ N(\mathfrak{m})\leq X}}^{\#} C(\mathbf{f},\mathfrak{m}) \ll_{\epsilon} X^{1/2+\epsilon}. \tag{3.9}$$

Proof. From [26] we know that $L(s,\mathbf{f})$ has polynomial growth in $\mathrm{Im}(s)$ for the vertical strip $\frac{1}{2} < \mathrm{Re}(s) < 1$. Therefore by Lemma 3.3.2 we conclude that $\mathrm{Sq}(s)$ has polynomial growth in the same strip. Consider the following integral I

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{(1+\epsilon)} \mathrm{Sq}(s) X^s \Gamma(s) ds \\ &= \frac{1}{2\pi i}\int_{(1+\epsilon)} \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K \\ \mathfrak{m}\neq\{0\}}}^{\#} \frac{C(\mathbf{f},\mathfrak{m})}{N(\mathfrak{m})^s} X^s \int_0^\infty e^{-t} t^s \frac{dt}{t} dt ds \\ &= \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K \\ \mathfrak{m}\neq\{0\}}}^{\#} C(\mathbf{f},\mathfrak{m}) \frac{1}{2\pi i}\int_{(1+\epsilon)} \left(\frac{N(\mathfrak{m})}{X}\right)^{-s} \int_0^\infty e^{-t} t^s \frac{dt}{t} dt ds. \end{aligned}$$

Let us apply the well known inverse Mellin transform, we see that

$$\frac{1}{2\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \mathbf{Sq}(s)\Gamma(s) X^s ds = \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K \\ \mathfrak{m}\neq\{0\}}}^{\#} C(\mathbf{f},\mathfrak{m}) e^{-\mathrm{N}(\mathfrak{m})/X}. \tag{3.10}$$

By setting $B(n,\mathbf{f}) := \sum_{\substack{\mathfrak{m} \\ N(\mathfrak{m})=n}}^{\#} C(\mathbf{f},\mathfrak{m})$, then (3.10) becomes

$$I = \sum_{n=1}^{\infty} B(n,\mathbf{f}) e^{-n/X}.$$

We now shift the line of integration in (3.10) to $\mathrm{Re}(s) = 1/2+\epsilon$. Since $\mathbf{Sq}(s)$ has at most polynomial growth in $\mathrm{Im}(s)$ inside the critical strip, this growth is taken care of by the exponential decay of the Γ-function. Furthermore, since the integrand is analytic inside the vertical strip $\frac{1}{2} < \mathrm{Re}(s) < 1$, we do not encounter with any pole. Thus, we have

$$\sum_{n=1}^{\infty} B(n,\mathbf{f})e^{-n/X} \ll_\epsilon X^{1/2+\epsilon}. \tag{3.11}$$

Since $B(n,\mathbf{f}) \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$, we can see that $\sum\limits_{n\geq X} B(n,\mathbf{f})e^{-n/X} = O(1)$, for large enough X. Finally this gives us $\sum\limits_{\substack{\mathfrak{m}\\ N(\mathfrak{m})\leq X}}^{\#} C(\mathbf{f},\mathfrak{m}) \ll_\epsilon X^{1/2+\epsilon}$. □

Lemma 3.3.4 *Let* $\mathbf{f}$ *be as above in Lemma 3.3.3. Then*

$$X \ll_{\epsilon,K} \sum_{N(\mathfrak{m})\leq X}^{\#} C^2(\mathbf{f},\mathfrak{m}).$$

Proof. From the Proposition 2.4.6

$$\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ (\mathfrak{m},\mathfrak{n})=\mathcal{O}_K}}^{\#} |C(\mathbf{f},\mathfrak{m})|^2 g\Big(\frac{N(\mathfrak{m})}{X}\Big) = A(\mathbf{f},g)X + O\big(X^a k_0^{d(1-a)+\epsilon} N(\mathfrak{n})^{2(\frac{1-a}{2}+\epsilon)}\big).$$

Note that g is a positive smooth compactly supported function on $[1/2,1]$. Therefore we get that

$$X \ll_{\epsilon,K} \sum_{N(\mathfrak{m})\leq X}^{\#} C^2(\mathbf{f},\mathfrak{m}).$$

□

3.3.2 Proof of the Theorem 3.2.1

In order to prove the theorem we first state a result of Murty et al. [34] regarding sign changes in a sequence of real numbers.

Theorem 3.3.5 (Meher-Murty [34]) *Let* $\{C(n)\}_{(n\geq 1)}$ *be a sequence of real* *numbers satisfying* $C(n) = O(n^\alpha)$ *such that*

$$\sum_{n \le X} C(n) \ll X^\beta$$

and

$$\sum_{n \le X} C(n)^2 = cX + O(X^\gamma)$$

where α, β, γ *and* c *are non-negative constants. If* $\alpha + \beta < 1$, *then for any* r *satisfying* $\max\{\alpha+\beta, \gamma\} < r < 1$, *the sequence* $C(n)_{(n\ge1)}$ *has at least one sign change for* $n \in (X, X+X^r]$. *In particular, the sequence* $C(n)$ *has infinitely many sign changes and the number of sign changes for* $n \le X$ *is* $\gg X^{1-r}$ *for sufficiently large* X.

Since the Fourier coefficients are not indexed by natural numbers, we cannot apply Theorem 3.3.5 directly. Nevertheless, we can modify the proof of Theorem 3.3.5 for our purposes.

Without loss of generality assume the contrary that finitely many of $C(\mathbf{f}, \mathfrak{m})$ are negative. Therefore, for large enough X, we have $C(\mathbf{f}, \mathfrak{m}) > 0$ whenever $N(\mathfrak{m}) \in (X, X+X^r]$ for some $1 > r > 1/2$. Therefore we have

$$\sum_{N(\mathfrak{m}) \in (X, X+X^r]} C^2(\mathbf{f}, \mathfrak{m}) \ll X^\epsilon \sum_{N(\mathfrak{m}) \in (X, X+X^r]} C(\mathbf{f}, \mathfrak{m}) \ll X^{1/2+\epsilon}. \tag{3.12}$$

The first inequality follows from the Ramanujan bound and the second inequality follows from (3.9).

On the other hand, since $r > 1/2$ we have

$$X^r \ll \sum_{N(\mathfrak{m}) \in (X, X+X^r]} C^2(\mathbf{f}, \mathfrak{m}) \tag{3.13}$$

from Lemma 3.3.4. Then (3.12) contradicts to (3.13). Therefore we have infinitely many sign changes in the sequence $\{C(\mathbf{f}, \mathfrak{m})\}_{\mathfrak{m}}$. The quantitative assertion of Theorem 3.2.1 follows easily from Theorem 3.3.5. □

3.4 Sign changes in arithmetic progression

In 1837, using the properties of L-function, Dirichlet proved a result on arithmetic progressions that states "there are infinitely many primes of the form $\{a + jm \mid j \in \mathbb{N}\}$ whenever $(a, m) = 1$". In order to generalize Dirichlet's result to a number field F, one must define the term "arithmetic progression" and "modulus". So one can ask what is an arithmetic progression for a number field F? Perhaps one could interpret it as a question about ideal class in the class group, but by this interpretation we miss the term "modulus". If we follow Dirichlet, we must replace m by an integral ideal $\mathfrak{m}$ of $\mathcal{O}_F$, and we must consider congruences modulo $\mathfrak{m}$. This naturally leads us to define a generalized ideal class group, called the *ray class group*.

Let I_F be the group of fractional ideals of F. Let $\mathfrak{m}$ be an integral ideal of $\mathcal{O}_F$. We define

$$I_F(\mathfrak{m}) := \{\mathfrak{a} \in I_F \;\big|\; (\mathfrak{a}, \mathfrak{m}) = 1\},$$

and

$$P_F^+(\mathfrak{m}) := \left\{ \left(\frac{\alpha}{\beta}\right) \mid \alpha, \beta \in \mathcal{O}_F \text{ prime to } \mathfrak{m}; \alpha \equiv \beta \pmod{\mathfrak{m}} \right\}.$$

The strict ray class group of F with modulus $\mathfrak{m}\mathfrak{m}_\infty$ is

$$\mathfrak{R}_{F,\mathfrak{m}}^+ := I_F(\mathfrak{m})/P_F^+(\mathfrak{m}),$$

where $\mathfrak{m}_\infty$ is the formal product of all the real embeddings of F into $\mathbb{C}$. It is well known that $\mathfrak{R}^+_{F,\mathfrak{m}}$ is a finite abelian group. Let $h_\mathfrak{m} = |\mathfrak{R}^+_{F,\mathfrak{m}}|$, the cardinality of the ray class group modulo $\mathfrak{m}$. We now define the notion of an arithmetic progression. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in $\mathcal{O}_F$, we say that $\mathfrak{a}$ and $\mathfrak{b}$ are in *arithmetic progression* modulo $\mathfrak{m}$ if $\mathfrak{a}$ and $\mathfrak{b}$ lie in same class in the ray class group for $\mathfrak{m}$, we shall denote this by $\mathfrak{a} \equiv \mathfrak{b} \pmod{\mathfrak{m}}$.
Let $\mathfrak{a}$ be an integral ideal of F and $[\mathfrak{a}]$ denotes its class in $\mathfrak{R}^+_{F,\mathfrak{m}}$. We define an indicator function $\delta_\mathfrak{a}$ on the group $I_F(\mathfrak{m})$ as follows

$$\delta_\mathfrak{a}(\mathfrak{b}) = \begin{cases} 1 & \mathfrak{a} \equiv \mathfrak{b} \pmod{\mathfrak{m}}, \\ 0 & \text{otherwise}. \end{cases} \tag{3.14}$$

Note that in a finite abelian group orthogonality of characters holds, and hence we get

$$\delta_\mathfrak{a}(\mathfrak{b}) = \frac{1}{|\mathfrak{R}^+_{F,\mathfrak{m}}|} \sum_{\Phi} \overline{\Phi([\mathfrak{a}])} \Phi([\mathfrak{b}]), \tag{3.15}$$

where the sum runs over all the Hecke characters Φ of $\mathfrak{R}^+_{F,\mathfrak{m}}$. Let $\mathbf{f} \in S_{\boldsymbol{k}}(\mathfrak{n}, \Psi)$ be a primitive form and Φ be a Hecke character of conductor $\mathfrak{m}$ coprime to level $\mathfrak{n}$. Then $\mathbf{f}|_\Phi$ is a primitive form in the space $S_{\boldsymbol{k}}(\mathfrak{n}\mathfrak{m}^2, \Psi\Phi^2)$ [see Theorem(1.3.6)]. Also, the relation between the Fourier coefficients $\mathbf{f}$ and $\mathbf{f}|_\Phi$ is given by

$$C(\mathbf{f}|_\Phi, \mathfrak{l}) = \Phi^*(\mathfrak{l}) C(\mathbf{f}, \mathfrak{l}),$$

where $\Phi^*(\mathfrak{l}) = 0$, whenever $(\mathfrak{l}, \mathfrak{m}) \neq 1$, and Φ^* is defined as

$$\Phi^*(\mathfrak{a}) = \begin{cases} \Phi([\mathfrak{a}]) & (\mathfrak{a}, \mathfrak{m}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Put $\mathbf{g}_\mathfrak{a} = \sum_\Phi \overline{\Phi([\mathfrak{a}])} \mathbf{f}|_\Phi$, now by (3.14) and (3.15), it is clear that the Fourier coefficients of $\mathbf{g}_\mathfrak{a}$ has the following property

$$C(\mathbf{g}_\mathfrak{a}, \mathfrak{l}) = \begin{cases} C(\mathbf{f}, \mathfrak{l}) & [\mathfrak{l}] = [\mathfrak{a}] \text{ inside } \mathfrak{R}^+_{K,\mathfrak{m}} \\ 0 & \text{otherwise} \end{cases},$$

for all integral ideals $\mathfrak{l}$ of $\mathcal{O}_K$. Consider the L-function associated to $\mathbf{g}_\mathfrak{a}$, we see that

$$L(s, \mathbf{g}_\mathfrak{a}) = \sum_\Phi \overline{\Phi([\mathfrak{a}])} L(s, \mathbf{f}|_\Phi).$$

Initially the above inequality holds for some half-plane, and by analytic continuation equality holds for the entire complex plane.

3.4.1 Two lemmas

To obtain the sign changes in arithmetic progression, we need the following two lemmas.

Lemma 3.4.1 *The function $L(s, \mathbf{g}_\mathfrak{a})$ has an analytic continuation to the whole complex plane.*

Proof. By Theorem 1.3.6, the function $L(s, \mathbf{f}|_\Phi)$ has an analytic continuation to the whole complex plane for all Hecke characters Φ whenever $\mathbf{f}$ is a primitive cusp form, and hence the finite sum. □

Next lemma claims that even $L(s, \mathbf{g}_{\mathfrak{a}})$ has an analytic continuation to the whole complex plane, but its abscissa of convergence is finite.

Lemma 3.4.2 *If $\mathbf{g}_{\mathfrak{a}} \neq 0$, then $L(s, \mathbf{g}_{\mathfrak{a}})$ has finite abscissa of convergence.*

Proof. Let $\{t_1, t_2, \ldots, t_{h_{\mathfrak{m}}}\}$ be a subset of $\mathbb{I}_K$ such that the set $\{t_1\mathcal{O}_K, t_2\mathcal{O}_K, \ldots, t_{h_{\mathfrak{m}}}\mathcal{O}_K\}$ forms a set of representative for the ray class group $\mathfrak{R}^+_{K,\mathfrak{m}}$. By abuse of notation we write t_λ for $t_\lambda\mathcal{O}_K$. Since our main goal is to study the sign changes in the class $[\mathfrak{a}]$, without loss of generality we assume that $[t_1] = [\mathfrak{a}]$. The L-function associated to $\mathbf{g}_{\mathfrak{a}}$ is given by

$$L(s, \mathbf{g}_{\mathfrak{a}}) = \sum_{\substack{\mathfrak{l} \in [\mathfrak{a}] \\ \mathfrak{l} \subset \mathcal{O}_K}} \frac{C(\mathbf{f}, \mathfrak{l})}{N(\mathfrak{l})^s}.$$

To show that $L(s, \mathbf{g}_{\mathfrak{a}})$ has finite abscissa of convergence we will prove that $\sum_{\substack{\mathfrak{l} \in [\mathfrak{a}] \\ \mathfrak{l} \subset \mathcal{O}_K}} |C(\mathbf{f}, \mathfrak{l})|$ is not finite. Thereafter we argue that the sum is not finite. We now consider $[t_1]$ as an equivalence class in strict class group. From section 1.2.3 we note that if $\mathbf{f}|_\Phi \in S_{\boldsymbol{k}}(\mathfrak{n}\mathfrak{m}^2, \Psi\Phi^2)$, then by the adelic correspondence due to Shimura, we have an h-tuple $((f|_\Phi)_1, (f|_\Phi)_2, \ldots, (f|_\Phi)_h)$ of classical Hilbert modular forms such that

$$C(\mathbf{f}|_\Phi, \mathfrak{l}) = \begin{cases} N(\mathfrak{l})^{\frac{1}{2}} a_\lambda(\xi) \xi^{\frac{-\boldsymbol{k}}{2}} & \text{if } \mathfrak{l} = \xi \mathfrak{I}_\lambda^{-1} \subset \mathcal{O}_K \\ 0 & \text{otherwise} \end{cases}, \tag{3.16}$$

and

$$(f|_\Phi)_\lambda(z) = \sum_{0 \ll \xi \in \mathfrak{I}_\lambda} a_\lambda(\xi) \exp(2\pi i \operatorname{Tr}(\xi z)) \in S_{\boldsymbol{k}}(\Gamma(\mathfrak{I}_\lambda, \mathfrak{n}\mathfrak{m}^2), \psi\phi^2) \subset M_{\boldsymbol{k}}(\Gamma_{N(\mathfrak{n}\mathfrak{m}^2)}). \tag{3.17}$$

Let us consider the sum $\sum_{\Phi} \overline{\Phi[\mathfrak{a}]}(f|_{\Phi})_1$ which is a classical Hilbert modular form in the space $M_{\boldsymbol{k}}(\Gamma_{N(\mathfrak{n}\mathfrak{m}^2)})$. Now using (3.16) and (3.17), we have

$$\begin{aligned}
\sum_{\Phi} \overline{\Phi[\mathfrak{a}]}(f|_{\Phi})_1 &= \sum_{\Phi} \overline{\Phi[\mathfrak{a}]} \sum_{0 \ll \xi \in t_1} a_{\lambda}(\xi) \exp(2\pi i \operatorname{Tr}(\xi z)) \\
&= \sum_{\Phi} \overline{\Phi[\mathfrak{a}]} \sum_{\substack{\xi \in t_1 \\ \mathfrak{l} = \xi t_1^{-1} \subset \mathcal{O}_K}} C(\mathbf{f}|_{\Phi}, \mathfrak{l}) N(\mathfrak{l})^{\frac{-1}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)) \\
&= \sum_{\Phi} \overline{\Phi[\mathfrak{a}]} \sum_{\substack{\xi \in t_1 \\ \mathfrak{l} = \xi t_1^{-1} \subset \mathcal{O}_K}} \Phi^*(\mathfrak{l}) C(\mathbf{f}, \mathfrak{l}) N(\mathfrak{l})^{\frac{-1}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)) \\
&= \sum_{\substack{\xi \in t_1 \\ \mathfrak{l} = \xi t_1^{-1} \subset \mathcal{O}_K}} C(\mathbf{f}, \mathfrak{l}) N(\mathfrak{l})^{\frac{-1}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)) \sum_{\Phi} \overline{\Phi[\mathfrak{a}]} \Phi([\mathfrak{l}]) \\
&= \sum_{\substack{\xi \in t_1 \\ \mathfrak{l} = \xi t_1^{-1} \subset \mathcal{O}_K \\ (\mathfrak{l}, \mathfrak{m}) = 1}} C(\mathbf{f}, \mathfrak{l}) N(\mathfrak{l})^{\frac{-1}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)) \\
&= \sum_{\substack{\mathfrak{l} \in [\mathfrak{a}] \in \mathfrak{R}^{+}_{K,\mathfrak{m}} \\ \mathfrak{l} \subset \mathcal{O}_K}} C(\mathbf{f}, \mathfrak{l}) N(\xi t_1^{-1})^{\frac{-1}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \exp(2\pi i \operatorname{Tr}(\xi z)).
\end{aligned}$$

In the last three equalities we have used the following facts, $C(\mathbf{f}, \mathfrak{l}) = 0$ whenever $\mathfrak{l} \not\subseteq \mathcal{O}_K$ and

$$\sum_{\Phi} \overline{\Phi([\mathfrak{a}])} \Phi([\mathfrak{l}]) = \begin{cases} 1 & \mathfrak{l} \equiv \mathfrak{a} \pmod{\mathfrak{m}} \text{ and } (\mathfrak{l}, \mathfrak{m}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $N(\xi t_1^{-1})^{\frac{-1}{2}} \xi^{\frac{\boldsymbol{k}}{2}} = N(t_1^{-1})^{\frac{1}{2}} \xi^{\frac{(\boldsymbol{k}-\mathbf{1})}{2}}$, where $\mathbf{1} = (1, 1, \ldots, 1)$. Therefore the sum

$$\sum_{\substack{\mathfrak{l} \in [\mathfrak{a}] \in \mathfrak{R}^{+}_{K,\mathfrak{m}} \\ \mathfrak{l} \subset \mathcal{O}_K}} \left| C(\mathbf{f}, \mathfrak{l}) N(\xi t_1^{-1})^{\frac{-1}{2}} \xi^{\frac{\boldsymbol{k}}{2}} \right| = \sum_{\substack{\mathfrak{l} \in [\mathfrak{a}] \in \mathfrak{R}^{+}_{K,\mathfrak{m}} \\ \mathfrak{l} \subset \mathcal{O}_K}} \left| C(\mathbf{f}, \mathfrak{l}) N(t_1^{-1})^{\frac{1}{2}} \xi^{\frac{(\boldsymbol{k}-\mathbf{1})}{2}} \right|$$

$$\leq N(t_1^{-1})^{\frac{1}{2}} \sum_{\substack{\mathfrak{l}\in[\mathfrak{a}]\in\mathfrak{R}_{K,\mathfrak{m}}^{+} \\ \mathfrak{l}\subset\mathcal{O}_K}} \big|C(\mathbf{f},\mathfrak{l})\big|.$$

To show that the sum $\sum_{\substack{\mathfrak{l}\in[\mathfrak{a}]\in\mathfrak{R}_{K,\mathfrak{m}}^{+} \\ \mathfrak{l}\subset\mathcal{O}_K}} \big|C(\mathbf{f},\mathfrak{l})\big|$ is not finite it is enough to show that left hand side of the above inequality is not finite. Note that $\sum_{\Phi} \overline{\Phi[\mathfrak{a}]}(f|_{\Phi})_1$ is a classical Hilbert modular form and $C(\mathbf{f},\mathfrak{l})N(\xi t_1^{-1})^{\frac{-1}{2}}\xi^{\frac{\boldsymbol{k}}{2}}$ are its Fourier coefficients where $\xi \in t_1$. The next Proposition guarantees that the following sum

$$\sum_{\substack{\mathfrak{l}\in[\mathfrak{a}]\in\mathfrak{R}_{K,\mathfrak{m}}^{+} \\ \mathfrak{l}\subset\mathcal{O}_K}} \left|C(\mathbf{f},\mathfrak{l})N(\xi t_1^{-1})^{\frac{-1}{2}}\xi^{\frac{\boldsymbol{k}}{2}}\right|$$

is not finite. □

Proposition 3.4.1 *Let $g(\neq 0) \in M_{\boldsymbol{k}}(\Gamma_{N(\mathfrak{n}\mathfrak{m}^2)})$ be a classical Hilbert modular form with the Fourier expansion*

$$g(\boldsymbol{z}) = a(0) + \sum_{\xi\gg 0} a(\xi)\exp(2\pi i Tr(\xi\boldsymbol{z})).$$

Then the sum $\sum_{\substack{\xi\gg 0 \\ \xi=0}} |a(\xi)|$ is not finite.

Proof. We prove the proposition by the method of contradiction. If possible, let

$$\sum_{\substack{\xi\gg 0 \\ \xi=0}} |a(\xi)| < \infty.$$

Then there exist a natural number M such that

$$|g(\mathbf{z})| \leq M < \infty, \tag{3.18}$$

for all $\mathbf{z} \in \mathcal{H}^n$. Let $\kappa\mathbb{Z} = \mathfrak{n}\mathfrak{m}^2 \cap \mathbb{Z}$. We observe that for every n, $\gamma_n = \begin{pmatrix} 1 & 0 \\ n\kappa & 1 \end{pmatrix} \in \Gamma_{N(\mathfrak{n}\mathfrak{m}^2)}$, and

$$g(\gamma_n(\mathbf{z})) = (n\kappa\mathbf{z} + 1)^{\boldsymbol{k}} g(\mathbf{z}).$$

Now observing (3.18) we see that

$$|g(\mathbf{z})| = |n\kappa\mathbf{z} + 1|^{-\boldsymbol{k}} |g(\gamma(\mathbf{z}))| \leq M |n\kappa\mathbf{z} + 1|^{-\boldsymbol{k}}.$$

If we let $n \to \infty$, we see that $g \equiv 0$. This contradicts our assumption that g is non-zero. Therefore

$$\sum_{\substack{\xi \gg 0 \\ \xi = 0}} |a(\xi)| \to \infty.$$

This completes the proof. □

3.4.2 Proof of the Theorem 3.2.2

In order to prove Theorem 3.2.2 we first state the following well-known result due to Landau (see [38]).

Theorem 3.4.3 (Landau [38]) *Let $f(s) = \sum_{n \in \mathbb{N}} a_n n^{-s}$ be an absolutely convergent Dirichlet series on some half plane and suppose that $a_n \geq 0$ for all but finitely many n. Then either $f(s)$ is absolutely convergent everywhere or $f(s)$ has a singularity at its abscissa of convergence.*

We make use of the Theorem 3.4.3 for the Dirichlet series $L(s, \mathbf{g}_{\mathfrak{a}})$. From Lemma 3.4.1, $L(s, \mathbf{g}_{\mathfrak{a}})$ is entire and from Lemma 3.4.2, $L(s, \mathbf{g}_{\mathfrak{a}})$ is not absolutely convergent everywhere. Therefore $L(s, \mathbf{g}_{\mathfrak{a}})$ must not satisfy the hypothesis of the theorem. In other words, we conclude that $B'(n_1) < 0$ and $B'(n_2) > 0$ for infinitely many choices of $n_1, n_2 \in \mathbb{N}$, where we have defined $B'(\kappa)$ as

$$B'(\kappa) = \sum_{\substack{[\mathfrak{b}]=[\mathfrak{a}] \\ N(\mathfrak{b})=\kappa}} C(\mathfrak{b}, \mathbf{g}_{\mathfrak{a}}).$$

This completes the proof of Theorem 3.2.2. □

CHAPTER 4

Lambert series associated to Hilbert modular forms

In 1981, Zagier studied the Lambert series associated with the weight 12 cusp form Δ and conjectured that this Lambert series should have an asymptotic expansion in terms of the non-trivial zeros of the zeta function [50]. This conjecture was proven by Hafner and Stopple [20]. In 2017 and 2019, Chakraborty et al. [10], [13] and [14] established an asymptotic relation between Lambert series associated to any primitive cusp form (for the full modular group, congruence subgroup and also in the case of Maass forms) and the non-trivial zeros of the zeta function. This chapter investigates the Lambert series associated with primitive adelic Hilbert modular cusp form and establishes a similar kind of asymptotic expansion. The content of this chapter has been published in [1].

4.1 Introduction

Let $\Delta(z)$ be the unique cusp form of weight 12 on the full modular group $SL(2,\mathbb{Z})$. This cusp form has the following Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where $q = \exp(2\pi i z)$, z belongs to upper half-plane $\mathcal{H}$, and τ is the well known Ramanujan τ-function. In 1981, Zagier [50] studied the Lambert series $x^{12}\sum_{n=1}^{\infty} \tau^2(n)\exp(-nx)$ associated to $\Delta(z)$, where x is a positive real number and conjectured that $x^{12}\sum_{n=1}^{\infty} \tau^2(n)\exp(-nx)$ should have an asymptotic expansion in terms of the non-trivial zeros of $\zeta(s)$, the Riemann zeta function. In 2000, Hafner and Stopple [20] proved Zagier's conjecture assuming the Riemann hypothesis. In particular they proved.

Theorem 4.1.1 (Hafner and Stopple [20]) *If all the non-trivial zeros of $\zeta(s)$ are simple, then we have,*

$$\sum_{n=1}^{\infty} \tau^2(n)\exp(-ny) = 12\Gamma(11)y^{-12} + y^{-11-\frac{1}{4}} \sum_{\rho} y^{\frac{1}{4}-\frac{\rho}{2}} \Gamma\left(\frac{\rho}{2}+11\right) \frac{\zeta(\rho/2)}{\zeta'(\rho)} L(\rho/2) + O(y^{-11+\frac{1}{2}}),$$

as $y \to 0^+$.

Note that, if the Riemann hypothesis is true, then the term $y^{\frac{1}{4}-\frac{\rho}{2}}$ is purely oscillatory in nature and hence the above result verifies Zagier's conjecture.

It is natural to investigate whether Zagier's conjecture holds only for the Δ function, or one can extend it to any arbitrary normalized Hecke eigenforms on the full modular group. This was considered by Chakraborty et al. in a series of papers [13] and [14] where they extended the result of Hafner and Stopple to any arbitrary normalized Hecke eigenform for $SL(2,\mathbb{Z})$ and its congruence subgroups too. In the case of congruence subgroups, they established the following result.

Theorem 4.1.2 (Chakraborty et al. [14]) *Let N and k be positive integers and χ be a primitive Dirichlet character modulo N. Let $g \in S_k(\Gamma_0(N),\chi)$ be a normalized Hecke eigenform with Fourier expansion*

$$g = \sum_{n=1}^{\infty} a_g(n) q^n.$$

Assume the Riemann hypothesis holds and all the non-trivial zeros of $\zeta(s)$ are simple. Then for any positive real number $\alpha \to 0^+$

$$\sum_{n=1}^{\infty} |a_g(n)^2| \exp(-n\alpha) = R_1 + \mathcal{P}(\alpha) + (\alpha^{-k+1+\epsilon}),$$

where

$$R_1 = \begin{cases} \frac{\Gamma(k)\phi(N)D(k)}{NL(2,\chi_0^2)\alpha^k} & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0 \end{cases},$$

and

$$\mathcal{P}(\alpha) = \sum_{\rho} \frac{\Gamma(\rho/2 + k - 1) L(\rho/2, \chi) D(\rho/2 + k - 1)}{L'(\rho, \chi^2)\alpha^{\rho/2+k-1}},$$

where $\rho = x + iy$ is running through all the non-trivial zeros of $L(s,\chi^2)$. This sum involves bracketing the terms so that the terms for which

$$|y_1 - y_2| < \exp\left(-A\frac{y_1}{\log y_1}\right) + \exp\left(-A\frac{y_2}{\log y_2}\right),$$

where A is suitable positive constant, are included in the same bracket.

In 2019, Banerjee and Chakraborty [10] further studied Lambert series associated to a Maass cusp form and proved the following theorem.

Theorem 4.1.3 (Banerjee and Chakraborty [10]) *Let f and g be Maass cusp forms which are normalized Hecke eigen forms over the full modular group with Fourier coefficients $\lambda_f(n)$ and $\lambda_g(n)$, respectively. Assume the Riemann hypothesis holds and all the non-trivial zeros of $\zeta(s)$ are simple. Then for any positive real number α,*

$$\sum_{n=1}^{\infty} \lambda_f(n)\overline{\lambda_g(n)} \exp(-n\alpha) = \begin{cases} R_1 + \mathcal{P}(\alpha) + O(\alpha^{\epsilon}) & \text{if } f = g \\ \mathcal{P}(\alpha) + O(\alpha^{\epsilon}) & \text{if } f \neq g \end{cases},$$

and the residual terms are

$$R_1 = \frac{24}{\pi^2 \alpha} \sin(\pi/2(1 + 2ir))\langle f, f\rangle,$$

and

$$\mathcal{P}(\alpha) = \sum_{\rho} \frac{\Gamma(\rho/2)L(\rho/2, f \otimes g)}{\zeta'(\rho)y^{\rho/2}},$$

where $\rho = x + iy$ is running through the non-trivial zeros of the zeta function. This sum is decomposed into pieces so that the terms for which

$$|y_1 - y_2| < \exp\left(-A\frac{y_1}{\log y_1}\right) + \exp\left(-A\frac{y_2}{\log y_2}\right),$$

where A is suitable positive constant, are included in the same piece.

Some recent work on Lambert series is due to Maji et al. and can be found in [27], [28] in that they study Lambert series associated to the Möbius function and symmetric square L-function.

4.2 Main results

In this chapter our main goal is to prove the following theorem which can be thought of as a generalization of Theorem 4.1.2 and the analouge of Theorem 4.1.3 in the context of Hilbert modular forms on the full modular group.

Theorem 4.2.1 ([1]) *Let* $\mathbf{f}$ *and* $\mathbf{g} \in S_{\boldsymbol{k}}(\mathcal{O}_K)$ *be two primitive adelic Hilbert cusp forms. Let* $\zeta_K(s)$ *be the Dedekind zeta function associated to the totally real number field* K *of degree* d *over* $\mathbb{Q}$*. Suppose that all the non-trivial zeros of* $\zeta_K(s)$ *are simple. Then for positive real number* $x \to 0$*,*

$$\sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ \mathfrak{m} \neq \{0\}}} C(\mathbf{f}, \mathfrak{m}) C(\mathbf{g}, \mathfrak{m}) e^{-N(\mathfrak{m})x} = \begin{cases} C_2 + B(x) + O(x^{\epsilon}) & \text{if } \mathbf{f} = \mathbf{g}, \\ B(x) + O(x^{\epsilon}) & \text{if } \mathbf{f} \neq \mathbf{g}, \end{cases}$$

where

$$C_2 = \frac{2^{d-1}(4\pi)^{\boldsymbol{k}} \Gamma(\boldsymbol{k})^{-1} R_K [\mathcal{O}_K^{\times +} : \mathcal{O}_K^{\times 2}]^{-1} \langle \mathbf{f}, \mathbf{f} \rangle}{x}$$

and

$$B(x) = \sum_{\rho} \Gamma\left(\frac{\rho}{2}\right) \frac{L(\frac{\rho}{2}, \mathbf{f} \otimes \mathbf{g})}{2\zeta_K'(\rho) x^{\frac{\rho}{2}}}$$

where $\rho = \sigma + it$ *is running through all the non-trivial zeros of* $\zeta_K(s)$*.*

An immediate corollary of the above theorem is as follows.

Corollary 4.2.2 ([1]) *Let* $\mathbf{f}, \mathbf{g}$ *and* $\zeta_K(s)$ *as in above Theorem 4.2.1. Assume the GRH and that all the non-trivial zeros of* $\zeta_K(s)$ *are simple. Then for positive real number* $x \to 0$*,*

$$\sum_{\substack{\mathfrak{m} \subset \mathcal{O}_K \\ \mathfrak{m} \neq \{0\}}} C(\mathbf{f}, \mathfrak{m}) C(\mathbf{g}, \mathfrak{m}) e^{-N(\mathfrak{m})x} = \begin{cases} C_2 + B(x) + O(x^{\epsilon}) & \text{if } \mathbf{f} = \mathbf{g}, \\ B(x) + O(x^{\epsilon}) & \text{if } \mathbf{f} \neq \mathbf{g}, \end{cases}$$

where

$$C_2 = \frac{2^{d-1}(4\pi)^{\boldsymbol{k}}\Gamma(\boldsymbol{k})^{-1}R_K[\mathcal{O}_K^{\times +} : \mathcal{O}_K^{\times 2}]^{-1}\langle \mathbf{f}, \mathbf{f}\rangle}{x}$$

and

$$B(x) = 2x^{-\frac{1}{4}} \sum_{\substack{\rho=\frac{1}{2}+it_n \\ t_n>0}} r_n \cos(\theta_n - \frac{t_n}{2}\log(x)),$$

with $r_n e^{i\theta_n} = \Gamma(\frac{\rho_n}{2})\frac{L(\frac{\rho_n}{2},\mathbf{f}\otimes\mathbf{g})}{2\zeta_F'(\rho_n)}$, *and* $\rho_n = \frac{1}{2} + it_n$ *is the* n*-th non-trivial zero of the Dedekind zeta function* $\zeta_K(s)$.

4.3 Preliminaries

4.3.1 Dedekind zeta function $\zeta_F(s)$

Let F be a finite extension of $\mathbb{Q}$ of degree n. Let r_1, r_2 be the number of real and complex embeddings of F into $\mathbb{C}$, respectively and $r_1 + 2r_2 = n$. The Dedekind zeta function associated with the number field F is defined as follows

$$\zeta_F(s) = \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_F \\ \mathfrak{m}\neq\{0\}}} \frac{1}{N(\mathfrak{m})^s},$$

where the sum runs over all the integral ideals $\mathfrak{m}$ of $\mathcal{O}_F$. Then $\zeta_F(s)$ converges absolutely for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to the whole complex plane except for a simple pole at $s = 1$. Set

$$\gamma_F(s) := \left(\frac{|D_F|}{2^{2r_2}\pi^n}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}, \tag{4.1}$$

where D_F denotes the discriminant of F over $\mathbb{Q}$. Define the completed

Dedekind zeta function $\Lambda_F(s)$ by

$$\Lambda_F(s) = \gamma_F(s)\zeta_F(s). \tag{4.2}$$

We have the following theorem, which establishes the meromorphic continuation by yielding a functional equation for the completed Dedekind zeta function [39].

Theorem 4.3.1 ([39]) *The completed Dedekind zeta function $\Lambda_F(s)$ has an analytic continuation to the whole complex plane as a meromorphic function and satisfies the following functional equation*

$$\Lambda_F(s) = \Lambda_F(1-s).$$

It has two poles one at $s = 0$ and the other at $s = 1$, both of which are simple. Residue of $\Lambda_F(s)$ at $s = 1$ is

$$\frac{2^{r_1} R_F |Cl_F|}{w_F},$$

where $R_F, |Cl_F|$ and w_F denote the regulator of F, cardinality of the class group of F and the number of roots of unity in F, respectively.

4.3.2 Bound on $\zeta_F(s)$

Lemma 4.3.2 *In the vertical strip $-1 \leq \sigma \leq 2$, we have*

$$\frac{1}{|\zeta_F(\sigma + iT)|} < e^{A_2 T}, \tag{4.3}$$

for some positive constant A_2.

Proof. The function $s \to s(s-1)\Lambda_F(s)$ is holomorphic on the whole complex plane and its zeros are the non-trivial zeros ρ of the Dedekind zeta function $\zeta_F(s)$.

Therefore by Weierstrass-Hadamard product formula, Λ_F can be expanded as,

$$s(s-1)\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\left(\frac{|D_F|}{2^{2r_2}\pi^n}\right)^{\frac{s}{2}}\zeta_F(s) = \exp(a+bs)\prod_{\rho}\left(1-\frac{s}{\rho}\right)\exp\left(\frac{s}{\rho}\right), \quad (4.4)$$

where a, b are constant. Now taking the logarithmic derivative of (4.4), we get

$$\frac{1}{s}+\frac{1}{s-1}+r_1\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+r_2\frac{\Gamma'(s)}{\Gamma(s)}+\log(C_F)+\frac{\zeta_F'(s)}{\zeta_F(s)} = b+\sum_{\rho}\frac{\frac{-1}{\rho}}{1-\frac{s}{\rho}}+\frac{1}{\rho},$$

and hence

$$\frac{\zeta_F'(s)}{\zeta_F(s)}+\frac{2s-1}{s(s-1)}+r_1\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+r_2\frac{\Gamma'(s)}{\Gamma(s)}+\log(C_F) = b+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$
$$\frac{\zeta_F'(s)}{\zeta_F(s)} = \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)+O(\log t). \quad (4.5)$$

Last equality follows from property of Γ-function. Let $N_F(T)$ be the number of zeros $\rho=\beta+i\gamma$ of $\zeta_F(s)$ in the critical strip $0\le\beta\le 1$ with $|\gamma|\le T$. For $T\ge 2$, from [26] we have,

$$N_F(T) = \frac{T}{\pi}\log\frac{|D_F|T^n}{(2\pi e)^n}+O(\log|D_F|T^n). \quad (4.6)$$

From (4.6) we see that

$$N_F(T+1)-N_F(T) = O(\log T). \quad (4.7)$$

Now using (4.5), (4.7) and following a similar method as in [page 217, [49]] we get

$$\log|\zeta_F(s)| \geq \sum_{|T-\gamma|\leq 1} \log|T-\gamma| + O(\log T).$$

Now let us take a sequence of positive real numbers $T \to \infty$ such that $|T-\gamma| > \exp(\frac{-A_1\gamma}{\log\gamma})$ for every ordinate γ of a zero of $\zeta_F(s)$, where A_1 is some positive constant. Then

$$\log|\zeta_F(\sigma+iT)| \geq -\sum_{|T-\gamma|\leq 1} \log|T-\gamma| + O(\log T) > A_2 T,$$

where $A_2 < \frac{\pi}{4}$ if A_1 is small enough and T large enough. This completes the proof. □

The next lemma is about the growth of the Γ-function in a vertical strip.

Lemma 4.3.3 (Stirling's formula [26]) *In a vertical strip $a_1 \leq \sigma \leq a_2$,*

$$|\Gamma(\sigma+iT)| = \sqrt{2\pi}|T|^{\sigma-\frac{1}{2}} e^{\frac{-1}{2\pi|T|}} \left(1 + O\left(\frac{1}{|T|}\right)\right)$$

as $|T| \to \infty$.

Lemma 4.3.4 *The function $L(s, \mathbf{f}\otimes\mathbf{g})$ is polynomially bounded in vertical strips $s = \sigma + it$ with $a_1 \leq \sigma \leq a_2$, $|t| \geq 1$. More precisely, given any $\epsilon > 0$ and for large enough T there exists a positive constant $A(\sigma)$ such that*

$$|L(\sigma+iT, \mathbf{f}\otimes\mathbf{g})| \ll |T|^{A(\sigma)+\epsilon}.$$

Proof. We refer [26] for proof. □

4.4 Proof of the Theorem 4.2.1

From the inverse Mellin transform of the Γ-function, we have the following identity

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)x^{-s}ds=\begin{cases}e^{-x} & \alpha>0,\\ e^{-x}-1 & -1<\alpha<0.\end{cases} \tag{4.8}$$

Let

$$L_1(s,\mathbf{f}\otimes\mathbf{g})=\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}}\frac{C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m})}{N(\mathfrak{m})^s}. \tag{4.9}$$

Take $\alpha>1$ and consider the following integral

$$\begin{aligned}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)x^{-s}L_1(s,\mathbf{f}\otimes\mathbf{g})ds &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)x^{-s}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}}\frac{C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m})}{N(\mathfrak{m})^s}ds\\ &= \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}}\frac{C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m})}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)(xN(\mathfrak{m}))^{-s}ds\\ &= \sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}}C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m})e^{-N(\mathfrak{m})x}\end{aligned}$$

By using (1.7), we get,

$$\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_K\\ \mathfrak{m}\neq\{0\}}}C(\mathbf{f},\mathfrak{m})C(\mathbf{g},\mathfrak{m})e^{-N(\mathfrak{m})x}=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds. \tag{4.10}$$

Now our aim is to compute the integral $\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds$. In order to do this, we consider a contour C oriented anti clock-wise, whose line segments are $[\alpha-iT,\alpha+iT],[\alpha+iT,\beta+iT],[\beta+iT,\beta-iT]$ and $[\beta-iT,\alpha-iT]$ for the real numbers $\beta<0,\alpha>1$ and for large enough positive real number T.

Consider the following integral I

$$I := \frac{1}{2\pi i}\int_C \Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds. \tag{4.11}$$

We break the proof into two cases namely $\mathbf{f}\neq\mathbf{g}$ and $\mathbf{f}=\mathbf{g}$.

Case 1. $\mathbf{f}\neq\mathbf{g}$

It is well known that all the non-trivial zeros of $\zeta_K(s)$ lie inside the vertical strip $0\leq \mathrm{Re(s)}\leq 1$. Consequently the integrand in (4.11) has infinitely many poles as ($T\to\infty$) with an additional pole at $s=0$, since $\zeta_K(0)=0$. Therefore by Cauchy residue Theorem equation (4.11) will become

$$\frac{1}{2\pi i}\int_C \Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds = C_1 + B(x), \tag{4.12}$$

where $C_1=\mathrm{Res}_{s=0}\left(\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}\right)$, and $B(x)$ is the residual function consisting of infinitely many terms contributed by the non-trivial zeros of $\zeta_K(2s)$ as $T\to\infty$. As

$$\int_C ds = \int_{\alpha-iT}^{\alpha+iT} ds + \int_{\alpha+iT}^{\beta+iT} ds + \int_{\beta+iT}^{\beta-iT} ds + \int_{\beta-iT}^{\alpha-iT} ds. \tag{4.13}$$

Put

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{\alpha+iT}^{\beta+iT} \Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds,\\
I_2 &= \frac{1}{2\pi i}\int_{\beta-iT}^{\alpha-iT} \Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds,\\
I_3 &= \frac{1}{2\pi i}\int_{\beta+iT}^{\beta-iT} \Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds.
\end{aligned}$$

Letting $T \to \infty$ in (4.13), we get

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds = -\lim_{T\to\infty}\Bigg(I_1+I_2+I_3+ \frac{1}{2\pi i}\int_C\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds\Bigg). \tag{4.14}$$

Using (4.12), the above equation (4.14) becomes,

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds = -\lim_{T\to\infty}\Bigg(I_1+I_2+I_3+C_1+B(x)\Bigg). \tag{4.15}$$

Now we compute each term of the right hand side of (4.15). We claim that $|I_1|$ and $|I_2| \to 0$ as $T \to \infty$. Note that

$$I_1 = \frac{1}{2\pi i}\int_{\alpha+iT}^{\beta+iT}\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds = \frac{1}{2\pi i}\int_{\alpha}^{\beta}\Gamma(\sigma+it)\frac{L(\sigma+iT,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2(\sigma+iT))}x^{-(\sigma+iT)}d\sigma$$

Using Lemma 4.3.2, 4.3.3 and 4.3.4, above equality becomes

$$|I_1| \ll |T|^A\exp\left(A_2T-\frac{1}{2}\pi|T|\right), \tag{4.16}$$

where A is constant. From Lemma 4.3.2, we set $A_2 < \frac{\pi}{4}$. Thus $|I_1| \to 0$ as $T \to \infty$. Similarly $|I_2| \to 0$ as $T \to \infty$. Let us compute I_3. To compute I_3, we will use the functional equation. Recall from (1.8) and (1.9),

$$\Lambda(s,\mathbf{f}\otimes\mathbf{g}) = \Lambda(1-s,\mathbf{f}\otimes\mathbf{g}),$$
$$N(\mathfrak{D}_K^2)^sL_\infty(s,\mathbf{f}\otimes\mathbf{g})L(s,\mathbf{f}\otimes\mathbf{g}) = N(\mathfrak{D}_K^2)^{1-s}L_\infty(1-s,\mathbf{f}\otimes\mathbf{g})L(1-s,\mathbf{f}\otimes\mathbf{g}). \tag{4.17}$$

Since the γ-factor of the $L(s,\mathbf{f}\otimes\mathbf{g})$ is given by

$$L_\infty(s,\mathbf{f}\otimes\mathbf{g}) = \prod_{j=1}^{d}(2\pi)^{-2s-k_j}\Gamma(s)\Gamma(s-1+k_j),$$

(4.17) reduces to the following identity,

$$L(s,\mathbf{f}\otimes\mathbf{g}) = \frac{N(\mathfrak{D}_K^2)^{1-2s}\prod_{j=1}^{d}(2\pi)^{-2(1-2s)}\prod_{j=1}^{d}\Big(\Gamma(1-s)\Gamma(-s+k_j)\Big)}{\prod_{j=1}^{d}\Big(\Gamma(s)\Gamma(s-1+k_j)\Big)}L(1-s,\mathbf{f}\otimes\mathbf{g}). \tag{4.18}$$

The completed Dedekind zeta function for a totally real field K is given by (4.2)

$$\Lambda_K = |D_K|^{\frac{s}{2}}\pi^{\frac{-sd}{2}}\Gamma\left(\frac{s}{2}\right)^d\zeta_K(s).$$

On applying the functional equation from Theorem 4.3.1, we have the following functional equation for the Dedekind zeta function $\zeta_K(s)$

$$\zeta_K(2s) = |D_K|^{\frac{1-4s}{2}}\pi^{\frac{4sd-d}{2}}\frac{\Gamma\left(\frac{1-2s}{2}\right)^d}{\Gamma(s)^d}\zeta_K(1-2s). \tag{4.19}$$

Now by a combined application of (4.18) and (4.19), we have

$$\frac{\Gamma(s)L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s} = N(\mathfrak{D}_K^2)^{1-2s}2^{-2d(1-2s)}\pi^{\frac{4sd-3d}{2}}|D_K|^{\frac{4s-1}{2}}B_\gamma(s)\frac{L(1-s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(1-2s)}x^{-s}, \tag{4.20}$$

where

$$B_\gamma(s) = \frac{\Gamma(s)\Gamma(1-s)^d\prod_{j=1}^{d}(\Gamma(-s+k_j))}{\Gamma\left(\frac{1-2s}{2}\right)^d\prod_{j=1}^{d}(\Gamma(s-1+k_j))}.$$

Now we compute $|I_3|$

$$\begin{aligned}|I_3| &= \left|\frac{-1}{2\pi i}\int_{\beta-iT}^{\beta+iT}\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds\right| \\ &\ll \int_{\beta-iT}^{\beta+iT}\left|B_\gamma(s)\frac{L(1-s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(1-2s)}x^{-s}ds\right| \\ &\ll \int_{\beta-iT}^{\beta+iT}\left|B_\gamma(s)\zeta_K(2-2s)\frac{L(1-s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2-2s)\zeta_K(1-2s)}x^{-s}ds\right|.\end{aligned}$$

Note that $L(s,\mathbf{f}\otimes\mathbf{g}) = \zeta_K(2s)L_1(s,\mathbf{f}\otimes\mathbf{g})$ [see eq.(1.7)], and hence the above inequality reduces to

$$\begin{aligned}|I_3| &\ll \int_{\beta-iT}^{\beta+iT}\left|B_\gamma(s)\zeta_K(2-2s)\frac{L_1(1-s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(1-2s)}x^{-s}ds\right| \\ &\ll \int_{\beta-iT}^{\beta+iT}\left|B_\gamma(s)x^{-s}ds\right|.\end{aligned}$$

To obtain the last inequality we use the fact that $\zeta_K(s)$ and $L_1(s,\mathbf{f}\otimes\mathbf{g})$ are absolutely convergent for $\mathrm{Re}(s) > 1$. Let $\omega = 1-s$. Since $\mathrm{Re}(s) < 0$, so we see that $\mathrm{Re}(\omega) > 1$. Thus

$$\begin{aligned}|I_3| \ll \int_{1-\beta-iT}^{1-\beta+iT}\left|B_\gamma(1-\omega)x^{\omega-1}d\omega\right| &\ll \int_{-T}^{T}\left|B_\gamma(1-1+\beta-it)x^{1-\beta+it-1}\right|dt \\ &\ll \int_{-T}^{T}\left|B_\gamma(\beta-it)x^{\beta+it}\right|dt.\end{aligned}$$

Now by Lemma 4.3.3 and letting $T\to\infty$, the above inequality becomes

$$\lim_{T\to\infty}|I_3| \ll \lim_{T\to\infty}\int_{-T}^{T}|t|^{1-\beta}e^{-\frac{1}{2}\pi|t|}x^{-\beta}dt \ll \Gamma(2-\beta)x^{-\beta}.$$

Thus $\lim_{T\to\infty}|I_3| = O(x^\epsilon)$ because we can take β to be an arbitrary small negative real number.

Now we calculate the quantities C_1 and $B(x)$.

From (4.12), we recall that $C_1 = \mathrm{Res}_{s=0}\Big(\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}\Big)$. It is not difficult to see that $C_1 \ll (\log(x))^r \ll x^\epsilon$, where r is the order of pole of $\Big(\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}\Big)$ at $s = 0$.

Let ρ be any arbitrary non-trivial zero of $\zeta_K(s)$ inside the verticle strip $0 \leq \mathrm{Re}(s) \leq 1$. If we assume the grand simplicity hypothesis for the non-trivial zeros of $\zeta_K(s)$, we get

$$
\begin{aligned}
B(x) = \sum_\rho \mathrm{Res}_{s=\frac{\rho}{2}}\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s} &= \sum_\rho \lim_{s\to\frac{\rho}{2}}\Big(s-\frac{\rho}{2}\Big)\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s} \\
&= \sum_\rho \Gamma\Big(\frac{\rho}{2}\Big)\frac{L(\frac{\rho}{2},\mathbf{f}\otimes\mathbf{g})}{2\zeta_K'(\rho)x^{\frac{\rho}{2}}}
\end{aligned}
$$

where sum runs over all the non-trivial zeros $\rho = \sigma + it$ of $\zeta_K(s)$, and the above sum over ρ is to be taken in the sense of $\lim_{T\to\infty}\sum_{|t|\leq T}$.

Case 2. $\mathbf{f} = \mathbf{g}$

If $\mathbf{f} = \mathbf{g}$, then the integrand in (4.11) has a simple pole at $s = 1$ contributed by $L(s,\mathbf{f}\otimes\mathbf{g})$. Due to the presence of the pole at $s = 1$ we have an extra term C_2 in (4.12). Therefore

$$
\frac{1}{2\pi i}\int_C \Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}ds = C_1 + C_2 + B(x) \tag{4.21}
$$

where

$$
C_2 = \mathrm{Res}_{s=1}\Big(\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}\Big) = \lim_{s\to 1}\Big((s-1)\Gamma(s)\frac{L(s,\mathbf{f}\otimes\mathbf{g})}{\zeta_K(2s)}x^{-s}\Big).
$$

Then by Theorem 1.3.7, we have,

$$
\begin{aligned}
C_2 &= \frac{2^{d-1}(4\pi)^{\boldsymbol{k}}\zeta_K(2)\Gamma(\boldsymbol{k})^{-1}R_K[\mathcal{O}_K^{\times +}:\mathcal{O}_K^{\times 2}]^{-1}\langle \mathbf{f},\mathbf{f}\rangle}{\zeta_K(2)x} \\
&= \frac{2^{d-1}(4\pi)^{\boldsymbol{k}}\Gamma(\boldsymbol{k})^{-1}R_K[\mathcal{O}_K^{\times +}:\mathcal{O}_K^{\times 2}]^{-1}\langle \mathbf{f},\mathbf{f}\rangle}{x}.
\end{aligned} \tag{4.22}
$$

This completes the proof. □

4.5 Proof of the Corollary 4.2.2

If we assume GRH, then all the non-trivial zeros of Dedekind zeta function $\zeta_K(s)$ lie on the verticle line $\mathrm{Re}(s) = \frac{1}{2}$. From the functional equation for $\zeta_K(s)$, we notice that if $\rho_n = \frac{1}{2} + it_n$ is n-th non-trivial zero of $\zeta_K(s)$, then $\frac{1}{2} - it_n$ is also a zero of $\zeta_K(s)$. Thus $B(x)$ will become

$$
B(x) = \sum_{\substack{\rho=\frac{1}{2}+it_n \\ t_n>0}} 2\mathrm{Re}\Big(\Gamma\left(\frac{\rho}{2}\right)\frac{L(\frac{\rho}{2},\mathbf{f}\otimes\mathbf{g})}{2\zeta_K'(\rho)x^{\frac{\rho}{2}}}\Big).
$$

Let $r_n e^{i\theta_n}$ be the polar representation of the complex number $\Gamma(\frac{\rho_n}{2})\frac{L(\frac{\rho_n}{2},\mathbf{f}\otimes\mathbf{g})}{2\zeta_K'(\rho_n)}$. Then

$$
\begin{aligned}
B(x) = \sum_{\substack{\rho=\frac{1}{2}+it_n \\ t_n>0}} 2\mathrm{Re}(r_n e^{i\theta_n} x^{-\frac{1}{4}-i\frac{t_n}{2}}) &= 2x^{-\frac{1}{4}} \sum_{\substack{\rho=\frac{1}{2}+it_n \\ t_n>0}} \mathrm{Re}\big(r_n e^{i\theta_n} e^{-i\frac{t_n}{2}\log(x)}\big) \\
&= 2x^{-\frac{1}{4}} \sum_{\substack{\rho=\frac{1}{2}+it_n \\ t_n>0}} r_n \cos\left(\theta_n - \frac{t_n}{2}\log(x)\right).
\end{aligned}
$$

This completes the proof of the corollary. □

Bibliography

[1] R. Agnihotri, *Lambert series associated to Hilbert modular form*, Int. J. of Number Theory, To appear.

[2] R. Agnihotri and K. Chakraborty, *On the Fourier coefficients of certain Hilbert modular forms*, Ramanujan J. (2021), To appear.

[3] R. Agnihotri, K. Chakraborty and K. Krishnamoorthy, *Sign change in restricted coefficients of Hilbert modular forms*, submitted. https://www.researchgate.net/publication/348590420

[4] P. Anamby and S. Das, *Sturm-like bound for square-free Fourier coefficients*, L-functions and automorphic forms, 1-7, Contrib. Math. Comput. Sci., **10** (2017), Springer, Cham.

[5] P. Anamby and S. Das, *Hecke-Siegel type threshold for square-free Fourier coefficients: an improvement*, Analytic and Arithmetic Theory of Automorphic Forms. Vol. **2100** (2019), RIMS Kokyuroku, pp. 35-42.

[6] T. M. Apostol, Introduction to Analytic Number Theory, Undergraduate Texts in Mathematics, *Springer-Verlag,* New York-Heidelberg, 1976.

[7] S. Baba, K. Chakraborty and Y. N, Petridis, *On the number of Fourier coefficients that determine a Hilbert modular for*, Proc. Amer. Math. Soc. **130:9** (2002), 2497-2502.

[8] A. Balog and K. Ono, *The Chebotarev density theorem in short intervals and some questions of Serre*, J. Number Theory, **91** (2), (2001), 356-371.

[9] S. Banerjee, *A note on signs of Fourier coefficients of two cusp forms*, Proc. Indian Acad. Sci. Math. Sci., **128** (2018), no 4, Paper No. 43, 6pp.

[10] S. Banerjee and K. Chakraborty, *Asymptotic behaviour of a Lambert series à la Zagier: Maass case,* Ramanujan J. **48** (2019), no. 3, 567-575.

[11] D. Blasius, *Hilbert modular forms and Ramanujan conjecture*, Non-commutative geometry and number theory, Aspects math., E37, Vieweg, Wiesbaden, (2006), pp. 35-56.

[12] D. Bump, Automorphic forms and representations, Cambridge Studies in Advanced Mathematics, **vol. 55**, *Cambridge University Press*, Cambridge, 1997.

[13] K. Chakraborty, A. Juyal, S. D. Kumar and B. Maji, *An asymptotic expansion of a Lambert series associated to cusp forms.* Int. J. Number Theory **14** (2018), no 1, 289-299.

[14] K. Chakraborty, S. Kanemitsu and B. Maji, *Modular-type relations associated to the Rankin-Selberg L-function.* Ramanujan J. **42** (2017), 285-299.

[15] N. Childress, Class field theory, Universitext, *Springer*, New York, 2009.

[16] P. Deligne, *La Conjecture de Weil, I*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273-307.

[17] E. Freitag, Siegelsche Modulfunktionen. Grundl, Math. Wiss., **254** (1983), *Springer-Verlag.*

[18] P. B. Garrett, Holomorphic Hilbert Modular Forms, Wadsworth and Brooks Cole Advanced Books and Software, *Pacific Grove*, 1990.

[19] S. Gun, W. Kohnen and P. Rath, *Simultaneous sign change of Fourier coefficients of two cusp forms*, Arch. Math. (Basel) **105** (5) (2015), 413-424.

[20] J. Hafner and J. Stopple, *A heat kernel associated to Ramanujan's tau function*, Ramanujan J. **4** (2000), 123-128 .

[21] A. Hamieh and N. Tanabe, *Determining Hilbert modular forms by central values of Rankin-Selberg convolutions: The weight aspect*, Ramanujan J. **45(3)** (2018), 615-637.

[22] G. Harcos, *Uniform apporximate functional equation for principal L-function*, Int. Math. Res. Not. IMRN **18** (2002), 923-932.

[23] E. Hecke, *Mathematishe Werke*, herausgegeben im Auftrage der Akademie der Wisswnschaften (1970).

[24] J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegel zero, with an appendix by D. Goldfeld, J. Hoffstein, D. Lieman,* Ann. of Math (2), **140(1)** (1994), 161-181.

[25] T. A. Hulse, E. M. Kiral, C. I. Kuan and L. M. Lim, *The sign of Fourier coefficients of half-integral weight cusp forms*, Int. J. Number Theory **8** (2012), no. 3, 749-762.

[26] H. Iwaniec and E. Kowalski, Analytic Number Theory, vol. **53**, *American Mathematical Society Colloquium Publications*, Chicago, 2004.

[27] A. Juyal, B. Maji and S. Sathyanarayana, *An exact formula for a Lambert series associated to a cusp form and the Möbius function.* To appear in Ramanujan J.

[28] A. Juyal, B. Maji and S. Sathyanarayana, *An asymptotic expansion for a Lambert series associated to the symmetric square L-function*, arXiv:2105.07130, 2021.

[29] S. Kaushik and N. Kumar, *Simultaneous behaviour of the Fourier coefficients of two Hilbert modular cusp forms*, Arch. Math. (Basel) **112** (2019), no. 3, 241-248.

[30] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Graduate Texts in Mathematics, vol. **97**, *Springer-Verlag,* New York, 1984.

[31] W. Kohnen and J. Sengupta, *On the first sign change of Hecke eigenvalues of newforms*, Math. Z. **254** (2006), 173-184.

[32] W. Kohnen and Y. Martin, *Sign Changes of Fourier Coefficients of Cusp forms supported on prime power indices*, Int. J. of Number Theory **10** (8) (2014), 1921-1927.

[33] E. Kowalski, Y. K. Lau, K. Soundararajan and J. Wu, *On modular signs,* Math. Proc. Cambridge Philos. Soc. **149** (2010), no. 3, 389-411.

[34] J. Meher and M. R. Murty, *Sign Change of Fourier coefficients of half-integral weight cusp forms*, Int. J. Number Theory **10** (04) (2014), 905-914.

[35] J. Meher, K. Shankhadhar and G. K. Viswanadham, *A Short Note On Sign Changes*, Proc. Indian Acad. Sci. Math. Sci. **123** (2013), 315-320.

[36] J. Meher and N. Tanabe, *Sign changes of Fourier coefficients of Hilbert modular forms*, J. Number Theory **145** (2014), 230-244.

[37] M. R. Murty, *Oscillations of Fourier coefficients of Modular forms*, Math. Ann. **262** (4) (1983), 431-446.

[38] M. R. Murty, Problems in Analytic Number Theory, second ed., Graduate Texts in Mathematics, vol. **206**, *Springer*, New York, 2008.

[39] M. Overholt, A course in analytic number theory, Graduate Studies in Mathematics, **160**, *American Mathematical Society, Providence,* RI, 2014.

[40] R. Pal, *On the signs of Fourier coefficients of Hilbert cusp forms*, Ramanujan J. **53** (2020), no. 2, 467-481.

[41] Y. Qu, *Linnik-type problems for automorphic L-function*, J. Number Theory **130** (2) (2010), 786-802.

[42] D. Ramakrishnan, *A refinement of the strong multiplicity one theorem for* $GL(2)$, Invent. Math. **116.1** (1994), 645-649.

[43] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* $SL(2)$, Ann. of Math. (2) **152** (2000), no. 1, 45-111.

[44] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Mathematical Journal, **45** (3) (1978), 637-679.

[45] G. Shimura, *The arithmetic of certain zeta functions and automorphic forms on orthogonal groups*, Ann. of Math **111** (1980), 313-375.

[46] J. Shurman, *Hecke characters classically and idelically.*

[47] R. Thomas Shemanske and H. Lynne Walling, *Twists of Hilbert modular forms*, Trans. Amer. Math. Soc. **338** (1993), no. 1, 375-403.

[48] Y. Takai, *An analogue of Sturm's theorem for Hilbert modular forms*, Algebra Number Theory **7** (2013), no. 4, 1001-1018.

[49] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed. Oxford: *Oxford University Press,* 1986.

[50] D. Zagier, *The Rankin-Selberg method for automorphic functions which are not of rapid decay.* J. Fac. Sci. Univ. Tokyo IA Math. **28** (1981), 415-437.